Default and Aggregate Income*

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Abstract

This paper studies how default varies with aggregate income. We analyze a model in which optimal contracts enable risk sharing of privately observed, idiosyncratic income by allowing for default. Default provisions allow agents with low idiosyncratic income realizations to repay less and thus provide insurance. Default penalties ensure that only these agents default. We show that default can occur under the optimal contract and that default provisions vary with aggregate income. We provide conditions such that both the amount of default and default penalties vary countercyclically with aggregate income and show that the default rate can be discontinuous. *Journal of Economic Literature* Classification Numbers: D82, E32, E44, G33. Keywords: Default, Asymmetric Information, Aggregate Fluctuations, Optimal Contracts.

1 Introduction

How does default vary with aggregate income? We analyze a model in which optimal contracts enable risk sharing of privately observed, idiosyncratic income by allowing default. We show that under natural assumptions about preferences, both the fraction of agents defaulting and the amount that defaulting agents default by vary countercyclically with aggregate income. Importantly, this is despite the fact that we allow contracts to be fully contingent on aggregate income and thus the countercyclical nature of default is not due to a lack of indexation.

In the model, default is a costly way of providing insurance contingencies. Default provisions provide insurance by allowing agents with low idiosyncratic income to repay less. Default is costly since to ensure that only agents with low idiosyncratic income default, default penalties need to be imposed. We show that, if agents preferences exhibit decreasing absolute risk aversion, the return to providing insurance increases by more when aggregate income is low than the cost in terms of default penalties. Thus, more default contingencies are optimally provided and there is more default in bad times. This also means that more default penalties have to be imposed in bad times. If one were to look only at default penalties, it would seem that default exacerbates aggregate shocks. The model hence predicts that a significant contraction in real activity, such as the great depression, might result in a large amount of foreclosure on mortgages, despite the fact that this seems to make matters worse.

In addition, we find that the default rate can be discontinuous in aggregate income. If aggregate income falls sufficiently for default to arise, the default rate can jump from no default to a strictly positive fraction of agents defaulting. Thus, the fraction of agents defaulting under optimal contracts can be sensitive to aggregate conditions. Furthermore, we show that whether an agent defaults or not can depend on aggregate income even when controlling for the agent's income. Notice that these effects are induced by the incentive compatibility constraints and we do not assume that there are any direct linkages between agents in the model.[1]

This paper studies a one period endowment economy with ex ante identical, risk averse agents whose stochastic consumption good endowments are the sum of a privately observed idiosyncratic component and a publicly observed aggregate component. In addition to their consumption good endowment agents are endowed with

[1]For models in which the default rate is sensitive to aggregate conditions due to direct linkages between agents see Rochet and Tirole (1996) and Kiyotaki and Moore (1997).

one unit of an agent-specific good, i.e., a good that is worth more to the specific agent than to any other agent, which will be used to impose default penalties. We refer to the agent-specific good as leisure in the model but several other interpretations are possible. In particular, the agent-specific good can be thought of as the indirect utility derived from participation in financial markets. Default penalties are then imposed by excluding agents from financial markets.[2,3] For simplicity, the value of the agent-specific good to other agents is normalized to zero, i.e., the agent-specific good is only valued by the agent who is endowed with it.

Because agents' consumption good endowments are not observable, contracts need to induce truth-telling which implies the following structure for an optimal contract: Each agent can choose to either pay an amount specified by the contract or default, in which case he incurs default penalties according to how much his payment falls short of the specified amount. The default penalties are imposed by seizing an agent's asset, i.e., his agent-specific good, or parts thereof.[4] In a sense, the agent-specific good plays the role of collateral in the optimal contract. Since additional consumption goods are worth less to the agents with high idiosyncratic endowments, they will choose to pay rather than forfeit their asset while agents with low idiosyncratic endowments will default. Default contingencies are thus a risk sharing mechanism and default penalties are used as a revelation device.

For given aggregate income, the optimal contract specifies an amount that agents need to pay to not be penalized. Contracts are debt like since a non-trivial subset of agents with high endowment realizations pay the same amount. Penalties make providing contingencies costly and hence it turns out not to be optimal to distinguish between agents with different high endowments. Thus, debt like contracts arise endogenously as in Townsend (1979) and Gale and Hellwig (1985) but for a different reason. In addition, it turns out that the optimal contract does not distinguish between agents with very low endowment realizations either and thus there is a

[2]See the comparison to models à la Kehoe and Levine (1993) below.

[3]Other interpretations of the agent-specific good include any collateral good which is at least partially specific to the current owner, such as the assets of a business, consumer durables, or houses, and social capital which can be destroyed by stigmatization.

[4]Models with non-pecuniary penalties for default include Diamond (1984), Rea (1984), who considers contracts involving 'arm-breaking,' Zame (1993), Welch (1999), who studies debtors' prison, Lacker (2001), who studies collateral which the borrower values more, Dubey, Geanakoplos, and Shubik (2002), and Geanakoplos and Zame (2002).

second range where contracts are not contingent on idiosyncratic income. Agents with income in this range all default by the same amount which implies that there is an endogenous limit on the amount that agents can default by.[5]

The main result of the paper is that if agents have decreasing absolute risk aversion the fraction of agents defaulting and the amount by which defaulting agents default is decreasing in the aggregate state. The benefits from providing insurance of consumption risk increase by more in bad times than the costs in terms of default penalties, since the dispersion of marginal utilities increases and, intuitively, the benefits scale with the marginal utility of the agents with low endowments while the costs scale with the marginal utility of agents with high endowments. This implies that optimally agents are allowed to default by more and moreover a larger fraction of agents is allowed to default. However, to keep agents with high idiosyncratic endowments from defaulting, default penalties have to increase in bad times as well. There are two reasons for this: First, in bad times all agents are relatively worse off and hence higher penalties are required to keep agents with high realizations from defaulting. Second, agents are allowed to default by more, which means that penalties have to be higher to compensate for this, too. In terms of default penalties then it would seem that default makes bad times worse. Optimal contracts are thus on the one hand tougher in bad times because of higher penalties but are on the other hand more lenient by allowing more agents to default and allowing agents to default by more. Importantly, the increase of default in bad times obtains despite the fact that we allow for contracting that is fully contingent on aggregate conditions and is not due to insufficient renegotiation of contracts which are not contingent on aggregates. Finally, if default penalties are imposed by excluding agents from financial markets, then the intuition from this model implies that more agents could be excluded from financial markets for a longer period of time when times are bad. This could make aggregate shocks more persistent.

Default as modeled here is an insurance device of last resort. Default allows risk sharing but comes at a first order cost since default penalties must be imposed by seizing agent-specific goods. If the agent-specific good or 'collateral' were valued also by other agents albeit less than by the original owner, the cost of insurance would be reduced. This may mitigate the effects discussed in this paper, but they would

[5] Formally, the non-negativity constraint on penalties is binding for high types and the monotonicity constraint corresponding to the second-order condition for truth-telling is binding for low types in our mechanism design problem, i.e., there is 'bunching.'

not disappear.

This paper is related to the extensive literature on optimal risk sharing with private information in dynamic environments where intertemporal tie-ins are used to provide incentives (see, e.g., Townsend (1982), Green (1987), Thomas and Worrall (1990), and Atkeson and Lucas (1993, 1995)). In contrast, we study optimal risk sharing with private information in a one period environment where the only way to provide incentives is by imposing penalties which implies that there is a first order cost of providing incentives. The cost of this approach is that we have to restrict attention to a one period problem. The benefit is that we can accommodate stochastic aggregate income and a continuum of idiosyncratic states. Moreover, the absence of intertemporal mechanisms for incentive provision leads to the empirically relevant implication that default occurs and default penalties are imposed in equilibrium.

The paper is also related to the literature on environments with limited commitment (see, e.g., Kehoe and Levine (1993)). In these models, endowments are observable but risk sharing is limited since there is an assumed limit to the harshest penalty for default, usually exclusion from financial markets. In these models, default does not occur in equilibrium. Here, risk sharing is limited because endowments are not observable, and limits to penalties arise endogenously since penalties must actually be imposed on agents with low endowments in equilibrium. However, one could interpret the second good in our model in the spirit of Kehoe and Levine (1993) as the indirect utility from access to financial markets and default penalties as the disutility due to exclusion from financial markets. Exclusion from financial markets would then occur in equilibrium.

In terms of methodology, we use standard tools from the literature on contract theory with incomplete information (see, e.g., Guesnerie and Laffont (1984)). The application to an optimal risk sharing problem and general equilibrium is, to the best of our knowledge, new. Related problems have been studied by Mirrlees (1971) and Lollivier and Rochet (1983) in the literature on optimal taxation, however. Moreover, we provide a characterization of the optimal contract which is sufficient to obtain comparative statics results of the optimal contract and allow numerical computation of the solution.

The paper proceeds as follows: In Section 2 we describe the economic environment. In Section 3 we characterize the solution to the planning problem when agents' idiosyncratic endowment is a Bernoulli random variable, i.e., either high or low. We show that the amount that defaulting agents default by is countercyclical, but the fraction of agents who default is constant when there is default. In Section 4 the case of continuous idiosyncratic uncertainty is analyzed. We show that in this case both the fraction of agents defaulting and the amount that defaulting agents default by are countercyclical. We compute the optimal contract in several example economies numerically. In Section 5 we discuss how the optimal allocation can be decentralized by competitive financial intermediaries as well as several extensions. We conclude in Section 6.

2 A Model of Default

In this section we describe the model of default. There are two dates in the economy, date 0 and date 1. Date 0 is a contracting date only. At date 1 agents get their endowment, trade, and consume. There is a continuum of agents with unit mass. Each agent's utility from consumption c and leisure h is separable in the two goods and is given by

$$\mathcal{U}(c,h) = u(c) + v(h).$$

We assume that both u and v are strictly increasing, strictly concave, and continuously differentiable and that $\lim_{h \searrow 0} v(h) = -\infty$. Each agent's stochastic endowment e is the sum of an aggregate endowment component ω and an independent and identically distributed idiosyncratic endowment component θ, i.e.,

$$e = \omega + \theta,$$

where ω is distributed with cumulative distribution $F(\omega)$ over $\Omega = [\underline{\omega}, \bar{\omega}]$ and θ is distributed with cumulative distribution $G(\theta)$ over $\Theta = [\underline{\theta}, \bar{\theta}]$. The fixed endowment of leisure is H for all agents. Leisure is inalienable in the sense that it can not be transfered to other agents but an agent can, at most, be prevented from consuming it. Such reductions in leisure can be used to impose (utility) penalties on agents as we discuss below.

The information structure is as follows. The economic environment is common knowledge. The realized aggregate endowment component ω is perfectly observable by all agents. The realized idiosyncratic endowment component of each agent is private information. Agents observe θ and ω contemporaneously.

The sole aim of trading in the model is risk sharing. Since there is private information about the realization of the idiosyncratic endowment component trading can not be contingent on these realizations directly. However, trading can be contingent on announcements by the agents. By the revelation principle, we can restrict the space of announcements to the type space, i.e., Θ. Thus, the trading mechanisms that we are led to study are assignments of a net repayment of consumption goods r and a level of leisure h as a function of an agent's reported endowment (or type) $\hat{\theta}$ and the aggregate endowment ω, i.e.,

$$U(\hat{\theta}, \theta, \omega) = u(\omega + \theta - r(\hat{\theta}, \omega)) + v(h(\hat{\theta}, \omega)),$$

where θ is the agent's true type and $U(\hat{\theta}, \theta, \omega)$ is the indirect utility as a function of the reported type $\hat{\theta}$, the true type θ and the aggregate state ω.[6] Define the net loss of utility from leisure $t(\hat{\theta}, \omega)$, i.e., the penalty given $\hat{\theta}$ and ω, as

$$t(\hat{\theta}, \omega) \equiv v(H) - v(h(\hat{\theta}, \omega)).$$

Normalize $v(H) = 0$. We assume that $h(\hat{\theta}, \omega) \leq H$ and hence $t(\hat{\theta}, \omega) \geq 0$, $\forall \hat{\theta}, \omega \in \Theta \times \Omega$, i.e., it is not possible to increase an agent's allocation of leisure. In other words, while wasting an agent's time through a lot of red tape is feasible, making his days longer is not. Rewrite the agent's utility function as

$$U(\hat{\theta}, \theta, \omega) = u(\omega + \theta - r(\hat{\theta}, \omega)) - t(\hat{\theta}, \omega),$$

and define $U(\theta, \omega) \equiv U(\theta, \theta, \omega)$ as the indirect utility with truth-telling.

Thus we study the following program:

$$\max_{r(\theta,\omega), t(\theta,\omega)} \int_{\underline{\omega}}^{\bar{\omega}} \int_{\underline{\theta}}^{\bar{\theta}} U(\theta, \omega) dG(\theta) dF(\omega)$$

subject to

$$\theta \in \arg\max_{\hat{\theta} \in \Theta} \left\{ U(\hat{\theta}, \theta, \omega) \right\}, \quad \forall \theta, \omega \in \Theta \times \Omega,$$

$$\int_{\underline{\theta}}^{\bar{\theta}} r(\theta, \omega) dG(\theta) \geq 0, \quad \forall \omega \in \Omega,$$

and $t(\theta, \omega) \geq 0$, $\forall \theta, \omega \in \Theta \times \Omega$. Hence, we maximize the ex ante expected utility of each agent (or a representative agent, for that matter) subject to three sets of

[6] Notice that we restrict attention to ex post deterministic allocations. In the next section we show that this is without loss of generality in the case of idiosyncratic Bernoulli uncertainty (Lemma 1).

constraints: (i) a set of incentive or truth telling constraints $\forall \theta, \omega \in \Theta \times \Omega$, (ii) a set of resource constraints $\forall \omega \in \Omega$, and (iii) non-negativity constraints on penalties, $\forall \theta, \omega \in \Theta \times \Omega$. The resource constraints restrict the aggregate net repayment to be non-negative for each ω, i.e., resources can only be transferred between agents and cannot be produced.

We call a solution to this program an optimal contract or optimal trading arrangements. We will characterize the solution in Section 3 and 4. However, it seems appropriate to point out the following important property of the above program here: The program can be solved pointwise in ω. Preferences are separable across aggregate states and so are the resource and non-negativity constraints and, hence, there is no room for tie-ins across aggregate states.

Finally, notice that given the assumption of strict concavity, the utility function $u(\omega+\theta-r)$ satisfies the single crossing property, i.e., $\frac{\partial^2 u(\omega+\theta-r)}{\partial r \partial \theta} = -u''(\omega+\theta-r) > 0$. Furthermore, we make the crucial assumption that preferences exhibit decreasing absolute risk aversion throughout the paper:

Assumption 1 *The utility function u satisfies* $\frac{\partial}{\partial c}\left(-\frac{u''(c)}{u'(c)}\right) \leq 0$.

To simplify the analysis, we study a model in which resources are exchanged only at date 1 which allows us to focus on the risk sharing role of default provisions exclusively. Borrowing and lending are left implicit. However, we could explicitly allow for borrowing and lending in the model by assuming that agents above borrow at date 0 from a second class of agents, the lenders, and repay these lenders at date 1. The average repayment at date 1 would then be strictly positive, but the implications for optimal default provisions would be the same. The comparative statics of default would also be equivalent since the lenders would not absorb all the aggregate risk assuming they are risk averse. The model studied here can hence be interpreted as a model of *net* repayments at date 1, i.e., repayments net of the average repayment. In this spirit, we refer to the payments made by agents at date 1 as net repayments (r) throughout and interpret contingencies provided by the optimal contract as default contingencies.

3 Discrete Idiosyncratic Uncertainty

Here we analyze the planning problem of Section 2 for an economy where the idiosyncratic endowment component is a simple random variable with two realizations

$\theta_1 > \theta_0$ and probabilities p_1 and p_0, respectively. The main result in this case is that the amount that agents who default default by, i.e., the intensive margin of default, varies countercyclically with aggregate income. However, the fraction of agents who default, i.e., the extensive margin of default, varies with aggregate income only very coarsely. In particular, the fraction of agents defaulting is constant in the range where a positive fraction of agents default and is zero when aggregate income is sufficiently high. In the next section, we will see that in the case of continuous idiosyncratic uncertainty default varies countercyclically both at the extensive margin and the intensive margin, so both in terms of the fraction of agents defaulting and in terms of the amount that agents default by.

Let us introduce the following notation: $\mathcal{I} = \{0, 1\}$, $r_i(\omega) = r(\theta_i, \omega)$, $\forall i \in \mathcal{I}$, and $t_i(\omega) = t(\theta_i, \omega)$, $\forall i \in \mathcal{I}$. Since we can solve the planning problem pointwise in ω, we are led to study the following program for each ω:

$$\max_{\{r_i, t_i\}_{i \in \mathcal{I}}} \sum_{i \in \mathcal{I}} p_i (u(\omega + \theta_i - r_i) - t_i) \tag{1}$$

subject to

$$\begin{aligned} u(\omega + \theta_i - r_i) - t_i &\geq u(\omega + \theta_i - r_j) - t_j, \quad i \neq j, \quad \forall i, j \in \mathcal{I}, && (2) \\ \sum_{i \in \mathcal{I}} p_i r_i &\geq 0, && (3) \end{aligned}$$

and

$$t_i \geq 0, \quad \forall i \in \mathcal{I}, \tag{4}$$

where the dependence of the choice variables on ω is left implicit to conserve on notation.

Notice that we restrict attention to ex post deterministic allocations. The following lemma states that this is without loss of generality in the case considered here:

Lemma 1 *The optimal solution to the planning problem with idiosyncratic Bernoulli uncertainty is ex post deterministic.*

The proof of this lemma, as well as all other proofs, are in the appendix unless noted otherwise.

We can rearrange the incentive compatibility constraints to get

$$\begin{aligned} u(\omega + \theta_1 - r_1) - u(\omega + \theta_1 - r_0) &\geq t_1 - t_0 \\ &\geq u(\omega + \theta_0 - r_1) - u(\omega + \theta_0 - r_0). \end{aligned} \tag{5}$$

If $t_0 = t_1$ the first inequality implies that $r_0 \geq r_1$ and the second inequality implies that $r_0 \leq r_1$. Hence, $r_0 = r_1$ and no contingencies are provided. This implies that $t_0 \neq t_1$ whenever contingencies are provided. Rearranging the incentive compatibility constraints again we have

$$u(\omega + \theta_1 - r_1) - u(\omega + \theta_0 - r_1) \geq u(\omega + \theta_1 - r_0) - u(\omega + \theta_0 - r_0). \qquad (6)$$

Strict concavity of u together with equation (6) imply $r_0 \leq r_1$ whenever $\theta_0 \leq \theta_1$. The interpretation is that the net repayment schedule will effectively transfer resources from agents with high endowment realizations to agents with low endowment realizations, i.e., provide partial insurance. An immediate consequence of $r_0 \leq r_1$ is that $t_0 \geq t_1$. Furthermore, it can not be optimal to penalize the high type, i.e., $t_1 = 0$, since if $t_1 > 0$ the penalties for both types of agents could be reduced by t_1 without violating any of the constraints and resulting in an increase in the value of the objective. Throughout the paper we will interpret the net repayment associated with a zero penalty as the amount that an agent needs to repay and say that agents who choose this net repayment repay in full. We will say an agent 'defaults' whenever the net repayment that the agent chooses is less than the net repayment associated with zero penalty which will imply that the agent incurs a strictly positive penalty. Thus, in the model default contingencies allow some agents to repay less than the average repayment while others repay more which amounts to an implicit transfer from the agents who repay in full to the agents who default.

Note that it is possible that no contingencies are provided at all ($r_i = t_i = 0, \forall i \in \mathcal{I}$) since default as modeled here is a risk sharing technology with a first order cost. To see this consider increasing the allocation of the low type by $+\varepsilon/p_0$ and decreasing the allocation of the high type by $-\varepsilon/p_1$ which satisfies the resource constraint. The benefit from this redistribution is approximately $(u'(\omega + \theta_0) - u'(\omega + \theta_1))\,\varepsilon > 0$. However, to keep the high type from defaulting (i.e., to induce truth-telling) we require $u(\omega + \theta_1 - \varepsilon/p_1) = u(\omega + \theta_1 + \varepsilon/p_0) - t_0$ which implies that $t_0 \approx u'(\omega + \theta_1)\varepsilon/(p_0 p_1)$, i.e., a first order cost of risk sharing. Risk sharing occurs only if the marginal benefit exceeds the marginal cost, i.e., $u'(\omega + \theta_0) > (1 + \frac{1}{p_1})u'(\omega + \theta_1)$. The left hand side captures the marginal benefit, namely the increase in utility from consumption of the low types. The right hand side captures the marginal cost which has two components: The decrease in utility from consumption of the high types and the disutility from penalties imposed on the low types.

We assume in the following that contingencies are provided, i.e., that $u'(\omega+\theta_0) >$

$(1+\frac{1}{p_1})u'(\omega+\theta_1)$, and hence $r_0 < r_1$ and $t_0 > 0$. Then strict concavity of the utility function implies that the inequality in equation (6) is strict. In other words, at most one incentive compatibility constraint is binding. Hence, at most one of the inequalities in equation (5) is binding. Indeed, the first inequality in equation (5) must be binding since otherwise t_0 could be reduced leaving the constraints unaffected while improving the objective. Thus, if contingencies are provided we are left with the following program:

$$\max_{r_0,r_1,t_0,t_1} p_0(u(\omega+\theta_0-r_0)-t_0)+p_1(u(\omega+\theta_1-r_1)-t_1) \tag{7}$$

subject to

$$u(\omega+\theta_1-r_1)-t_1 \geq u(\omega+\theta_1-r_0)-t_0, \tag{8}$$

$$p_0r_0+p_1r_1 \geq 0 \tag{9}$$

and

$$t_1 \geq 0. \tag{10}$$

On inspection of this program we have the following result:

Proposition 1 *Assume u satisfies $\frac{\partial}{\partial c}\left(-\frac{u''(c)}{u'(c)}\right)=0$ and $u'(\omega+\theta_0) > (1+\frac{1}{p_1})u'(\omega+\theta_1)$. Then the net repayment schedule $\{r_i(\omega)\}_{i\in\mathcal{I}}$ is independent of ω and $t_0(\omega)$ is decreasing in ω.*

With constant absolute risk aversion the amount of default is thus independent of the aggregate state and default penalties are decreasing in the aggregate state.

To proceed, notice that the above program is not a convex program in general. We can however convert it into a convex program by changing variables as follows: Let $U_0 \equiv u(\omega+\theta_0-r_0)-t_0$ and $U_1 \equiv u(\omega+\theta_1-r_1)-t_1$, i.e., the indirect utilities from truth-telling given θ_0 and θ_1, respectively. By substituting indirect utilities for penalties the program in equations (7-10) can be rewritten as

$$\max_{r_0,r_1,U_0,U_1} p_0U_0+p_1U_1 \tag{11}$$

subject to

$$u(\omega+\theta_0-r_0)-u(\omega+\theta_1-r_0)+U_1-U_0 \geq 0, \tag{12}$$

$$p_0r_0+p_1r_1 \geq 0 \tag{13}$$

and

$$u(\omega + \theta_1 - r_1) - U_1 \geq 0. \tag{14}$$

Notice that the low type's incentive constraint and non-negativity constraint are slack and we can hence ignore them. Given Assumption 1 this is a convex program since the objective function is linear and the constraints define a convex set. The constraints are linear, and hence obviously convex, in U_0 and U_1. Furthermore, strict concavity of u implies that (14) defines a convex set and Assumption 1 implies that (12) defines a convex set.[7,8] Thus, the Kuhn-Tucker theorem gives necessary and sufficient conditions for optimality.

We can now prove the main result for utility functions with decreasing absolute risk aversion:

Proposition 2 *Assume u satisfies $\frac{\partial}{\partial c}\left(-\frac{u''(c)}{u'(c)}\right) < 0$ and $u'(\omega + \theta_0) > (1 + \frac{1}{p_1})u'(\omega + \theta_1)$. Then the net repayment $r_0(\omega)$ is increasing in ω and $t_0(\omega)$ is decreasing in ω.*

When aggregate income is low, the return to providing insurance is higher. However, the incentive to misreport is higher as well which makes insurance more costly since higher penalties are needed. Proposition 2 says that with decreasing absolute risk aversion the first effect dominates. The gains from insurance increase by more than the cost. The amount by which the types with low idiosyncratic endowment default is higher the lower the aggregate endowment. Thus, penalties imposed on the types with low endowments, i.e., the 'defaulting' agents, are higher when the aggregate endowment is lower for two reasons: First, the lower the aggregate endowment the higher default penalties need to be to keep the agents with high endowment from defaulting. Second, the lower the aggregate endowment the higher the amount agents with low endowment are allowed to default by, which requires yet higher penalties to keep the high endowment agents from defaulting. Interestingly, the optimal contract is more lenient in bad times since it allows agents to default by more but is at the same time harsher since it imposes higher default penalties.

[7] Let $\Delta \equiv u(c) - u(c + \theta_1 - \theta_0)$ and recall that $\theta_1 - \theta_0 > 0$. Then $\frac{\partial}{\partial c}\Delta = u'(c) - u'(c + \theta_1 - \theta_0) > 0$ and $\frac{\partial^2}{\partial c^2}\Delta = u''(c) - u''(c + \theta_1 - \theta_0) < 0$ since Assumption 1 implies that $u''' > 0$. Hence, Δ is strictly concave.

[8] Assumption 1 is thus crucial for the convexity of the set defined by the constraints and provides the rationale for restricting attention to (ex post) deterministic allocations. In contrast, when absolute risk aversion is increasing there can be a role for ex post stochastic allocations or 'lotteries' (see Prescott and Townsend (1984)).

To see intuitively that the benefits to insurance decrease faster than the costs as ω increases, consider the benefits and costs from providing insurance at all. Insurance is provided only if $u'(\omega+\theta_0) > (1+\frac{1}{p_1})u'(\omega+\theta_1)$. Notice that the marginal benefits on the left hand side scale with the marginal utility of the low type while the marginal costs on the right hand side, which include the disutility from penalties imposed on the low types, scale with the marginal utility of the high types. Penalties scale with the marginal utility of the high types since they are set to keep the high types from defaulting. To see how the benefits change relative to the costs, consider the ratio $\frac{u'(\omega+\theta_0)}{u'(\omega+\theta_1)}$ and notice that

$$\frac{\partial}{\partial\omega}\left(\frac{u'(\omega+\theta_0)}{u'(\omega+\theta_1)}\right) = \frac{u'(\omega+\theta_0)}{u'(\omega+\theta_1)}\left(-\frac{u''(\omega+\theta_1)}{u'(\omega+\theta_1)} - \left(-\frac{u''(\omega+\theta_0)}{u'(\omega+\theta_0)}\right)\right) < 0$$

given decreasing absolute risk aversion, which means that gains decrease faster than costs as ω increases. Furthermore, this implies that if ω is sufficiently high, no contingencies are provided.

4 Continuous Idiosyncratic Uncertainty

In this section we solve the planning problem for the economy with continuous idiosyncratic uncertainty. We will show that both the fraction of agents defaulting and the amount that agents default by vary countercyclically with aggregate income which is the main result of the paper. In addition, we show that the fraction of agents defaulting can be discontinuous in aggregate income: There is a level of aggregate income, such that above that level no agent defaults and below a non-negligible fraction of agents defaults.

We start by providing a characterization of the optimal contract which is sufficient to obtain comparative statics results as well as allow numerical computation of the solution. Since we can solve the problem pointwise in ω, we can rewrite the planning problem for given ω as follows:

$$\max_{r(\theta),t(\theta)} \int_{\underline{\theta}}^{\bar{\theta}} U(\theta)dG(\theta)$$

subject to

$$\theta \in \arg\max_{\hat{\theta}\in\Theta} \left\{U(\hat{\theta},\theta)\right\}, \quad \forall\theta\in\Theta,$$

$$\int_{\underline{\theta}}^{\bar{\theta}} r(\theta) dG(\theta) \geq 0,$$

and $t(\theta) \geq 0, \forall \theta \in \Theta$, where we simplified notation by suppressing the dependence on ω. The following definition of implementability and monotonicity result are standard.

Definition 1 *A net repayment schedule $r : \Theta \rightarrow \Re$ is implementable if there exists a non-negative penalty function t such that the allocation $(r(\theta), t(\theta))$ for $\theta \in \Theta$ satisfies the incentive compatibility constraint $\theta \in \arg\max_{\hat{\theta} \in \Theta} U(\hat{\theta}, \theta)$, $\forall \theta \in \Theta$.*

Lemma 2 *A necessary condition for r to be implementable is that it be non-decreasing in θ, i.e., $\dot{r}(\theta) \geq 0$ a.e.*

To preview the characterization results, we show below that the net repayment schedule has the following characterization: $\Theta = [\underline{\theta}, \bar{\theta}]$ is partitioned into three intervals: A lower interval $[\underline{\theta}, \tilde{\theta})$ on which the monotonicity constraint binds and the net repayment is constant, an intermediate interval $[\tilde{\theta}, \theta^*)$ on which the net repayment is contingent and an upper interval $[\theta^*, \bar{\theta}]$ where the non-negativity constraint binds and the net repayment is again constant. Agents on the lower and the intermediate interval default, with agents on the lower interval defaulting the maximum amount, while agents on the upper interval repay in full.

The next proposition shows that the non-negativity constraint is binding on an upper interval only.

Proposition 3 *The non-negativity constraint on the penalty function t is binding on an interval $[\theta^*, \bar{\theta}]$ and nowhere else.*

Furthermore, we have the following classic result of Mirrlees (1971).

Lemma 3 *An allocation (r, t) is implementable iff $\dot{U}(\theta) = u'(\omega + \theta - r(\theta))$, a.e., and $\dot{r}(\theta) \geq 0$, a.e.*

The proof is standard and is omitted. Note that the characterization lemma applies despite the presence of a non-negativity constraint.

For the rest of this section we make the following assumption about the distribution of idiosyncratic uncertainty, except where explicitly stated:

Assumption 2 *θ is uniformly distributed on $\Theta = [\underline{\theta}, \bar{\theta}]$.*

Furthermore, we restrict the analysis by assuming either constant absolute risk aversion or constant relative risk aversion to allow a characterization that is explicit enough to obtain comparative statics results and allow numerical computation.

Assumption 3 *u satisfies either $\frac{\partial}{\partial c}\left(-\frac{u''(c)}{u'(c)}\right) = 0$ or $\frac{\partial}{\partial c}\left(-\frac{u''(c)c}{u'(c)}\right) = 0$.*

We can now write the problem as follows: Let $v(\theta) = \dot{r}(\theta)$ be the control variable[9] and $r(\theta)$, $U(\theta)$ and $R(\theta) = \int_{\underline{\theta}}^{\theta} r(\vartheta)d\vartheta$ be the state variables. Then the control problem is

$$\max_{U(\theta),R(\theta),r(\theta),v(\theta)} \int_{\underline{\theta}}^{\bar{\theta}} U(\theta)d\theta$$

subject to

$$\begin{aligned} \dot{U}(\theta) &= u'(\omega + \theta - r(\theta)), \\ \dot{R}(\theta) &= r(\theta), \\ \dot{r}(\theta) &= v(\theta), \\ u(\omega + \theta - r(\theta)) - U(\theta) &\geq 0, \quad \forall \theta \in \Theta, \\ v(\theta) &\geq 0, \quad \forall \theta \in \Theta, \end{aligned}$$

and the boundary conditions $R(\underline{\theta}) = 0$ and $R(\bar{\theta}) \geq 0$. Note that the state variables $U(\theta)$ and $r(\theta)$ are free at both boundaries. $U(\theta)$ is free at both boundaries since we solve a planning problem and there are no participation constraints. $r(\theta)$ is free since we choose $\dot{r}(\theta)$ as the control to formally impose monotonicity as a constraint on the control. Neustadt (1976) provides necessary conditions for this type of problem and Seierstad and Sydsæter (1987) provide a Mangasarian (1966) type sufficiency result.[10] Kahn and Scheinkman (1985) solve a similar problem with linear laws of motion using Rockafellar's (1972) duality results for convex problems of Bolza.

The Hamiltonian for the above control problem, then, is

$$\mathcal{H} = U + \lambda u'(\omega + \theta - r) + \mu r + \eta v$$

where λ, μ and η are the costates for U, R and r respectively. The Lagrangian with multiplier functions ν and ζ is

$$\mathcal{L} = \mathcal{H} + \nu\left(u(\omega + \theta - r) - U\right) + \zeta v$$

[9] We choose $\dot{r}(\theta)$ as the control to characterize the solution since it turns out that the monotonicity constraint is binding in our model. See also Mussa and Rosen (1978), Lollivier and Rochet (1983), and Guesnerie and Laffont (1984).

[10] See Seierstad and Sydsæter (1987), section 6.7.

$$= \quad U + \lambda u'(\omega + \theta - r) + \mu r + \eta v + \nu\left(u(\omega + \theta - r) - U\right) + \zeta v.$$

Note that given Assumption 1, v, $u(\omega + \theta - r) - U$, $\mathcal{H}$, $R(\underline{\theta})$ and $R(\bar{\theta})$ are all differentiable and concave in (U, R, r, v) as long as $\lambda \leq 0$. Hence theorem 6.13 of Seierstad and Sydsæter (1987) implies that the following are sufficient conditions for optimality:

$$\begin{aligned}
0 &= \frac{\partial \mathcal{L}}{\partial v} = \eta + \zeta \\
\dot{\lambda} &= -\frac{\partial \mathcal{L}}{\partial U} = -1 + \nu \\
\dot{\mu} &= -\frac{\partial \mathcal{L}}{\partial R} = 0 \\
\dot{\eta} &= -\frac{\partial \mathcal{L}}{\partial r} = \lambda u''(\omega + \theta - r) - \mu + \nu u'(\omega + \theta - r) \\
\nu &\geq 0, \qquad \nu\left(u(\omega + \theta - r) - U\right) = 0 \\
\zeta &\geq 0, \qquad \zeta v = 0 \\
\lambda(\bar{\theta}) &\leq 0, \qquad \lambda(\bar{\theta})(u(\omega + \bar{\theta} - r(\bar{\theta})) - U(\bar{\theta})) = 0 \\
\mu(\bar{\theta}) &\geq 0, \qquad \mu(\bar{\theta})R(\bar{\theta}) = 0 \\
\eta(\bar{\theta}) &= \lambda(\bar{\theta})u'(\omega + \bar{\theta} - r(\bar{\theta})) \\
\lambda(\underline{\theta}) &\geq 0, \qquad \lambda(\underline{\theta})(u(\omega + \underline{\theta} - r(\underline{\theta})) - U(\underline{\theta})) = 0 \\
\eta(\underline{\theta}) &= \lambda(\underline{\theta})u'(\omega + \underline{\theta} - r(\underline{\theta})).
\end{aligned}$$

The law of motion for the costate μ implies that $\mu = \bar{\mu}$, i.e., μ is a constant. $\bar{\mu}$ can be interpreted as the shadow cost of resources in the economy. The differential equation for the costate λ implies that $\lambda(\theta) = \lambda(\underline{\theta}) - (\theta - \underline{\theta})$ for $\theta \in [\underline{\theta}, \theta^*]$. If the solution is non-trivial we have $u(\omega + \underline{\theta} - r(\underline{\theta})) - U(\underline{\theta}) > 0$ and hence $\lambda(\underline{\theta}) = 0$. Note also that $\bar{\mu} \geq 0$ and $\bar{\mu}R(\bar{\theta}) = 0$ from the boundary conditions. Assume that there is a neighborhood $\mathcal{N}$, say $\mathcal{N} = [\tilde{\theta}, \tilde{\theta} + \varepsilon]$, where neither the non-negativity nor the monotonicity constraint is binding. For $\theta \in \mathcal{N}$ we can then write the fourth necessary condition as

$$-(\theta - \underline{\theta})u''(\omega + \theta - r) = \bar{\mu}. \tag{15}$$

This is the crucial equation characterizing the optimal contract. The intuition is as follows. Consider a marginal increase in the allocation of type θ which would raise his utility from consumption by $u'(\omega + \theta - r)$. But to keep type $\theta + d\theta$ from misreporting, the penalty on type θ would have to be increased by $u'(\omega + \theta + d\theta - r)$. Thus, type θ's utility would increase by $u'(\omega + \theta - r) - u'(\omega + \theta + d\theta - r)$ or, taking $d\theta$ to zero, by $-u''(\omega+\theta-r)$. An increase in the indirect utility of type θ in turn would allow an equivalent reduction of the penalties for all types lower than θ, which implies an overall increase in indirect utility of $-(\theta-\underline{\theta})u''(\omega+\theta-r)$. At an optimum, this return has to equal the resource cost $\bar{\mu}$.

Assumption 1 implies that $u''' \geq 0$ and hence u'' is monotone. We can thus solve equation (15) for the net repayment schedule on $\mathcal{N}$:

$$r(\theta) = \omega + \theta - g\left(-\frac{a}{\theta-\underline{\theta}}\right) \tag{16}$$

where $g(\cdot) \equiv (u'')^{-1}(\cdot)$ and $a \equiv \bar{\mu} \geq 0$.

The next proposition asserts that given Assumption 3 monotonicity is at most binding on a lower interval.

Proposition 4 *Assume u satisfies Assumption 3. Then $g\left(-\frac{a}{\theta-\underline{\theta}}\right)$ is concave in θ and the monotonicity constraint is at most binding on an interval $[\underline{\theta},\tilde{\theta}]$.*

To see this result, totally differentiate equation (15) and notice that $dr/d\theta$ would be negative for small θ, violating monotonicity. Intuitively, while the return to giving type θ additional resources, $-u''(\omega+\theta-r)$, is decreasing in θ which seems to suggest that net repayments should be increasing in θ, the overall return $-(\theta-\underline{\theta})u''(\omega+\theta-r)$, which includes the savings from reducing the penalties on all types below θ, is increasing in θ for low θ. This would suggest that net repayments should initially decrease, which is not implementable. Thus, types on a lower interval all make the same payment, i.e., there is 'bunching' on a lower interval.

The characterization in equation (16) applies to the entire interval $[\tilde{\theta},\theta^*]$. If we assume that $r(\theta)$ is continuous and not just piecewise continuous, we can characterize the solution further. In the range where the non-negativity constraint is binding, i.e., for $\theta \in [\theta^*,\bar{\theta}]$, we have $r(\theta)=r(\theta^*)$. Similarly, in the range where monotonicity is binding, i.e., for $\theta \in [\underline{\theta},\tilde{\theta}]$, we have $r(\theta)=r(\tilde{\theta})$. Note also that $U(\theta^*) = u(\omega+\theta^*-r(\theta^*)) = u\left(g\left(-\frac{a}{\theta^*-\underline{\theta}}\right)\right)$, since we know that penalties are zero on $[\theta^*,\bar{\theta}]$. Using this expression, the law of motion for indirect utility $\dot{U}(\theta) = u'(\omega+\theta-r(\theta))$ and the characterization of $r(\theta)$ above, we can also characterize the indirect utility $U(\theta)$ for the three intervals $[\underline{\theta},\tilde{\theta})$, $[\tilde{\theta},\theta^*)$ and $[\theta^*,\bar{\theta}]$. Thus we are left with the following finite dimensional constrained maximization problem:

$$\max_{\{a,\tilde{\theta},\theta^*\}\in\Re_+\times\Theta^2} u\left(g\left(-\frac{a}{\theta^*-\underline{\theta}}\right)\right)(\bar{\theta}-\underline{\theta}) - \int_{\underline{\theta}}^{\tilde{\theta}}\int_{\tilde{\theta}}^{\theta^*} u'\left(g\left(-\frac{a}{\vartheta-\underline{\theta}}\right)\right)d\vartheta d\theta$$

$$-\int_{\underline{\theta}}^{\tilde{\theta}}\int_{\theta}^{\tilde{\theta}} u'\left(g\left(-\frac{a}{\tilde{\theta}-\underline{\theta}}\right)+\vartheta-\tilde{\theta}\right)d\vartheta d\theta-\int_{\tilde{\theta}}^{\theta^*}\int_{\theta}^{\theta^*} u'\left(g\left(-\frac{a}{\vartheta-\underline{\theta}}\right)\right)d\vartheta d\theta$$

$$+\int_{\theta^*}^{\bar{\theta}}\int_{\theta^*}^{\theta} u'\left(g\left(-\frac{a}{\theta^*-\underline{\theta}}\right)+\vartheta-\theta^*\right)d\vartheta d\theta \tag{17}$$

subject to

$$\int_{\underline{\theta}}^{\tilde{\theta}}\left(g\left(-\frac{a}{\tilde{\theta}-\underline{\theta}}\right)-\tilde{\theta}\right)d\theta+\int_{\tilde{\theta}}^{\theta^*}\left(g\left(-\frac{a}{\theta-\underline{\theta}}\right)-\theta\right)d\theta$$

$$+\int_{\theta^*}^{\bar{\theta}}\left(g\left(-\frac{a}{\theta^*-\underline{\theta}}\right)-\theta^*\right)d\theta-\omega(\bar{\theta}-\underline{\theta})\leq 0. \tag{18}$$

This problem can be solved numerically given a function u and parameters ω, $\underline{\theta}$ and $\bar{\theta}$. We compute several examples below, but before doing so we characterize the dependence of the solution to the contracting problem on aggregate endowment for two cases. Proposition 5, the equivalent of Proposition 1 for the case of continuous idiosyncratic uncertainty, asserts that with constant absolute risk aversion the amount of default is independent of the aggregate endowment.

Proposition 5 *Assume u satisfies $\frac{\partial}{\partial c}\left(-\frac{u''(c)}{u'(c)}\right)=0$. Then the amount of default $\theta^*(\omega)$ and the net repayment schedule $r(\theta,\omega)$ are independent of ω.*

Notice that this proposition does not require Assumption 2. An example illustrating Proposition 5 is provided below.

Proposition 6 asserts that with logarithmic utility and $\Theta=[0,1]$, the amount of default varies inversely with aggregate endowment, i.e., the amount of default is higher in bad aggregate states. Furthermore, the amount that agents are allowed to default by, i.e., $r(\theta^*,\omega)-r(\tilde{\theta},\omega)$, is decreasing in ω as well. Thus, agents are allowed to default by more when aggregates are low.

Proposition 6 *Assume u is logarithmic and $\Theta=[0,1]$. Given Assumption 2, the amount of default $\theta^*(\omega)$ and the amount that agents are allowed to default by $(r(\theta^*,\omega)-r(\tilde{\theta},\omega))$ are decreasing in ω.*

An example illustrating Proposition 6 is provided below.

Using equations (24) to (26) in the proof of Proposition 6 one can show that there exists a unique level of aggregate endowment ω°, say, with $\tilde{\theta},\theta^*\in(0,1)$ at which $\tilde{\theta}$ and θ^* coincide. Thus there is a discontinuity in $\theta^*(\cdot)$: For $\omega\geq\omega^\circ$, the costly risk sharing technology is not used at all and hence there is no default in this range, whereas for $\omega<\omega^\circ$ the measure of agents defaulting is larger than

$\theta^*(\omega^\circ) \equiv \lim_{\omega \nearrow \omega^\circ} \theta^*(\omega)$. In other words, when ω is sufficiently high the optimal contract does not provide implicit contingencies. As ω decreases the optimal contract starts to provide contingencies. When default first occurs, i.e., contingencies are first provided, the measure of agents who default is bounded away from zero. Thus, the measure of agents defaulting is discontinuous in aggregates. Intuitively, there is bunching on a lower interval whenever contingencies are provided here, and thus when the threshold aggregate income is reached, the default rate goes from no contingencies and hence no default to contingencies and strictly positive default due to bunching.

To illustrate these results we provide two numerical examples. The idiosyncratic endowment θ is chosen to be uniformly distributed on the unit interval, i.e., $\Theta = [0, 1]$. We graph the net repayment schedule for ω between 0.005 and 0.1 below.

Figure 1 shows the net repayment schedule for an economy with constant absolute risk aversion (equal to 4) as a function of the aggregate endowment ω and the idiosyncratic endowment θ. As asserted in Proposition 5 the net repayment schedule does not vary with the aggregate endowment. The projection of the net repayment schedule illustrates this fact even more clearly. Only default penalties vary with the aggregate endowment. The characterization derived above can also easily be seen from the figure: There are two intervals where net repayments are constant across types. In the lower interval $[0, \tilde{\theta}]$ agents make (negative) minimal net repayments and incur maximal penalties.[11] This is the interval where the monotonicity constraint is binding, i.e., there is 'bunching.' In the upper interval $[\theta^*, 1]$ agents make positive (maximal) net repayments. These agents pay in full and are not penalized, i.e., the non-negativity constraint on penalties is binding here.

The example with logarithmic utility in Figure 2 shows the main result of the paper: With decreasing absolute risk aversion the number of agents defaulting and the amount by which defaulting agents default is decreasing in the aggregate endowment ω. When aggregate endowment is low more agents default, i.e., the interval of agents who pay in full $[\theta^*(\omega), 1]$ is shrinking. Thus optimal contracting implies higher default rates when aggregate endowment is low. Furthermore, the agents who default, default by more. In other words, the difference between the maximal net repayment and the minimal net repayment is increasing as ω decreases which means that more contingencies are provided. Net repayments become more variable (across

[11] Recall that we study *net* repayments, which we interpret as the repayments net of the average repayment, and hence net repayments may be negative.

θ) as ω decreases which means that more contingencies are provided. Notice also the discontinuity in the default rate when ω gets sufficiently low.

Finally, notice that if an agent's endowment is held fixed at, say .15, he does not default when the aggregate endowment is high and no other agent defaults but does default when the aggregate endowment is low and many agents default. Hence, whether an agent defaults or not depends on aggregate income even when controlling for the agent's income.

In addition to the example with logarithmic utility and hence constant relative risk aversion with $\sigma = 1$, we have computed several examples with constant relative risk aversion $\sigma \neq 1$. The numerical results are very similar to those reported for logarithmic utility (including the discontinuity of $\theta^*(\cdot)$). While we do not have an analytical characterization of the comparative statics for the more general case, the examples seem to confirm that more generally, with decreasing absolute risk aversion, default varies countercyclically with aggregate income.

5 Discussion

In this paper we study the solution to a planning problem to characterize the efficient allocation. The optimal allocation can be decentralized by competitive financial intermediaries as in Prescott and Townsend (1984). Intermediaries offer allocations or contracts to agents that specify a repayment and penalties which are imposed if the agent repays less than this amount. Both the repayment and the penalties are a function of the aggregate state. If intermediaries are competitive ex ante but can ensure exclusivity ex post, they will offer the optimal contract characterized above in equilibrium and make zero profits.

The optimal contract is not surprisingly quite non-linear. However, it turns out that optimal defaultable contracts have similar properties even if the contracts are restricted to a class of (piecewise) linear contracts in which default penalties (in utiles) are a linear function of the amount an agent defaults by (see Rampini (1998) for details). In particular, the fraction of agents defaulting and the amount that agents are allowed to default by are decreasing in ω.

We study an environment in which the endowment of each agent is the sum of an aggregate and an idiosyncratic component. Suppose instead that the aggregate state affects not just the mean but also the cross sectional dispersion of idiosyncratic endowment. If preferences exhibit constant relative risk aversion and aggregate shocks are multiplicative in the sense that $e = \omega\theta$, where the distribution of θ is not affected by ω, then the default rate is independent of aggregate conditions (see Rampini (1998)). Thus, default is countercyclical as long as the dispersion of endowment is not too procyclical. Note that the literature finds that the cross sectional dispersion of income is countercyclical[12] and the assumptions of our model seem hence reasonable.

6 Conclusions

This paper studies an environment in which default is motivated as an integral part of optimal contracts that enables risk sharing. We show that the extent to which optimal contracts provide default contingencies depends on aggregate income even if contracts are contingent on aggregate income. Under the assumption of decreasing absolute risk aversion, the fraction of agents defaulting and the amount that defaulting agents default by varies countercyclically with aggregate income and the fraction of agents defaulting can be discontinuous. One possible interpretation of this finding is that with decreasing absolute risk aversion, the return on marginally completing the market increases by more than the cost in bad times. Thus, default contingencies are used more heavily, making markets endogenously more complete in bad times. This however means that optimally default rates are sensitive to aggregate income and indeed that default is at times widespread.

The model also has predictions about the extent to which costly default contingencies should be used across countries with different income levels and as economies grow. In the cross section, we should see more default in poorer countries. Furthermore, we would expect the provision of costly default contingencies to decrease as economies grow.

[12]See Storesletten, Telmer, and Yaron (2002).

Appendix

Proof of Lemma 1. Allowing for ex post randomization, the planning problem can be stated as follows:

$$\max_{r_0, r_1, U_0, U_1} p_0 U_0 + p_1 U_1$$

subject to

$$\begin{aligned} E[u(\omega + \theta_1 - r_1)] - E[u(\omega + \theta_0 - r_1)] - (U_1 - U_0) &\geq 0, & (19) \\ E[u(\omega + \theta_0 - r_0)] - E[u(\omega + \theta_1 - r_0)] + U_1 - U_0 &\geq 0, & (20) \\ p_0 E[r_0] + p_1 E[r_1] &\geq 0 & (21) \end{aligned}$$

and

$$E[u(\omega + \theta_i - r_i)] - U_i \geq 0, \quad i = 0, 1, \tag{22}$$

where $U_i \equiv E[u(\omega+\theta_i-r_i)]-t_i$, $i = 1, 2$ and $E[\cdot]$ denotes the expectation with respect to ex post randomization. Assume for the rest of the proof that the solution is non-degenerate, i.e., $r_1 \neq r_0 \neq 0$, since otherwise the result is obvious. The proof follows from the following four lemmata:

Lemma 4 *At a solution to the program r_0 is deterministic.*

Proof. Suppose $\{(r_0, r_1), (U_0, U_1)\}$ solve the program where $\{(r_0, r_1)\}$ is a pair of random variables. Let $\hat{r}_0 = E[r_0]$ and

$$\begin{aligned} \hat{U}_0 = U_0 \quad &+ \quad \{(u(\omega + \theta_0 - E[r_0]) - E[u(\omega + \theta_0 - r_0)]) \\ &- \quad (u(\omega + \theta_1 - E[r_0]) - E[u(\omega + \theta_1 - r_0)])\}. \end{aligned}$$

Notice that $\hat{U}_0 > U_0$ since

$$\begin{aligned} & \frac{\partial}{\partial \theta} (u(\omega + \theta - E[r]) - u(\omega + \theta - E[r] - \pi(\theta))) \\ = \; & u'(\omega + \theta - E[r]) - u'(\omega + \theta - E[r] - \pi(\theta))(1 - \pi'(\theta)) \\ \leq \; & u'(\omega + \theta - E[r]) - u'(\omega + \theta - E[r] - \pi(\theta)) < 0, \end{aligned}$$

where $\pi(\theta)$ denotes the risk premium, i.e., solves $u(\omega+\theta-E[r]-\pi(\theta)) = E[u(\omega+\theta-r)]$ and using the fact that $\pi'(\theta) \leq 0$ given Assumption 1. $\{(\hat{r}_0, r_1), (\hat{U}_0, U_1)\}$ clearly improves the objective and satisfies (19), (21) and the non-negativity constraint for $i = 1$. The downward incentive compatibility constraint (20) is satisfied by construction and the non-negativity constraint for $i = 0$ is satisfied since

$$\begin{aligned} u(\omega + \theta_0 - \hat{r}_0) - \hat{U}_0 \quad &= \quad E[u(\omega + \theta_0 - r_0)] - U_0 \\ &+ \quad (u(\omega + \theta_1 - E[r_0]) - E[u(\omega + \theta_1 - r_0)]) \\ &> \quad E[u(\omega + \theta_0 - r_0)] - U_0 \geq 0. \end{aligned}$$

This contradicts optimality of $\{(r_0, r_1), (U_0, U_1)\}$. □

Lemma 5 *At a solution to the program at most one of the incentive compatibility constraints is binding.*

Proof. Suppose not. Then

$$\begin{aligned} u(\omega + \theta_1 - r_0) - u(\omega + \theta_0 - r_0) &= E[u(\omega + \theta_1 - r_1)] - E[u(\omega + \theta_0 - r_1)] \\ &> u(\omega + \theta_1 - E[r_1]) - u(\omega + \theta_0 - E[r_1]), \end{aligned}$$

where the inequality follows from the convexity of $u(x+\Delta)-u(x)$. But since $u(x+\Delta)-u(x)$ is decreasing this implies that $E[r_1] < r_0$. This implies for the value of the objective that

$$\begin{aligned} p_0 U_0 + p_1 U_1 &\leq p_0 u(\omega + \theta_0 - r_0) + p_1 u(\omega + \theta_1 - E[r_1]) \\ &< p_0 u(\omega + \theta_0) + p_1 u(\omega + \theta_1), \end{aligned}$$

since the contract is at best a mean-preserving spread and hence optimality is contradicted. □

Corollary 1 *At a solution to the program* $r_0 < E[r_1]$.

Lemma 6 *At a solution to the program the downward incentive compatibility constraint is binding.*

Proof. Suppose not. Then it must be the case that $U_0 = u(\omega + \theta_0 - r_0)$. But, given this, the downward incentive compatibility constraint implies

$$\begin{aligned} U_1 > u(\omega + \theta_1 - r_0) &> u(\omega + \theta_1 - E[r_1]) \\ &\geq E[u(\omega + \theta_1 - r_1)] \geq U_1, \end{aligned}$$

i.e., a contradiction. □

Lemma 7 *At a solution to the program* r_1 *is deterministic.*

Proof. Suppose not. Choose $\hat{r}_1$ such that r_1 is a mean-preserving spread of $\hat{r}_1$ and let

$$\hat{U}_1 = U_1 + (E[u(\omega + \theta_1 - \hat{r}_1)] - E[u(\omega + \theta - r_1)]).$$

$\hat{r}_1$ can be chosen such that the upward incentive compatibility constraint (19) is satisfied at $\{(r_0, \hat{r}_1), (U_0, \hat{U}_1)\}$. Clearly, $\hat{U}_1 > U_1$ and the constraints are satisfied, contradicting optimality. □

This completes the proof of the lemma. □

Proof of Proposition 1. By the hypothesis of the proposition we have $u(c) = -\exp(-Rc)$. Suppose $\{r_0, r_1, t_0, 0\}$ solve the maximization problem at ω. Then $\{r_0, r_1, t_0 \exp(-R(\omega' - \omega)), 0\}$ solve the maximization problem at ω' since multiplication of the objective function and the incentive constraint by $\exp(-R(\omega - \omega'))$ maps the problem at ω' into the problem at ω except for a change of variables. Hence, the net repayment schedule is independent of ω and $t_0(\omega)$ is decreasing in ω. □

Proof of Proposition 2. The first order conditions of the program in equations (11)-(14) are:

$$\begin{aligned} p_0 &= \lambda, \\ p_1 &= \nu - \lambda, \\ (u'(\omega + \theta_0 - r_0) - u'(\omega + \theta_1 - r_0))\lambda &= \mu p_0, \\ p_1 \mu &= u'(\omega + \theta_1 - r_1)\nu \end{aligned}$$

and

$$\begin{aligned} u(\omega + \theta_0 - r_0) - u(\omega + \theta_1 - r_0) + U_1 - U_0 &= 0, \\ p_0 r_0 + p_1 r_1 &= 0, \\ u(\omega + \theta_1 - r_1) - U_1 &= 0, \end{aligned}$$

where λ, μ and ν are the multipliers associated with equation (12), (13) and (14), respectively.

Totally differentiating the non-negativity constraint that is satisfied with equality we have

$$\frac{dU_1}{d\omega} = u'(\omega + \theta_1 - r_1)\left(1 - \frac{dr_1}{d\omega}\right)$$

which is strictly positive if $\frac{dr_1}{d\omega} < 0$. Since the budget constraint is satisfied with equality $\frac{dr_1}{d\omega} < 0$ implies $\frac{dr_0}{d\omega} > 0$. Finally,

$$\begin{aligned} \frac{dt_0}{d\omega} &= \frac{d}{d\omega}(u(\omega + \theta_0 - r_0) - U_0) \\ &= \frac{d}{d\omega}(u(\omega + \theta_1 - r_0) - U_1) \\ &= u'(\omega + \theta_1 - r_0)\left(1 - \frac{dr_0}{d\omega}\right) - u'(\omega + \theta_1 - r_1)\left(1 - \frac{dr_1}{d\omega}\right), \end{aligned}$$

which is strictly negative if $\frac{dr_1}{d\omega} < 0$. Thus it is sufficient to show that $\frac{dr_1}{d\omega} < 0$.

We can simplify the system of first order conditions to get

$$\begin{aligned} u'(\omega + \theta_1 - r_1) &= p_1(u'(\omega + \theta_0 - r_0) - u'(\omega + \theta_1 - r_0)) \\ 0 &= p_0 r_0 + p_1 r_1. \end{aligned}$$

For notational simplicity, let $y_{ij} \equiv \omega + \theta_j - r_i$, $f_{ij} \equiv f(y_{ij})$ and $R_{ij} \equiv -\frac{u''_{ij}}{u'_{ij}}$. We can then write

$$0 = u'^{-1}_{00} u'^{-1}_{01} - p_1 u'^{-1}_{11}(u'^{-1}_{01} - u'^{-1}_{00}). \tag{23}$$

Noting that $dr_0 = -\frac{p_1}{p_0} dr_1$ we can write the total differential of equation (23) as

$$\begin{aligned} 0 &= (\frac{p_1}{p_0} dr_1 + d\omega) \left(R_{00} u'^{-1}_{00} u'^{-1}_{01} + R_{01} u'^{-1}_{00} u'^{-1}_{01} \right) \\ &\quad - p_1(-dr_1 + d\omega) \left(R_{11} u'^{-1}_{11} (u'^{-1}_{01} - u'^{-1}_{00}) \right) \\ &\quad - p_1 u'^{-1}_{11} (\frac{p_1}{p_0} dr_1 + d\omega) \left(R_{01} u'^{-1}_{01} - R_{00} u'^{-1}_{00} \right). \end{aligned}$$

The terms multiplying $d\omega$ are

$$(R_{00} + R_{01}) u'^{-1}_{00} u'^{-1}_{01} - p_1 R_{11} u'^{-1}_{11} (u'^{-1}_{01} - u'^{-1}_{00}) - p_1 u'^{-1}_{11} \left(R_{01} u'^{-1}_{01} - R_{00} u'^{-1}_{00} \right)$$

$$= R_{00} u'^{-1}_{00} u'^{-1}_{01} - p_1 R_{11} u'^{-1}_{11} (u'^{-1}_{01} - u'^{-1}_{00}) + R_{01} u'^{-1}_{00} u'^{-1}_{01} - p_1 u'^{-1}_{11} R_{01} u'^{-1}_{01} + p_1 u'^{-1}_{11} R_{00} u'^{-1}_{00}.$$

Note that $y_{00} < y_{11} < y_{01}$ and thus $R_{00} > R_{11} > R_{01}$. Using equation (23) we see that the first two terms in above expression together are positive and that the last three terms can be written as

$$R_{01} p_1 u'^{-1}_{11} (u'^{-1}_{01} - u'^{-1}_{00}) - p_1 u'^{-1}_{11} R_{01} u'^{-1}_{01} + p_1 u'^{-1}_{11} R_{00} u'^{-1}_{00}$$

$$= p_1 u'^{-1}_{11} u'^{-1}_{00} (R_{00} - R_{01}) > 0.$$

Thus, the terms multiplying $d\omega$ together are strictly positive. The terms multiplying dr_1 are

$$\begin{aligned} & p_1 R_{11} u'^{-1}_{11} (u'^{-1}_{01} - u'^{-1}_{00}) + \frac{p_1}{p_0} R_{00} u'^{-1}_{00} u'^{-1}_{01} \\ & + \frac{p_1}{p_0} R_{01} u'^{-1}_{00} u'^{-1}_{01} - p_1 \frac{p_1}{p_0} u'^{-1}_{11} \left(R_{01} u'^{-1}_{01} - R_{00} u'^{-1}_{00} \right). \end{aligned}$$

The first two terms are clearly positive. Using equation (23) we have for the last two terms

$$p_1 \frac{p_1}{p_0} R_{01} u'^{-1}_{11} (u'^{-1}_{01} - u'^{-1}_{00}) - p_1 \frac{p_1}{p_0} u'^{-1}_{11} R_{01} u'^{-1}_{01} + p_1 \frac{p_1}{p_0} u'^{-1}_{11} R_{00} u'^{-1}_{00}$$

$$= p_1 \frac{p_1}{p_0} u'^{-1}_{01} u'^{-1}_{00} (R_{00} - R_{01}) > 0.$$

Thus, the terms multiplying dr_1 together are strictly positive. Hence, we have $\frac{dr_1}{d\omega} < 0$ which implies both assertions. □

Proof of Lemma 2. Incentive compatibility requires

$$U(\theta) \geq U(\hat{\theta}, \theta) = U(\hat{\theta}) + \Big(u(\omega + \theta - r(\hat{\theta})) - u(\omega + \hat{\theta} - r(\hat{\theta})) \Big)$$

and thus

$$U(\theta) - U(\hat{\theta}) \geq u(\omega + \theta - r(\hat{\theta})) - u(\omega + \hat{\theta} - r(\hat{\theta})).$$

Reversing the role of θ and $\hat{\theta}$ and combining the results we have

$$u(\omega + \theta - r(\theta)) - u(\omega + \hat{\theta} - r(\theta)) \geq u(\omega + \theta - r(\hat{\theta})) - u(\omega + \hat{\theta} - r(\hat{\theta}))$$

which by strict concavity of u implies $r(\theta) \geq r(\hat{\theta})$ whenever $\theta \geq \hat{\theta}$. □

Proof of Proposition 3. Monotonicity of the net repayment schedule r implies $t(\hat{\theta}) \geq t(\theta)$ whenever $\theta \geq \hat{\theta}$. Hence, if the constraint is not binding at θ^* it is not binding for $\hat{\theta} < \theta^*$. To show that the constraint is satisfied at $\bar{\theta}$ with equality assume the opposite, i.e., $t(\bar{\theta}) > 0$. But then replacing $t(\theta)$ by $\tilde{t}(\theta) \equiv t(\theta) - t(\bar{\theta})$, $\forall \theta \in \Theta$, leaves incentive compatibility and resource constraints unaffected, satisfies non-negativity and increases the value of the objective, which contradicts optimality of t. □

Proof of Proposition 4. It is straightforward to show that g is concave in θ for utility functions with constant absolute risk aversion and constant relative risk aversion. Assume monotonicity is not binding in some neighborhood $\mathcal{N} = [\tilde{\theta}, \tilde{\theta} + \varepsilon]$, i.e., $\dot{r}(\theta) \geq 0$, $\forall \theta \in \mathcal{N}$. If $g\left(-\frac{a}{\theta - \underline{\theta}}\right)$ is concave in θ we have that $\dot{r}(\theta) \geq 0$, $\forall \theta \geq \tilde{\theta}$, and thus monotonicity is non-binding on $[\tilde{\theta}, \theta^*]$. I.e., monotonicity is at most binding on an interval $[\underline{\theta}, \tilde{\theta}]$. □

Proof of Proposition 5. By the hypothesis of the proposition we can write $u(c) = -\exp(-Rc)$ where R is the coefficient of absolute risk aversion. Suppose the pair $\{r(\theta), t(\theta)\}$ solves the maximization problem at ω. Then $\{r(\theta), \exp(-R(\omega' - \omega))t(\theta)\}$ solves the maximization problem at ω' since multiplication by $\exp(-R(\omega - \omega'))$ maps the problem at ω' into the problem at ω except for a change of variables. Hence, the net repayment schedule and, thus, the amount of default θ^* are independent of ω and penalties are decreasing in ω. □

Proof of Proposition 6. Taking $\Theta = [0, 1]$ and noting that $u(\cdot) = \log(\cdot)$ implies $g(z) = (-z)^{-\frac{1}{2}}$ we have the following explicit characterization of the objective function and constraint in equation (17) and (18), respectively:

$$\begin{aligned}\mathcal{U} \equiv \int_0^1 U(\theta)d\theta &= \theta^* - \tilde{\theta} - 1 - \frac{2}{3}a^{\frac{1}{2}}\theta^{*\frac{3}{2}} + \frac{2}{3}a^{\frac{1}{2}}\tilde{\theta}^{\frac{3}{2}} \\ &- \left(a^{-\frac{1}{2}}\theta^{*\frac{1}{2}} - \theta^*\right)\log\left(a^{-\frac{1}{2}}\theta^{*\frac{1}{2}}\right) + \left(a^{-\frac{1}{2}}\tilde{\theta}^{\frac{1}{2}} - \tilde{\theta}\right)\log\left(a^{-\frac{1}{2}}\tilde{\theta}^{\frac{1}{2}}\right) \\ &+ \left(a^{-\frac{1}{2}}\theta^{*\frac{1}{2}} + 1 - \theta^*\right)\log\left(a^{-\frac{1}{2}}\theta^{*\frac{1}{2}} + 1 - \theta^*\right) \\ &- \left(a^{-\frac{1}{2}}\tilde{\theta}^{\frac{1}{2}} - \tilde{\theta}\right)\log\left(a^{-\frac{1}{2}}\tilde{\theta}^{\frac{1}{2}} - \tilde{\theta}\right)\end{aligned}$$

and

$$\mathcal{X} \equiv \left(a^{-\frac{1}{2}}\theta^{*\frac{1}{2}} - \theta^*\right)(1-\theta^*) + \left(a^{-\frac{1}{2}}\tilde{\theta}^{\frac{1}{2}} - \tilde{\theta}\right)\tilde{\theta} + \frac{2}{3}a^{-\frac{1}{2}}\theta^{*\frac{3}{2}} - \frac{2}{3}a^{-\frac{1}{2}}\tilde{\theta}^{\frac{3}{2}} - \frac{1}{2}\theta^{*2} + \frac{1}{2}\tilde{\theta}^2 - \omega.$$

The optimization problem is then to maximize $\mathcal{U}$ subject to $\mathcal{X} \leq 0$ with the first order conditions:

$$\begin{aligned}0 &= -\frac{1}{3}a^{-\frac{1}{2}}\left(\theta^{*\frac{3}{2}} - \tilde{\theta}^{\frac{3}{2}}\right) - \frac{1}{2}a^{-\frac{3}{2}}\theta^{*\frac{1}{2}}\log\left(1 + a^{\frac{1}{2}}\theta^{*-\frac{1}{2}}(1-\theta^*)\right) + \frac{1}{2}a^{-\frac{3}{2}}\tilde{\theta}^{\frac{1}{2}}\log\left(1 - a^{\frac{1}{2}}\tilde{\theta}^{\frac{1}{2}}\right) \\ &- \frac{1}{2}a^{-1}\theta^* + \frac{1}{2}a^{-1}\tilde{\theta} - \kappa(-a^{-\frac{3}{2}})\left(\frac{1}{2}\tilde{\theta}^{\frac{1}{2}}\tilde{\theta} + \frac{1}{2}\theta^{*\frac{1}{2}}(1-\theta^*) + \frac{1}{3}\left(\theta^{*\frac{3}{2}} - \tilde{\theta}^{\frac{3}{2}}\right)\right), \\ 0 &= -\left(\frac{1}{2}a^{-\frac{1}{2}}\tilde{\theta}^{-\frac{1}{2}} - 1\right)\left(\log\left(1 - a^{\frac{1}{2}}\tilde{\theta}^{\frac{1}{2}}\right) + a^{\frac{1}{2}}\tilde{\theta}^{\frac{1}{2}}\right) - \kappa\left(\frac{1}{2}a^{-\frac{1}{2}}\tilde{\theta}^{-\frac{1}{2}} - 1\right)\tilde{\theta}, \\ 0 &= \left(\frac{1}{2}a^{-\frac{1}{2}}\theta^{*-\frac{1}{2}} - 1\right)\left(\log\left(1 + a^{\frac{1}{2}}\theta^{*-\frac{1}{2}}(1-\theta^*)\right) + a^{\frac{1}{2}}\theta^{*\frac{1}{2}}\right) - \kappa\left(\frac{1}{2}a^{-\frac{1}{2}}\theta^{*-\frac{1}{2}} - 1\right)(1-\theta^*)\end{aligned}$$

and

$$\left(a^{-\frac{1}{2}}\theta^{*\frac{1}{2}} - \theta^*\right)(1-\theta^*) + \left(a^{-\frac{1}{2}}\tilde{\theta}^{\frac{1}{2}} - \tilde{\theta}\right)\tilde{\theta} + \frac{2}{3}a^{-\frac{1}{2}}\theta^{*\frac{3}{2}} - \frac{2}{3}a^{-\frac{1}{2}}\tilde{\theta}^{\frac{3}{2}} - \frac{1}{2}\theta^{*2} + \frac{1}{2}\tilde{\theta}^2 - \omega = 0 \quad (24)$$

where κ is the multiplier on the constraint. The second and third first order conditions can be simplified to:

$$\begin{aligned}\log\left(1 - a^{\frac{1}{2}}\tilde{\theta}^{\frac{1}{2}}\right) + a^{\frac{1}{2}}\tilde{\theta}^{\frac{1}{2}} + \kappa\tilde{\theta} &= 0, \\ \log\left(1 + a^{\frac{1}{2}}\theta^{*-\frac{1}{2}}(1-\theta^*)\right) + a^{\frac{1}{2}}\theta^{*\frac{1}{2}} - \kappa(1-\theta^*) &= 0.\end{aligned}$$

Using these two expressions the first first order condition can be reduce to $a = \kappa$, i.e., the multiplier on the resource constraint in the control problem and in the maximization problem coincide. We are thus left with a system of equations in the three variables a, $\tilde{\theta}$ and θ^*:

$$\log\left(1 - a^{\frac{1}{2}}\tilde{\theta}^{\frac{1}{2}}\right) + a^{\frac{1}{2}}\tilde{\theta}^{\frac{1}{2}} + a\tilde{\theta} = 0, \quad (25)$$

$$\log\left(1 + a^{\frac{1}{2}}\theta^{*-\frac{1}{2}}(1-\theta^*)\right) + a^{\frac{1}{2}}\theta^{*\frac{1}{2}} - a(1-\theta^*) = 0, \quad (26)$$

and equation (24).

Totally differentiating this system of equations we obtain

$$M\begin{pmatrix}\frac{da}{d\omega}\\ \frac{d\tilde{\theta}}{d\omega}\\ \frac{d\theta^*}{d\omega}\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}$$

where $M=[m_{ij}]$ is the Jacobian of the system of equations (25), (26) and (24). Notice that $m_{13}=m_{22}=0$. Specifically, we have:

$$\begin{aligned}m_{31} &= -a^{-\frac{3}{2}}\left(\frac{1}{2}\tilde{\theta}^{\frac{1}{2}}\tilde{\theta}+\frac{1}{2}\theta^{*\frac{1}{2}}(1-\theta^*)+\frac{1}{3}\left(\theta^{*\frac{3}{2}}-\tilde{\theta}^{\frac{3}{2}}\right)\right),\\ m_{32} &= \left(\frac{1}{2}a^{-\frac{1}{2}}\tilde{\theta}^{-\frac{1}{2}}-1\right)\tilde{\theta},\\ m_{33} &= \left(\frac{1}{2}a^{-\frac{1}{2}}\theta^{*-\frac{1}{2}}-1\right)(1-\theta^*).\end{aligned}$$

$0\le\tilde{\theta}\le\theta^*\le 1$ and $a>0$ imply that $m_{31}<0$. Monotonicity implies that $\dot{r}(\theta)\ge 0$ and hence $\frac{1}{2}a^{-\frac{1}{2}}\theta^{-\frac{1}{2}}-1<0$ at $\tilde{\theta}$ and θ^*. Thus, $m_{32}<0$ and $m_{33}<0$. Furthermore, we get

$$\begin{aligned}m_{11} &= \tilde{\theta}\left(1-\frac{1}{2}\frac{1}{1-a^{\frac{1}{2}}\tilde{\theta}^{\frac{1}{2}}}\right),\\ m_{12} &= a\left(1-\frac{1}{2}\frac{1}{1-a^{\frac{1}{2}}\tilde{\theta}^{\frac{1}{2}}}\right).\end{aligned}$$

Monotonicity implies that $a^{\frac{1}{2}}\theta^{\frac{1}{2}}>1/2$ and hence $m_{11}<0$ and $m_{12}<0$. Finally, we have

$$\begin{aligned}m_{21} &= \frac{-\frac{1}{2}(1-\theta^*)-a^{\frac{1}{2}}\theta^{*-\frac{1}{2}}(1-\theta^*)^2+\frac{1}{2}a^{-\frac{1}{2}}\theta^{*-\frac{1}{2}}}{1+a^{\frac{1}{2}}\theta^{*-\frac{1}{2}}(1-\theta^*)},\\ m_{23} &= \frac{a}{2\theta^*\left(1+a^{\frac{1}{2}}\theta^{*-\frac{1}{2}}(1-\theta^*)\right)}\left(1+\theta^*+2a^{\frac{1}{2}}\theta^{*\frac{1}{2}}(1-\theta^*)-a^{-\frac{1}{2}}\theta^{*-\frac{1}{2}}\right).\end{aligned}$$

Monotonicity implies that $a^{-\frac{1}{2}}\theta^{-\frac{1}{2}}<2$ and thus

$$m_{23}>\frac{a}{2\theta^*\left(1+a^{\frac{1}{2}}\theta^{*-\frac{1}{2}}(1-\theta^*)\right)}\left(2-a^{-\frac{1}{2}}\theta^{*-\frac{1}{2}}\right)>0.$$

To sign m_{21} notice that equation (26) implies that

$$a(1-\theta^*)-a^{\frac{1}{2}}\theta^{*\frac{1}{2}}=\log\left(1+a^{\frac{1}{2}}\theta^{*-\frac{1}{2}}(1-\theta^*)\right)>0 \tag{27}$$

where the inequality obtains because the logarithm of a term larger than 1 is taken. Equation (27) implies that $a^{\frac{1}{2}}\theta^{*-\frac{1}{2}}(1-\theta^*)>1$ as well as $\theta^{*-\frac{1}{2}}(1-\theta^*)>a^{-\frac{1}{2}}$. Thus, on the one hand note that

$$m_{21}<\frac{-\frac{3}{2}(1-\theta^*)+1}{1+a^{\frac{1}{2}}\theta^{*-\frac{1}{2}}(1-\theta^*)}$$

which is negative as long as $\theta^* \leq 1/3$. On the other hand, observe that

$$\begin{aligned} m_{21} &< \frac{-\frac{3}{2}(1-\theta^*) + \frac{1}{2}\theta^{*-1}(1-\theta^*)}{1 + a^{\frac{1}{2}}\theta^{*-\frac{1}{2}}(1-\theta^*)} \\ &= \frac{(1-\theta^*)}{2\left(1 + a^{\frac{1}{2}}\theta^{*-\frac{1}{2}}(1-\theta^*)\right)}\left(\frac{1}{\theta^*} - 3\right) \end{aligned}$$

which is negative as long as $\theta^* \geq 1/3$. Thus, m_{21} is unambiguously negative. We have signed all elements of matrix M. To sign the determinant $|M|$ notice that

$$|M| = m_{23}(m_{12}m_{31} - m_{11}m_{32}) - m_{12}m_{21}m_{33}$$

which is positive as long as the term in parenthesis $m \equiv m_{12}m_{31} - m_{11}m_{32}$ is. After some rearrangement we get

$$m = \left(1 - \frac{1}{2}\frac{1}{1 - a^{\frac{1}{2}}\tilde{\theta}^{\frac{1}{2}}}\right)\left(-a^{-\frac{1}{2}}\left(\frac{1}{2}\theta^{*\frac{1}{2}}(1-\theta^*) + \frac{1}{3}\left(\theta^{*\frac{3}{2}} - \tilde{\theta}^{\frac{3}{2}}\right)\right) - \left(a^{-\frac{1}{2}}\tilde{\theta}^{-\frac{1}{2}} - 1\right)\tilde{\theta}^2\right)$$

which is positive since $a^{-\frac{1}{2}}\tilde{\theta}^{-\frac{1}{2}} > 1$ for equation (25) to be well defined. Hence, we can now apply Kramer's rule to obtain

$$\begin{aligned} \frac{da}{d\omega} &= |M|^{-1}m_{12}m_{23} < 0, \\ \frac{d\tilde{\theta}}{d\omega} &= -|M|^{-1}m_{11}m_{23} > 0, \\ \frac{d\theta^*}{d\omega} &= -|M|^{-1}m_{12}m_{21} < 0. \end{aligned}$$

Recall that for $\theta \in [\tilde{\theta}, \theta^*]$, $r(\theta) = \omega + \theta - a^{-\frac{1}{2}}\theta^{\frac{1}{2}}$ and thus

$$\frac{dr(\theta)}{d\omega} = 1 + \frac{1}{2}a^{-\frac{3}{2}}\theta^{\frac{1}{2}}\frac{da}{d\omega} - \left(\frac{1}{2}a^{-\frac{1}{2}}\theta^{-\frac{1}{2}} - 1\right)\frac{d\theta}{d\omega}.$$

Hence,

$$\begin{aligned} \frac{d}{d\omega}\left(r(\theta^*) - r(\tilde{\theta})\right) &= \left(\left(-\frac{1}{2}a^{-\frac{3}{2}}\tilde{\theta}^{\frac{1}{2}}\right) - \left(-\frac{1}{2}a^{-\frac{3}{2}}\theta^{*\frac{1}{2}}\right)\right)\frac{da}{d\omega} \\ &+ \left(\frac{1}{2}a^{-\frac{1}{2}}\tilde{\theta}^{-\frac{1}{2}} - 1\right)\frac{d\tilde{\theta}}{d\omega} - \left(\frac{1}{2}a^{-\frac{1}{2}}\theta^{*-\frac{1}{2}} - 1\right)\frac{d\theta^*}{d\omega} \\ &< 0. \end{aligned}$$

This completes the proof of the proposition. □

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Figure 1: Net Repayment Schedule and its Projection for an Economy with Constant Absolute Risk Aversion

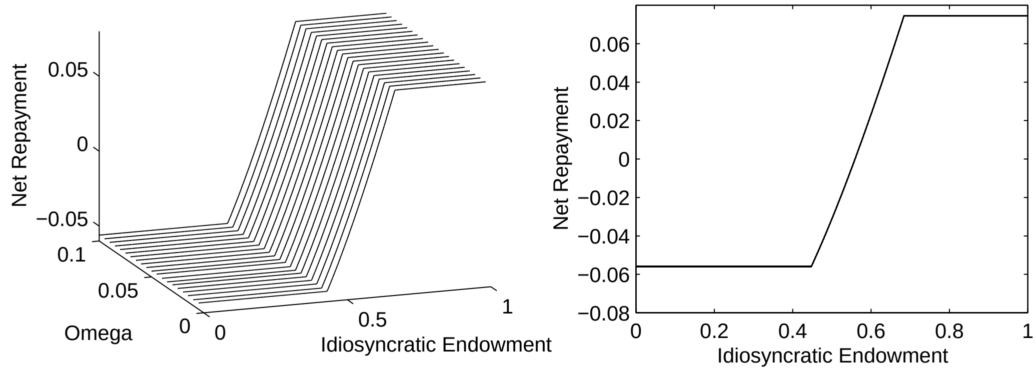

Figure 2: Net Repayment Schedule and its Projection for an Economy with Logarithmic Utility