The Correlation Effect

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Abstract

We examine how the correlations of bank loan defaults depend on the correlations of asset returns and how correlations and diversification are affected by macroeconomic risks. We highlight the main properties of the relationship between asset returns and default correlations, illustrating how adverse macroeconomic shocks raise not only the likelihood of defaults, but also the correlation of defaults. The latter effect, called *correlation effect,* may account for more than 50% of the increase in the credit risk.

Keywords: Credit Portfolio Management, Default Correlations, Macroeconomic Shocks, Correlation Effect, Monte-Carlo Simulation.

JEL Classification: F47, G11, G33.

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1 Introduction

In dealing with traded assets, portfolio considerations have been a core component of modern finance. However, diversification considerations in loan portfolios of banks have only received attention in the last two decades, due to increasing problems with losses on credits. The crucial ingredients for any portfolio consideration are the correlations of returns on loans, which determine the achievable degree of diversification and the pricing. In this paper we examine how correlations of bank loans can be determined from correlations of asset returns and how correlations and diversification in loan portfolios will be affected by macroeconomic risks.

Loans are modeled in a standard way as a claim on the value of a firm. The market value of its business is thus equal to the value of its assets. The value of the firm's assets is measured by the price at which the total of the firm's liabilities (equity and debt) can be purchased. Thus, the total value of the firm's assets is equal to the value of the stock plus the value of the debt. Loan default occurs if the market value of the firm's assets falls below the amount due to the loan. Then lenders take over the assets and realize a loss equal to the difference between the face value of the loan and the market value of assets. Thus, the default distribution of a firm is a Bernoulli distribution derived from the distribution of the value of firm's assets and hence from the distribution of the firm's asset returns. Firms' asset return correlations thus translate into correlations of default distributions. We assume that firms' asset values are log normal distributed.

In the first part of the paper, we establish some of the properties of the relationship between asset return correlations and default correlations. We use ρ^{def} to denote the correlation of defaults and ρ^{ret} for the correlation of the asset returns of two firms. First, for $\rho^{ret} > 0$, $\rho^{def}(\rho^{ret})$ is always smaller than ρ^{ret}. Second, $\rho^{def}(\rho^{ret}) = \frac{2}{\pi}\arcsin(\rho^{ret})$ is the lowest upper bound of default correlations and occurs if and only if both firms' default probability is 50 percent. Third, if the default probability of one firm tends to zero, $\rho^{def}(\rho^{ret})$ converges to zero. Hence, for highly solvent firms, its correlation of defaults with other firms can be approximated by zero. Fourth, for sufficiently high asset return correlations, ρ^{def} can be approximated by a simple formula containing only the default probabilities. Fifth, macroeconomic risks, such as interest rate hikes or negative aggregate productivity shocks, raise both default probabilities and default correlations, therefore increasing credit risk from two sides.

In the second part we simulate loan portfolios and derive expected losses and credit risk, i.e. the standard deviation of losses. We examine how macroeconomic risks affect the diversification potential. Macroeconomic risks affect default probabilities and default correlations. We therefore isolate the impact of default correlation by adjusting asset correlations after macroeconomic shocks such that default correlations return to their original levels. Since default probabilities and expected losses are not affected by the adjustment, the relative reduction of credit risk measures the impact of macroeconomic risk on credit risk through changing default correlations. The latter effect is called *correlation effect.* Our major conclusions from the simulations are: First, macroeconomic shocks raise the likelihood of defaults and the correlation of defaults. The order of magnitude of the correlation effect is increasing in the number of firms in the portfolio. Second, the correlation effect can account for 50% of the increase in the credit risk. Typically the correlation effect tends to be high for low asset correlations, and low for high asset correlations. Third, the correlation effect is significant except for very high asset correlations. Accounting for the correlation effect must therefore be an integral part of credit risk management.

The paper is organized as follows. In the next section we relate our analysis to the literature. In section 3 we derive the main properties of default correlations. In section 4 we outline the simulation approach. The results are presented in section 5, followed by a discussion of robustness in section 6. Section 7 concludes.

2 Review of the Literature

Credit risk measurement and diversification in loan portfolios have evolved dramatically over the last two decades. A comprehensive summary is given in Altman and Saunders 1996. Stulz 1998 provides a comprehensive and general overview of institutions' risk management practices and incentives.[1] As a starting point, we employ the risk of ruin or option-pricing model developed in Wilcox 1973, Merton 1974, Scott 1981, Santomero and Vinso 1977 and applied commercial by KMV (see e.g. Kealhofer 1998). The probability of a firm going bankrupt depends on the market value of the firm's assets relative to its outside debt, as well as on the volatility of the market value of the assets. The firm goes bankrupt when the market value of its assets' falls below its debt obligations to outside creditors. In our paper, we explore the relationship between the correlation of assets and the correlation of de-

1 There is a growing academic literature discussing the underlying theory and empirical factors motivating firms have to hedge market risk (see Allayannis and Ofek 1996, DeMarzo and Duffie 1995, Froot, Scharfstein and Stein 1993, May 1995, Mian 1996, Smith and Stulz 1985, Stulz 1990 and Tufano 1996 among others). Most recently Ahn, Boudoukh, Richardson and Whitelaw 1999 have provided an analytical solution to the problem of an institution optimally managing the market risk by using options.

faults in the presence of macroeconomic shocks and derive its implications for the diversification of loan portfolios.

Our approach is useful for examining the impact of macroeconomic shocks on loan portfolios. In both the credit risk and management literature (see e.g. Wahrenburg and Niethen 1999 and Crouhi, Galai and Mark 2000) and the economic literature (see e.g. Hellwig 1998) the question of how financial institutions should cope with macroeconomic risk has been recognized as one of the central questions of credit risk management. We indicate that negative macroeconomic shocks, such as the rise of interest rates or an economic downturn, increase not only default rates but also the correlation of default probabilities and hence increases the credit risk from both quarters. Credit risk management practices must, therefore, incorporate the latter effect as well, which, to our knowledge, is not the case in standard credit portfolio management tools.

3 Correlation of Defaults

3.1 Model

We begin with a discussion of how correlations of firms' asset returns translate into correlations of defaults and use the risk of ruin or option-pricing model (Wilcox 1973, Merton 1974, Scott 1981, Santomero and Vinso 1977, Kealhofer 1998). We assume that the asset values of two firms are jointly log normal distributed, which translates into normal distributions of asset returns. We denote the asset values of two firms 1 and 2 by X and Y and the distributions of the asset returns of firm 1 and firm 2 by $x \sim N(\mu_1, \sigma_1^2)$ and $y \sim N(\mu_2, \sigma_2^2)$ with probability distributions ϕ_x and ϕ_y.[2] The correlation coefficient is denoted by ρ^{ret}. Default occurs if the market value of a firm's assets falls below its debt obligations. This happens when asset returns are sufficiently low. We denote the value of asset return triggering default for firms 1 and 2 by a and b, respectively. Hence, we can derive the following binomial distributions describing default or non-default:

$$\tilde{x} = \begin{cases} 1 & : \text{ if } x \leq a, \text{ i.e., default with probability } \phi_x(a) \\ 0 & : \text{ if } x > a, \text{ i.e., no default with probability } 1 - \phi_x(a) \end{cases}$$

2 By the standard transformation asset returns $x = \frac{X - X_0}{X_0}$ and $y = \frac{Y - Y_0}{Y_0}$ for some initial values X_0 and Y_0 translate into normal distributions if X and Y are log normal distributed.

$$\tilde{y} = \begin{cases} 1 & : \text{ if } y \le b, \text{ i.e., default with probability } \phi_y(b) \\ 0 & : \text{ if } y > b, \text{ i.e., no default with probability } 1 - \phi_y(b) \end{cases}$$

We use $\phi_{xy}(.,.)$ to denote the bivariate normal distribution of the firms asset returns. $\phi_{xy}(a,b)$ is given as

$$\phi_{xy}(a,b) = \int_{-\infty}^{a}\int_{-\infty}^{b} \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^{ret2}}} \quad \exp\Big\{-\frac{1}{2(1-\rho^{ret2})}\big[\frac{(x-\mu_1)^2}{\sigma_1^2} \tag{1}$$

$$-2\rho^{ret}\frac{(x-\mu_2)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\big]\Big\}dydx \tag{2}$$

We calculate

$$\mathrm{Cov}(\tilde{x},\tilde{y}) = E(\tilde{x}\tilde{y}) - E(\tilde{x})E(\tilde{y}) \quad \text{using}$$

$$E(\tilde{x}\tilde{y}) = \phi_{xy}(a,b) \quad \text{and} \quad E(\tilde{x})E(\tilde{y}) = \phi_x(a)\phi_y(b). \tag{3}$$

And we obtain the standard expression

$$\rho^{def} = \frac{\phi_{xy}(a,b) - \phi_x(a)\phi_y(b)}{\sqrt{\phi_x(a)(1-\phi_x(a))}\sqrt{\phi_y(b)(1-\phi_y(b))}}. \tag{4}$$

3.2 Characteristics of Default Correlations

In this section we summarize some of the most important properties of ρ^{def}. In the appendix we establish an upper bound on ρ^{def}:

Property 1
Suppose $a = \mu_1, b = \mu_2$ *then* $\rho^{def}(\rho^{ret}) = \frac{2}{\pi}\arctan(\frac{\rho^{ret}}{\sqrt{1-\rho^{ret2}}}) = \frac{2}{\pi}\arcsin\rho^{ret}$.

Figure 1 is a visualization of the first property for homogeneous firms when the default probability varies from 0.1% to 50%. It illustrates that all default correlations are below the upper bound $\frac{2}{\pi}\arcsin\rho^{ret}$. When both firms have a default probability of 50%, the upper bound $\frac{2}{\pi}\arcsin\rho^{ret}$ is reached. In figure 2 we illustrate property 1 when one firm has a default probability of 50% and the default probability of the other firm varies from 0.1% to 50%, where again the upper bound is reached. The essential difference is that the default correlation does not converge to 1 for inhomogeneous firms when the asset return correlation approaches 1.

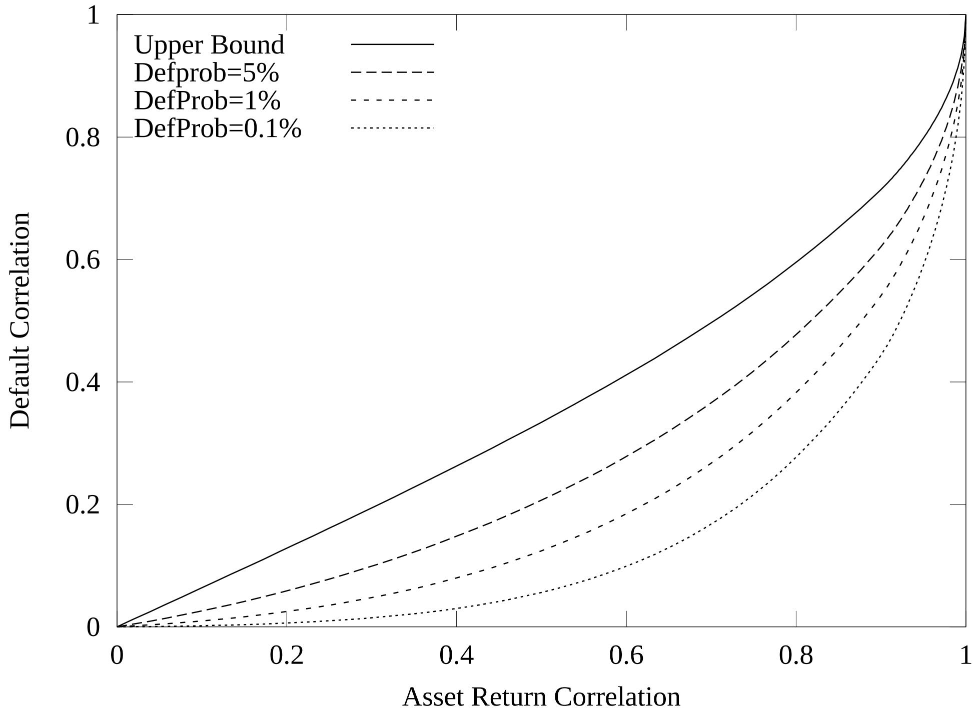

Figure 1: Two equal firms

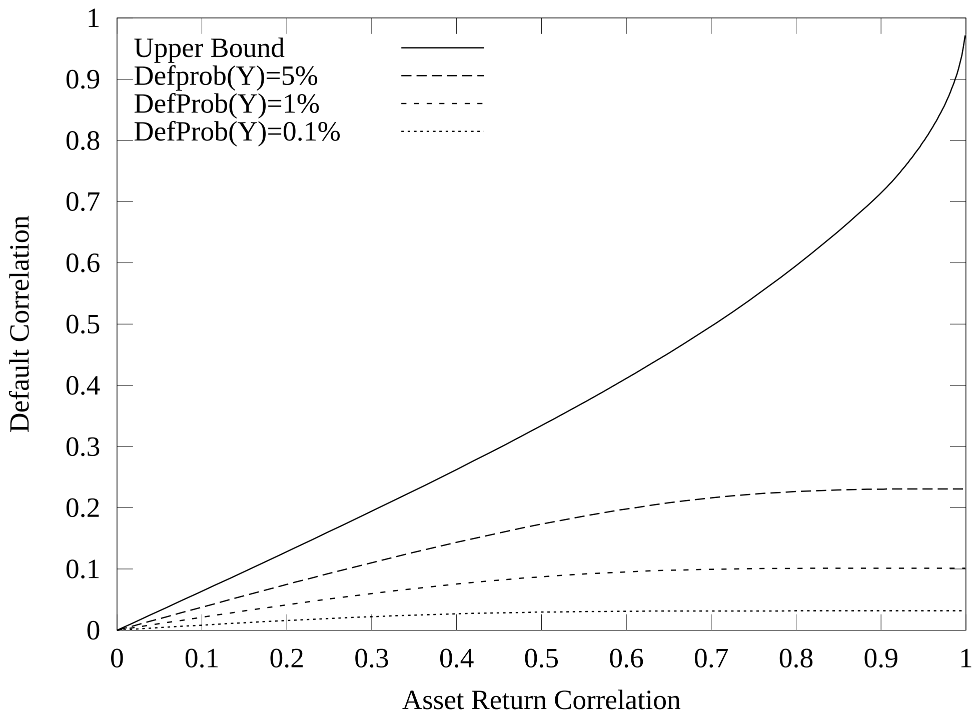

Figure 2: Two different firms (Default Probability of firm X is always 50%)

We next establish the limit properties of $\rho^{def}(\rho^{ret})$. To simplify the notation we work directly with standard normal distributions since $\text{Cov}(\widetilde{x}\widetilde{y})$ can always be transformed into expressions using only standard normal distributions. Consider $\rho^{def}(\rho^{ret})$ as a function of the cut-off points a and b, denoted by $\rho^{def}(a, b, \rho^{ret})$. In the limit, we obtain

Property 2
$\lim_{a\to-\infty} \rho^{def}(a, b, \rho^{ret}) = 0.$

The result follows from using equation (3) with $\phi_{xy}(a, b) = \phi_x(a) \cdot \phi_{y|x}(b|a)$ and by letting $\phi_x(a)$ go to zero for fixed b.

Obviously, the same result holds for $b \to -\infty$, i.e. $\lim\limits_{b\to-\infty} \rho^{def}(a, b, \rho^{ret}) = 0$. The preceding result highlights the important approximation property of default correlations:

Property 3
If a firm has a small default probability, its default correlation with other firms can be approximated by zero.

The above results imply the following relationship between $\rho^{def}(a, b, \rho^{ret})$ and ρ. Since ρ^{def} is continuous in a, b and ρ^{ret}, $\rho^{def}(a, b, \rho^{ret})$ can take all values between $\frac{2}{\pi} \arcsin \rho^{ret}$ and 0 for a fixed ρ^{ret}, depending on the default cut-off points a and b. Hence:

Property 4
By varying the cut-off points a and b, $\rho^{def}(a, b, \rho^{ret})$ can be any value between $\frac{2}{\pi} \arcsin \rho^{ret}$ and 0.

We briefly illustrate the property for $\rho = 1$ and thus $Y = X$. We obtain

$$E(\widetilde{x}\widetilde{y}) = \phi(\min(a, b)),$$

$$\rho^{def}(a, b, 1) = \frac{\phi(\min(a, b)) - \phi(a)\phi(b)}{\sqrt{\phi(a)(1 - \phi(a))}\sqrt{\phi(b)(1 - \phi(b))}}.$$

Asset Return Correlation	0.40	0.80
Default Probability	Default Correlation	
1%	0.08	0.37
5%	0.14	0.47
10%	0.18	0.51
15%	0.21	0.54
20%	0.22	0.56
25%	0.24	0.57
30%	0.25	0.58
35%	0.25	0.58
40%	0.26	0.58
45%	0.26	0.59
50%	0.26	0.59

Table 1: Default Correlations and Increasing Default Probability

Property 5
Suppose $\rho^{ret} = 1$. *Then*

(i) $\rho^{def}(a, a, 1) = 1$,

(ii) $\rho^{def}(a, b, 1) = \frac{\sqrt{\phi(a)}\sqrt{1-\phi(b)}}{\sqrt{1-\phi(a)}\sqrt{\phi(b)}}$ *for* $a < b$,

(iii) $\rho^{def}(a, b, 1) = \frac{\sqrt{1-\phi(a)}\sqrt{\phi(b)}}{\sqrt{\phi(a)}\sqrt{1-\phi(b)}}$ *for* $b < a$.

Again, we observe that for fixed b

$$\lim_{a \to -\infty} \rho^{def}(a, b, 1) = 0.$$

The last property demonstrates that for $\rho^{ret} = 1$, the relative levels of the default cut-off points determine the size of the correlation. The absolute levels of the default probabilities are less important. The expressions in property 5 can be used to approximate default correlations if asset return correlations are sufficiently high.

The above properties of default correlations can be extended. In particular, one can show that $\rho^{def}(a, b, \rho^{ret})$ is monotonically increasing in a and b for $a < 0$ and $b < 0$, respectively for any given asset return correlation (see Erlenmaier and Gersbach 1999).

Hence, not only asset return correlations but also default probabilities are important for the value of default correlations. This is illustrated in Table 1, which shows default correlations when default probabilities of two firms vary from 1% to 50%, given an asset return correlation of 0.4 or 0.8. The details of the calculation are explained in section 4. While the asset return correlations determine the upper bound, increasing default probabilities raise the default correlations considerably.

The positive relationship between default probabilities and default correlations has important implications for the behavior of credit risk in the event of macroeconomic shocks. Consider an increase of an interest rate risk when the interest rates of loan contracts between a firm and a bank are flexible and will be adjusted to new circumstances. An increase in the interest rates will raise the debt obligations and thus a. Therefore, both the default probability and the default correlation between the firm and all other firms will rise. Hence, the interest rate raise will lower the potential for diversification through an increase in default correlation. The same occurs with negative aggregate productivity shocks. We summarize this observation in the following property.

Property 6
A negative aggregate productivity shock or an increase of interest rates when loan contracts have flexible interest rate clauses increase both default probabilities and default correlations.

4 Simulation of Loan Portfolios

4.1 Approach

To examine a loan portfolio and compare the increase in default probabilities and default correlations in the presence of macroeconomic shocks, we simulate loan portfolios using a Monte-Carlo approach. We define loan recovery as the percentage of a firm's outstanding debt obligation that can be recovered by the creditor in the event of bankruptcy. To simplify the comparison between the different simulation outcomes, we work with a fixed recovery rate. Additionally, we split debt obligation into a face value and an interest component to differentiate between interest rate adjustments and loan size changes.

In the following, we describe the inputs, procedures and outputs of the portfolio management tool in more detail, presenting the results directly in terms of asset values and expected losses.

Variable	Explanation
μ, σ^2	parameters of asset distribution
D	outstanding loan size
i	interest rates on loan
R	recovery rate

Table 2: Firm-Specific Data

4.2 Inputs

We require firm-specific data, the number of firms denoted by n, the covariance matrix of firms' asset values denoted by $\Sigma = (\sigma_{ij})_{i,j=1\ldots n}$ and assumptions about the recoveries and the asset distribution. As noted, we operate with a fixed recovery rate. For compatibility with our model in chapter 3.1, we use normal distributed asset returns, which implies log normal distributed asset values. The data for a specific firm is summarized in Table 2.

It is obvious that loan contracts in practical situations are much more complicated, including provisions for durations and seniority clauses. We operate with a simple model to highlight the problems of macroeconomic risks.

4.3 Procedure

We first need a set of n stochastically independent normally distributed random variables. For that purpose, we use the standard Box-Muller transformation[3] to generate normal distributions from uniformly distributed random variables. The latter are generated by the standard random number generator in the GNU C-Library. The Box-Muller transformation is given by

$$
\begin{aligned}
x &= \sqrt{-2\ln(u_1)}\sin(2\pi u_2), \\
y &= \sqrt{-2\ln(u_1)}\cos(2\pi u_2),
\end{aligned}
$$

where u_1, u_2 denote realizations of uniformly distributed random variables.

We denote the vector of statistically independent normal distributions by $\mathbf{z} = (z_1, \ldots, z_n)^T$. In order to generate a multivariate normal distribution with covariance matrix Σ and a vector of expected values $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)^T$ we need to find a root of Σ, denoted by $\sqrt{\Sigma}$, which satisfies $\sqrt{\Sigma}^T\sqrt{\Sigma} = \Sigma$. With $\sqrt{\Sigma}$ we can construct

3 Box and Muller 1958

$$\mathbf{x} = \sqrt{\Sigma}\mathbf{z} + \boldsymbol{\mu}.$$

The covariance matrix of $\mathbf{x}$ is given by $\sqrt{\Sigma}^T\sqrt{\Sigma} = \Sigma$ and the vector of expected values is equal to $\boldsymbol{\mu}$.

We use the Cholesky decomposition of Σ in order to find the root $\sqrt{\Sigma}$, because this procedure can be used for any positive semi-definite and symmetric matrix and thus for any covariance matrix. Moreover, we use it because its recursive calculation[4] allows easy implementation in a computer program. In order to calculate the covariance (and correlation) matrix of the default distribution, we simulate the model implemented in the computer language C++. To achieve an error lower than 0.01 for the correlation matrix, we need a sample of half a million runs. This number of runs also assures simulated default probabilities that differ from analytically derived default probabilities by no more than 0.0005.[5]

4.4 Output

From the large number of runs we can calculate the repayments to banks for each firm. We derive expected losses and the standard deviation of losses. If a firm cannot pay back the face value of its loan and the interest rate, it has defaulted, and the bank only obtains the recovery of the loan. We calculate the default correlation from the default / no default entries.

4.5 Isolation of the Default Correlation Effect

For the examination of macroeconomic shocks we assume that asset value correlations remain constant. This is usually not the case over the business cycle. But by assuming constant asset value correlations we can isolate the impact of changing

4 For $i \leq j \leq n$ calculate

$$c_{jj} = \sqrt{\sigma_{jj} - \sum_{k=1}^{j-1} c_{jk}^2} \text{ and } c_{ij} = \begin{cases} 0 & \text{for } i < j \\ \frac{\sigma_{ij} - \sum_{k=1}^{j-1} c_{ik}c_{jk}}{c_{jj}} & \text{for } i > j \end{cases}$$

to derive the Cholesky matrix $\sqrt{\Sigma} = (c_{ij})_{i,j=1,\dots,n}$ column by column.

5 We check the default probabilities for the simulation by directly calculating the default probabilities from the firm's parameters using Hasting's approximation procedure, which generates a precision of $7.5 \cdot 10^{-8}$. See e.g. Hastings 1957.

default correlations in the event of macroeconomic shocks, which is the focus of this paper. One could always add a change of asset value correlations for a complete examination of business-cycle or interest-cycle effects.

In order to study the impact of macroeconomic shocks, we raise the interest rate by a certain value. Again, we calculate expected losses and the standard deviation of losses. Since the interest rate shock shifts default probabilities and default correlations simultaneously, we employ the following adjustment procedure to isolate the correlation effect.

After the interest rate increase, we adjust the asset correlations downwards until default correlations are back to their initial values. Since default probabilities are not affected by the adjustment procedure (and therefore expected losses also remain constant), the relative change of the standard deviation due to the adjustment only measures the impact of macroeconomic shocks on default correlations and the increase in the risk of bank losses. Therefore we have isolated the default correlation effect of macroeconomic shocks from the default probability effect.

5 Results of the Simulation

5.1 Portfolio Description

We use portfolios consisting of 1 to 10 firms. Moreover, we provide the results for portfolios with a very large number of firms where all potential diversification has been exhausted as a benchmark. In order to concentrate on the impact of default probabilities and asset and default correlations firms will be completely homogeneous with respect to firm-specific data. We calculate expected loss, standard deviation of losses and default correlations for four different examples, providing a first sample set of results without any comprehensive coverage. On the one hand, we have high (5%) and low (2%) initial default probabilities for each firm. On the other, we vary between low (0.4) and high (0.8) correlation of the firms' asset values. For all cases we calculate our parameters for a) the initial scenario, b) a scenario with a macroeconomic shock where the interest rate rises from 5% to 10% (or equivalently the expected asset value is lowered by a corresponding amount), and c) the default correlation adjustment scenario. There, the macro shock is still present, but in order to isolate the default correlation effect the asset correlations are adjusted so that default correlations are back to their initial values.

We choose expected asset value, the variance of the asset values and loan size such that the default probabilities and the asset correlations are realized. Since the loss distribution only depends on default probabilities, default correlations, recovery rates and repayment obligation, the initial data are irrelevant for the results. However, since the repayment obligation of credit exposure (face value of loan and the interest) differs in various scenarios, our results can only be meaningfully compared after we normalize the credit exposure. Therefore, all our results are normalized for a total credit exposure of 1.

With the simulated default correlations it is possible to derive the standard deviations and thus the magnitude of the correlation effect analytically. For a given number n of firms which have default probability p, the standard deviation of losses is given by

$$\sigma_{loss} = \sqrt{p(1-p)(1-R)^2\left[(1-\frac{1}{n})\rho^{def} + \frac{1}{n}\right]}.$$

For $n \to \infty$ the formula simplifies to

$$\sigma_{loss} = \sqrt{p(1-p)(1-R)^2\rho^{def}}.$$

The magnitude of the correlation effect denoted by C can, therefore, be calculated directly as

$$C = \frac{\sqrt{p_{after}(1-p_{after})\rho^{def}_{after}} - \sqrt{p_{after}(1-p_{after})\rho^{def}_{before}}}{\sqrt{p_{after}(1-p_{after})\rho^{def}_{after}} - \sqrt{p_{before}(1-p_{before})\rho^{def}_{before}}}.$$

p_{before} and p_{after} denote the default probabilities before and after the shock and ρ^{def}_{before} and ρ^{def}_{after} denote the default correlations before and after the shock, respectively.

5.2 Results

In Tables 3 and 4 and in Figure 3, we give the output for three different scenarios: a) when the initial default probability is equal to 5%, b) when a large macro shock occurs and the interest rate jumps from 5% to 10% or equivalently, when the expected asset value is lowered such that the same increase of default probability occurs and, c) when we employ the default correlation adjustment. The tables illustrate the

Number of Firms	1	2	6	10	∞
Expected Loss*	0.025	0.025	0.025	0.025	0.025
dto. after macro shock	0.059	0.059	0.059	0.059	0.059
dto. after adjustment		0.059	0.059	0.059	0.059
Std. Dev. of Loss*	0.109	0.093	0.081	0.079	0.075
dto. after macro shock	0.162	0.142	0.126	0.123	0.118
dto. after adjustment		0.139	0.121	0.117	0.111
Correlation Effect		6%	12%	13%	15%
Default Correlation		0.469	0.469	0.469	0.469
dto. after macro shock		0.526	0.526	0.526	0.526
dto. after adjustment		0.469	0.469	0.469	0.469

Table 3: Asset Value Correlation $\rho^{value} = 0.8$, Initial Default Probability 5%, i.e. CCC Rating, *normalized with Total Credit Exposure.

following effects:

- The macroeconomic shock raises positive default correlations. The lower the initial default correlations are the greater the relative change.
- For given asset correlations and default probabilities, the correlation effect is monotonically increasing in the number of firms.
- For initial default correlations of 0.146, i.e. for an asset correlation of 0.4, the correlation effect accounts for about 30% of the increase in the credit risk when default probabilities are 5%.
- When default correlations are high, diversification, i.e. the decrease of standard deviation with an increasing number of firms, is quickly exhausted. For instance, when initial default correlations amount to 0.469 more than 85% of the diversification potential is achieved with 10 firms in the portfolio.

In Tables 5 and 6 and in Figure 4 we perform the same exercise but with much lower initial default probabilities. They confirm the insights from the first simulations. The exhaustion of the diversification potential is even more pronounced. Although the increase in the number of firms is slightly different the correlation effect for a large number of firms has a similar magnitude.

Number of Firms	1	2	6	10	∞
Expected Loss*	0.025	0.025	0.025	0.025	0.025
dto. after macro shock	0.059	0.059	0.059	0.059	0.059
dto. after adjustment		0.059	0.059	0.059	0.059
Std. Dev. of Loss*	0.109	0.082	0.059	0.052	0.042
dto. after macro shock	0.162	0.125	0.093	0.085	0.072
dto. after adjustment		0.123	0.087	0.078	0.062
Correlation Effect		6%	17%	22%	32%
Default Correlation		0.146	0.146	0.146	0.146
dto. after macro shock		0.195	0.195	0.195	0.195
dto. after adjustment		0.146	0.146	0.146	0.146

Table 4: Asset Value Correlation $\rho^{value} = 0.4$, Initial Default Probability 5%, i.e. CCC Rating, *normalized with Total Credit Exposure.

Number of Firms	1	2	6	10	∞
Expected Loss*	0.010	0.010	0.010	0.010	0.010
dto. after macro shock	0.028	0.028	0.028	0.028	0.028
dto. after adjustment		0.028	0.028	0.028	0.028
Std. Dev. of Loss*	0.070	0.059	0.050	0.048	0.045
dto. after macro shock	0.115	0.099	0.087	0.084	0.080
dto. after adjustment		0.097	0.082	0.079	0.074
Correlation Effect		6%	12%	14%	17%
Default Correlation		0.411	0.411	0.411	0.411
dto. after macro shock		0.477	0.477	0.477	0.477
dto. after adjustment		0.411	0.411	0.411	0.411

Table 5: Asset Value Correlation $\rho^{value} = 0.8$, Initial Default Probability 2%, i.e. B Rating, *normalized with Total Credit Exposure.

Number of Firms	1	2	6	10	∞
Expected Loss*	0.010	0.010	0.010	0.010	0.010
dto. after macro shock	0.028	0.028	0.028	0.028	0.028
dto. after adjustment		0.028	0.028	0.028	0.028
Std. Dev. of Loss*	0.070	0.052	0.035	0.031	0.022
dto. after macro shock	0.115	0.087	0.062	0.056	0.045
dto. after adjustment		0.085	0.058	0.050	0.037
Correlation Effect		6%	17%	23%	37%
Default Correlation		0.101	0.101	0.101	0.101
dto. after macro shock		0.152	0.152	0.152	0.152
dto. after adjustment		0.101	0.101	0.101	0.101

Table 6: Asset Value Correlation $\rho^{value} = 0.4$, Initial Default Probability 2%, i.e. B Rating, *normalized with Total Credit Exposure.

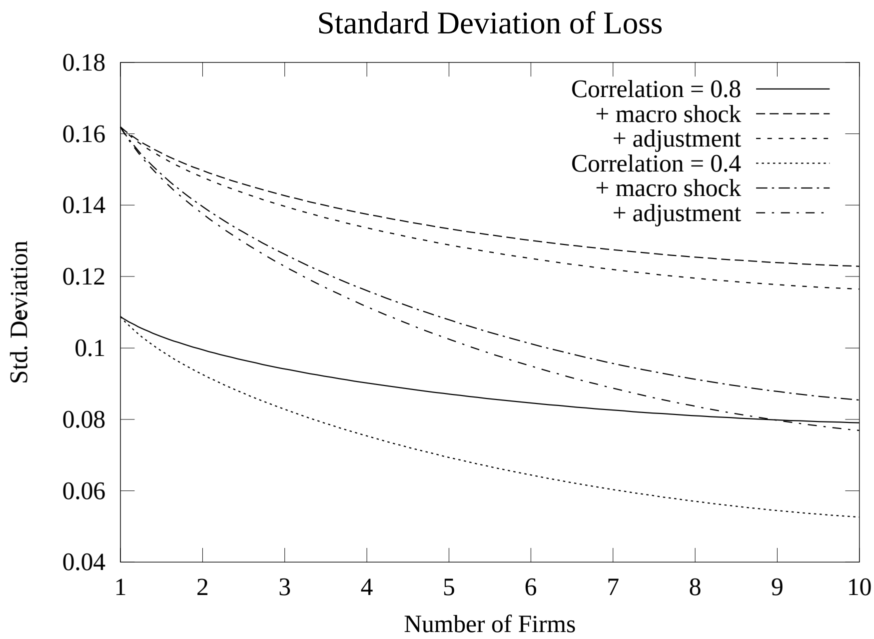

Figure 3: Initial Default Probability=5%

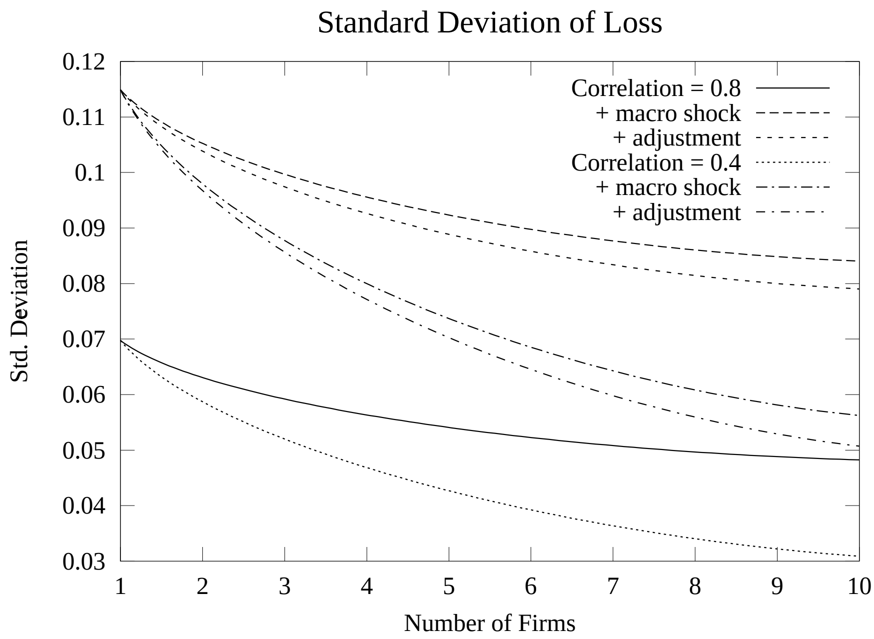

Figure 4: Initial Default Probability=2%

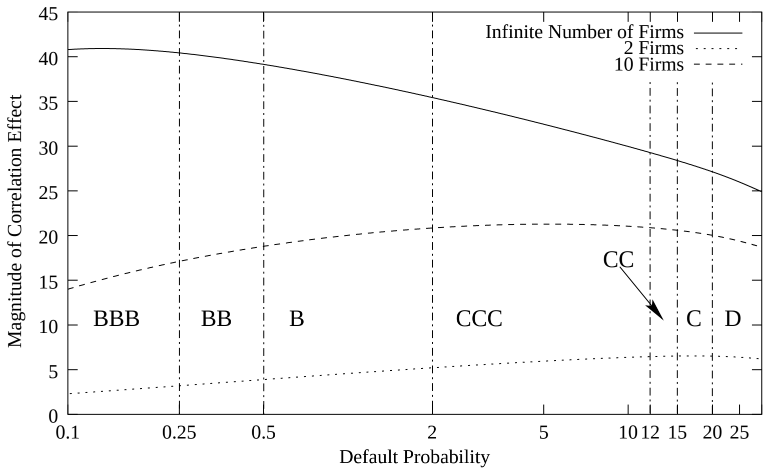

Figure 5: Correlation Effect for Asset Correlation = 0.4

6 Robustness

While the results of the simulations establish a range for the correlation effect, examining the robustness of the findings requires a much more comprehensive look at credit portfolios.

There are three main dimensions to be considered for establishing a robust pattern for the correlation effect. We have performed a variety of simulations covering a grid of a three-dimensional parameter span: number of firms, default probabilities and default correlations.[6] Since the correlation effect is monotonically increasing in the number of the firms, the upper bound on the magnitude of the correlation effect along the other two dimensions can always be found by choosing a sufficiently high number of firms.

In the next step we have varied the default probability between 0.1% and 30% for a given asset correlation. This yields the correlation effect as a function of default probabilities, shown in Figure 5 for an asset correlation of 0.4. Figure 5 also contains the rating class associated with a certain bracket of default probabilities.[7] Figure 5 demonstrates that for a given number of firms, the correlation effect is single-peaked as a function of default probabilities. This property holds for all values of asset and corresponding default correlations. However the maximum depends on the number of firms. While for an asset correlation of 0.4 the function reaches its maximum 42%

6 A further dimension which we have checked is that of endogenous recovery rates, where the recovery is calculated based on the simulated asset values in case of default. Endogenous recovery rates have no effect on our key findings.

7 The allocation of rating classes to default probabilities follows e.g. Crouhi, Galai and Mark (2000).

Rating	p in %	Asset Correlation .001	.05	.1	.2	.3	.4	.5	.6	.7	.8	.9	.95	1.0
AAA	0.02	65	63	61	56	50	44	38	32	25	19	11	7	0
AA	0.05	63	61	59	54	49	43	37	31	25	18	11	7	0
A	0.1	61	59	57	53	48	42	37	31	25	18	11	7	0
BBB	0.25	59	57	55	51	46	41	35	30	24	18	11	7	0
BB	0.5	57	55	53	49	44	39	34	29	23	18	11	7	0
B	2	52	50	48	44	40	36	31	26	22	16	10	7	0
CCC	5	48	46	44	40	36	33	28	24	20	15	10	6	0
CCC	10	44	42	41	37	34	30	26	22	18	14	9	6	0
CC	15	42	40	39	35	32	28	25	21	17	13	9	6	0
C	20	40	39	37	34	30	27	24	20	17	13	8	6	0
D	25	39	38	36	33	29	26	23	20	16	12	8	5	0
D	30	38	37	35	32	29	25	22	19	16	12	8	5	0

Table 7: Correlation Effect in %, infinite number of firms

at the default probability of 0.1% in the limit for infinite firms, the maximum shifts to 5% when there are only 10 firms in the portfolio.

Moreover varying the asset correlations from 0 to 1, it transpires that the maximum value for the correlation effect is reached for low asset correlations and there at low default probabilities.[8] This is confirmed by table 7 which shows results along the asset correlation dimension for the default probabilities from 0.02% to 30%, i.e. for the rating classes AAA to D with the maximum value of the correlation effect at asset correlations 0.001 and default probability 0.02%.

We have not yet considered variations in the size of the macroeconomic shock, which is a fourth dimension of our robustness analysis. However, variations in the size of the macroeconomic shock have relatively little impact on the size of the correlation effect. When moving to small or marginal macroeconomic shocks, the correlation effect tends to become smaller, but remains significant.[9]

While our results are robust for any homogeneous sample of firms, it is not a priori clear how asymmetries of firms with respect to default probabilities, asset and default correlations affect our results. However, since the standard deviations of losses for inhomogeneous portfolios are the weighted mean of the standard deviations of losses for homogeneous subportfolios, the conclusions about the significant magnitude of the correlation effect are not affected.

8 An analytical deviation of the maximum value of the correlation effect is not available yet.

9 For instance, for AAA-rated firms and asset correlations of 0.001 the correlation effect shrinks to 47% when interest rates are raised marginally by one base point.

The most important conclusion from the whole exercise is that the correlation effect is significant except for very high asset correlations.[10] Accounting for the correlation effect must therefore be an integral part of any approach to credit risk management.

7 Conclusions

In this paper, we have examined a standard model to determine defaults of firms and bank loans. Our main results indicate that the credit risk of banks experiencing macroeconomic shocks is affected both by changing default probabilities and changing default correlations even if asset correlations remain unchanged. The present analysis may not only be relevant for credit portfolio management in banks but also for regulatory purposes. The insight that negative macroeconomic shocks usually increase default probabilities and default correlations calls for a cautious approach in the calculation of the capital required by banks to buffer negative macroeconomic shocks.

Our analysis could and should be extended in various directions. First, a more complete account of the correlation effect for inhomogeneous portfolios is desirable. Second, how the correlation effect should be built into loan pricing remains an open issue. Since a higher volatility of default correlations imposes higher costs in terms of higher credit capital, the expected volatility of default correlations due to macroeconomic shocks might justify loan price differences for firms which are equal in all other respects. Third, the static nature of our exercise could be extended to cases where asset values of firms are described by a diffusion process in which the stochastic component is a geometric Brownian motion and, therefore, asset returns are normally distributed. Fourth, should banks try to hedge macroeconomic risk in general and credit risk due to the volatility of the default correlations in particular and if so, how should they do it? The answers to these and other issues could be relevant for a better understanding of credit risk.

10 In the limit case, when the asset values are fully correlated, the correlation effect disappears.

8 Appendix

Proof of Property 1:
Let us investigate the function:

$$E(\widetilde{X}\widetilde{Y}) = \int_{-\infty}^{a}\int_{-\infty}^{b} \frac{dxdy}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^{ret^2}}} \exp\Big\{-\frac{1}{2(1-\rho^{ret^2})}\Big[\frac{(x-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\Big]\Big\}. \tag{5}$$

Using $a = \mu_1$ and $b = \mu_2$ and substituting

$$x = \mu_1 + \sigma_1 u,\ y = \mu_2 + \sigma_2\sqrt{1-\rho^{ret^2}}v + \rho^{ret}\sigma_1 u,$$

we obtain

$$\begin{aligned} E(\widetilde{X}\widetilde{Y}) &= \int_{-\infty}^{0}\int_{-\infty}^{\frac{-\rho^{ret}}{\sqrt{1-\rho^{ret^2}}}u} \frac{1}{2\pi}\cdot\exp\left\{-\frac{1}{2}(u^2+v^2)\right\} dvdu, \\ &= \int_{-\infty}^{0}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{u^2}{2}\right)\int_{-\infty}^{\frac{-\rho^{ret}}{\sqrt{1-\rho^{ret^2}}}u}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{v^2}{2}\right)dvdu. \end{aligned} \tag{6}$$

Hence, we have transformed $\phi_{xy}(\mu_1,\mu_2)$ into two independent standard normal distributions. Applying rotation symmetry for a two-dimensional standard normal distribution we obtain $E(\widetilde{X}\widetilde{Y}) = \frac{|\beta|}{2\pi}$ where β is the angle in the (u,v) plane between the two half straight lines

$$u = 0, v < 0,$$

$$v = -\frac{\rho^{ret}}{\sqrt{1-\rho^{ret^2}}}u,\ u < 0.$$

Hence,

$$
\begin{aligned}
|\beta| &= \frac{\pi}{2} + \left| \arctan\left(- \frac{\rho^{ret}}{\sqrt{1-\rho^{ret^2}}} \right) \right|, \\
&= \frac{\pi}{2} + \arctan\left(\frac{\rho^{ret}}{\sqrt{1-\rho^{ret^2}}} \right), \\
&= \frac{\pi}{2} + \arcsin(\rho^{ret}).
\end{aligned}
$$

Therefore, we obtain

$$
E(\tilde{X}\tilde{Y}) = \frac{1}{4} + \frac{1}{2\pi} \arcsin(\rho^{ret}),
$$

$$
\begin{aligned}
\rho^{def}(\rho^{ret}) &= \frac{Cov(\tilde{X}\tilde{Y})}{\sigma_{\tilde{x}}\sigma_{\tilde{y}}} = \frac{\frac{1}{4} + \frac{1}{2\pi}\arcsin(\rho^{ret}) - \frac{1}{4}}{\frac{1}{4}}, \\
&= \frac{2}{\pi} \arcsin(\rho^{ret}).
\end{aligned}
$$

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