The Term Structure of Default in Copula Models

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Abstract

We introduce a measure for the term structure of multiple default events and investigate the term structure induced by two families of copula functions, the extreme value copulas and the $\mathcal{C}^1$ copulas. We construct a family of copula functions that are term structure free under this measure, including a term structure free version of the normal copula model.

1 Introduction

In Merton's firm model for corporate default a firm is considered to be in default if its outstanding liabilities exceed its total assets. This model has been widely used for valuing multi-name credit derivatives such as first to default baskets and CDO's. In practice, a single name default swap spread is used to calibrate the "liability curve" (which is in general time dependent) below which a firm is considered to be in default. Multi-name credit derivatives, which typically depend on multiple default events, can then be valued from a multi-asset distribution and a set of individual credit liability curves. Such a model for multiple default events still allows for the following modeling choices to be made:

1. Does a default occur at time t if the asset value has crossed the liability curve between $[0, t]$ or only if it is below the liability curve at time t?

In the standard approach a firm is considered to have defaulted by time t only if its asset value is below the liability curve at time t, regardless of whether or not the firm's asset value crossed the liability curve before time t. In [3] an approach is explored in which a firm is considered to have defaulted by time t if its asset value crossed the liability curve at least once before time t.

2. What is to be used for the multi-asset distribution? If a multivariate normal distribution is assumed, then what value should be used for the correlation?

Although Brownian motion or Geometric Brownian motion is usually assumed for the multi-asset distribution, the Student's T copula has also been investigated in a number of reports. A correlation parameter usually refers to the correlation of the multivariate normal distribution used to generate a normal copula, but may also refer to the correlation of the default time distribution, or the correlation of default events (see [7]).

3. Is a deterministic or a stochastic default intensity assumed for the individual credit default process? Should the liability curve itself be considered a random variable?

Stochastic intensity copula models have been analyzed in [9] and [5]. Models with a random default liability curve for valuing single name default swaps have been explored recently in a number of papers, for example [2]. Such models have not yet been extended to multi-name credit derivatives.

A widely used and particularly simple approach is to consider a model in which:

1. Default occurs at time t if the asset value is below the liability curve at time t.
2. A multivariate normal distribution, or equivalently a multivariate normal copula with normal marginals, is used for the multi-asset distribution.
3. The default intensities and the liability curves are considered to be deterministic and time dependent.

We will consider a model as specified above to be the standard model for multi-name credit derivatives. In this paper we will primarily explore models in which assumptions 1 and 3 hold, but assumption 2 is weakened.

Li shows in [6] that Merton's firm model as specified above is equivalent to a model in which joint default times are computed using a normal copula, i.e. if $S_1(t_1), ..., S_n(t_n)$ are the probabilities that credits $1, ..., n$ survive to times $t_1, ..., t_n$ then the joint survival probability is $c_\rho(S_1(t_1), ..., S_n(t_n))$, where c_ρ is a normal copula with correlation matrix ρ. In fact, assumption 2 is more general than is immediately apparent; a basic property of copula functions is that they are invariant under transformations which modify the underlying univariate distributions by composition with monotonic increasing functions. In particular, if it is assumed that the multi-asset distribution is log normal, rather than normal, then the same copula c_ρ governs the joint default times.

The probability of a default event occurring, whether the default of an individual credit, or the n'th default in a basket of credits can be characterized by a default intensity function. Let

$$S_i(t) \quad = \quad \exp(-\int_0^t \lambda_i(u)du) \equiv Pr(\tau_i > t)$$

denote the survival probability for credit i. The first to default intensity, $\lambda^1(u)$ is defined through the following equation:

$$Pr(\tau_1, \ldots, \tau_N > t) = \exp(-\int_0^t \lambda^1(u)du)$$

or equivalently

$$\lambda^1(t) \quad = \quad -\frac{d}{dt}\log(Pr(\tau_1, \ldots, \tau_N > t)) \tag{1}$$

where $(\tau_1, \ldots, \tau_N)$ is the joint default time distribution. If a copula function $c(x_1, x_2, \ldots, x_N)$ determines the probability of joint default events then:

$$\lambda^1(t) \quad = \quad -\frac{d}{dt}\log(c(S_1(t), S_2(t), \ldots, S_N(t))). \tag{2}$$

It can be observed numerically that if a normal copula is used to model joint default events, even if all individual credit default intensities are flat (we say that an intensity curve is *flat* if $\lambda_i(t) = \lambda_i$ for a fixed value λ_i), the first to default intensity curve will be downward sloping; in other words the first to default intensity is decreasing as a function of time. For a normal copula with a correlation of 60%, the first to default intensity for a basket of two credits with constant spreads of 300 b.p. and 500 b.p. is plotted in Figure 1. We refer to the shape of the first to default intensity curve as the *term structure* for the first default event. We will argue in Section 2 that if the individual intensity curves $\lambda_i(t)$ are flat then a model *imposes* a term structure for the first default event if the first to default intensity curve is not flat.

An immediate consequence of a downward sloping first to default intensity curve is that par spreads for the corresponding first to default swaps decrease with maturity; this follows immediately from the fact that the par spread for a default swap with maturity T can be expressed approximately in terms of the default intensity as follows:

$$s_{par} \simeq \frac{(1-R)\int_0^T \lambda(t)D(t)\exp(-\int_0^t \lambda(u)du)dt}{\int_0^T D(t)\exp(-\int_0^t \lambda(u)du)dt},$$

where R is the recovery and $D(t)$ is the discount factor at time t (this approximation is well known, see for example [8]).

The term structure of the first to default intensity curve as imposed by a normal copula model is a cause for concern for two separate but related reasons. One problem is that even if the correlation is a fixed constant and the individual credit curves are flat, forward starting swaps will price with less risk than swaps of the same maturity starting today (in a forward starting default swap contract if a default occurs before the forward start date the trade is terminated without any economic consequence to either the protection seller or buyer). In other words the par spread of a forward starting N year first to default swap will be less than the par spread of an N year first to default swap starting today. Another effect of a downward sloping term structure is that first to default swaps will appear more risky as they mature. For similar reasons the equity tranche of a synthetic

CDO valued using a normal copula will appear more risky as the deal matures and the super-senior tranche will appear to be less risky.

The two questions that we explore in this paper are:
1. How do we measure the term structure of the default intensity of multiple default events — in particular the n'th default in a basket of credits — as imposed by a copula model?
In Section 4 we introduce two measures for the term structure of multiple default events (see equations (13) and (15)). We also define the notion of a *completely term structure free* model for multiple default events.
2. Which copula models, unlike a normal copula model, do not impose a term structure for the default intensity of joint default events?
In Section 5 we introduce a new family of copula functions which are completely term structure free. In particular we construct a "flat" version of a normal copula (see equations (24) and (26)):

$$\begin{aligned} c_{\text{flat}}(u_1,\ldots,u_N) &= \exp(-\int_0^{t_1} f(\lambda_1(t),\lambda_2(t),\ldots,\lambda_N(t);t)dt - \int_{t_1}^{t_2} f(0,\lambda_2(t),\lambda_3(t),\ldots,\lambda_N(t);t)dt \\ &\quad \ldots - \int_{t_{N-2}}^{t_{N-1}} f(0,\ldots,0,\lambda_{N-1}(t),\lambda_N(t);t)dt - \int_{t_{N-1}}^{t_N} f(0,\ldots,0,\lambda_N(t);t)dt) \end{aligned}$$

where $u_i = \exp(-\int_0^{t_i} \lambda_i(u)du), t_1 \le t_2 \le \ldots \le t_N$ (we provide a more general definition in the sequel that does not assume increasing survival times),

$$f(x_1,x_2,\ldots,x_N;t) = -\log(c_{\rho(t)}(e^{-x_1},e^{-x_2},\ldots,e^{-x_N}))$$

and $c_{\rho(t)}$ is a normal copula with correlation matrix $\rho(t)$. If the correlation $\rho(t)$ is flat then such a model satisfies the completely term structure free measure as defined by equation (14).

2 First and Second to Default Intensities

In a world in which the default intensity curves $\lambda_i(t)$ are flat the forward default time distribution (conditional on survival to the forward date) is identical to the default time distribution as measured from today: let τ_i denote the default time distribution for credit i, then for $\lambda_i(t)$ flat (recall that a parameter is said to be flat if it is not time dependent) and $t > s$,

$$Pr(\tau > t|\tau_i > s) = Pr(\tau_i > t - s).$$

Ignoring the term structure of the risk-free discount curve it follows that forward starting single credit default swaps and swaps starting today of the same maturity will have the same par spreads.

The basic data for a first to default swap are the individual intensity curves, $\lambda_i(t)$ and a set of dependence parameters $\theta_k(t), k = 1,\ldots,m$ which determine the dependence term structure. For example, in the normal copula model the correlation $\rho(t)$ is the dependence parameter that completely determines the dependence term structure. Similar to the case of a single name default swap we expect that if all of the basic data is flat then a forward starting first to default swap and one starting today should have the same par spread. In particular the first to default intensity should be flat; as noted in Section 1 this is not the case for the normal copula model. For this reason we believe that the term structure imposed by a normal copula model is a defect of the model. For completely term structure free models as defined in Section 5, the first to default intensity curve is flat if all individual intensity curves and dependence parameters are flat.

Unlike the first to default intensity, we expect to see an intrinsic term structure for the second to default intensity (or the n'th to default intensity for $n > 1$) in any reasonable model in which the underlying credits are positively correlated and the individual credit default intensity curves are flat. This can be explained roughly as follows. Let $\lambda^2(u)$ denote the intensity for the second default event, i.e. $S(t) = \exp(-\int_0^t \lambda^2(u)du)$ is the probability that the second default event occurs after time t, or equivalently $\lambda^2(t)\delta t$ is the probability that the second default event occurs at time t (conditional on the second default event not occurring before time t). Let

$t_1 > t_0$ denote two possible default times; then it is more likely that the second default event occurs at time t_1 than at time t_0 since it is more likely that the *first* default event has occurred by time t_1 than by time t_0. Therefore, $\lambda^2(t_1)\delta t > \lambda^2(t_0)\delta t$, or equivalently $\lambda^2(t_1) > \lambda^2(t_0)$.

We can verify this argument analytically in the special case where default events occur independently. Let $S(t)$ denote the probability that a second to default basket with two underlying credits survives to time t. Then, assuming that $\lambda_i(t) \equiv \lambda_i$ (where λ_i is the default intensity of credit i),

$$\begin{aligned} S(t) &= Pr(\tau_1, \tau_2 > t) + Pr(\tau_1 > t, \tau_2 < t) + Pr(\tau_2 > t, \tau_1 < t) \\ &= e^{-(\lambda_1+\lambda_2)t} + (1 - e^{-\lambda_1 t})e^{-\lambda_2 t} + (1 - e^{-\lambda_2 t})e^{-\lambda_1 t} \\ &\simeq 1 - \lambda_1\lambda_2 t^2 + o(t^3). \end{aligned} \tag{3}$$

We can compute the default intensity for the second to default event, $\lambda^2(u)$ using the relation $S(t) = \exp(-\int_0^t \lambda^2(u)du)$,

$$\begin{aligned} S(t) &= \exp(-\int_0^t \lambda^2(u)du \\ &\simeq 1 - \int_0^t \lambda^2(u)du. \end{aligned} \tag{4}$$

Equating equations (3) and (4), for small t we find that $\lambda^2(t) = 2\lambda_1\lambda_2 t + o(t^2)$; in other words the second to default intensity curve is upward sloping. We will return to computing second to default intensities in Section 4.

3 General Results for $\mathcal{C}^1$ copulas

For simplicity of exposition we will limit the discussion in this section to two dimensional copula functions and assume that the individual default intensities are constant in time: $\lambda_1(t) \equiv \lambda_1$ and $\lambda_2(t) \equiv \lambda_2$. Let $c(x, y)$ be a $\mathcal{C}^1$ copula, i.e. a copula function for which the partial derivatives c_x and c_y exist and are continuous on $[0, 1]^2$. The partial derivatives can be interpreted as conditional distribution functions: if X, Y are random variables with cumulative distributions $F_1(x), F_2(y)$ such that the dependence structure is given by the copula $c(x, y)$ then $c_x(x, y) = Pr(Y < F_2^{-1}(y)|X = F_1^{-1}(x))$ and $c_y(x, y) = Pr(X < F_1^{-1}(x)|Y = F_2^{-1}(y))$. In particular, $c_x(x, 1) = c_y(1, y) = 1$ for all $x, y \in [0, 1]$. An important technical point is that the random variables X and Y need not be supported on a bounded subset of $\mathcal{R}$. In this case one or more of $F_1^{-1}(1), F_1^{-1}(0), F_2^{-1}(1), F_2^{-1}(0)$ may be infinite and $c_x(1, 1), c_x(0, 1)$ or $c_y(1, 1), c_y(0, 1)$ are defined by continuity. For example

$$c_x(1, 1) \equiv \lim_{x\to 1} c_x(x, 1) = 1.$$

The class of $\mathcal{C}^1$ copulas includes the three most widely used copulas: the normal copula, the Students' T copula and the trivial copula of independence (the trivial copula is defined as $c(x, y) = xy$, or equivalently a normal copula with zero correlation). It should be noted that these three copulas are all examples of the Students T copula; the normal copula is the limit of the Students' T copula with infinite degrees of freedom and the trivial copula of independence is a normal copula with zero correlation . For any copula c, if c_x and c_y exist we can compute the first to default intensity using equation (2) as follows:

$$\begin{aligned} \lambda^1(t) &= \frac{d}{dt}\log(c(S_1(t), S_2(t))) \\ &= \frac{c_x(S_1(t), S_2(t))S_1(t)\lambda_1 + c_y(S_1(t), S_2(t))S_2(t)\lambda_2}{c(S_1(t), S_2(t))}. \end{aligned} \tag{5}$$

For $\mathcal{C}^1$ copulas, by evaluating the above expression we find that defaults occur independently at $t = 0$:

$$\lambda^1(0) = c_x(1, 1)\lambda_1 + c_y(1, 1)\lambda_2 = \lambda_1 + \lambda_2.$$

A consequence of this observation is the following fact:
Any $\mathcal{C}^1$ copula with a flat first to default intensity curve will necessarily generate the same first to default intensity curve as the trivial copula . This follows immediately from the fact that by the above argument for any $\mathcal{C}^1$ copula the first to default intensity at $t = 0$ is $\lambda_1 + \lambda_2$. Therefore, any copula model with a flat first to default intensity curve that deviates from that of the trivial copula cannot be a $\mathcal{C}^1$ copula. We will show that a large class of such copulas exist in Section 5.
Question: Is the trivial copula the only $\mathcal{C}^1$ copula which generates a flat first to default intensity curve?

We should note that a copula function can be $\mathcal{C}^1$ and have tail dependence as defined in [4]. For a bivariate copula c the measure of tail dependence α is defined as:

$$\lim_{u \to 1} \frac{c(u,u) - 2u + 1}{1 - u} = \alpha.$$

A copula is said to have tail dependence if $\alpha > 0$. A Student's T copula with finite degrees of freedom is an example of a $\mathcal{C}^1$ copula function that has tail dependence (see [1] for a numerical computation of the tail dependence).

Let $X_i(t)$ denote the Bernoulli random variable for the survival of credit i:

$$Pr(X_i(t) = 1) = Pr(\tau_i > t) = S_i(t).$$

We can compute the correlation of $X_1(t), X_2(t)$ assuming that the joint survival probability is determined by a copula function $c(x, y)$:

$$\frac{E[X_1(t)X_2(t)] - E[X_1(t)]E[X_2(t)]}{\sqrt{var(X_1(t))var(X_2(t)}} = \frac{c(S_1(t), S_2(t)) - S_1(t)S_2(t)}{\sqrt{S_1(t)(1 - S_1(t))S_2(t)(1 - S_2(t))}}.$$

Not surprisingly we find that at $t = 0$, by taking a limit, that the correlation is zero for a $\mathcal{C}^1$ copula.

It is interesting as a comparison with the case of $\mathcal{C}^1$ copulas to compute the first to default intensity curve for two Bernoulli random variables $X_1(t), X_2(t)$ defined as above, but with fixed correlation ρ. We define the default intensity as in equation (1):

$$\begin{aligned} \lambda(t) &= -\frac{d}{dt}\log(E[X_1(t)X_2(t)]) \\ &= -\frac{d}{dt}\log(E[X_1(t)]E[X_2(t)] + \rho\sqrt{var(X_1(t))var(X_2(t)}) \\ &= -\frac{d}{dt}\log(S_1(t)S_2(t) + \rho\sqrt{S_1(t)(1 - S_1(t))S_2(t)(1 - S_2(t))}) \end{aligned} \tag{6}$$

At $t = 0$, by taking a limit we find that the default intensity reduces to:

$$\lambda_1 + \lambda_2 - \rho\sqrt{\lambda_1\lambda_2}.$$

This model exhibits asymptotic dependence unlike the case of dependence structures determined by $\mathcal{C}^1$ copulas.

4 Default Intensities and a Measure of Term Structure

Let $\lambda_i(t)$ denote the default intensity curve for credit i (this intensity may be inferred from a series of credit default swap quotes). In a non-stochastic intensity setting, a copula function and a collection of default intensity curves $\lambda_i(t), i = 1, \ldots N$ completely determines the joint default time distribution. Let $(\tau_1, \ldots, \tau_N)$ denote the joint default time distribution. Let $\tau_{(n)}$ denote the n'th to default time distribution, i.e. if the default times are listed in ascending order, $\tau_{i_1} \leq \tau_{i_2} \leq \ldots \leq \tau_{i_N}$ then $\tau_{(n)} = \tau_{i_n}$. We define the density for the n'th to default time distribution, $\rho^n(t)$ as

$$\rho^n(t) = Pr(\tau_{(n)} \in [t, t + dt])$$

and the survival probability to time t for the n'th default event, $S^n(t)$ as

$$S^n(t) = 1 - \int_0^t \rho^n(u)du.$$

Define the n'th to default intensity $\lambda^n(t)$ by

$$S^n(t) = \exp(-\int_0^t \lambda^n(u)du).$$

Then

$$S^n(t)\lambda^n(t) = \rho^n(t),$$

or equivalently

$$\lambda^n(t)dt = Pr(\tau_{(n)} \in [t, t+dt] | \tau_{(n)} \geq t).$$

Define $\lambda_i^n(t)$ to be the n'th to default intensity due to credit i:

$$\lambda_i^n(t) = Pr(\tau_{(n)} \in [t, t+dt] \text{ and } \tau_{(n)} = \tau_i | \tau_{(n)} \geq t).$$

Then

$$\begin{aligned} \lambda^n(t) &= Pr(\tau_{(n)} \in [t, t+dt] | \tau_{(n)} \geq t) \\ &= \Sigma_{i=1}^N Pr(\tau_{(n)} \in [t, t+dt] \text{ and } \tau_{(n)} = \tau_i | \tau_{(n)} \geq t) \\ &= \Sigma_{i=1}^N \lambda_i^n(t). \end{aligned} \tag{7}$$

We can explicitly compute $\lambda_i^n(t)$ in terms of a given copula function $c(x_1, x_2, \ldots, x_N)$; for example (for $n = 1$),

$$\lambda_i^1(t) = \frac{c_i(S_1(t), S_2(t), \ldots, S_N(t))S_i(t)\lambda_i}{c(S_1(t), S_2(t), \ldots, S_N(t))} \tag{8}$$

where c_i is the partial derivative of c with respect to x_i.
For a $\mathcal{C}^1$ copula, by continuity as computed in equation (5),

$$\lambda_i^1(0) = \lim_{t \to 0} \lambda_i^1(t) = \lambda_i \tag{9}$$

where λ_i is the default intensity for credit i and therefore

$$\lambda^1(0) = \lambda_1 + \lambda_2 + \ldots + \lambda_N. \tag{10}$$

We can decompose the default intensity $\lambda_i(t)$ into a weighted sum of the expected intensity conditional on at least one default in the basket, which we will denote by $\gamma_i(t)$ and the intensity conditional on the survival of all credits, $\lambda_i^1(t)$. Let A denote the set of events where credit i defaults after time t and at least one of the credits $1, \ldots, i-1, i+1, \ldots, N$ default before time t:

$$A = \{\tau_i \geq t, \min(\tau_1, \ldots, \tau_{i-1}, \tau_{i+1}, \ldots, \tau_N) < t\}.$$

Notice that by the above definition,

$$S_i(t) = c(S_1(t), \ldots, S_N(t)) + Pr(A)$$

where $Pr(A)$ is the probability that any event in the set A occurs. Let $\gamma_i(t)$ denote the instantaneous default probability for credit i conditional on being in A:

$$\gamma_i(t)dt = Pr(\tau_i \in [t, t+dt] | A).$$

Then

$$\begin{aligned} S_i(t)\lambda_i(t)dt &= Pr(\tau_i \in [t, t+dt]) \\ &= Pr(\tau_i \in [t, t+dt] | \tau_1, \dots, \tau_N \geq t) \cdot Pr(\tau_1, \dots, \tau_N \geq t) + Pr(\tau_i \in [t, t+dt] | A) \cdot Pr(A) \\ &= \lambda_i^1(t) c(S_1(t), \dots, S_N(t)) dt + \gamma_i(t)(S_i(t) - c(S_1(t), \dots, S_N(t))) dt \end{aligned} \tag{11}$$

By the above calculation we can write the default intensity $\lambda_i(t)$ as a weighted sum of $\lambda_i^1(t)$ and $\gamma_i(t)$:

$$\lambda_i(t) = w_1 \lambda_i^1(t) + w_2 \gamma_i(t)$$

where $w_1 + w_2 = 1$. We define a "positively correlated default model" to be a model in which $\lambda_i^1(t) < \lambda_i(t) < \gamma_i(t)$, i.e. default is more likely for credit i if at least one credit has defaulted and less likely if all credits have survived. (In [9] it is shown that in a normal copula model the default intensity for credit i decreases conditional on the survival of credit j if the correlation parameter is positive.) Taken together with equation (9) we find that if all $\lambda_i(t)$ are flat in a positively correlated model:

$$\lambda_i^1(t) < \lambda_i^1(0)$$

and therefore $\lambda^1(t) < \lambda^1(0)$ for $t > 0$. The above result does not show that the default intensity curve $\lambda^1(t)$ is downward sloping, but it does indicate that this is plausible in a postively correlated model.
Question: What are the sufficient and necessary conditions that a copula must satisfy to ensure that the conditional default intensity curves, $\lambda_i^1(t)$ are downward sloping if the individual intenstiy curves $\lambda_i(t)$ are flat?

As noted in Section 2 for n'th to default events, where $n \geq 2$, if default times are positively correlated or indepedent then we expect that the default intensity curve $\lambda^n(t)$ is necessarily upward sloping. Therefore $\lambda^n(t)$ for $n \geq 2$ does not measure the term structure imposed by a copula model. In a term structure free model if the state of the world is the same at time s – in the future – as it is today then an n'th to default basket has the same forward price at time s as it does today. More precisely, let

$$Q^n(s, s+T)$$

denote the probability that the n'th to default basket survives to time $s+T$ conditional on the survival of all credits to time s. If all credit curves $\lambda_i(t)$ are flat then the state of the world at any future time (conditional on the survival of all credits) is the same as it is today. Assuming that the curves $\lambda_i(t)$ are flat, comparing the survival probabilities $Q^n(0,T)$ and $Q^n(s, s+T)$ is a measure of the term structure imposed by the copula model at time s; for example if $Q^n(s, s+T) > Q^n(0,T)$ then a forwarding starting n'th to default basket of term T (starting at time s) is less risky than the same basket starting from today. Assuming that the default intensity curves $\lambda_i(t)$ are flat, a copula model is term structure free for the n'th default event if at each future time s,

$$Q^n(0,T) = Q^n(s, s+T). \tag{12}$$

As in equation (2) it is helpful to express $Q^n(s, s+T)$ in terms of a default intensity:

$$Q^n(s, s+T) = \exp(-\int_0^T \alpha_s^n(u) du).$$

In Figure 2 we have graphed $\alpha_s^2(u)$ for a basket of two credits with spreads 300 b.p. and 500 b.p., assuming a normal copula with correlation 60% for $s = 0, 3, 5$, i.e. the default intensity curves for a second to default swap starting today, in 3 year from today and in 5 years from today. As can be seen from the graph, even when all curves $\lambda_i(t)$ are flat in a normal copula model a forward starting second to default basket starting in the future is less risky than one starting today.

For $n = 1$, $Q^n(s,t)$ reduces to the survival probability generated by the first to default intensity $\lambda^1(u)$:

$$\begin{aligned} Q^1(s,t) &= \frac{c(S_1(t), \dots, S_N(t))}{c(S_1(s), \dots, S_N(s))} \\ &= \exp(-\int_s^t \lambda^1(u) du) \end{aligned}$$

where $\lambda^1(u)$ was defined by equation (7).

We can also compute $Q^n(s,t)$ explicitly in terms of a copula function; for a basket of two credits and copula $c(x,y)$, $Q^2(s,t)$ reduces to:

$$\begin{aligned} Q^2(s,t) &= Pr(\tau_1, \tau_2 > t | \tau_1, \tau_2 > s) + Pr(\tau_1 > t; s < \tau_2 < t | \tau_1, \tau_2 > s) + Pr(\tau_2 > t, s < \tau_1 < t | \tau_1, \tau_2 > s) \\ &= \frac{c(S_1(t), S_2(t)) + (c(S_1(t), S_2(s)) - c(S_1(t), S_2(t))) + (c(S_1(s), S_2(t)) - c(S_1(t), S_2(t)))}{c(S_1(s), S_2(s))} \end{aligned}$$

Although we would expect the n'th to default intensity curve $\lambda^n(t)$ to be upward sloping for $n \geq 2$ in any positively correlated model, it is at least of academic interest to consider copulas in which the underlying credits are anti-correlated.

Question: Is it possible to construct a (negatively correlated) copula function for which some or all of $\lambda^N(t)$ are flat for $n \geq 2$?

If the default intensity curves $\lambda_i(t)$ are not flat then comparing $Q^n(0,T)$ and $Q^n(s, s+T)$ is no longer a valid measure of the term structure imposed by the copula. For example if $\lambda_i(t+s) > \lambda_i(t)$ then it is more likely that the first default event occurs between times s and $s+T$ (conditional on the survival of all credits to time s) than between today and time T; in this case we expect that for a term structure free copula model, $Q^1(s, s+T) < Q^1(0,T)$. A solution to this problem is to translate the intensity curves, $\lambda_i(t)$ from the future date s to today and then apply the measure as defined above. In the sequel we will use the notation $Q^n(s, s+T, \lambda_i(t), c)$ to denote the dependence of the measure Q^n on the intensity curves $\lambda_i(t)$ and the copula c. We can now generalize the definition given in equation (12) to the case where the intensity curves $\lambda_i(t)$ are not flat; a copula model is said to be *term structure free for the n'th default event* if

$$Q^n(0, T, \lambda_i(s+t), c) \quad = \quad Q^n(s, s+T, \lambda_i(t), c). \tag{13}$$

In a universe of N credits at each future time there are 2^N possible states of survival and default; equation (13) as defined above only measures the term structure of the N distinct states corresponding to the survival of the n'th to default basket for $n = 1, \ldots, N$. For a copula function c and a set of curves $\lambda_i(t)$, let

$$R(s, t_i, \lambda_i(t), c) \quad = \quad Pr(\tau_1 > s + t_1, \ldots, \tau_N > s + t_N | \tau_1, \ldots, \tau_N > s) \tag{14}$$

where τ_i is the default time for the i'th credit. Then a copula c is said to be *completely term structure free* if for all $s, t_1, \ldots, t_N$,

$$R(s, t_i, \lambda_i(t), c) \quad = \quad R(0, t_i, \lambda_i(s+t), c). \tag{15}$$

It is easy to check that any copula model that satisfies equation (15) is necessarily term structure free for the n'th default event as defined by equation (13). In Section 5 we will construct a family of copula functions that are term structure free as defined by the above equation.

It should be noted that a copula c which is term structure free as defined above for a set of default intensity curves $\lambda_i(t)$ is not necessarily term structure free for another set of curves $\lambda_i(t)$.

Conjecture: The trivial copula function is the only copula function that is term structure free in the sense of equation (15) for an arbitrary set of intensity curves $\lambda_i(t)$.

5 Extreme Value Copulas and the Refinement Operation

An extreme value copula is defined in [1] as a copula function $c(x_1, x_2, \ldots, x_N)$ for which

$$c(x_1^t, x_2^t, \ldots, x_N^t) = c(x_1, x_2, \ldots, x_N)^t.$$

We refer the reader to [1] for a table of extreme value copulas. Using equation (2), and assuming that the individual intensities are flat, we find that for an extreme value copula the first to default intensity is flat:

$$\lambda^1(\lambda_1, \ldots, \lambda_N; t) = -\log(c(e^{-\lambda_1}, e^{-\lambda_2}, \ldots, e^{-\lambda_N})), \tag{16}$$

where $\lambda_i(t) \equiv \lambda_i$ (we are following the notation of Section 4 for $\lambda^n(t)$ and $\lambda_i^n(t)$). Assuming that default intensities, $\lambda_i(t)$ are flat equation (8) takes the form:

$$\lambda_i^1(t) = \frac{c_i(e^{-\lambda_1 t}, e^{-\lambda_2 t}, \ldots, e^{-\lambda_N t})e^{-\lambda_i t}\lambda_i}{c(e^{-\lambda_1 t}, e^{-\lambda_2 t}, \ldots, e^{-\lambda_N t})}.$$

For an extreme value copula this simplifies to:

$$\lambda_i^1(t) = \frac{c_i(e^{-\lambda_1}, e^{-\lambda_2}, \ldots, e^{-\lambda_N})e^{-\lambda_i}\lambda_i}{c(e^{-\lambda_1}, e^{-\lambda_2}, \ldots, e^{-\lambda_N})}.$$

We see from the above expression that for an extreme value copula the default intensity λ_i^1 is also flat for $i = 1, \ldots, N$.

We should note that the only extreme value copula that is $\mathcal{C}^1$ is the trivial copula. We can check this by combining equations (10) and (16):

$$\begin{aligned} -\log(c(e^{-\lambda_1}, e^{-\lambda_2}, \ldots, e^{-\lambda_N})) &= \lambda^1(0) \\ &= \lambda_1 + \lambda_2 + \ldots + \lambda_N. \end{aligned} \tag{17}$$

As the above expression holds for all values of $\lambda_i \geq 0$ it follows that $c(e^{-\lambda_1}, e^{-\lambda_2}, \ldots, e^{-\lambda_N}) = e^{-\lambda_1}e^{-\lambda_2}\ldots e^{-\lambda_N}$, or equivalently $c(x_1, x_2, \ldots, x_N) = x_1 x_2 \ldots x_N$ for $x_i \in [0, 1]$.

It is easy to check that if the intensity curves $\lambda_i(t)$ are flat then an extreme value copula is term structure free for the first default event as defined by equation (12):

$$\begin{aligned} Q^1(0,T) &= c(e^{-\lambda_1 T}, e^{-\lambda_2 T}, \ldots, e^{-\lambda_N T}) \\ &= c(e^{-\lambda_1}, e^{-\lambda_2}, \ldots, e^{-\lambda_N})^T \\ &= c(e^{-\lambda_1}, e^{-\lambda_2}, \ldots, e^{-\lambda_N})^{s+T-s} \\ &= \frac{c(e^{-\lambda_1(s+T)}, e^{-\lambda_2(s+T)}, \ldots, e^{-\lambda_N(s+T)})}{c(e^{-\lambda_1 s}, e^{-\lambda_2 s}, \ldots, e^{-\lambda_N s})} \\ &= Q^1(s, s+T). \end{aligned} \tag{18}$$

If the credit intensity curves $\lambda_i(t)$ are not flat then an extreme value copula is not, in general, term structure free as defined by equation (13). For example, assuming that the credit intensity curves $\lambda_i(t), i = 1, 2$ are piecewise constant,

$$\lambda_i(t) = \lambda_{ij} \text{ for } t \in [t_j, t_{j+1}], j = 1, 2$$

and $t_{j+1} - t_j = T$ we can measure the term structure of the Gumbel copula (an example of an extreme value copula),

$$c_\alpha(x_1, x_2) = \exp(-(-\log(x_1))^\alpha + (-\log(x_2))^\alpha)^{1/\alpha})$$

for the first default event by computing

$$Q^1(t_1, t_1 + T, \lambda_i(t), c_\alpha) = \frac{\exp(-(\lambda_{11}^\alpha + \lambda_{12}^\alpha + \lambda_{21}^\alpha + \lambda_{22}^\alpha)^{1/\alpha}T)}{\exp(-(\lambda_{11}^\alpha + \lambda_{21}^\alpha)^{1/\alpha}T)}$$

and

$$Q^1(0, T, \lambda_i(t_1 + t), c_\alpha) = \exp(-(\lambda_{12}^\alpha + \lambda_{22}^\alpha)^{1/\alpha}T).$$

For $\alpha \neq 1$, the copula is not term structure free in the sense of equation (13) as $Q^1(t_1, t_1 + T, \lambda_i(t), c_\alpha) \neq Q^1(0, T, \lambda_i(t_1 + t), c_\alpha)$ for arbitrary values of λ_{ij}.

A simple approach to constructing a term structure free model for first to default valuation is as follows: given N credits that follow a Poisson process with intensity $\lambda_i(t), i = 1, \ldots, N$ we assume that the first to default process is Poisson with intensity $f(\lambda_1(t), \lambda_2(t), \ldots, \lambda_N(t))$. If defaults occur independently then $f(\lambda_1(t), \lambda_2(t), \ldots, \lambda_N(t)) = \lambda_1(t) + \lambda_2(t) + \ldots + \lambda_N(t)$; if there is a positive dependence between credits then $f(\lambda_1(t), \lambda_2(t), \ldots, \lambda_N(t)) < \lambda_1(t) + \lambda_2(t) + \ldots + \lambda_N(t)$. The survival probability for the first default event is:

$$\exp(-\int_0^T f(\lambda_1(t), \lambda_2(t), \ldots, \lambda_N(t))dt). \tag{19}$$

It is easy to check that this model is term structure free for the first default event as specified by equation (13). There are three main problems with this approach:

1. It is not necessarily true that valuation using the above approach is consistent with a copula model (for example the value of a basket with exactly one credit may not coincide with that of a single credit default swap).

2. It is not apparent how to extend the above approach to n'th to default baskets for $n > 1$ or to CDO valuation.

3. The function $f(\lambda_1(t), \lambda_2(t), \ldots, \lambda_N(t))$ is a "black box" and does not have an obvious financial interpretation in terms of Merton's model.

In an attempt to follow the spirit of the above approach but avoid the limitations listed above we introduce the *refinement* operation for a copula model. The refinement operation may be viewed as the discretization of a multivariate Poisson process in which the dependence structure is determined by a copula. Let $\xi(u, x, y)$ be defined for $x > y$ as follows

$$\xi(u, x, y) = \begin{cases} 1 & \text{for } u > x \\ u/x & \text{for } x \geq u > y \\ y & \text{for } u < y. \end{cases}$$

Given a sequence of partitions, $x_{ij}, i = 1, \ldots, N, j = 1, \ldots, m$ of $[0, 1]$:

$$1 = x_{i1} > x_{i2} > \ldots > x_{im} > 0, i = 1, \ldots, N$$

and a copula c, the *refinement* is defined as

$$c^{\text{ref}}(x_1, \ldots, x_N) = \prod_{i=1}^{m-1} c(\xi(x_1, x_{1i}, x_{1i+1}), \ldots, \xi(x_N, x_{Ni}, x_{Ni+1})).$$

It is easy to check that the refinement of a copula function is a copula function. It is not difficult to show that the every $\mathcal{C}^1$ copula converges to the trivial copula as the mesh size tends to zero.

Assuming a set of piecewise constant intensity curves, $\lambda_i(t) = \lambda_{ij}$ on $[t_j, t_{j+1})$, define $x_{i1} = 1$ and for $j > 0$,

$$x_{ij} = \prod_{k=1}^{j} e^{-\lambda_{ik}(t_{k+1}-t_k)}. \tag{20}$$

Let c^{ref} denote the refinement of the copula c under the partition x_{ij} defined above. Then

$$c^{\text{ref}}(S_1, \ldots, S_N) = c(e^{-\lambda_{1j}(T-t_j)}, \ldots, e^{-\lambda_{Nj}(T-t_j)}) \prod_{k=1}^{j-1} c(e^{-\lambda_{1k}(t_{k+1}-t_k)}, \ldots, e^{-\lambda_{Nk}(t_{k+1}-t_k)})$$

where $T \in [t_j, t_{j+1}]$ and $S_i(T) = \exp(-\int_0^T \lambda_i(t)dt)$. If c is an extreme value copula then the above expression takes the particularly simple form:

$$c^{\text{ref}}(S_1, \ldots, S_N) = \exp(-\int_0^T \lambda^1(\lambda_1(t), \ldots, \lambda_N(t))dt) \tag{21}$$

where $\lambda^1(x_1, \ldots, x_N) = -\log(c(e^{-x_1}, \ldots, e^{-x_N}))$ is defined as in equation (16). Comparing equations (21) and (19) we see that the refinement copula has the same functional form for the survival probability for the first default event as the simple Poisson model with the "black box" function f replaced by the first to default intensity λ^1. Unlike the simple Poisson model it is a copula model that can be used in a consistant manner to value n'th to default swaps for $n > 1$ and CDOs.

The refinement model as defined above is a discrete model for piecewise constant intensity curves and only satisfies the completely term structure free condition of equation (15) on the time scale of the discretization. To construct a copula that is completely term structure free as defined by equation (15) we choose a partition as in equation (20) — with $\lambda_i(t)$ no longer necessarily piecewise linear — and let the mesh size shrink to zero. For an extreme value copula as the mesh size tends to zero, the limit copula $c^{\text{ref}}_{\text{lim}}$, takes the form

$$\begin{aligned} c^{\text{ref}}_{\text{lim}}(u_1,\ldots,u_N) &= \exp(-\int_0^{t_1} \lambda^1(\lambda_1(t),\ldots,\lambda_N(t))dt - \int_{t_1}^{t_2} \lambda^1(\pi_{i_1}(\lambda_1(t),\ldots,\lambda_N(t)))dt \\ &\quad \ldots - \int_{t_{N-2}}^{t_{N-1}} \lambda^1(\pi_{i_1 i_2\ldots i_{N-2}}(\lambda_1(t),\ldots,\lambda_N(t)))dt - \int_{t_{N-1}}^{t_N} \lambda^1(\pi_{i_1 i_2\ldots i_{N-1}}(\lambda_1(t),\ldots,\lambda_N(t)))dt) \end{aligned} \tag{22}$$

where $u_i = \exp(-\int_0^{t_i} \lambda_i(u)du), t_{i_1} \leq t_{i_2} \leq \ldots \leq t_{i_N}$ and the projection maps π are defined as follows:

$$\begin{aligned} \pi_i(x_1,x_2,\ldots,x_N) &= (x_1,x_2,\ldots,x_{i-1},0,x_{i+1},\ldots,x_N) \\ \pi_{i_1 i_2\ldots i_k} &= \pi_{i_1}\circ\pi_{i_2}\circ\ldots\circ\pi_{i_k} \end{aligned} \tag{23}$$

Inspired by the above expression, for $u_i = \exp(-\int_0^{t_i} \lambda_i(u)du), t_{i_1} \leq t_{i_2} \leq \ldots \leq t_{i_N}$, we define the class of *completely term structure free copulas* as follows:

$$\begin{aligned} c_{\text{flat}}(u_1,\ldots,u_N) &= \exp(-\int_0^{t_1} f(\lambda_1(t),\ldots,\lambda_N(t))dt - \int_{t_1}^{t_2} f(\pi_{i_1}(\lambda_1(t),\ldots,\lambda_N(t)))dt \\ &\quad \ldots - \int_{t_{N-2}}^{t_{N-1}} f(\pi_{i_1 i_2\ldots i_{N-2}}(\lambda_1(t),\ldots,\lambda_N(t)))dt - \int_{t_{N-1}}^{t_N} f(\pi_{i_1 i_2\ldots i_{N-1}}(\lambda_1(t),\ldots,\lambda_N(t)))dt) \end{aligned} \tag{24}$$

where the *dependence* function $f(x_1,x_2,\ldots,x_N) : (\mathcal{R}^+)^N \to \mathcal{R}^+$ must satisfy the following conditions to ensure that c_{flat} is a copula function:

1. $f(x_1,x_2,\ldots,x_N) \geq 0$.
2. $f(0,\ldots,0,x_i,0,\ldots,0) = x_i$.
3. $f(x_1,\ldots,x_N) \geq f(\pi_{i_1}(x_1,\ldots,x_N)) \geq f(\pi_{i_1 i_2}(x_1,\ldots,x_N)) \geq \ldots \geq f(\pi_{i_1 i_2\ldots i_N}(x_1,\ldots,x_N))$ for any choice of $i_1,\ldots,i_N$.

In particular the first to default intensity function, λ^1, satisfies the above conditions.

(In order for equations (22) and (24) to be well defined we require that each equation $u_i = \exp(-\int_0^{t_i} \lambda_i(u)du)$ can be inverted to solve for t_i; in other words $\lambda_i(t)$ must be strictly positive and $\int_0^{+\infty} \lambda_i(t)dt = +\infty$.) We can check that c_{flat} as defined above is completely term structure free as defined in equation (15):

$$\begin{aligned} R(s,t_i,\lambda_i(t),c_{\text{flat}}) &= \frac{\exp(-\int_0^{s+t_1} f(\lambda_1(t),\lambda_2(t),\ldots,\lambda_N(t))dt \cdots - \int_{s+t_{N-1}}^{s+t_N} f(\pi_{i_1 i_2\ldots i_{N1}}(\lambda_1(t),\ldots,\lambda_N(t)))dt)}{\exp(-\int_0^s f(\lambda_1(t),\lambda_2(t),\ldots,\lambda_N(t))dt)} \\ &= \exp(-\int_s^{s+t_1} f(\lambda_1(t),\lambda_2(t),\ldots,\lambda_N(t))dt \cdots - \int_{s+t_{N-1}}^{s+t_N} f(\pi_{i_1 i_2\ldots i_{N1}}(\lambda_1(t),\ldots,\lambda_N(t)))dt) \\ &= \exp(-\int_0^{t_1} f(\lambda_1(s+t),\lambda_2(s+t),\ldots,\lambda_N(s+t))dt \\ &\quad \cdots - \int_{t_{N-1}}^{t_N} f(\pi_{i_1 i_2\ldots i_{N1}}(\lambda_1(s+t),\ldots,\lambda_N(s+t)))dt) \\ &= R(0,t_i,\lambda_i(s+t),c_{\text{flat}}) \end{aligned} \tag{25}$$

Given an extreme value copula c we can use equation (22) to compute the survival probability for the first default event under its corresponding refinement limit copula $c^{\text{ref}}_{\text{lim}}$. For example, for the Gumbel copula with parameter α,

$$c(x_1, x_2, \ldots, x_N) = \exp(-(\sum_{i=1}^{N}(-\log x_i)^\alpha)^{1/\alpha})$$

with default intensity function $\lambda^1(x_1, \ldots, x_N) = (\sum_{i=1}^N x_i^\alpha)^{1/\alpha}$ the survival probability to time t for the first default event under $c^{\text{ref}}_{\text{lim}}$ is (compare with equation (19)):

$$c^{\text{ref}}_{\text{lim}}(S_1(t), \ldots, S_N(t)) = \exp(-\int_0^t (\sum_{i=1}^{N} \lambda_i(u)^\alpha)^{1/\alpha} du.$$

Two advantages of using a normal copula model are that a well defined financial interpretation for joint default events exists, namely Merton's model for corporate default, and that a correlation matrix ρ can be used as an input to the model or inferred as a market observable. Extreme value copulas do not share these properties. In light of this we propose to use equation (24) and a choice of a representative time scale t_0 (for example $t_0 = 1$ year) to define a "completely term structure free normal copula model" in one of the two following ways. One approach is to define the dependence function $f(x_1, \ldots, x_N)$ in terms of the survival probability to time t_0 of the the normal copula:

$$f(x_1, x_2, \ldots, x_N) = -\frac{1}{t_0}\log(c_\rho(e^{-x_1 t_0}, e^{-x_2 t_0}, \ldots, e^{-x_N t_0}))$$

where c_ρ is a normal copula with correlation ρ. If the intensity curves are flat then the survival probability to time t_0 for the first default event of the completely term structure free copula, with dependence function as above, matches that of the normal copula, i.e. $c_\rho(S_1(t_0), \ldots, S_N(t_0)) = c_{\text{flat}}(S_1(t_0), \ldots, S_N(t_0))$. Another approach is to define $f(x_1, \ldots, x_N)$ to be the first to default intensity at time t_0 under a normal copula c as in equation (2):

$$f(x_1, \ldots, x_N) = -\frac{d}{dt}\log(c_\rho(e^{-x_1 t}, e^{-x_2 t}, \ldots, e^{-x_N t}))|_{t=t_0}.$$

In this approach the resulting term structure free copula has the same first to default intensity as the normal copula at time $t = t_0$ if the individual intensity curves λ_i are flat.

The purpose of this paper has been to measure the term structure of copula models for default and construct term structure free models for default. Models for default, however, should be adaptable to arbitrageur's views on the term structure of multi-credit dependence. The simplest parametrization of such a view on default is to consider a correlation matrix $\rho(t)$ that is time dependent (such a matrix may be roughly inferred from market quotes on baskets or CDOs of varying maturity). We can adapt equation (24) to model time varying dependence by allowing the dependence function $f(x_1, \ldots, x_N)$ to be time dependent. For example, a term structure free normal copula model with time varying correlation is given by using the dependence function

$$f(x_1, x_2, \ldots, x_N; t) = -\frac{1}{t_0}\log(c_{\rho(t)}(e^{-x_1 t_0}, e^{-x_2 t_0}, \ldots, e^{-x_N t_0})). \quad (26)$$

Such a model satisfies the completely term structure free condition of equation (15) when the correlation parameter $\rho(t)$ is flat. A broader definition of a completely term structure free model is a model that satisfies equation (15) when all dependence parameters are flat.

While it is possible to simulate default events for a normal copula or Students' T copula model using standard methods (see for example [6]), there is no general method that works for an arbitrary copula function. For CDO valuations especially, the analytic formulas presented in Section 4 become combinatorially intractable and a Monte-Carlo approach seems unavoidable. For an Archimedean copula, $c(x_1, \ldots, x_N)$ there are known algorithms for sampling u_{i+1} conditional on u_i, where $u_1, u_2, \ldots, u_N$ are dependent uniform random variables with cumulative distribution function $c(x_1, \ldots, x_N)$. In practice this may be difficult unless the copula partial derivatives can be computed analytically. A better approach for sampling from an Archimedean copula is proposed in [10]. A Metropolis Monte-Carlo scheme may prove useful for sampling from a term structure free copula function (as defined by equation (24)); a new multivariate uniform sample $y_1, y_2, \ldots, y_N$ is accepted or rejected with probability

$$\frac{p(y_1, y_2, \ldots, y_N)}{p(x_1, x_2, \ldots, x_N)}$$

where $x_1, x_2, \ldots, x_N$ is the last accepted draw and $p(u_1, u_2, \ldots, u_N)$ is the copula density function. For a term structure free copula we can compute the copula density explicitly by differentiating equation (24) with respect to $u_1, u_2, \ldots, u_N$ (the default time t_i is defined through the equation $u_i = \exp(\int_0^{t_i} \lambda_i(u)du)$):

$$c(u_1, u_2, \ldots, u_N)[f(\pi_{i_1}(\lambda_1(t_1), \ldots, \lambda_N(t_1))) - f(\lambda_1(t_1), \ldots, \lambda_N(t_1))] \cdot [f(\pi_{i_1 i_2}(\lambda_1(t_2), \ldots, \lambda_N(t_2)))] \\ \cdots [f(\pi_{i_1 i_2 \ldots i_{N-1}}(\lambda_1(t_{N-1}), \ldots, \lambda_N(t_{N-1})))] \frac{1}{u_1 \ldots u_N \lambda_1(t_1) \ldots \lambda_N(t_N)} \tag{27}$$

The convergence speed of this algorithm for multi-name credit modelling needs to be investigated.

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My investigation of the term structure of copula models began with Martin St. Pierre's observation that the first to default intensity curve is downward sloping even if all individual credit curves are flat. I would like to thank Jeff Boland and Niall Whelan for many helpful comments and corrections to earlier versions of this report.

References

[1] Bouye, E., Durrleman, V., Nikeghbali, A., Riboulet, G., Roncalli, T. (July 2000) Copulas for Finance: A Reading Guide and Some Applications, *Working Paper, Groupe de Recherche Operationnelle, Credit Lyonnais, Paris* (availiable at http://gro.creditlyonnais.fr/content/wp/copula-survey.pdf)

[2] Finger, C. C. Editor (2002) CreditGrades Technical Document, *RiskMetrics Group.*

[3] Hull, J. and White, A. (2001) Valuing Credit Default Swaps II: Modelling Default Correlations, *Journal of Derivatives*, Vol. **8** No. **3**, 12–22.

[4] Joe, H. (1997) Multivariate Models and Dependence Concepts, *Monographs of Statistics and Applied Probability*, Vol. **73**, Chapman & Hall, London.

[5] Jouanin, J.-F., Rapuch, G., Riboulet, G. and Roncalli, T. (August 2001) Modelling Dependence for Credit Derivatives with Copulas, *Working Paper, Groupe de Recherche Operationnelle, Credit Lyonnais, Paris* (availiable at http://gro.creditlyonnais.fr/content/wp/copula-intensity.pdf)

[6] Li, D. X. (2000) On Default Correlation: A Copula Function Approach, *Journal of Fixed Income*, Vol. **9** No. **4**, 43–54.

[7] Pugachevsky, D. (2003) Correlations in Multi-Credit Models, *Working Paper, Bear Stearns and Co., New York*

[8] Pugachevsky, D. (2002) Notes on Default Swap Valuation, *Working Paper, Bear Stearns and Co., New York*

[9] Schonbucker, P.J. and D. Schubert (April 2001) Copula-Dependent Defaults in Intensity Models, Department of Mathematics, ETH Zurich, *Working Paper* (availiable at http://www.math.ethz.ch/ degiorgi/CreditRisk/ftp/schonb_copula.pdf)

[10] Whelan, N. (May 2003) Sampling from Archimedean Copulas, *Working Paper, Risk Management and Control, Bank of Nova Scotia*

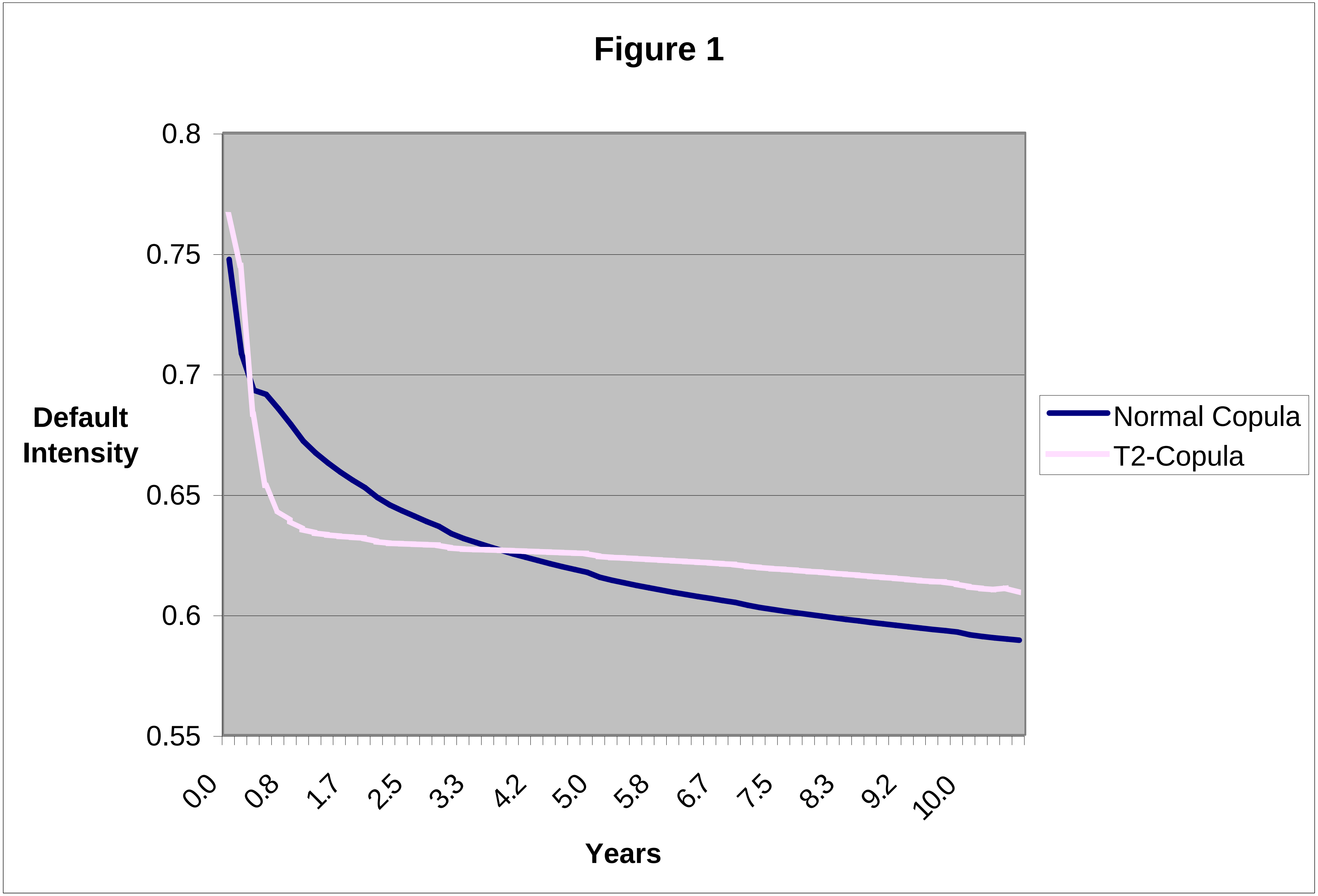

Figure 1
0.8
0.75
0.7
0.65
0.6
0.55
Default Intensity
0.0
0.8
1.7
2.5
3.3
4.2
5.0
5.8
6.7
7.5
8.3
9.2
10.0
Years
Normal Copula
T2-Copula

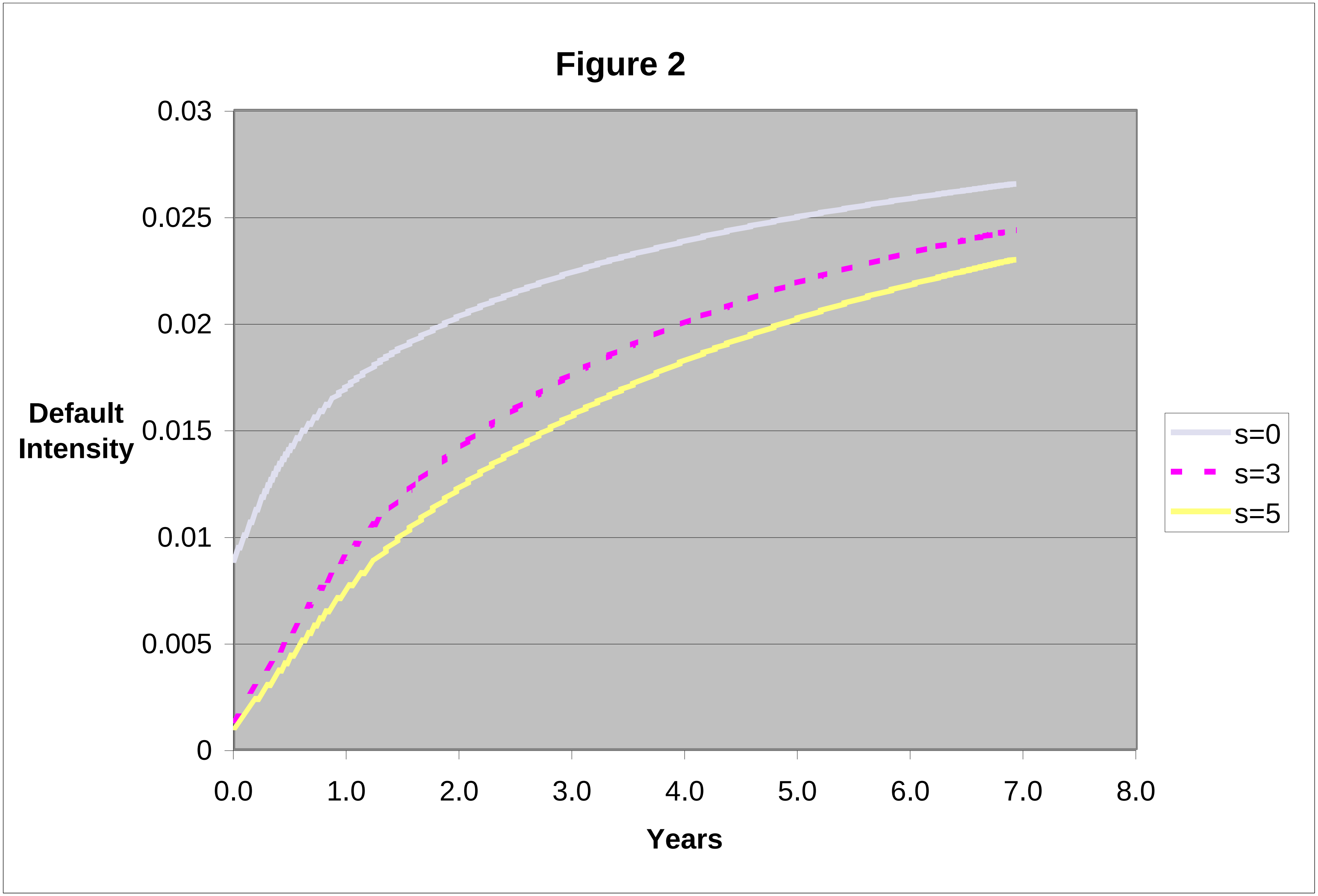
Figure 2
Default Intensity
0.03
0.025
0.02
0.015
0.01
0.005
0
0.0
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
Years
s=0
s=3
s=5