Models of Joint Defaults in Credit Risk Management: An Assessment

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Abstract

In this paper we review the models of joint defaults of the current major industry-sponsored credit risk frameworks. Recognizing the need for further improvements of these models, we address the following issues. First, we identify the most important modeling drawbacks that could be fixed on a short-term basis. Second, we analyze which of the proposed models is the conceptually most promising basis for next-generation models. Concluding that the KMV methodology is the most suitable to go forward, we set out a research agenda aiming at further improvements and at extending the KMV model to non-quoted firms.

Keywords: Credit portfolio management, Credit risk models, Joint defaults.

JEL: G11, G21, G28.

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1 Introduction

The last years have seen a rapid growth of interest in credit risk modeling from banking regulators, practitioners and academic researchers. Since the first generation of models have been developed, big banks have used these models for risk management purposes. Moreover, regulators have started to explore the potential of the models to determine regulatory capital (Basel Committee on Banking Supervision 1999).

The models' accuracy is the major factor in determining the success of both types of applications. It has therefore become an important point of focus in academic research on credit risk modeling. The findings up to now can be summarized as follows. First, many empirical and theoretical objections regarding the modeling assumptions and the models' calibration have been raised.[1] Second, it has been pointed out that the models' outputs are very sensitive with respect to parameter estimates and differ quite significantly across models (see e.g. Gordy (2000)). Finally, a first backtesting study (Nickell, Parraudin, and Varotto 1999) for two of the currently available models found that - for portfolios of Eurobonds - the models yielded far more exceptions than they would if they were accurately measuring risk.

The one conclusion emerging from this academic discussion is that current credit risk models cannot yet be used to determine regulatory capital and that new, more sophisticated models have to be developed to measure credit risk more accurately. Moreover, it has been argued that once sufficiently elaborated models are available, extensive backtesting will be necessary before these models can be used to determine regulatory capital.

Some general suggestions for important roads of improvements have been made. Most prominently it has been argued that credit risk models should employ stochastic interest rates instead of deterministic ones (see e.g. Crouhy, Galai, and Mark (2000)). It has also been put forward that they should take into account that default probabilities and rating transition probabilities vary through the business cycle or between obligors belonging to different industries (see e.g. Nickell, Parraudin, and Varotto (2000)). However, to our knowledge there are no contributions that take a broader view, aimed at the description of a detailed research agenda for the development of next-generation models from today's existing methodologies.

This paper intends to contribute to the closure of this gap by focusing on one particular area of credit risk modeling, *joint default probabilities.* In doing so, we will also present a broad picture of modeling drawbacks and suggestions for the improvement of existing

[1] For an overview see Jackson and Perraudin (2000).

methodologies in this area. We will analyze the four major currently available industry-sponsored credit risk models: CreditMetrics (by JP Morgan), Portfolio Manager (by KMV), CreditRisk$^+$ (by Credit Suisse Financial Products) and CreditPortfolioView (by McKinsey). In the following we will also refer to these models as CM (CreditMetrics), KMV, CR (CreditRisk$^+$) and McK (McKinsey).[2]

We start our analysis by identifying the most important modeling problems with respect to joint defaults. While repeating some of the arguments that have already been made, we will be able to point to quite a few drawbacks that have not been discussed yet. We then make suggestions on how the models' performance could be improved.

Reviewing the identified modeling problems and the improvements that could be achieved, we finally set out to assess which of the currently proposed approaches to joint-default modeling is the *conceptually* most promising basis for next-generation models.[3] We argue that a "mixed" model blending modeling features of KMV and of a model recently employed in the literature (NICKELL, PARRAUDIN, AND VAROTTO 2000) is the most suitable to go forward. Moreover, we try to set out a research agenda aiming at further improvements and at extending the scope of KMV-type model to non-quoted firms.

In doing so we stress that - besides backtesting *complete* models - it is important to assess (theoretically and empirically) the adequateness of the model's different building blocks with respect to modeling assumptions and parameter estimation. Once sufficiently well-performing blocks have been developed, backtesting of complete models can determine how wide remaining error margins are. This method has not only the advantage that bad performance can be tracked more specifically to single model components but also makes it possible to combine successful parts of different models.

Finally, while we advocate to use the mixed KMV model as a starting point for next-generation models, we recognize that *each* of the currently proposed models will be applied in banking practice in the near future. We think that the suggestions made for CM, CR and McK can help to fix some important drawbacks on a short-term view.

The paper is organized as follows. In section 2 we present a detailed description of the currently proposed joint-default models. In section 3 we discuss the major drawbacks

[2]A comprehensive description of all of these models can be found in CROUHY, GALAI, AND MARK (2000). More specific references are JP MORGAN (1997) (for CM), CREDIT SUISSE (1997) (for CR) and MCKINSEY (1998) (for McK). To our knowledge, the best documentation available for the KMV model is the paper of CROUHY, GALAI, AND MARK (2000). Some interesting information can also be found on the KMV homepage (http://www.kmv.com).

[3]"Non-conceptual" issues have also been put forward in the discussion of the models' relative merits, in particular computational simplicity (see e.g. CROUHY, GALAI, AND MARK (2000)). Such issues will not be taken up here.

of the models, argue which conceptual issues should be clarified, and suggest how the models' performance could be improved. The suggestions contain short-term fixes as well as a longer-term research agenda. In section 4 we present our conclusions. A summary of the major insights from section 3 builds the basis for the identification of the most promising joint-default approach for next-generation models.

2 Presentation of the Models

Despite the fact that a comprehensive survey for all of the credit risk models discussed in this paper has recently been provided by Crouhy, Galai, and Mark (2000), we will present a detailed description of each proposed joint-default methodology in this section. This is done for the following reasons. First, since Crouhy, Galai, and Mark (2000) capture all aspects of credit risk modeling, the description of the joint-default models is not as detailed as necessary. In particular, the presentation of the multi-year horizon (for CR and McK), the construction of the KMV country and industry indices, and KMV's estimation of the relative size of the systematic risk component are important modeling features that are not described in Crouhy, Galai, and Mark (2000). Second, focusing on joint-default models allows us to present all models within a unified framework which makes similarities and differences more transparent and provides a clear background for the analysis. Finally, while reviewing the literature, we found two interesting new versions of current credit risk models, one for CR (Gordy 2000), which hereafter will be referred to as CR-GO, and one for KMV (Nickell, Parraudin, and Varotto 1999), hereafter referred to as NPV. Both were not presented in their own rights but as part of studies attempting to compare model outputs and backtest currently proposed models. Nevertheless, we think that these versions represent important alternatives to the initial proposals, and we will therefore discuss them as well.

To focus on joint-default modeling, we employ the most simple credit risk management framework. We consider a bank that holds a loan portfolio and does risk management in $t = 0$. The loans are due in $t = T$ and we assume that the bank's risk management horizon is identical with the date at which the loans mature, i.e. the bank is interested in the distribution of the value of its portfolio at $t = T$. For each loan, two states are possible in $t = T$. Either the firm has not defaulted and the loan's principal and interest are paid back; or the firm has defaulted and the bank receives nothing.[4] Denoting the sum of principal and interest due from firm i by L_i and the default event indicator variable by

[4]It is easy to integrate fixed recovery rates.

$B_{i,T}$, we can describe the $t = T$ portfolio value by

$$\sum_{i=1}^{n} L_i B_{i,T}.$$

In general, $B_{i,t}$ is an indicator variable which is zero if firm i has not yet defaulted until time t and which is equal to 1 otherwise ($i = 1, .., n$). Moreover, we use $\mathbf{B}_t$ to denote the vector $(B_{i,t})_{i=1}^n$ of default indicators at time t. Obviously, the distribution of the above portfolio value is completely described by the distribution of the vector $\mathbf{B}_T$.

We follow the literature and divide the credit risk models into two classes, labeled "structural models" (CM, KMV and NPV) and "reduced-form models" (CR, CR-GO and McK) respectively. This terminology can be found for example in JARROW AND TURNBULL (2000). It refers to the fact that structural models build on a microeconomic description of firm defaults while reduced-form models employ general heuristics predicting how default probabilities change due to changes in systematic factors such as the business cycle. Since the reduced-form models anualize the credit risk horizon, we will measure time in years and will - for simplicity of representation - assume that $T \in \{1, 2, 3, ...\}$. Negative time indices will be employed to describe historical observations that are used to calibrate the models. T_h denotes the length of the time span from which these observations are taken. Figure 1 illustrates the time horizon.

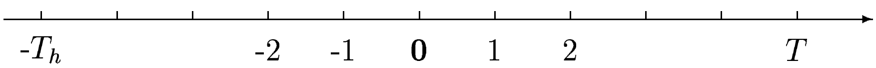

Figure 1: Time horizon.

Note that all models derive joint default probabilities by defining a vector $\mathbf{S}$ representing the systematic risk in the economy. Default events are assumed to be independent given the realization of $\mathbf{S}$. The probability of a certain realization $\mathbf{b} = (b_1, ..., b_n)$ of the vector $\mathbf{B}_T$ can then be derived by integrating conditional default probabilities $p_{i,T}(\mathbf{S})$ over the distribution of $\mathbf{S}$:[5]

$$\mathbb{P}\{\mathbf{B}_T = \mathbf{b}\} = \mathbb{E} \prod_{i=1}^{n} \Big\{ 1\{b_i = 1\} p_{i,T}(\mathbf{S}) + 1\{b_i = 0\} \Big(1 - p_{i,T}(\mathbf{S})\Big) \Big\}.$$

[5]Note how the indicator function $1\{\cdot\}$ is defined. $1\{A\}$ is equal to 1 if statement A holds and equal to 0 if statement A does not hold.

2.1 Reduced-Form Models

For both reduced-form models, the vector $\mathbf{S}$ consists of T systematic vectors: $\mathbf{S} = (\mathbf{S}_1, .., \mathbf{S}_T)'$.[6] The entries $\mathbf{S}_1, ..., \mathbf{S}_T$ can be thought of as realizations of a systematic vector at different points in time. At each point $t-1$, the probability that a firm that has survived until $t-1$ will also survive until t is analyzed ($t = 1, .., T$). Default events are assumed to be independent given $\mathbf{S}_t$ and the conditional one-year default probabilities

$$p_i(\mathbf{S}_t) := \mathbb{P}\Big\{B_{i,t} = 1 \,|\, B_{i,t-1} = 0, \mathbf{S}_t\Big\}$$

are assumed to be stable over time. The multi-year conditional default probabilities $p_{i,T}(\mathbf{S})$ are then specified in the following way:[7]

$$p_{i,T}(\mathbf{S}) := 1 - \prod_{t=1}^{T}\Big(1 - p_i(\mathbf{S}_t)\Big). \tag{1}$$

In the following sections we discuss how the conditional one-year default probabilities $p_i(\cdot)$ and the distribution of the vector $\mathbf{S}$ are determined for the different models. The unconditional one-year default probability of firm i will be denoted by $\bar{p}_i$ ($i = 1, .., n$). It is important to note that the term unconditional refers to the fact that the default probabilities are integrated over the distribution of the complete time series $(\mathbf{S}_t)_{t=-\infty}^{+\infty}$ of the systematic factors.

2.1.1 CreditRisk$^+$ (CR)

To develop the reasoning behind the CR model, we start with the most simple version of CR. CR does not model the distribution of the systematic factors $\mathbf{S}$ *directly* but only the distribution of the resulting portfolio default rate, denoted by μ_t^{P}:[8]

$$\mu_t^{\mathrm{P}} := \mu^{\mathrm{P}}(\mathbf{S}_t) := \frac{1}{n}\sum_{i=1}^{n} p_i(\mathbf{S}_t). \tag{2}$$

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[6] We will use the symbol "$'$" to describe a transposed vector or matrix.

[7] In the appendix, we show that this specification rests on the implicit assumption that

$$\mathbb{P}\{B_{i,t} = b_i \,|\, \mathbf{S}_t, ..., \mathbf{S}_T\} = \mathbb{P}\{B_{i,t} = b_i \,|\, \mathbf{S}_t\} \quad (t = 1, ..., T-1,\ i = 1, ..., n).$$

We will comment on this assumption in section 3.1.1 in the paragraph dedicated to multi-year default probabilities (page 27).

[8] To be more precise, $\mu_t^{\mathrm{P}}(\mathbf{S}_t)$ is the *expected* portfolio default rate conditioned on $\mathbf{S}_t$. However, in this paper we will - for the sake of brevity and to keep track with the terminology used in the CR manual - use the loose vocabulary "portfolio default rate".

It is assumed that the series μ_t^{P} is gamma-distributed and independent, identically distributed (iid) over time.[9] Given the realization of the default rate μ_t^{P}, the default events of all obligors are assumed to be independent. The conditional default probability for each firm i is specified by

$$p_i(\mu_t^{\mathrm{P}}) = \bar{p}_i \frac{\mu_t^{\mathrm{P}}}{\bar{\mu}^{\mathrm{P}}}, \tag{3}$$

where $\bar{\mu}^{\mathrm{P}}$ is the mean of μ_t^{P}. Note that this definition of conditional default probabilities guarantees (a) that $\mu_t^{\mathrm{P}} = (1/n)\sum_{i=1}^n p_i(\mu_t^{\mathrm{P}})$ for all realizations of μ_t^{P} and (b) that $\bar{p}_i = \mathbb{E}\, p_i(\mu_t^{\mathrm{P}})$. Moreover, note that by equations (2) and (3) the mean $\bar{\mu}^{\mathrm{P}}$ and the standard deviation σ^{P} of μ_t^{P} are given by

$$\bar{\mu}^{\mathrm{P}} = \frac{1}{n}\sum_{i=1}^n \bar{p}_i \tag{4}$$

$$\sigma^{\mathrm{P}} = \frac{1}{n}\sum_{i=1}^n \sigma_i. \tag{5}$$

σ_i is the standard deviation of the random variable $p_i(\mu_t^{\mathrm{P}})$, i.e.

$$\sigma_i = \sqrt{\mathrm{Var}\Big(p_i(\mu_t^{\mathrm{P}})\Big)} = \sqrt{\mathrm{Var}\Big(p_i(\mathbf{S}_t)\Big)}.$$

To determine mean and standard deviation of μ_t^{P} from equations (4) and (5), CR suggests to estimate the parameters $\bar{p}_i$ and σ_i $(i = 1, ..., n)$ from rating class default-rate series by mapping each obligor i to a rating class $\zeta(i)$.[10] The parameter estimates have to be provided by the user. An estimation technique is outlined in section 2.1.4. We illustrate the model setup as produced up to now with the following example.

Example 1:

> Suppose that the conditional default probabilities are given by $p_i(\mathbf{S}_t) = c_i G(\mathbf{S}_t)$ where $G(\cdot)$ is an arbitrary function mapping into the interval $[0, \min_{i=1,..,n}\{1/c_i\}]$.

[9] The only motivation given in the CR manual for choosing the Gamma distribution, is that it can be paramatrized solely by its mean and standard deviation.

[10] Rating class default rate data can be taken from rating agencies such as Moody's or Standard&Poor's.

Then

$$\mu_t^{\mathrm{P}} = \frac{G(\mathbf{S}_t)}{n} \sum_{i=1}^{n} c_i,$$
$$\bar{\mu}^{\mathrm{P}} = \frac{\mathbb{E}\, G(\mathbf{S}_t)}{n} \sum_{i=1}^{n} c_i$$

and $\bar{p}_i = c_i \,\mathbb{E}\, G(\mathbf{S}_t)$. Hence, formula (3) indeed specifies conditional default probabilities correctly.

To account for diversification effects, the framework outlined above can be extended by dividing the obligors among different subsets $\mathcal{S}_1, .., \mathcal{S}_m$ where each subset is a collection of obligors under the common influence of a systematic factor. These subsets are called "sectors" in the CR manual but do not necessarily represent *industry* sectors. An other example might be the division of a portfolio according to the country of domicile of each obligor. The default rates

$$\mu_{j,t}^{P} := \frac{1}{n} \sum_{i=1}^{n} 1\{i \in \mathcal{S}_j\} p_i(\mathbf{S}_t) \tag{6}$$

of each sector are assumed to be gamma distributed, independent of each other and iid in time. Moreover, defaults are assumed to be independent given the realization of the vector $\boldsymbol{\mu}_t^{\mathbf{P}}$ of all sector default rates. Conditional default probabilities are specified by

$$p_i(\boldsymbol{\mu}_t^{\mathbf{P}}) := \bar{p}_i \sum_{j=1}^{m} 1\{i \in \mathcal{S}_j\} \frac{\mu_{j,t}^{P}}{\bar{\mu}_j^{\mathrm{P}}}, \tag{7}$$

where $\bar{\mu}_j^{\mathrm{P}}$ is the mean of $\mu_{j,t}^{P}$. This implies that the mean $\bar{\mu}_j^{\mathrm{P}}$ and the standard deviation σ_j^{P} of the default rate in sector j are given by

$$\bar{\mu}_j^{\mathrm{P}} = \frac{1}{n} \sum_{i=1}^{n} 1\{i \in \mathcal{S}_j\} \bar{p}_i \tag{8}$$
$$\sigma_j^{\mathrm{P}} = \frac{1}{n} \sum_{i=1}^{n} 1\{i \in \mathcal{S}_j\} \sigma_i. \tag{9}$$

The most general version of CR allows for the mapping of each obligor i into more than one sector. This is achieved by assigning obligor-specific weights $(\theta_{i1}, ..., \theta_{im})$ to each sector that are required to sum to 1 ($\sum_{j=1}^{m} \theta_{ij} = 1$). The weights are specified by the user. By formal analogy (i.e. by substituting $1\{i \in \mathcal{S}_j\}$ with θ_{ij}), the formulas (7) - (9)

are generalized to

$$p_i(\boldsymbol{\mu}_t^{\mathbf{P}}) = \bar{p}_i \sum_{j=1}^{m} \theta_{ij} \frac{\mu_{j,t}^{P}}{\bar{\mu}_j^{\mathrm{P}}} \tag{10}$$

$$\bar{\mu}_j^{\mathrm{P}} = \frac{1}{n} \sum_{i=1}^{n} \theta_{ij} \bar{p}_i \tag{11}$$

$$\sigma_j^{\mathrm{P}} = \frac{1}{n} \sum_{i=1}^{n} \theta_{ij} \sigma_i. \tag{12}$$

It should be noted that the CR manual gives no definition of the variables $\mu_{j,t}^{P}$ $(j = 1, .., m)$ in this general version of the model. Consequently, it is also unclear what formulas (11) and (12) actually mean.[11]

Finally, note that CR models idiosyncratic risk by employing an additional sector (called idiosyncratic or specific sector). A model with $m-1$ systematic risk factors and obligor-specific idiosyncratic risk can be parametrized as a m-sector model in which one of the sectors (without loss of generality sector 1) represents the idiosyncratic risk components. The default rate associated with this sector is assumed to have zero volatility and thus $\mu_1^P/\bar{\mu}_1^{\mathrm{P}} = 1$.[12] θ_{i1} is supposed to describe the size of the idiosyncratic component of default risk.

In the sequel we will use the following terminology to refer to the different CR versions: *single-sector version* (to denote the simple version which is built on only one default rate), *multi-sector version without weights* (to refer to the version with m sectors where each obligor can only be mapped to one sector) and *weighted multi-sector version* (to refer to the most general specification where each obligor can be mapped to all sectors by assigning obligor-specific weights to each sector).

2.1.2 CreditRisk$^+$: The Modified Version of Gordy (CR-GO)

Gordy (2000) compares the model outputs of CR and CM for a variety of hypothetical portfolios. In doing so he presents a more abstract form of the weighted two-sector version of the CR model (which will hereafter be referred to as CR-GO). He abstracts from the modeling of portfolio default rates and, hence, replaces the default rate $\mu_{2,t}^{P}$ of the second (systematic-risk-) sector by a general systematic factor $\mathbf{S}_t$. $\mathbf{S}_t$ is a scalar, and the series $(\mathbf{S}_t)_{t=-T_h}^{T}$ is assumed to be gamma distributed and iid in time. By convention, the first

[11]We discuss this problem in more detail in section 3.1.1.

[12]See the CR manual (Credit Suisse 1997), section A 12.3. Because the specific sector represents diversifiable risk, it is assumed to contribute no volatility to a well diversified portfolio.

sector represents the idiosyncratic risk (i.e. $\mu^P_{1,t}/\bar{\mu}^{\mathrm{P}}_{1,t} = 1$), and the weight on this sector is given by $\theta_{i1} = 1 - \theta_{i2}$. $\theta_i := \theta_{i2}$ is called the weight on the systematic factor and conditional default probabilities are described in analogy to equation (10):

$$p_i(\mathbf{S}_t) = \bar{p}_i\Big((1-\theta_i) + \theta_i \frac{S_t}{\mathbb{E}\, S_t}\Big). \tag{13}$$

Gordy then maps obligors to rating classes ($i \to \zeta(i)$). Within each rating grade ζ, obligors are assumed to be *statistically identical,* i.e. they have the same unconditional default probability $\bar{p}_\zeta$ and the same weight θ_ζ on the systematic factor. To calibrate the model, Gordy estimates the unconditional default probabilities $\bar{p}_\zeta$ and the variances of default probabilities $\sigma^2_\zeta := \mathrm{Var}\big(p_\zeta(\mathbf{S}_t)\big)$ from rating class default-rate series. The estimation method is outlined in section 2.1.4. The parameters θ_ζ are then obtained by inserting the estimates into the formula for default-probability variances that follows from equation (13):

$$\sigma^2_\zeta = \bar{p}^2_\zeta \theta^2_\zeta \,\mathrm{Var}(\mathbf{S}_t).$$

However, these estimates are not sufficient to fully specify the model, since $\mathrm{Var}(\mathbf{S}_t)$ is still undetermined. Gordy argues that there is no obvious information to estimate this parameter, and (for illustration purposes) considers three different values for $\sqrt{\mathrm{Var}(\mathbf{S}_t)}$, namely 1, 1.5 and 4.0.

2.1.3 CreditPortfolioView (McK)

McK models the distribution of certain industry or country (one-year) default rates $(\hat{\mu}^{\mathrm{C}}_{1,t}, ..., \hat{\mu}^{\mathrm{C}}_{m,t})$ and then assumes that default events are independent given the realization of the vector $\hat{\boldsymbol{\mu}}^{\mathbf{C}}_t := (\hat{\mu}^{\mathrm{C}}_{j,t})^m_{j=1}$ of these default rates. The default-rate model is constructed using macroeconomic variables $\mathbf{S}_t := (S_{1,t}, ..., S_{k,t})$. For each default rate $\hat{\mu}^{\mathrm{C}}_{j,t}$ employed, a macro index $\bar{S}_{j,t}$ is constructed as a linear combination of the macroeconomic variables:

$$\bar{S}_{j,t} = \sum_{l=1}^{k} \tau_{jl} S_{l,t},$$

where the weights τ_{ij} on each of the macroeconomic variables are fitted to the respective default-rate series using a logit model:

$$\hat{\mu}^{\mathrm{C}}_{j,t} = \frac{1}{1 + \exp(\bar{S}_{j,t} + \bar{\nu}_{j,t})}.$$

The variables $(\bar{\nu}_{1,t}, ..., \bar{\nu}_{m,t})$ are assumed to be normally distributed zero-mean error terms that are independent in time, while the macroeconomic variables are assumed to follow an AR(2) process with jointly normally distributed innovations $(\nu_{j,t})_{j=1}^{k}$. Hence, by estimating the parameters of the AR process and the covariance matrix

$$\begin{pmatrix} \Sigma_{\bar{\nu}} & \Sigma_{\bar{\nu},\nu} \\ \Sigma_{\nu,\bar{\nu}} & \Sigma_{\nu} \end{pmatrix}$$

of innovation and error terms,[13] the joint distribution of the vectors $\hat{\boldsymbol{\mu}}_1^{\mathbf{C}}, ..., \hat{\boldsymbol{\mu}}_T^{\mathbf{C}}$ is specified conditional on the realization $\mathbf{S}_0$ of the macroeconomic variables in $t = 0$.

To derive the conditional default probabilities $p_i(\hat{\boldsymbol{\mu}}_t^{\mathbf{C}})$, the model draws on the intuition that with default rates higher than average, default probabilities should also be higher than unconditional default probabilities while the contrary should be true if default rates are lower than average. The formal specification used in McK is

$$p_i(\hat{\boldsymbol{\mu}}_t^{\mathbf{C}}) = \begin{cases} \Big(1 - \exp(-\kappa_i)\Big)(1 - \bar{p}_i) + \bar{p}_i & \text{if } \kappa_i = \kappa_i(\hat{\boldsymbol{\mu}}_t^{\mathbf{C}}) \geq 0 \\ \exp(\kappa_i)\bar{p}_i & \text{if } \kappa_i = \kappa_i(\hat{\boldsymbol{\mu}}_t^{\mathbf{C}}) < 0, \end{cases}$$

where

$$\kappa_i(\hat{\boldsymbol{\mu}}_t^{\mathbf{C}}) := \Big[\sum_{j=1}^{m} \theta_{ij} \frac{\hat{\mu}_{j,t}^{\mathrm{C}}}{\mathbb{E}\,\hat{\mu}_{j,t}^{\mathrm{C}})}\Big] - 1.$$

θ_{ij} describes the sensitivity of firm i with respect to the default rate $\hat{\mu}_{j,t}^{\mathrm{C}}$. In the current version of McK, conditional default probabilities of a firm i can only depend on the default rate of one industry/country $j = j(i)$ (hence $\theta_{ij} = 0$ for all $j \neq j(i)$) and the intensity of exposure to the respective default rate is assumed to be identical for each firm ($\theta_{ij(i)} = \theta$ for all i). Hence, the expression for κ_i simplifies to

$$\kappa_i(\hat{\boldsymbol{\mu}}_t^{\mathbf{C}}) = \theta \frac{\hat{\mu}_{j(i)}^{\mathrm{C}}}{\mathbb{E}\,\hat{\mu}_{j(i)}^{\mathrm{C}}} - 1,$$

implying that if two firms i and $\tilde{i}$ are mapped to the same default rate ($j(i) = j(\tilde{i}) = j$), then the defaults of those firms are assumed to be independent given $\hat{\mu}_{j,t}^{\mathrm{C}}$. To calibrate the model, McK estimates the AR process that describes the stochastic of the macroeconomic variables, and the logit model that links those variables to firm default rates. Moreover,

[13] $\Sigma_{\bar{\nu}}$ is the covariance matrix of the error terms $\bar{\nu}_i$, Σ_{ν} is the covariance matrix of the innovations ν_i, and $\Sigma_{\bar{\nu},\nu}$ is the matrix describing the covariances between error terms and innovations.

as for CR, unconditional default probabilities are determined using average rating class default frequencies. The sensitivity parameter θ has to be set by the user. It is suggested to estimate θ from default data published by rating agencies.[14]

2.1.4 Estimating Mean and Variance of Default Probabilities

In this section we present a method proposed by GORDY (2000), to estimate both mean and variance of annual default probabilities from default rates. It can be used to calibrate both CR versions we have presented. To describe this method, we first introduce some notation. We denote the number of corporate obligors in the rating class ζ in $t-1$ by $\hat{n}_{\zeta,t}$ and the number of obligors in ζ who have defaulted until t by $\hat{d}_{\zeta,t}$. Gordy makes the following assumptions:

1. There is a vector of systematic factors driving default probabilities: for each year $t-1$, obligor defaults until t are independent given the realization $\mathbf{S}_t$ of this systematic vector in t.

2. Conditional one-year default probabilities are identical for all obligors in one rating grade ζ and are stationary over time. They are denoted by $p_\zeta(\mathbf{S}_t)$.

3. The systematic factor series $(\mathbf{S}_t)_{t=-T_h}^{T}$ and the series $(\hat{n}_{\zeta,t})_{t=-T_h}^{-T}$ are iid and independent of each other.

4. The idiosyncratic components of the default risk (which are modeled implicitly) are serially independent.

Under these assumptions, the observed rating class default frequencies $\hat{\mu}_{\zeta,t}^R := \hat{d}_{\zeta,t}/\hat{n}_{\zeta,t}$ are iid in time. By dropping the time index, it can be shown that the mean $\bar{p}_\zeta$ and the variance σ_ζ^2 of $p_\zeta(\mathbf{S}_t)$ are given by $\bar{p}_\zeta = \mathbb{E}\,\hat{\mu}_\zeta^R$ and

$$\sigma_\zeta^2 = \frac{\mathrm{Var}(\hat{\mu}_\zeta^R) - \mathbb{E}[1/\hat{n}_\zeta]\bar{p}_\zeta(1-\bar{p}_\zeta)}{1-\mathbb{E}[1/\hat{n}_\zeta]} \tag{14}$$

respectively.[15] Hence, estimates for $\bar{p}_\zeta$ and σ_ζ^2 can be derived by estimating $\mathbb{E}\,\hat{\mu}_\zeta^R$, $\mathrm{Var}(\hat{\mu}_\zeta^R)$, and $\mathbb{E}[1/\hat{n}_\zeta]$ from the series $(\hat{\mu}_{\zeta,t}^R)_{t=-1}^{-T_h}$ and $(\hat{n}_{\zeta,t})_{t=-1}^{-T_h}$ and by inserting these results in

[14] See MCKINSEY (1998), p. 56. Unfortunately, an estimation procedure is not described.

[15] The formula for $\bar{p}_\zeta$ is obvious, and the formula for σ_ζ^2 is derived in GORDY (2000).

equation (14).[16]

2.2 Structural Approach

The structural approach is employed by CM, KMV and NPV. Under this approach, the specification of joint defaults is derived from an option-pricing model as developed in Merton (1974). A firm goes bankrupt if the value of its assets falls below a certain threshold that depends on the firm's liability structure. Models of this type consist of two building blocks: (a) assumptions about the joint dynamics of the firms' asset values and (b) the firms' liability structures. We will start with the description of the former and denote firm i's asset value in t by $V_{i,t}$. The process V_i is assumed to follow the stochastic differential equation

$$dV_{i,t} = \mu_i^v V_{i,t}dt + \sigma_i^v V_{i,t}dW_{i,t} \quad (i = 1,..,n) \tag{15}$$

where the vector process $(W_1,..,W_n)$ is a multidimensional standard Brownian motion.[17] The correlation structure of this vector process is described using aggregate asset value processes V_j^S $(j = 1,..,k)$ that represent the systematic risk in the economy. In analogy to equation (15), those processes are assumed to follow

$$dV_{j,t}^S = \mu_j^S V_{j,t}^S dt + \sigma_j^S V_{j,t}^S dW_{j,t}^S \quad (j = 1,..,k). \tag{16}$$

The vector process $(W_1^S,..,W_k^S)$ is assumed to be a multidimensional standard Brownian motion with covariance matrix Σ^S. The firms' asset value processes are linked to the systematic processes by the joint log returns

$$Z_{i,t}(\Delta t) := \log(V_{i,t}/V_{i,t-\Delta t}) \quad (i = 1,..,n) \tag{17}$$

$$Z_{j,t}^S(\Delta t) := \log(V_{j,t}^S/V_{j,t-\Delta t}^S) \quad (j = 1,...,k) \tag{18}$$

[16]Since both series are iid, standard techniques can be used for estimation. If $(X_i)_{i=1}^n$ is an iid series, then the standard estimates for mean and variance are $\bar{X} := \frac{1}{n}\sum_{i=1}^n X_i$ and

$$\frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2$$

respectively.

[17]The term "standard" refers to the fact that the means and the standard deviations of the instantaneous returns of the processes $W_1,..,W_n$ are assumed to be equal to 0 and 1 respectively.

which are normally distributed.[18] More precisely, it is assumed that a firm's returns can be described as a linear combination of the systematic returns and an idiosyncratic component:

$$Z_{i,t}(\Delta t) = \sum_{j=1}^{k} \theta_{ij} Z_{j,t}^{S}(\Delta t) + \epsilon_{i,t}(\Delta t) \quad (i = 1, .., n), \tag{19}$$

where the random variables $\epsilon_{1,t}(\Delta t), .., \epsilon_{n,t}(\Delta t)$ are normally distributed for fixed Δt, mutually independent and independent of the systematic processes $V_1^S, .., V_k^S$.

To determine joint default probabilities, the firms' liability structures have to be specified and parameters have to be estimated. We first present the approaches of CM and KMV, and then the alternative approach suggested by Nickell, Parraudin, and Varotto (1999).

2.2.1 CreditMetrics and KMV

Both CM and KMV assume that default occurs if a firm's asset value drops below a certain level at the *end of the risk management horizon,* i.e in $t = T$.[19] In this case the default event can be equivalently expressed as the event that the standard normally distributed variable

$$Z_i := \frac{Z_{i,T}(T) - \mathbb{E}\, Z_{i,T}(T)}{\sqrt{T}\, \sigma_i^v}$$

hits some low level z_i (called default point). To derive the distribution of the vector $\mathbf{B}$ of default indicators, it is therefore sufficient to specify the default points and the joint distribution of the normalized returns Z_i $(i = 1, .., n)$. Default points are determined by $z_i := \Phi^{-1}(\bar{p}_{i,T})$, where $\Phi^{-1}(\cdot)$ is the inverse standard-normal distribution function and $\bar{p}_{i,T}$ is the default probability of firm i conditional on the information available in $t = 0$ but averaged over all realizations of Z_i $(i = 1, ..., n)$.

The joint distribution of standardized returns is determined in the following way. First, for each firm i, weights $\tilde{\theta}_{i1}, ..., \tilde{\theta}_{ik}$ are chosen that determine the extent to which firm i is exposed to the systematic processes $V_1^S, ..., V_k^S$. These weights are required to sum to 1

[18] Δt is the time interval on which the returns are reported.

[19] Note that - in our description of the KMV model - we primarily refer to the version described in the paper of Crouhy, Galai, and Mark (2000). To our knowledge, this paper contains the most precise publicly available documentation of the KMV model. We do not know whether KMV has also implemented other versions of its model.

($\sum_{j=1}^{k}\tilde{\theta}_{ij}=1$). We denote the matrix of all weights by $\tilde{\Theta} := (\tilde{\theta}_{ij})_{1\le i\le n;\,1\le j\le k}$. Second, to allow for the ratio of systematic return risk to overall return risk to take all values between 0 and 1, the parameters θ_{ij} are supposed to be multiples of $\tilde{\theta}_{ij}$: $\theta_{ij}=\theta_i\tilde{\theta}_{ij}$. Instead of determining the parameters θ_i directly, the variables Z_i are written as

$$Z_i = \lambda_i^S \frac{(\tilde{\Theta}\mathbf{S})_i}{\sqrt{(\tilde{\Theta}\Sigma^S\tilde{\Theta}^T)_i}} + \epsilon_i, \tag{20}$$

where

$$\lambda_i^S := \frac{\theta_i\sqrt{(\tilde{\Theta}\Sigma^S\tilde{\Theta}^T)_i}}{\sigma^v}.$$

$\mathbf{S}$ is a random vector distributed according to $\mathcal{N}(0,\Sigma^S)$ and the variables ϵ_i are mutually independent, normally distributed, independent of $\mathbf{S}$, and have zero mean and variance $1-(\lambda_i^S)^2$. Obviously, the parameter λ_i^S represents the ratio of systematic to overall return risk for firm i (in terms of standard deviation). Moreover, from equation (20) we can conclude that conditional default probabilities are given by

$$p_{i,T}(\mathbf{S}) = \Phi\Big\{\frac{z_i - \lambda_i^S(\tilde{\Theta}\mathbf{S})_i/\sqrt{(\tilde{\Theta}\Sigma^S\tilde{\Theta}^T)_i}}{\sqrt{1-(\lambda_i^S)^2}}\Big\}.$$

Hence, the distribution of the vector $\mathbf{B}$ is completely specified by choosing the unconditional default probabilities $\bar{p}_i$, the matrices Σ^S and $\tilde{\Theta}$, and the systematic risk ratios λ_i^S ($i=1,..,n$). CM and KMV differ in the way these parameters are determined. Before we describe the different approaches, we note that given the modeling assumptions, the parameters Σ^S and λ_i^S can be estimated from time series data. To see this, define

$$\mathbf{Z}_t(\Delta t) := \Big(Z_{i,t}(\Delta t)\Big)_{i=1}^{n}, \quad \mathbf{Z}_t^S(\Delta t) := \Big(Z_{i,t}^S(\Delta t)\Big)_{i=1}^{n} \tag{21}$$

and note that the following lemma holds.

Lemma 1

Suppose that the set of assumptions presented up to now holds. Then the vector series $\mathbf{Z} := \Big(\mathbf{Z}_{l\cdot\Delta t}(\Delta t)\Big)_{l=-1}^{T_h/\Delta t}$ *and* $\mathbf{Z}^S := \Big(\mathbf{Z}_{l\cdot\Delta t}^S(\Delta t)\Big)_{l=-1}^{T_h/\Delta t}$ *are both iid. Moreover,*

$$\mathrm{Cov}\Big(\mathbf{Z}_t^S(\Delta t)\Big) = \Delta t\Sigma^S, \tag{22}$$

and for $Z_{t,i}$, the ratio of systematic to overall risk is constant over time and equal to λ_i^S ($i=1,..,n$).

Lemma 1 implies that if the vector series $\mathbf{Z}$ and $\mathbf{Z}^S$ are available, Σ^S and λ_i^S can be estimated from these series. Standard techniques can be used for the estimation of $\mathrm{Cov}\left(\mathbf{Z}_t^S(\Delta t)\right)$[20] which, by formula (22), also yield an estimate for Σ^S. An estimation technique for the parameters λ_i^S is provided by KMV and is described in section 2.2.3.

2.2.2 CreditMetrics: Rating Class Default Frequencies and Equity Index Returns

In CM unconditional default probabilities $\bar{p}_i$ are determined by mapping each obligor i into a rating class $\zeta(i)$ and by setting $\bar{p}_i$ to the average historical $T-$year default frequency in the rating class $\zeta(i)$. The underlying rating system can be Moody's, Standard&Poor's, or the internal rating system of the bank.

Concerning the correlation model, CM interprets the indices $V_1^S, ..., V_k^S$ as describing aggregate asset values of certain industries in specific countries. For example V_1^S may describe the assets in the German banking industry, V_2^S the assets in the German insurance, V_3^S the United States chemical industry and so on. The covariance matrix Σ^S of aggregate industry asset returns is approximated by the corresponding matrix for *equity* index returns, which in turn is estimated from widely available equity index time series. The ratio λ_i^S of systematic to total return risk and the weights $\tilde{\theta}_{i1}, ..., \tilde{\theta}_{ik}$ of each firm i on the different industries have to be specified on a judgmental basis by the user. We use the following example for illustration:

Example 2:

> Suppose that firm i participates in the German banking (30%) and the German insurance industry (70%). Assume that V_1^S represents the German banking industry while V_2^S is the index for the German insurance industry. The ratio of systematic risk to total risk is assessed to be 0.9. In this case we have $\lambda_i^S = 0.9$, $\tilde{\theta}_{i1} = 0.3$ and $\tilde{\theta}_{i2} = 0.7$; all other coefficients $\tilde{\theta}_{ij}$ $(j > 2)$ are set to zero.

The approaches of KMV and NPV avoid the approximation of asset returns with equity returns by deriving asset values from observable variables (equity and liability data) using option-pricing models similar to that of MERTON (1974).

[20] If $(\mathbf{X}_l)_{l=1}^n$ is an iid series of random vectors then the standard estimate for $\mathrm{Cov}(\mathbf{X}_{1i}, \mathbf{X}_{1j})$ is given by

$$\frac{1}{n-1}\sum_{l=1}^{n}(\mathbf{X}_{li} - \bar{\mathbf{X}}_i)(\mathbf{X}_{lj} - \bar{\mathbf{X}}_j)$$

where $\bar{\mathbf{X}}_i := \frac{1}{n}\sum_{l=1}^n \mathbf{X}_{li}$.

2.2.3 KMV: Distance to Default, Country- and Industry Indices

Table 1 gives an overview over the steps involved in the calibration of the KMV model.[21] Before we start to describe the calibration steps associated with the respective entries of the table, we note that KMV uses data of two firm pools to calibrate the model. The first pool consists of the firms $i = 1, .., n$ in the actual portfolio and the second consists of firms from a large KMV database. The database, which includes firms that have actually defaulted, is used to empirically determine default frequencies and to construct the aggregate asset value indices $V_1^S, ..., V_k^S$. A major input into all calibration steps are the historical asset value series of the firms in both pools (reported on a certain time grid $t = -T_h, -T_h + \Delta t, ..., 0$) and the parameters μ^v and σ^v of the firms' asset value processes. To derive both asset values and parameter estimates from observable variables, KMV employs an option pricing model which is outlined in the next paragraph.

	Database	Portfolio
Default Probabilities	Calculate DD Map firms to rating classes Determine EDFs	Calculate DD Map firms to rating classes $\bar{p}_{i,T} := \text{EDF}_{\zeta(i),T}$
Correlation Model	Choose $\tilde{\Theta}$ Construct indices V_i^S Estimate Σ^S	Choose $\tilde{\Theta}$ Estimate λ_i^S
Input	Asset value series, parameters μ^v and σ^v	

Table 1: Calibration of the KMV model.

Firm Model As already described in equation (15), a firm's asset value process is assumed to follow

$$dV_t = \mu^v V_t dt + \sigma^v V_t dW_t$$

where W is a standard Brownian motion. Furthermore, it is assumed that the capital structure of a firm is composed of equity, short-term debt D_t (which is considered to be equivalent to cash), and of long-term debt, which is assumed to be a perpetuity paying an average coupon c. Under these assumptions, V_t can be determined as function of the

[21] The terms DD and EDF will be explained during the description of the calibration steps on pages 18 and 19 respectively.

value X_t of the firm's equity, the leverage ratio $K_t := X_t/D_t$, and the instantaneous risk-free interest rate r by solving the corresponding option pricing model. This function is denoted by H_{KMV}:

$$V_t = H_{\text{KMV}}(\sigma^v;\, X_t, K_t, c, r). \tag{23}$$

To calibrate equation (23) for σ^v, KMV uses an iterative technique.[22] Once σ^v is determined the asset value series can be derived from equation (23) and the parameter μ^v can be estimated from the obtained series.

Distance to Default Like CM, KMV determines univariate default probabilities by mapping firms into different rating classes and by calculating the average default frequencies in these classes. In contrast to CM, however, KMV constructs its own rating classes, and default frequencies are derived using the database firms. To map firms into different rating classes, a cardinal measure for the respective firm's default probability is calculated. This measure is called "distance to default" (DD).

If T-year default frequencies shall be determined, then the DD of a firm at a point t in time is given by

$$\text{DD} = \text{DD}_{T,t} := \frac{\log(V_t/v_{T,t}) + \Big(\mu^v - 0.5(\sigma^v)^2\Big)T}{\sigma^v\sqrt{T}}.$$

Note that given the firms' asset value processes follow a geometric Brownian motion (as described in the previous paragraph) and given a firm defaults within a T- year horizon if and only if its asset value falls below a certain threshold $v_{T,t}$ at the end of the horizon T, all firms with the same $\text{DD}_{\text{T},t}$ would have the same T-year default probability in t, namely $\Phi(-\text{DD}_{T,t})$.[23]

The critical threshold $v_{T,t}$ is set at the par value of liabilities in t including short-term debt (STD) to be serviced over the time horizon, plus half the long-term debt (LTD): $v_{T,t} := \text{STD}_{T,t} + (1/2)\text{LTD}_{T,t}$.[24]

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[22] To our knowledge, this technique is not public. But most likely it proceeds by choosing a starting value σ_0^v for σ^v and calculating the asset value series derived from σ_0^v. The estimate for σ^v that is derived from the obtained asset value series would then provide the next value for the iteration.

[23] This follows directly from the fact that the event $V_T \leq v_{T,t}$ can be equivalently written as $\log(V_T/V_t) \leq \log(v_{T,t}/V_t)$ and that the variable $\log(V_T/V_t)$ is normally distributed with mean $\Big(\mu^v - 0.5(\sigma^v)^2\Big)T$ and standard deviation $\sigma^v\sqrt{T}$. See Crouhy, Galai, and Mark (2000), p. 75.

[24] Note that the formula for $v_{T,t}$ is an ad-hoc specification. Using a sample of several hundred companies,

Univariate Default Probabilities To determine T-year default frequencies, the DD measure is calculated for a time horizon of T years for each firm in the database at each point t on the time grid. Firms with a similar $\mathrm{DD}_{T,t}$ are mapped into one rating class ζ. We denote the set that contains all firms in the data base which are mapped to the rating class ζ in t if a T-year horizon is considered by $\mathcal{C}_{\zeta,T,t}$. Then the average T-year default frequency $\mathrm{EDF}_{\zeta,T}$[25] in each rating class ζ and for each time horizon T is calculated:

$$\mathrm{EDF}_{\zeta,T} := \frac{\sum_{t=-T_h}^{-T} \#(\text{Firms defaulted in } \mathcal{C}_{\zeta,T,t} \text{ within } T \text{ years})}{\sum_{t=-T_h}^{-T} \#\mathcal{C}_{\zeta,T,t}}.$$

To determine the unconditional T-year default probability $\bar{p}_{i,T}$ of firm i in the portfolio, the measure $\mathrm{DD}_{T,0}$ is calculated implying the mapping of the firm into a class $\zeta(i)$. Finally, $\bar{p}_{i,T}$ is set to the default frequency in class $\zeta(i)$: $\bar{p}_{i,T} = \mathrm{EDF}_{\zeta(i),T}$.

Correlation Model To specify the correlation model, KMV interprets the processes $V_1^S, .., V_k^S$ as describing aggregate asset values in certain industries and countries (industry and country indices). The processes $(V_1^S, .., V_{k_I}^S)$ may represent industry indices while the remaining processes $(V_{k_I+1}^S, .., V_{k_I+k_C}^S)$ represent the country indices $(k_I + k_C = k)$. The weights $\tilde{\Theta}$ for the firms in both pools are determined using accounting data (sales and assets) as is illustrated in the following example.

Example 2 (continued):

As in example 2 we assume that firm i participates in the German banking and the German insurance industry. Let V_1^S be the industry index for banking, V_2^S the index for insurance, and $V_{k_I+1}^S$ the country index for Germany. Furthermore assume that the following data have been extracted from a database providing financial information about firms:[26]

Business line	Assets (%)	Sales (%)
Banking	35	45
Insurance	65	55
Total	100	100

In this case we would set $\tilde{\theta}_{i1} = (0.35 + 0.45)/2 = 0.4$, $\tilde{\theta}_{i2} = (0.65 + 0.55)/2 = 0.6$. and $\tilde{\theta}_{k_I+1} = 1$.

KMV has observed that firms default when their asset value reaches a level somewhere between the value of total liabilities and the value of short-term debt.

[25]The term EDF is used by KMV and stands for Expected Default Frequency.

To derive the correlation structure Σ^S between the industry and country indices, KMV constructs the returns on these indices from the asset returns of the database firms. For each point on the time grid, a (general least square) *cross-section* regression on the database asset returns is estimated:

$$\mathbf{Z}_t(\Delta t) = \boldsymbol{\alpha} + \tilde{\Theta}\boldsymbol{\beta}_t + \boldsymbol{\epsilon}_t. \tag{24}$$

Note that, in order to avoid overstretching the notation, we have used the variables $\mathbf{Z}_t(\Delta t)$ and $\tilde{\Theta}$ to denote the vector of the *database* returns and the matrix of weights for the *database* firms respectively, despite the fact that elsewhere they are used to describe the *portfolio* firms.[27] Hence, if N denotes the number of firms in the database, then $\mathbf{Z}_t$ is an N-dimensional return vector and the vectors $\boldsymbol{\alpha}$ of constants and $\boldsymbol{\epsilon}_t$ of error terms also have the dimension N. Moreover, $\tilde{\Theta}$ is a $N \times k$ matrix and the vector $\boldsymbol{\beta}_t$ of regression coefficients has dimension k.

The estimates $\hat{\boldsymbol{\beta}}_t$ obtained for the regression coefficients are interpreted as index returns: $\mathbf{Z}^{\mathrm{S}}_t(\Delta t) := \hat{\boldsymbol{\beta}}_t$. The matrix Σ^S is estimated from the obtained vector *time series* $\left(\mathbf{Z}^{\mathrm{S}}_{l\cdot\Delta t}(\Delta t)\right)_{l=-1}^{T_h/\Delta t}$ as described in the conclusions from lemma 1. For explanatory purposes, industry and country indices are further decomposed into independent factors as illustrated in figure 2.[28]

Finally, the systematic risk ratios λ_i^S have to be determined for the portfolio firms $i = 1, .., n$. They are estimated from a *time series* regression of firm returns on index returns:

$$Z_{i,l\cdot\Delta t}(\Delta t) = \alpha_i + \beta_i \cdot (\tilde{\Theta}\mathbf{Z}^{\mathrm{S}}_{l\cdot\Delta t})_i \;\; (l = -1, ..., -T_h/\Delta t).$$

λ_i^S is set to the square root of the regression's R^2.

2.2.4 Nickell et. al.: Market Portfolio

The NPV model differs from the KMV model in the following points:

1. It employs only one systematic process ($k = 1$) which is interpreted as the asset value of the market portfolio.[29] Moreover, the returns on this systematic process

[26] An example for such a database would be Compustat.

[27] The definitions of the database variables are completely analogous to the definitions of the respective portfolio variables. See equations (17) and (21) for $\mathbf{Z}_t$ and page 15 for $\tilde{\Theta}$.

[28] Figure 2 is quoted from Crouhy, Galai, and Mark (2000).

[29] The term "market" refers to national or international equity markets.

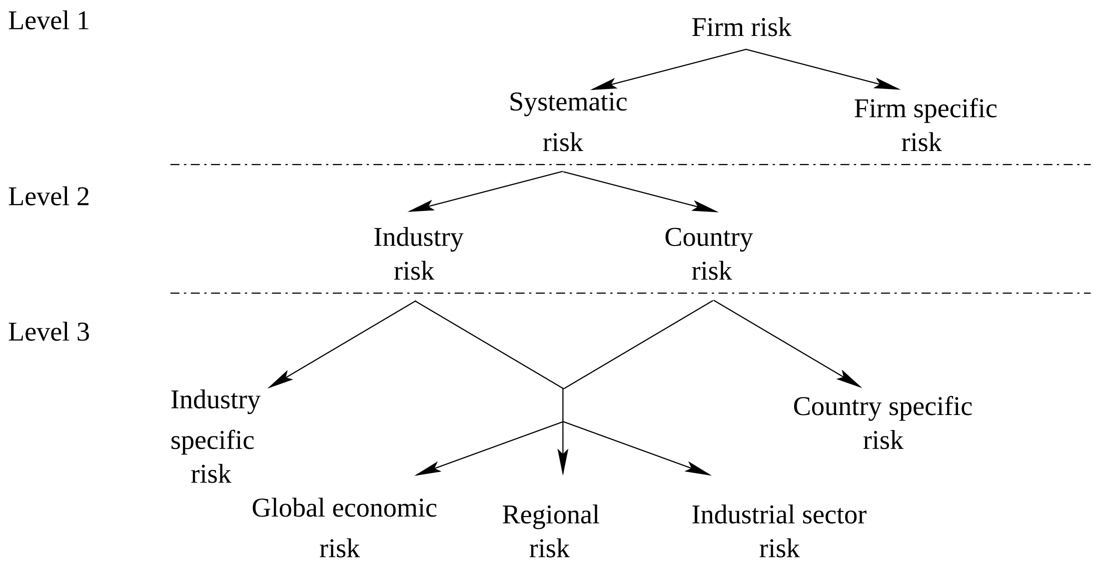

Figure 2: Risk decomposition in the KMV model (source: KMV Corporation).

are not constructed explicitly. Only the parameters μ_1^S and σ_1^S of this process are estimated.

2. It uses a slightly different liability structure to specify the firm model.

3. Firm default is assumed to occur if the firm's asset value falls below some low level at *any* point t in the risk management horizon T. Consequently, default probabilities are calculated as *absorbing-barrier* probabilities.

Firm Model Each firm is assumed to have an earning flow $\delta \cdot (V_t - D_t)$ where δ is the dividend payout rate and D_t denotes the firm's liabilities, assumed to follow $dD_t = \mu^d D_t dt$. Under the assumption that a firm is declared bankrupt when its *asset*-to-liability ratio $K_t := V_t/D_t$ first hits some low level k, the observable *equity*-to-liability ratio X_t/D_t can be expressed as (non-linear) function of K_t by solving the corresponding option-pricing model. We denote this function by H_{NPV}:

$$\frac{X_t}{D_t} = H_{\text{NPV}}(K_t;\, r, \delta, \sigma^v, \mu^d). \tag{25}$$

r is the instantaneous risk-free interest rate.

Parameter Estimation The parameters μ_i^v, σ_i^v, δ_i, μ_i^d and θ_{i1} for each firm i in the portfolio and the parameters μ_1^S, σ_1^S of the systematic process are determined by a Maximum-

Likelihood estimation of the complete model described by equations (15), (16), (19), and the specifications of equation (25) for each firm i in the portfolio:[30]

$$\frac{X_{i,t}}{D_{i,t}} = H_{\text{NPV}}(K_{i,t};\, r, \delta_i, \sigma_i^v, \mu_i^d) \qquad (i = 1, .., n). \tag{26}$$

Joint Default Probabilities For given parameter estimates, NPV derive a formula for the default probability of firm i conditional on the realization of the systematic return variable $Z_T^S(\Delta t)$.[31] In contrast to KMV and CM, NPV calculates an absorbing-barrier probability, i.e. the probability that the asset-to-liability ratio K_t falls below the default threshold k at any point t in the risk management horizon T. To stay consistent with our notation, we normalize $Z_T^S(\Delta t)$ with respect to mean and variance and denote the normalized variable by $\mathbf{S}$. Using the formulas in NPV, conditional default probabilities can be expressed as a function of $\mathbf{S}$: $p_{i,T} = p_{i,T}(\mathbf{S})$. The distribution of the default indicator vector $\mathbf{B}_t$ can then be derived by integrating $p_{i,T}(\mathbf{S})$ over the distribution of $\mathbf{S}$.

Finally note that, to ensure better fit to empirical data, observed default frequencies could also be used for calibration (as in the KMV model). In this case the default barriers k_i would be chosen in a way ensuring that default probabilities are equal to empirical default frequencies, instead of being explicitly derived from the theoretical model. We will discuss this more closely in section 3.2.3 where we present our suggestions for further improvements of the structural approach.

3 Assessment and Suggestions

In this section we try to identify the major problems of the models, argue which conceptual issues should be clarified, and make suggestions on how the models' performance could be improved. In sections 3.1 we discuss the reduced-form models while the structural models are analyzed in section 3.2. For both approaches we perform the assessment in two steps. In the first step we look at the model setup and in the second at the calibration procedure.

[30]To account for the non-linearity of equations (26) in K_t, NPV include a Jacobian term in the likelihood.

[31]Since there is only one systematic process, the subscript j can be dropped.

3.1 Reduced-Form Models

3.1.1 Model setup

We begin the analysis of the model setup by assessing how joint *one-year* default probabilities are determined. After that we examine how multi-year default probabilities are derived from the one-year ones.

One-Year Default Probabilities The major problem of all reduced-form models is that there is no adequate theoretical or empirical foundation of either the assumptions about the distribution of the vectors $\mathbf{S}_t$, $\boldsymbol{\mu}_t^{\mathbf{P}}$ and $\hat{\boldsymbol{\mu}}_t^{\mathbf{C}}$ describing the systematic risk in the portfolio, or of the respective conditional default probabilities' functional form. Moreover, the assumption of conditional independence given the realization of the systematic factors can also be problematic. Table 2 gives an overview.[32]

	CR	**McK**
$\mathcal{L}(\mathbf{S}_t)$, $\mathcal{L}(\boldsymbol{\mu}_t^{\mathbf{P}})$, $\mathcal{L}(\hat{\boldsymbol{\mu}}_t^{\mathbf{C}})$	No foundation	Monotonic in macro variables
$p_i(\mathbf{S}_t)$, $p_i(\boldsymbol{\mu}_t^{\mathbf{P}})$, $p_i(\hat{\boldsymbol{\mu}}_t^{\mathbf{C}})$	Monotonic in systematic factors	
Cond. Independence	Problematic for multi-factor systematic risk	

Table 2: Assessment of the reduced-form models' setup.

Starting with the first row of the table we note that while the CR-type models give no foundation for the employed distributions at all,[33] McK employs time series of macroeconomic indices for which the assumption of normally distributed innovation can be tested and might be adequate. However, except for a general intuition that country/industry default rates should be increasing functions of those indices, it cannot be justified why default rates should be constructed via a *logit* transformation of these indices. Nonetheless, this transformation, together with the indices' distribution, determines the distribution of $\hat{\boldsymbol{\mu}}_t^{\mathbf{C}}$. Turning to the second row of table 2, we observe that while all models build on the reasoning that default probabilities should be increasing in the systematic factors, none of the models delivers a foundation for the specific functional form employed. It is, however, worth noting that in the special case where conditional default probabilities

[32]Note that $\mathcal{L}(X)$ denotes the law (or distribution) of a random variable X.

[33]Except for the gamma distribution being a distribution that can be completely specified by choosing its mean and variance.

are identical for all obligors ($p_i(\mathbf{S}_t) = p(\mathbf{S}_t)$ for all i), they are specified correctly in the single sector-version of CR, since $\mu_t^{\mathrm{P}}(\mathbf{S}_t) = p(\mathbf{S}_t)$. In this case it does not make sense to introduce an additional idiosyncratic sector (as proposed by CR), because the diversification effect that stems from the presence of idiosyncratic risk is already accounted for by the fact that default events are independent given $\mathbf{S}_t$. Concerning CR-GO, it should be noted that the assumption of statistical homogeneity within rating classes contradicts the intuitive argument that a firms' exposure to macroeconomic risk should depend on the type of business it is in rather than on its default probability.

We finally turn to the last row of table 2. We ask what happens if the distribution of the systematic factors and the conditional default probabilities are specified correctly (i.e. if the concerns raised in the first two rows of the table are not justified), and want to analyze in which cases the assumption of conditional independence given the realization of the systematic factors may be problematic. For the sake of clarity, we will consider a portfolio where all obligors are mapped to the same default rate (sector). Hence, for CR, defaults are assumed to be independent given $\mu_t^{\mathrm{P}} = (1/n)\sum_{i=1}^{n} p_i(\mathbf{S}_t)$. The corresponding statistic for McK can, by the law of the large numbers, be approximated by $\hat{\mu}_t^C \approx (1/N)\sum_{i=1}^{N} p_i(\mathbf{S}_t)$ where N is the number of firms in the industry/country from which the default rate is taken. Assuming - for the sake of simplicity - that the portfolio is representative for the corresponding industry/country (with respect to conditional default probabilities), we have $\hat{\mu}_t^C \approx \mu_t^{\mathrm{P}}$ and, hence, conditioning on μ_t^{P} and $\hat{\mu}_t^C$ will have the same effect. Finally, since we analyze one-year default probabilities in this paragraph, we will consider a risk management horizon of one year ($T = 1$).

We start the analysis with the case where the vector $\mathbf{S}_1$ is a scalar (i.e. the dimension of $\mathbf{S}_1$ is 1). By our assumptions, CR-GO will be correct in that case and we obtain that this can also be said for CR and McK under the plausible assumption that the risk factor can be described in a way such that $\mu^{\mathrm{P}}(\cdot)$ is strictly de- or increasing in $\mathbf{S}_1$.

Lemma 2

Suppose that the distribution of μ_1^P and the conditional default probabilities $p_i(\cdot)$ are specified correctly, and that the vector $\mathbf{S}_1$ of systematic risk is a scalar. Then the one-year portfolio distribution produced by CR and McK is correct if $\mu^P(\cdot)$ is strictly de- or increasing in $\mathbf{S}_1$.

Now consider the case where the dimension of $\mathbf{S}_1$ is higher than 1. In this case it is unclear whether it is possible to construct a one-dimensional statistic of the systematic vector $\mathbf{S}_1$ that contains all the information relevant for the portfolio distribution (we call such a statistic *sufficient*). Before we present two cases where μ_1^{P} is indeed a sufficient statistic, we present a more precise definition of what sufficiency refers to in our context.

Definition 1

A statistic μ of the systematic variable $\mathbf{S}_1$ is called sufficient if the portfolio distributions derived under the assumption of conditional independence given $\mathbf{S}_1$ and under the assumption of independence given μ are the same.

Lemma 3

In the following cases μ_1^P is a sufficient statistic:

(i) *If all loans in the portfolio are homogeneous with respect to size.*

(ii) *If all obligors are homogeneous with respect to their exposure to the systematic vector, i.e. if $p_i(\mathbf{S}_t) = c_i G(\mathbf{S}_t)$ with some constants $c_1, ..., c_n$ and an arbitrary function G mapping into the interval $[0, \min_{i=1,..,n}\{1/c_i\}]$.*

In general, however, it will not be possible to construct sufficient statistics. In this case the assumption of conditional independence will understate portfolio risk. This is illustrated in the next example.

Example 3:

We consider a portfolio of 4 loans with face values $L_i = i$ $(i = 1, ..4)$. Unconditional default probabilities are assumed to be identical for all obligors ($\bar{p}_i = \bar{p} = 0.2\%$) and the systematic vector $\mathbf{S}_1$ has dimension two: $\mathbf{S}_1 = (S_{1,1}, S_{2,1})$. Furthermore, we assume that obligors can be divided into two groups, A (which consists of obligors 1 and 2) and B (obligors 3 and 4) so that conditional default probabilities are identical within groups:

$$p_g(\mathbf{S}_1) = \Phi\Big(z - \theta_{g1} S_{1,t} - \theta_{g2} S_{2,t}\Big), \quad g = A, B.$$

Note that the specification of $p_g(\cdot)$ is taken from the two-factor version of KMV. We choose $\theta_{A1} = \theta_{B2} = 0.5$ and $\theta_{A2} = \theta_{B1} = 1$ and $\mathbf{S}_1$ to have independent components with mean zero and variance 1. Figures 3 and 4 show the probability of one-year portfolio losses exceeding the quantiles 0 to 8 for the correct specification and the specification assuming conditional independence given μ_1^{P}, respectively. Note that the scaling of both graphs is different, which illustrates the misspecification caused by assuming conditional independence for our example.[34]

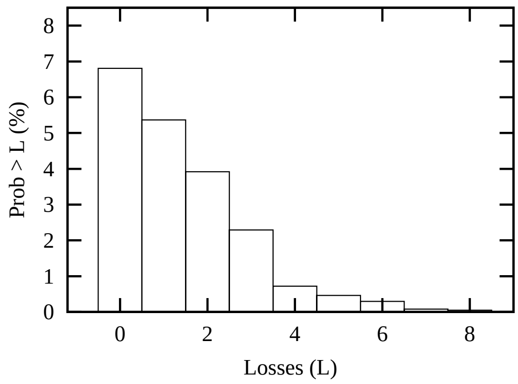

Figure 3: Correct specification.

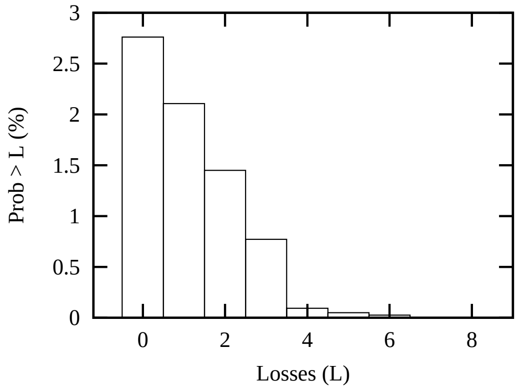

Figure 4: Assuming conditional independence given μ^{P}.

For McK, homogeneity with respect to conditional default probabilities might be realistic if industry default rates are used but it should be problematic for country default rates. Concerning CR, it is generally unrealistic that the conditions of lemma 3 hold for the entire portfolio and, hence, portfolio risk will be underestimated. Moreover, note that the one-sector version, which has been used for this discussion is the most prudential one, producing the highest quantile for a given confidence level.[35] Therefore, the underestimation of portfolio risk will be even worse if the multi-sector version is used instead of the single-sector one.

We conclude the discussion of the one-year framework by summarizing the conceptual critique of CR that has not already been presented. CR employs the economic intuition that the correlation between the defaults of single obligors can be accounted for by modeling the variation of the portfolio default rate due to changes in the macroeconomic environment. We have already argued that, using this intuition, it is inconsistent to introduce a second type of idiosyncratic risk on the sector level, since the diversification effect that stems from the presence of idiosyncratic risk is already accounted for by assuming that default events are independent given $\mathbf{S}_t$. Moreover, only the multi-sector version of the model can account for the fact that default probabilities of obligors depend on the systematic factors in different forms. The multi-sector version without weights is consistent with the default-rate intuition but would imply that default events of obligors in different sectors are independent of each other, which is not plausible. The multi-sector version with weights is constructed only by *formal* analogy to the non-weighted version: the expressions $1\{i \in \mathcal{S}_j\}$ are replaced by θ_{ij} in formulas (7) - (9). However, the CR manual does

[34]The formula for the conditional default probabilities is derived in the appendix.

[35]See Credit Suisse (1997), p. 23.

not mention how the variables $\mu_{j,t}^P$ $(j = 1, .., m)$ should be interpreted in this weighted multi-sector version.

It would be most natural to interpret the variables $\mu_{j,t}^P$ as weighted default rates by applying the same formal analogy as used to derive the formulas (7) - (9):[36]

$$\mu_{j,t}^P := \frac{1}{n} \sum_{i=1}^{n} \theta_{ij} p_i(\mathbf{S}_t). \tag{27}$$

This would also be consistent with formula (8) for the mean of $\mu_{j,t}^P$. However, it would not be consistent with formula (9) for the standard deviation of $\mu_{j,t}^P$. Moreover, inserting equation (10), which specifies the conditional default probabilities $p_i(\cdot)$, into the definition of $\mu_{j,t}^P$ given in (27) would imply that the following condition would have to hold if the model should be consistent:

$$\mu_{j,t}^P = \frac{1}{n} \sum_{i=1}^{n} \bar{p}_i \theta_{ij} \sum_{l=1}^{s} \theta_{il} \frac{\mu_{l,t}^P}{\bar{\mu}_l^{\mathrm{P}}} \qquad (j = 1, .., m).$$

This condition lacks a sensible economic interpretation. We are also not aware of any other plausible interpretation for the variables $\mu_{j,t}^P$ that is consistent with the formulas for conditional default probabilities and with the formulas for mean and standard deviation employed by CR (formulas 10 - 12). In summary, the economic interpretation of the variables μ_j^P, which drive the joint-default behavior, is unclear in the weighted multi-sector version of CR. Consequently, the formulas for the mean and the standard deviation of these variables are somewhat arbitrary. This is in particular problematic since these formulas are a key ingredient to the calibration procedure.

Multi-Year Default Probabilities Recall that formula (1) provides the link between the one-year and the multi-year default probabilities for all reduced-form models. In the appendix we show that formula (1) rests on the assumption that

$$\mathbb{P}\{B_{i,t} = b_i \,|\, \mathbf{S}_t, ..., \mathbf{S}_T\} = \mathbb{P}\{B_{i,t} = b_i \,|\, \mathbf{S}_t\} \quad (t = 1, ..., T-1, \; i = 1, ..., n).$$

Hence, it is assumed that future realizations $\{\mathbf{S}_{t+1}, ..., \mathbf{S}_T\}$ of the systematic factors will not affect default probabilities in t, and only $\mathbf{S}_t$ is decisive. However, this is in sharp contrast to the basic idea of the structural approach to firm defaults. Since a firm's value

[36] I.e. by replacing $1\{i \in \mathcal{S}_j\}$ with θ_{ij} in the definition of $\mu_{j,t}^P$ for the non-weighted multi-sector version (see equation 6).

reflects the discounted sum of all future cash flows, knowing that the macroeconomic environment will be bad in the future (e.g.) will affect the asset value of the firm today and hence will also affect the probability that equity holders choose to default on the obligations of the firm. Therefore, by projecting on the σ-algebra generated by the random vector $\mathbf{S}_t$ rather than on that generated by the vectors $\{\mathbf{S}_t, \mathbf{S}_{t+1}, ..., \mathbf{S}_T\}$, the volatility of $\mathbf{B}_T$ and therefore portfolio risk is underestimated.

Note also that while the implicit assumption expressed by equation (1) is consistent within the CR framework (since the systematic factors are assumed to be independent), this is not the case for McK (since McK explicitly stresses the importance of recognizing the autocorrelation of default-rate time series, which allows them to model changes of default probabilities over the business cycle). Of course this observation should not lead to the conclusion that CR dominates McK in this respect. Rather it should be stressed that the autocorrelation of default frequency series is an empirically well documented phenomenon,[37] which should be taken into account by CR and by McK.

3.1.2 Calibration

A major point of critique with respect to the calibration of all CR-type models is the following. While the model setup is built on the intuition that default probabilities change due to changes in systematic factors (of which the business cycle is the most important example), they do not condition parameter estimates on business cycle information. This will not only have the (rather obvious) effect that default probabilities will be underestimated in recessions. It will also - as has been pointed out by Erlenmaier and Gersbach (2001) - have the effect that the variances $\text{Var}(\hat{\mu}_\zeta^R)$ of rating-class default frequencies and, hence, the variances of default probabilities will be underestimated. Of course the opposite will be true during expansion states of the business cycle. Moreover, the rating class default-frequency series $(\hat{\mu}_{\zeta,t}^R)_{t=1}^{-T_h}$ is very likely serially dependent.[38] This is inconsistent with the assumptions posed by Gordy (2000) to set up an estimation technique for default-probability variances (see section 2.1.4).

Addressing the problems mentioned above is important, since the portfolio quantiles produced by CR-type models tend to be very sensitive with respect to changes in default probability variances as has been demonstrated for example by Gordy (2000). Additionally, to make CR-GO applicable, a method of determining the variance of the systematic factor $\mathbf{S}_t$ has to be developed.

[37] See e.g. Bär (2000).

[38] Default frequencies in t should be correlated with the respective frequencies in $t-1$ due to the common dependency on the business cycle. See for example Bär (2000).

Concerning McK, it remains unclear which statistical framework should be used for the estimation of the sensitivity parameter θ and whether this parameter is stable over the business cycle.

3.1.3 Suggestions

For all CR-type models, the most important issue is the adaptation of default probability moments to the business cycle. Recently, some authors have proposed statistical frameworks to predict default rates with macroeconomic variables.[39] They suggested that this might be a method of adjusting unconditional default probabilities in credit risk models. However, it is much more unclear how to adjust default-probability variances empirically. A starting point could be the framework of GORDY (2000), which has been presented in section 2.1.4. It could be extended by additionally modeling changes in conditional default probabilities $p_i(\mathbf{S}_t)$ over time while keeping the assumption that the systematic vector $\mathbf{S}_t$ is iid in time. A reasoning for this kind of modeling is that, as in the structural approach, $\mathbf{S}_t$ could represent returns on aggregate asset value indices. This would make the iid assumption for $\mathbf{S}_t$ look like a reasonable approximation. Explicitly constructing such indices - as has been done e.g. in ANDERSON AND SUNDARESAN (2000) - would also make it possible to determine the lacking estimate for $\mathrm{Var}(\mathbf{S}_t)$ in the CR-GO model.

Concerning CR, we think that a clarification of the theoretical model is necessary. Either the factors that drive joint defaults are not interpreted as portfolio default rates; then the model can be reduced to the CR-GO version. Or the portfolio default rate is still used as economic intuition; then it is necessary to construct a consistent model that (a) allows for a different dependence of obligors on systematic factors and (b) still has interpretable components. In particular, the role of the idiosyncratic sector should be clarified. We have demonstrated that by employing such a sector in the way it is done in the current CR framework, portfolio risk will be underestimated.[40]

Finally, the underestimation of portfolio risk arising in the context of multi-year default probabilities could be avoided for all reduced-form models if multi-year default probabilities were derived directly from multi-year default rates.

[39]Albeit only on the level of *country* default rates: see e.g. BÄR (2000).

[40]See the discussion in section 3.1.1.

3.2 Structural Models

3.2.1 Model Setup

The structural approach rests on the following two building blocks.

1. The *joint* distribution of the obligors' asset value processes.

2. The obligors' capital structure.

3. The default event definition employed to determine joint default probabilities.

We first turn to the theoretical justification of the distributional assumptions concerning the firms' asset value processes. If (unanticipated) asset returns are independent in time, identically distributed over each time interval with the same length and have finite variance, then - by the central limit theorem - the asset value processes will follow a multidimensional geometric Brownian motion.[41] Serial independence in time can be derived from the efficient market hypothesis (see FAMA (1970)). Identical distribution over time intervals with the same length builds on the idea that the nature of unexpected asset returns does not change over time.

Can this model be backed by empirical results? When trying to test hypotheses about the distribution of asset returns, one faces the problem that asset market values cannot be observed directly, since not all of a firm's obligations are traded in the market. As a first approximation, one can investigate *equity* returns of firms with a very low probability of default. In this case, equity and asset returns should perform sufficiently similar. In general, studies on equity returns tend to find that - while not being perfectly accurate - the normal distribution seems to be a first approximation. However, there are important deviations of empirical return distributions from the normal family (in particular skewness and fat tails). A next natural step towards a better fit to empirical data is to drop the (rather artificial) assumption of finite variances in the reasoning given above. This enlarges the family of potential distributions for returns to the (parametric) class of stable-law distributions. Stable-law distributions can account for fat tails and skewness.[42] The second important empirical objection against a Brownian motion model of equity returns is time-varying volatility.[43]

[41]See e.g. COX AND MILLER (1990).

[42]See FAMA (1970) for a discussion of the literature and RACHEV, SCHWARTZ, AND KHINDANOVA (2000) for a more recent contribution.

[43]See again RACHEV, SCHWARTZ, AND KHINDANOVA (2000).

These results on equity returns give first hints on whether the Brownian motion model might be a good description for a firm's asset value process. However, for a final assessment it is important to test this assumption more directly using asset-value series derived from stock prices and debt structures. In doing so, the two building blocks of the structural approach, distributional and capital structure assumptions, become interdependent. To our best knowledge, the only empirical evidence on the distribution of model-implied asset value returns has been provided by KMV. According to their studies, actual data conform quite well to this hypothesis for the univariate case,[44] but nothing is said about the accuracy of the multivariate distributional assumptions. Of course it also lies in the commercial interest of KMV to provide evidence in favor of their own approach; therefore these statements should be treated with some caution.

Concerning the second building block (capital structure assumptions), it should be noted that - compared to arrangements in the practice - KMV and NPV assume a quite simplistic capital structure. We will comment on this later when presenting our suggestions for the structural approach in section 3.2.3.

Finally, a major advantage of the NPV model compared to KMV and CM is that default is modeled as an absorbing barrier that can be reached at any time within the risk management horizon when equity holders choose to exercise their default option. KMV and CM on the other hand only allow for the default of a firm at the end of the risk management horizon which, of course, is unrealistic. On the other hand, the KMV technique of using theoretically derived univariate default probabilities not directly, but only as an intermediate measure for group-building, seems superior to the sole reliance on theoretical results. It is, however, completely unclear why the theoretical default probabilities are not employed *directly* as group-building indices. Using a different scaling, such as KMV's DD measure (which is the quantile of the default probability under the standard normal distribution), implies that the variation of default probabilities will be much higher in some classes than in others. This, of course, is unfavorable.

3.2.2 Calibration

The CM approach uses rating-class default frequencies to determine univariate default probabilities and equity-index return correlations as proxies for asset index return correlations. The former is problematic since it does not take into account that default probabilities vary through the business cycle. The latter might be an appropriate approximation for highly rated firms, for which equity and asset values should perform sufficiently

[44] See Crouhy, Galai, and Mark (2000), p. 87.

similar. However, it will be problematic for firms with a substantial probability of default. Additionally, equity volatility and therefore equity index volatility is relatively unstable over time,[45] which is not reflected in CM's estimation procedure. Finally, it is also unsatisfactory that the relative size λ_i^S of a firm's systematic return component is chosen by rules of thumb and is not based on a quantitative analysis.

The approach of KMV and NPV avoids these problems by relying on asset value data. However, the proposed methods are not practicable for non-quoted firms.[46] Comparing the methods of determining asset correlations of JPM and KMV on the one hand and of NPV on the other hand, it should be noted that empirical studies on the correlation structure of *equity* returns come to the very robust result that traditional-industry-index models are dominated by market-index models. Provided that these results can be generalized for asset returns, the NPV approach is superior to the one employed by KMV. In this context it should also be noted that it is straightforward to make the current NPV model suitable for the management of international portfolios with obligors placed in different markets (e.g. US and Europe): the single-index model can easily be generalized to a multi-index one. Finally, recent research seems also to suggest that fundamental models, relating equity returns to macroeconomic variables, might outperform market-index models;[47] this points to a potential source of further improvements.

3.2.3 Suggestions

Our suggestions for the structural approach can be summarized as follows. First, an appropriate mix of the KMV and the NPV firm model can be seen as a reasonable starting point for producing asset value data and modeling joint defaults. Using this model, the most important assumptions about the distribution of the firms' asset value processes should be tested empirically, i.e. multivariate normality of asset returns and the stability of the parameters μ^v and σ^v over time.[48] Second, to fully specify the mixed structural model, a method for determining asset-return correlations has to be chosen. Using the asset-value data obtained from the model, empirical research should be conducted to assess which of the discussed approaches is the most promising (market-index models, industry-index models or fundamental models). Third, we suggest how to extend the

[45] See e.g. Crouhy, Galai, and Mark (2000), p. 88.

[46] Note that KMV has extended its model to non-quoted firms. However, there is - to our knowledge - no publicly available documentation of this extension.

[47] For an overview of the empirical evidence on correlation models for equity returns see Elton and Gruber (1995).

[48] Remember that μ^v and σ^v denote the mean and the standard deviation of instantaneous asset returns respectively.

KMV model to non-quoted firms. We will discuss each of these points in turn.

Firm Model When deciding about the appropriate option-pricing model to be employed to produce asset value data, one can choose among quite a variety of models. Both models proposed in the credit risk context (KMV and NPV) assume a very simple capital structure, which might be regarded as unrealistic for many firms. More sophisticated modifications of the original Merton model have been discussed in the literature (see BOHN (1999b) for an overview). However, as Bohn notes:[49]

> "The cost of these modifications is tractability. The more realistic the model becomes, the more complex is the resulting valuation equation. In some of the more extreme cases we must rely on numerical solutions which can be unintutitve and computationally expensive. Even in the cases where we can find closed-form solutions, we may lose clarity regarding the factors driving the value. More often than not, however, we end up with equations characterized by numerous parameters that are difficult to estimate. Finding the appropriate balance between realism and tractability requires assumptions and approximations. Empirical research can illuminate the aspects of these models that can be simplified or even ignored."

Our own assessment is very much along these lines. We therefore advocate to start with firm models that are as simple as possible. Empirical support for such a position has been provided by (BOHN 1999A) who showed that credit spread data can be fitted reasonably well with a model even simpler than the original Merton framework.

Both models discussed here (KMV and NPV) are simple enough to provide a reasonable starting point. The NPV approach seems to be more accurate with respect to the theoretical derivation of default probabilities (since it can account for default prior to the risk management horizon). On the other hand, the KMV technique of using theoretically derived univariate default probabilities not directly, but only as an index for group-building, seems superior to the sole reliance on theoretical results. However, group-building should rely *directly* on theoretical default probabilities and not on a derived measure such as KMV's "distance to default".

We therefore suggest to use the theoretical NPV default probabilities as index for group-building. Group default frequencies can then provide the estimates for the univariate

[49]See BOHN (1999b), p. 20.

default probabilities of the group members. Finally, *joint* default probabilities can again be calculated from the NPV formula, using exogenous default barriers k_i derived from the empirically determined univariate default probabilities. [50]

Using this "mixed" firm model, the most important underlying distributional assumptions about the firms' asset value processes should be tested empirically: multivariate normally distributed returns and the stability of the parameters μ^v and σ^v over time. Concerning the latter assumption, it should be noted that the well documented phenomenon of time-varying *equity*-return volatility can be explained by a firm model with stable *asset* return distributions over time. As has been pointed out by BENSOUSSAN, CROUHY, AND GALAI (1994), stable asset return volatility would imply that equity-return volatility fluctuates with the firm's default probability, since the elasticity of the equity value with respect to the underlying asset value changes with a firm's leverage. Empirically investigating asset value models as proposed above could therefore also provide important insights for equity research.

Correlation Model To assess which of the proposed methods of determining asset return correlations works best, the empirical research on equity-return correlations should be reproduced for asset returns. The evidence of equity research suggests that it should be sensible to start with simple index models that explain correlations solely by the co-movements of the firms' asset values with the market. While performing robustly better than traditional-industry-index models for equity returns, such models would also have the charm of simplicity. However, it should be checked whether fundamental models might not further improve the correlation models' performance.

Finally, it should also be stressed that in risk management correlation models are intimately related to the topic of hedging undiversifiable risk. If the correlation models are constructed around publicly available and well documented indices, then a bank can hedge its exposure to these indices. Fundamental models rely on such indices per definition (e.g. interest rates, economic growth rates) and the method of KMV explicitly constructs such indices (which could be published). The approach of NPV would have to be refined in so far that an index for the asset value of the complete market would have to be constructed. This should, however, be achieved rather easily by aggregating microeconomic firm data.[51]

[50] If the KMV version is used nevertheless, then at least the sensitivity of joint default probabilities should be assessed with respect to the simplifying assumption that default can only occur at the end of the risk management horizon.

[51] For examples of how to construct such aggregates see ANDERSON AND SUNDARESAN (2000).

Extension to Non-Quoted Firms While CM can deal with non-quoted firms, we have already seen (a) that the determination of univariate default probabilities by CM is problematic, (b) that equity index returns might be too crude an approximation for asset returns and (c) that the grouping of firms according to traditional industries might not be optimal. If the KMV and NPV approach is extended to non-quoted firms, two issues have to be taken up. First, the determination of univariate default probabilities and second the derivation of asset return correlations. For both issues, group-building could be the method of choice. If groups of firms can be identified that are sufficiently homogeneous with respect to default probabilities and asset returns, and if the groups contain quoted as well as non-quoted firms, then the correlation and default probability results obtained for the former can be used as a proxy for the latter.[52]

For univariate default probabilities, rating classes may be a first obvious group choice which later on might be refined by industry- or country- specific rating classes.[53] Concerning asset-return correlations, remember that group building via traditional industries proved to be not particularly successful for equity-return correlations. It should therefore be thought of trying to construct pseudo industries as it is done in equity-return research. Started by FARELL (1974), who extracted four types of pseudo industries (growth stocks, cyclical stocks, stable stocks, and oil stocks), pseudo-industry models are applied in more sophisticated ways in today's investment banking practice.

To assess the relative performance of the different group-building techniques, estimates (of default probabilities and return correlations respectively) for quoted firms can be calculated once using the procedure that is only viable for quoted firms and once using group building. It can then be evaluated which group-building technique produces results that fit best to the estimates obtained under the quoted-firm procedure.

4 Conclusions

In this paper we have discussed the joint-default models of the four major, currently available credit risk frameworks: CreditMetrics (CM), Portfolio Manager (by KMV), CreditRisk$^+$ (CR) and CreditPortfolioView (by McKinsey, McK). Moreover, we have also included two new versions of current joint-default models, one for CR, presented by GORDY (2000) (CR-GO), and one for KMV, presented by NICKELL, PARRAUDIN,

[52]Of course it is necessary that "group membership" is identifiable for quoted and non-quoted firms.

[53]Note that the crucial difference to the JPM method is that - by determining average rating class default probabilities for quoted firms from a KMV- or NPV-type firm model - estimates for univariate default probabilities will reflect business cycle information contained in the stock prices.

and Varotto (1999) (NPV). Following the literature we have divided the six models into two classes, labeled "structural models" (CM, KMV and NPV) and "reduced-form models" (CR, CR-GO and McK) respectively.

We have attempted to identify the most severe drawbacks of the proposed models and suggested measures for improving them. These measures contained short-term fixes as well as a long-term research agenda. In sections 4.1 and 4.2 we summarize our major insights with respect to reduced-form models and structural models respectively. Drawing on these results, we finally compare the structural and the reduced-form approach to joint-default modeling in section 4.3. We conclude that the mixed structural model proposed in section 3.2.3 is the conceptually most convincing basis for second-generation models.

4.1 Reduced-Form Models

Our findings with respect to the reduced-form models are as follows. First, we have argued that the arbitrary nature of the assumptions about the systematic risk factors' distribution and about the functional form of conditional default probabilities is the most important drawback of the reduced-form approach to joint-default modeling.

Second, the assumption of conditional independence given portfolio default rates (CR) or industry- (country-) default rates (McK) is adequate as long as all systematic risk can be represented by a *one*-dimensional random variable. This could be appropriate for portfolios where all obligors are placed in one single (equity) market (e.g. the US market).[54] However, for multi-market portfolios (e.g. US and Europe), more factors should be necessary. We have argued that if this is the case, McK should still perform quite well while CR will underestimate portfolio risk.

Third, we have demonstrated that all reduced-form models underestimate portfolio risk for multi-year risk management horizons; when calculating multi-year default probabilities from the one-year formulas, they neglect the influence of future realization of the systematic variables on today's conditional default probabilities. We have suggested that this could be avoided by directly modeling multi-year default rates.

Concerning the reduced-form models' calibration, we have argued that it is important for the CR-type models to adapt the estimations of mean and variance of default probabilities to the business cycle. We have outlined that this may be achieved by using returns on

[54] Research on equity-return correlations seems to suggest that a market portfolio index is a quite good measure for the systematic-risk exposure of firms in a common market (see Elton and Gruber (1995)).

aggregate asset indices as systematic factors and by additionally modeling changes in conditional default probabilities over the business cycle. This would also allow to complete the calibration of CR-GO through a direct estimate of the systematic factor's variance. For McK, the estimation of the parameter that measures the sensitivity of default probabilities with respect to default-rate realizations seems to be the most problematic part of the model's calibration. It has to be assessed empirically whether a robust estimation technique can be found and whether this parameter is stable over the business cycle.

Finally, concerning CR, we think that a clarification of the theoretical model is necessary. Either the reduced version CR-GO is used or a consistent interpretation of the weighted multi-sector version should be developed. In particular, it has to be clarified why risk-reducing diversification effects enter the CR model via *two* channels. First, via the fact that default events are assumed to be independent given the systematic risk variable. Second, via an additional idiosyncratic sector. We have argued that the first channel is sufficient and that adding the second one will lead to an underestimation of portfolio risk.

4.2 Structural Models

The second part of the paper was concerned with structural models. We have pointed out that JPM has the important disadvantages that (a) univariate default probabilities are not sensitive to changes in the macroeconomic environment and (b) that equity index returns might be too crude an approximation for asset returns. Our analysis has therefore focused on the models of KMV and NPV, which avoid these problems. These models rely on asset value data which are produced by employing option-pricing. While recognizing that both models might be quite simplistic with respect to the specification of the capital structure, we have argued that it makes sense to start with such simple models.

Comparing both firm models, we have identified the most problematic assumption in the KMV model, namely that (as in CM) firms are assumed to default only at the end of the risk management horizon. NPV uses the more realistic approach of modeling default as an absorbing barrier of the firm's asset value process that can be reached at any time within the risk management horizon. We have therefore proposed that a "mixed" firm model employing the theoretical setup of NPV, and the group-building technique of KMV (to empirically determine univariate default probabilities) would be the best starting point for next-generation structural joint-default models. Using the asset value data produced by this mixed model, the following issues should be addressed empirically.

First, the adequacy of the assumptions about the distribution of asset returns should be assessed (multivariate normal distribution and stability of mean and standard deviation in time). Second, to fully specify the mixed structural model, a method of determining asset return correlations has to be chosen (market-index models, industry-index models or fundamental models). Using the asset value data obtained from the model, the best-performing method should be identified empirically.

Finally, we have suggested a method of extending the scope of structural models to non-quoted firms, which does not share the major deficits of the CM approach. Univariate default probabilities and asset return correlations could be determined by building groups of quoted and non-quoted firms. If group members are sufficiently homogeneous, results obtained for the quoted firms can then be used as a proxy for non-quoted ones. A case in point would be group-building via rating classes (for default probabilities) and empirically determined pseudo-industries (for asset correlations).

4.3 Comparison

Using the above observations and suggestions we finally want to argue why - in our view - the structural approach to the modeling of joint defaults should be favored. As has already been pointed out, the major drawback of the reduced-form models is the arbitrary nature of the distributional assumptions involved. When backtesting these models or comparing them with other models, it is completely unclear why the actually proposed distribution should be used and not *any* other distribution that does not violate the few qualitative restrictions that can be made. This question is especially hard to answer when recognizing that quantiles belonging to extreme probability levels are the most important output of credit risk models; these quantiles will very likely depend on the higher moments of the default rates' distribution[55] and on the specific functional form chosen for conditional default probabilities. Both cannot be estimated from empirical data.

Moreover, even if a potential default-rate distribution is determined, it will be difficult to test its accuracy empirically since default data are recorded on an annual rather than on a monthly or even shorter-term basis (as it is the case for stock returns). Aggravating this problem, there is strong theoretical and empirical evidence that default-rate distributions vary significantly over the business cycle.

In contrast, the structural approach can use the Brownian motion model for the asset-value process and a sufficiently simple capital structure as a starting point to test distributional assumptions and model performance. Data availability is much better and the case for

[55] Recall that default rates are the systematic factors used by CR and McK.

the distributional stability of unexpected asset returns over the business cycle is much stronger than for default probabilities and default-rate distributions.

If the starting point proves to be too simplistic, it would be possible to move to more sophisticated models that are already available in the theoretical literature. Of course it could well turn out that reality is too complex for sufficiently simple firm models to be a sensible first-order approximation. But we think that the successful application of very simple structural models for default prediction, estimation of univariate default probabilities, and in the area of bond pricing is a quite encouraging sign for structural models to become a practicable and reasonably accurate approach to the management of default risk.[56]

Do these conclusions stand up when extending the focus from the narrow view on joint-default modeling to the complete credit risk management framework where loans may not mature at the end of the risk horizon T and thus have to be repriced in T? In this general setting the case for structural models seems to be even stronger. Pricing loans is a natural application of the option-pricing models on which the structural approach is based. Indeed, these models have been developed to price bonds and were then extended to the field of credit portfolio management. The same cannot be said for the reduced-form models we have discussed. While CR does not allow for repricing of loans at all, McK can deal with repricing but employs an ad-hoc specification for pricing risky future cash flows that uses average credit spread data. KMV, in contrast, builds on a full-fledged risk-neutral-pricing framework.

It should, however, be noted that a new reduced-form model with a state-of-the-art contingent claims pricing framework and abstract systematic risk factors (instead of default rates) has been proposed by Jarrow and Turnbull (2000). While this model does not share the methodological problems of McK and CR, it does share the major drawback of reduced-form models, namely the arbitrary nature of distributional assumptions. Moreover, the model is calibrated using price data for publicly traded bonds. This raises the question of how the bond prices entering the calibration have been determined in the first place, or more generally, how the different securities issued by a firm should be priced.

While a financial analyst can be expected to assess expected future cash flows and derive a firm's asset value by risk-adjusted discounting, she will need the guidance of a theoretical model to split the asset value into the values of a firm's different contingent securities (such

[56] KMV has published several studies on the performance of its EDF measure with respect to default prediction and default probability estimation (for further assessments see also Crouhy, Galai, and Mark (2000), pp. 92-93). For the application to bond pricing see Bohn (1999a).

as bonds and equity). But the generally accepted and also most convincing approach to do this is the option-pricing framework as employed by the structural approach.

In summary, the structural approach - compared to the reduced-form approach - presents a unified, consistent and more complete framework for the pricing and management of portfolios of corporate securities. These properties will prove to be particularly important for the development of future models that should allow to manage *all* assets of a bank within a single risk management framework.

A Appendix

Proof of lemma 2.

If $\mathbf{S}_1$ is a scalar and $\mu^{\mathrm{P}}(\cdot)$ is strictly de- or increasing in $\mathbf{S}_1$, then the event $\{\mathbf{S}_1 = s\}$ can be equivalently described by $\{\mu_1^{\mathrm{P}} = \mu^{\mathrm{P}}(s)\}$. Hence, denoting the density of $\mathbf{S}_1$ and μ_1^{P} by f_S and $f_{\mu^{\mathrm{P}}}$ respectively, we obtain:

$$\begin{aligned}
\mathbb{P}\{\mathbf{B}_1 = \mathbf{b}\} &= \int \prod_{i=1}^{n} \mathbb{P}\{B_{i,1} = b_i \,|\mathbf{S}_1 = s\} f_S(s)\, ds \\
&= \int \prod_{i=1}^{n} \mathbb{P}\{B_{i,1} = b_i \,|\mu_1^{\mathrm{P}} = \mu^{\mathrm{P}}(s)\} f_S(s)\, ds \\
&= \int \prod_{i=1}^{n} \mathbb{P}\{B_{i,1} = b_i \,|\mu_1^{\mathrm{P}} = r\} f_{\mu^{\mathrm{P}}}(r)\, dr.
\end{aligned}$$

□

Proof of lemma 3.

(i) If loans are homogeneous with respect to size (i.e. $L_i = L$ for $i = 1, ..., n$), then

$$\begin{aligned}
\mathbb{P}\{\sum_{i=1}^{n} L_i B_{i,1} = q\} &= \mathbb{E}\,\mathbb{P}\Big\{\sum_{i=1}^{n} B_{i,1} = q/L \,|\, \mathbf{S}_1\Big\} \\
&= \mathbb{E}\,\mathbb{P}\Big\{X_{\mu^{\mathrm{P}}(\mathbf{S}_1)} = q/L\Big\}
\end{aligned}$$

where $X_{\mu^{\mathrm{P}}}$ is a random variable with $\mathcal{L}(X_{\mu^{\mathrm{P}}}) = \mathcal{L}(\sum_{i=1}^{n} X_i)$ and $X_1, .., X_n$ are independent Bernoulli variables with $\sum_{i=1}^{n} \mathbb{P}\{X_i = 0\} = \mu^{\mathrm{P}}$. The same formula is derived under the assumption of conditional independence given μ_1^{P}.

(ii) If conditional default probabilities are homogeneous, then we obtain

$$\mathbb{P}\{\mathbf{B}_1 = \mathbf{b}\} = \mathbb{E}\, G(\mathbf{S}_1)^n \prod_{i=1}^{n} c_i.$$

Moreover, $\mu_1^{\mathrm{P}} = G(\mathbf{S}_1)(\sum_{i=1}^{n} c_i)$, implying that conditional default probabilities are given by $p_i(\mu_1^{\mathrm{P}}) = c_i \mu_1^{\mathrm{P}}/(\sum_{i=1}^{n} c_i)$. This in turn shows that joint probabilities are calculated correctly under the assumption of conditional independence given μ_1^{P}.

□

Derivation of conditional default probabilities for example 3

To derive the conditional default probabilities, we assume that, for arbitrary s_1, $\mu^{\mathrm{P}}(s_1,\cdot)$ is a strictly decreasing, continuous function of the second systematic factor $S_{2,t}$ and that $\mu^{\mathrm{P}}(s_1,\cdot)$ takes all values between 0 and 1 if s_2 varies between $-\infty$ and $+\infty$.[57] This implies that if $\mu_t^{\mathrm{P}}=r$, then for an arbitrary realization s_1 of $S_{1,t}$ there is an unique value $S_2(s_1,r)$ so that $\mu^{\mathrm{P}}\Big(s_1, S_2(s_1,r)\Big)=r$ (Mean value theorem). Therefore

$$p_i(\mu_t^{\mathrm{P}}) = \frac{\int_{-\infty}^{+\infty} p_i\Big(s_1, S_2(s_1,\mu_t^{\mathrm{P}})\Big)\, f\Big(s_1, S_2(s_1,\mu_t^{\mathrm{P}})\Big)\, ds_1}{\int_{-\infty}^{+\infty} f\Big(s_1, S_2(s_1,\mu_t^{\mathrm{P}})\Big)\, ds_1}.$$

Multi-year default probabilities (reduced-form models)

Without loss of generality we assume that $T=2$ and that $B_{i,2}=0$ for $i=1,..,l$ and $B_{i,2}=1$ for $i=l+1,..,n$. In this case we obtain:

$$\begin{aligned}
\mathbb{P}\Big\{\mathbf{B}_2=\mathbf{b}_2\Big\} &= \mathbb{E}\,\mathbb{P}\Big\{\mathbf{B}_2=\mathbf{b}_2 \,|\, \mathbf{S}_2,\mathbf{B}_1\Big\} \\
&= \mathbb{E}\Bigg[\prod_{i=1}^{l} 1\{B_{i,1}=0\}[1-p_i(\mathbf{S}_2)] \\
&\qquad \cdot \prod_{i=l+1}^{n} \Big\{1\{B_{i,1}=0\}p_i(\mathbf{S}_2) + 1\{B_{i,1}=1\}\Big\}\Bigg] \\
&= \mathbb{E}\Bigg[\prod_{i=1}^{l} \mathbb{P}\{B_{i,1}=0|\mathbf{S}_1,\mathbf{S}_2\}[1-p_i(\mathbf{S}_2)] \\
&\qquad \cdot \prod_{i=l+1}^{n} \Big\{\mathbb{P}\{B_{i,1}=0|\mathbf{S}_1,\mathbf{S}_2\}p_i(\mathbf{S}_2) + \mathbb{P}\{B_{i,1}=1|\mathbf{S}_1,\mathbf{S}_2\}\Big\}\Bigg].
\end{aligned}$$

Note that the last equation can be derived by applying the conditional expectation operator $\mathbb{E}[\cdot|\mathbf{S}_1,\mathbf{S}_2]$. On the other hand, using formula (1) for $T=2$, we find that

$$\begin{aligned}
\mathbb{P}\Big\{\mathbf{B}_2=\mathbf{b}_2\Big\} &= \mathbb{E}\,\mathbb{P}\Big\{\mathbf{B}_2=\mathbf{b}_2 \,|\, \mathbf{S}_1,\mathbf{S}_2\Big\} \\
&= \mathbb{E}\Bigg[\prod_{i=1}^{l}[1-p_i(\mathbf{S}_1)]\cdot[(1-p_i(\mathbf{S}_2)] \\
&\qquad \cdot \prod_{i=l+1}^{n} \Big\{[1-p_i(\mathbf{S}_1)]p_i(\mathbf{S}_2) + p_i(\mathbf{S}_1)\Big\}\Bigg].
\end{aligned}$$

[57]Note that this condition is fullfilled for the specifications of $p_i(\cdot)$ given in example 3.

Comparing expressions leads to the statement that, by using formula (1) to determine multi-year default probabilities from the one-year framework, one assumes that $\mathbb{P}\{B_{i,1} = b_i \mid \mathbf{S}_1, \mathbf{S}_2\} = \mathbb{P}\{B_{i,1} = b_i \mid \mathbf{S}_1\}$ $(i = 1, .., n)$.

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