

Integrating Correlations

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Introduction

In the last few years the quantitative modelling of credit risk has received a lot of attention within the financial industry. Several models have been released to the public, notably *CreditRisk+*, *CreditMetrics*, *Credit Portfolio View* [CR97, CM97, CP98]. Although different approaches to credit risk are used, studies have shown that the models yield similar results if the parameters are set in a consistent way, see for example [KO98]. *CreditRisk+* in particular is ideal for practical implementation and has some nice features, namely: (a) few assumptions have to be made, (b) the methodology is very transparent and based on concepts already used in the insurance business and (c) the loss distribution can be calculated analytically in an efficient way using an iterative procedure.

One of the shortcomings of the *CreditRisk+* model is the assumption of independent sectors. The sectors represent for example different regions or industries within the economy and individual obligors can be assigned to particular sectors¹. In order to perform a sector analysis, the authors of *CreditRisk+* propose apportioning an obligor's systematic credit risk across a mixture of independent sectors. However, such an approach is difficult to realize in practice.

In this paper we present a framework for extending the *CreditRisk+* model by examining correlations between industries and derive a formula for the unexpected loss and risk contributions. The correct modelling of correlations of default risk between sectors is very important for examining the effects of diversification on active portfolio management. Note that only the loss from defaults of counterparties is modelled.

The article is organized as follows: we first derive the probability generating function of the loss distribution for one industry sector following the *CreditRisk+* approach. This is then generalized to several sectors to obtain the first two moments of the loss distribution, as well as a loss distribution consistent with the observed correlations between sectors.

Loss Distribution for one Industry Sector

We will express the loss distribution of a loan portfolio by its probability generating function, which is defined as a power series of the form

$$G(z) = \sum_{n \geq 0} p(n) \cdot z^n, \quad (1)$$

where $p(n)$ is the probability of losing the amount n and z is a formal variable. Generating functions possess some nice features (see Appendix) and are preferred to probability distributions in this paper, although both are just different representations of the same thing. In the following, we express all losses and exposures as integer multiples of some fixed chosen unit. Let p_A be the default probability of obligor A and v_A be its exposure net of recovery. The recovery rates are assumed to be constant. The loss distribution for a single obligor A , having two possible states with $n = 0$ (no default) and $n = v_A$ (default), can be expressed by the following probability generating function

$$G_A(z) = (1 - p_A) \cdot z^0 + p_A \cdot z^{v_A} = 1 + p_A \cdot (z^{v_A} - 1). \quad (2)$$

Assuming independence of default events between obligors - an assumption which is relaxed in equation (4) - we obtain the generating function for the portfolio loss distribution (compare appendix)

$$\begin{aligned} G(z) &= \prod_A G_A(z) = \exp \left[\sum_A \log(1 + p_A \cdot (z^{v_A} - 1)) \right] \approx \exp \left[\sum_A p_A \cdot (z^{v_A} - 1) \right] \\ &=: \exp[P(z) - P(1)] \end{aligned} \quad (3)$$

¹ Henceforth we are using sector and industry interchangeably.

where we use the approximation $\log(1+h) \approx h$, which is valid for small default probabilities - a good assumption for most loan portfolios. We call $P(z) = \sum_A p_A z^{v_A}$ the *portfolio polynomial*. This polynomial contains all of the relevant information about the portfolio for our purpose, i.e., the expected number of defaults per exposure. We remark that the distribution described by the generating function in equation (3) is *compound Poisson* (cf. [BU70]). In the special case of identical exposures $v_A = 1$, we have

$P(z) = \mu z$, where $\mu = \sum_A p_A$ is the expected number of defaults in the portfolio. The resulting distribution is Poisson. In fact, this is the distribution of the number of defaults.

In the next step, the model is refined to include the systematic risk of correlated changes in the default rates, thereby dropping the assumption of the independence of default events in equation (3). This is achieved by introducing a random scaling factor γ modeling changes in the economy. The variable γ describes the relative number of default events in the economy normalized to the mean equal to one. Let $g(\gamma)$ be the corresponding probability density function with variance σ^2 , which we call the *relative default variance*. Note that this quantity is independent of the portfolio.

To obtain the generating function $F(z)$ of the unconditional loss distribution we average the conditional generating function $G^{(\gamma)}(z)$ over the possible states of the economy, characterized by γ :

$$F(z) = \int_0^\infty G^{(\gamma)}(z) \cdot g(\gamma) d\gamma = \int_0^\infty \exp[\gamma \cdot (P(z) - P(1))] \cdot g(\gamma) d\gamma. \quad (4)$$

Here, it is assumed that the generating function of the loss distribution conditioned on the state of the economy γ is $G^{(\gamma)}(z) = \exp[\gamma \cdot (P(z) - P(1))]$. This means that the systematic default risk affects all obligors in the same way.

Remark:

If we assume $g(\gamma)$ follows a Gamma-distribution, then $F(z)$ can be expressed analytically, and the explicit form of the loss distribution is amenable to efficient computation (see [CR97]). When counting the number of defaults (i.e., formally putting $v_A = 1$, and consequently, $P(z) = \mu z$), the corresponding distribution is *negative binomial* (cf. [BU70]).

Moreover, the distribution of losses allows the following mathematical description: suppose that X_1, X_2, X_3, \dots are independent identically distributed having as generating function the polynomial $P(z)/\mu$, and let M be negative binomially distributed, independent of the X_i . Then the distribution of losses is given by $\sum_{i=1}^M X_i$.

The Moments of the Loss Distribution for one Industry Sector

The expectation (expected loss: EL) and the standard deviation (unexpected loss: UL) of the loss distribution can be expressed as follows:

$$EL = P'(1) = \sum_A p_A v_A, \quad UL^2 = P'(1)^2 \sigma^2 + P''(1) + P'(1) \quad (5)$$

The expressions $P'(1)$ and $P''(1)$ are the first and second derivatives of the portfolio polynomial $P(z)$ at $z = 1$. It is important to note that expected loss and variance of the losses only depend on the portfolio polynomial $P(z)$ and the relative default variance σ^2 . The shape of the default distribution $g(\gamma)$ is thus *irrelevant* for computing EL and UL . From equation (5) we obtain the variance as in *CreditRisk+*, relating the volatility in the losses UL to the relative default variance σ^2 , and the portfolio structure:

$$UL^2 = \sigma^2 EL^2 + \sum_A p_A v_A^2. \quad (6)$$

The first expression represents the risk due to systematic changes in the economy, reflected by the relative default variance. The second term is the risk contribution due to the statistical nature of default events,

which is only important for either small portfolios or for cases with low systematic risk. For large and homogeneous portfolios and values σ of the order of one, the credit risk is only driven by the EL , scaled with the systematic risk factor σ . The detailed structure of the portfolio does not affect the loss distribution. From equation (6) we also derive that the variance of the number of defaults in the portfolio equals $\sigma^2 \mu^2 + \mu$ (put $\nu_A = 1$). For large portfolios and $\sigma \neq 0$ this is roughly the variance of the number of defaults $\sigma^2 \mu^2$ observed in the economy.

Loss Distribution for Several Industry Sectors

For simplicity, we first restrict ourselves to two correlated industries. We now have a portfolio polynomial for each industry: $P_1(z)$ and $P_2(w)$ defined as before, by assigning each obligor to one industry. The probability generating function in the conditional case is given by the product of the individual generating functions, having a form as in equation (3):

$$G(z, w) = \exp[P_1(z) - P_1(1)] \cdot \exp[P_2(w) - P_2(1)] . \quad (7)$$

To derive the generating function for the unconditional case, we average over all possible states of the economy. These states are now modelled by the joint distribution of two scaling factors γ_1, γ_2 with density $g(\gamma_1, \gamma_2)$, describing the relative number of default events in the two industry sectors, each normalized to mean one. We thus incorporate the default correlations between the two industries. The loss distribution is now given by the following probability generating function

$$F(z, w) = \int_0^\infty \int_0^\infty G^{(\gamma_1, \gamma_2)}(z, w) g(\gamma_1, \gamma_2) d\gamma_1 d\gamma_2 , \quad (8)$$

where $G^{(\gamma_1, \gamma_2)}(z, w) = \exp[\gamma_1 \cdot (P_1(z) - P_1(1))] \cdot \exp[\gamma_2 \cdot (P_2(w) - P_2(1))]$ is the generating function of the losses conditioned on the state (γ_1, γ_2) .

At this point, it would be natural to take some suitable two-dimensional generalization of the Gamma-distribution for $g(\gamma_1, \gamma_2)$, which would allow the explicit calculation of the full loss distribution. However, as already seen in the case of one industry sector, the standard deviation of the loss does not depend on the specific form of the density function. The specification of g is thus only relevant for the overall loss distribution and the calculation of the risk capital. In addition, there are few data to support the exact shape of a multidimensional distribution for default events. That is why we proceed in a different way: we first calculate the UL of the overall portfolio and then use the single sector approach to calculate the full loss distribution.

The Moments of the Loss Distribution for Several Industries

We proceed directly to calculate the expected loss EL and the unexpected loss UL . The expected loss by industry sector is:

$$EL_1 = P_1'(1) , \quad EL_2 = P_2'(1) . \quad (9)$$

The variances of the losses per industry and the covariance of the losses between the two industries are given by (for further details see the appendix):

$$\begin{aligned} UL_1^2 &= P_1'(1)^2 \sigma_1^2 + P_1''(1) + P_1'(1) \\ UL_2^2 &= P_2'(1)^2 \sigma_2^2 + P_2''(1) + P_2'(1) \\ Cov_{1,2} &= P_1'(1) P_2'(1) \cdot Cov(\gamma_1, \gamma_2) . \end{aligned} \quad (10)$$

It is now easy to derive the desired formula for the unexpected loss UL of the overall portfolio using the relationship $UL^2 = UL_1^2 + UL_2^2 + 2Cov_{1,2}$, where we express the covariance by the corresponding correlation and the variances:

$$UL^2 = \sigma_1^2 EL_1^2 + \sigma_2^2 EL_2^2 + 2 \cdot Cor(\gamma_1, \gamma_2) \cdot \sigma_1 \sigma_2 \cdot EL_1 \cdot EL_2 + \sum_A p_A v_A^2 \quad (11)$$

In the general situation of N industry sectors $k = 1, \dots, N$ we obtain in analogy

$$UL^2 = \sum_k \sigma_k^2 EL_k^2 + \sum_{\substack{k, \ell \\ k \neq \ell}} Cor(\gamma_k, \gamma_\ell) \cdot \sigma_k \sigma_\ell \cdot EL_k \cdot EL_\ell + \sum_A p_A v_A^2 \quad (12)$$

Again, as in the single sector case, the first two sums represent the risk due to systematic changes in the industries, reflected by the relative default variances, and the correlations between them. The third sum is the risk contribution due to the statistical nature of default events, which - as in the single sector case - is only important for small portfolios or for cases with low systematic risk. The correlation between sectors - if positive as usually observed in the economy - increases the systematic risk contribution and becomes very important when working with many sectors.

Remark:

To estimate the risk parameters - the relative default variance and the correlation of default events between industries - one can choose between three approaches: (1) using industry specific time series of historically observed default events, (2) using asset values of firms to derive pairwise asset correlations, which are then linked to the relevant default correlations using the Merton model (cf. [ME74]), and (3) using a factor model that relates the relative number of default events to macro-economic drivers.

The method to be applied depends on the type of portfolio: while time-series of defaults are suited for middle market portfolios with mostly non-quoted firms, the asset value approach is best for portfolios with large corporates. A limiting factor for both methods is the time period over which data exists. Here a factor model has some advantage because time series of the drivers of default are available far back in time.

Equation (12) allows one to calculate the UL of the portfolio loss distribution in a manner consistent with the observed sector covariance matrix. It also lays the foundation to calculate UL risk contributions at the level of individual obligors, as shown below.

To calculate the full loss distribution, we match the UL from the *CreditRisk+* single sector approach (equation (6)) to the UL as from equation (12):

$$\sigma^2 EL^2 = \sum_k \sigma_k^2 \cdot EL_k^2 + \sum_{\substack{k, \ell \\ k \neq \ell}} Cor(\gamma_k, \gamma_\ell) \cdot \sigma_k \sigma_\ell \cdot EL_k \cdot EL_\ell \quad (13)$$

The *relative default variance* σ^2 is determined by equation (13). The parameter σ is now used for the Gamma-distribution in *CreditRisk+* in order to calculate the portfolio loss distribution and the overall economic capital.

Illustrative Example:

To illustrate the correct application of the formulae we provide a very simple example. We choose two industry sectors, each of them having 1000 obligors with identical exposures (net of recovery) and identical default probabilities per sector. The relative default variance is assumed to be $\sigma_k^2 = 0.75^2$ for both sectors. The detailed figures are given in Table 1 below.

	Sector 1	Sector 2
Number obligors	1000	1000
Exposure per obligor	1	2
Default probability p_A	4%	2%
Relative default variance	0.75^2	0.75^2
EL	40	40
UL^2 (see equation (6))	940	980
UL	30.7	31.3

Table 1: Parameters for the two sectors of the sample portfolio

Note that the major driver for the UL is the risk due to the systematic variation of the number of defaults, which contributes $0.75 * EL = 30$ to the sector UL 's.

If we consider the overall portfolio we have to take the correlation of default events between the two industry sectors into account, which is assumed to be 50%. We get the following results:

Overall Portfolio (Sector 1 + 2)	
Sector correlation	50%
Number obligors	2000
Relative default variance	0.65^2
EL	80
UL^2 (see equation (11))	2820
UL	53.1

Table 2: Overall portfolio parameters

If we neglect the correlation between the two sectors, the UL will be only 43.8 - about 20% smaller than with a correlation of 50%. Notice that a correlation of 50% between sectors is not unusually high, since it does not correspond to individual default events, but to the correlation on an aggregated level. Given the UL of the overall portfolio, we obtain from equation (13) the relative default variance of the combined portfolio $\sigma^2 = 0.65^2$. This is to be compared with a relative default variance of 0.53^2 , if we assume independence of the two sectors, thus underestimating the risk in the overall portfolio. Using the calculated value for σ , we can now derive the complete loss distribution assuming a Gamma-distributed number of default events in the economy, as done in *CreditRisk+*, see fig. 1.

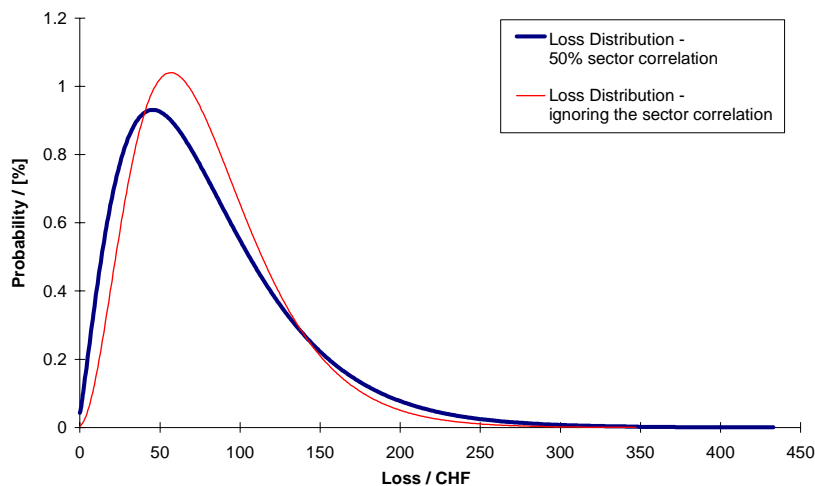


Fig. 1: The loss distributions of the overall portfolio for the case when correlation between the sectors is taken into account (fat line, 99% percentile 250 CHF) and when correlations are ignored (thin line, 99% percentile 214 CHF).

Calculating Risk Contributions

Most banks are changing their approach to their loan business, moving from a buy-and-hold strategy towards active portfolio management. The credit risk of complete sectors is sold to investors or credit risk is taken over from other banks with the goal of diversifying existing portfolios. These transactions can only be valued correctly if their relationship to the existing portfolio is taken into account. The sector approach is an efficient and practical way of doing this - we need to know only the risk inherent in the sector to be transferred (stand-alone) and its correlation to the rest of the portfolio. Given this information we can calculate the risk contributions of the sector to the overall portfolio risk, using the approach presented above. Therefore, the ability to take sector correlations into account when calculating risk contributions is a prerequisite for active portfolio management.

In general, the risk contribution of a single obligor A can be defined as follows:

$$RC_A = v_A \frac{\partial UL}{\partial v_A} = \frac{v_A}{2UL} \frac{\partial UL^2}{\partial v_A} \quad (14)$$

To clarify, RC_A is the sensitivity of the portfolio UL with respect to changes of the exposure v_A of obligor A times its exposure. The risk contributions of all obligors sum up to the UL of the portfolio.

Using the *UL*-formula (equation (11)) developed above, by a straightforward calculation, we obtain the following expression for the risk contribution of an obligor *A* in sector *k* :

$$RC_A = \frac{P_A V_A}{UL} \left[\sigma_k^2 EL_k + \sum_{\ell: \ell \neq k} Cor(\gamma_k, \gamma_\ell) \cdot \sigma_k \sigma_\ell \cdot EL_\ell + v_A \right]. \quad (15)$$

Again, the first two terms in the bracket represent the systematic risk, incorporating sector default correlations, whereas the last term is due to the statistical nature of default. The risk contribution of a complete sector is obtained by adding up the risk contribution all obligors in the sector.

Summary

For active portfolio management, sector analysis is an important ingredient to benefit from diversification. The standard *CreditRisk+* approach is based on independent industry sectors. In order to perform a sector analysis, the authors of *CreditRisk+* propose apportioning the default probability per obligor to different independent industries. However, such an approach is difficult to realize in practice.

We have extended the sector approach to a framework that takes the correlations between sectors directly into account. The procedure is easy to implement, and allows one to calculate the unexpected loss, individual risk contributions and the loss distribution of the full portfolio.

Appendix

Throughout our paper, rather than dealing with probability distributions directly, we use probability generating functions instead. The reason being that generating functions are a compact way of representing discrete probability distributions, which moreover have some useful properties, namely (see for instance [L168]):

- the generating function of the sum of two independent random variables is the product of the generating functions of the individual variables,
- the moments of the probability distribution can be expressed by the derivatives of the generating function.

As an illustration, we derive the covariance formula (equation (10)) in the two-industry case. In general, the probability generating function $F(z, w)$ of a two dimensional discrete distribution has the form

$$F(z, w) = \sum_{m,n} p(m,n) z^m w^n$$

where $p(m,n)$ denotes the probability of the state (m,n) . It is easy to check that the covariance can be expressed by

$$Cov_{1,2} = \left(\frac{\partial^2 F}{\partial z \partial w} - \frac{\partial F}{\partial z} \frac{\partial F}{\partial w} \right) (1,1).$$

We now compute the mixed derivative for our specific generating function $F(z, w)$ from equation (8). This implies

$$\frac{\partial^2 F}{\partial z \partial w} (1,1) = \int_0^\infty \int_0^\infty \frac{\partial^2 G^{(\gamma_1, \gamma_2)}}{\partial z \partial w} (1,1) g(\gamma_1, \gamma_2) d\gamma_1 d\gamma_2 = P_1'(1) P_2'(1) E(\gamma_1 \gamma_2),$$

where we have used that

$$\frac{\partial^2 G^{(\gamma_1, \gamma_2)}}{\partial z \partial w} (z, w) = \gamma_1 P_1'(z) \gamma_2 P_2'(w) G^{(\gamma_1, \gamma_2)}(z, w).$$

In a similar way, one can show that $\frac{\partial F}{\partial z} (1,1) = P_1'(1) E(\gamma_1)$ and $\frac{\partial F}{\partial w} (1,1) = P_2'(1) E(\gamma_2)$. Altogether, we obtain:

$$Cov_{1,2} = P_1'(1) P_2'(1) \cdot (E(\gamma_1 \gamma_2) - E(\gamma_1) E(\gamma_2)) = P_1'(1) P_2'(1) \cdot Cov(\gamma_1, \gamma_2)$$

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