A NOTE ON SURVIVAL MEASURES AND THE PRICING OF OPTIONS ON CREDIT DEFAULT SWAPS

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Abstract. In this note the pricing of options on credit default swaps using the *survival-measure*-pricing technique is discussed. In particular, we derive a modification of the famous Black (1976) futures pricing formula which applies to options on CDS, and show how other pricing formulae can be easily derived if the dynamics of the forward CDS rates are specified differently. The main tool in the derivation of the pricing formulae is to express prices and payoffs in terms of a defaultable numeraire asset, the fee stream of the underlying forward-starting CDS. As this numeraire becomes worthless in default, certain technical difficulties arise which can be solved using the mathematical tool of the T-*forward survival measure* (first introduced in Schönbucher (1999)), a pricing measure which is conditioned on survival until T. The properties of such pricing measures are a second focus of this note.

With increasing liquidity of the plain-vanilla CDS markets, the first derivatives on these basic credit derivatives have been introduced. In particular the market options on credit default swaps, or *credit default swaptions* is growing, be it as embedded options to extend or cancel an existing CDS, or as explicit options on the CDS. In the cover article of the May 2003 issue of Risk magazine, Patel (2003) discussed the growing market for options on credit default swaps, or *credit default swaptions*, and their practical applications.

In this note we explain in more detail some of the methods which were first used in Schönbucher (1999) to price such options. In particular, we derive a modification of the famous Black (1976) futures pricing formula which applies to options on CDS and show how the methods can be extended to other models if you do not like the lognormality assumption. The main tool in the derivation of this formula

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Comments and suggestions are very welcome. All errors are my own.

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is to express the payoff of the option in terms of a defaultable numeraire asset, the fee stream of the underlying forward-starting CDS. As this numeraire becomes worthless in default, certain technical difficulties arise which can be solved using the mathematical tool of the *T-forward survival measure* (this term was first introduced in Schönbucher (1999)), a pricing measure which is conditioned on survival until T. The range of applications of the survival measures is not restricted to the pricing of credit default swaptions, they are also useful in other complex pricing problems, e.g. the analysis of counterparty credit risk in OTC-transactions, or multi-currency CDS spreads.

The pricing of credit spread options has been addressed in a number of previous publications but usually these options were *not* explicitly options on CDS but rather contingent claims whose payoff depended on some sort of continuously-compounded yield spread. In particular in many intensity-based credit risk models spread-options can be priced by transfer of techniques from the pricing of default-free bond options, see e.g. Duffie and Singleton (1999) or Schönbucher (2002). More closely related to the results of this paper are the paper by Jamshidian (2002) who uses a similar setup but with numeraire assets with strictly positive value processes, and Hull and White (2003) who also explain the derivation of the Black (1976) pricing formula for CDS options and who give numerical examples using historical data on quoted CDS spreads. For this note, the fundamental reference is Schönbucher (1999) where the CDS option pricing formula (and other results) are derived in a full defaultable Libor market model. As for default-free interest-rates, using a LMM setup based upon forward rates has the advantage that it allows a consistent specification of volatilities and hedging strategies for multiple options on CDS on the same reference entity, in particular if the terms of the underlying CDS overlap.

1. Forward CDS Rates

The underlying asset of a CDS option is a *forward starting Credit Default Swap.* This is a CDS which starts its life not immediately at the trade date but at some later date in the future[1], the forward-start date. If the reference entity should default before the forward-start date, the contract is null and void and no payments are made. Like every CDS, the payoffs of a forward CDS can be separated in a *fee leg* which is paid by the *protection buyer*, and a *protection leg* which is paid by the *protection seller.*

In order to describe the payoffs let us call $t \geq 0$ the trade date, $T_0 \geq t$ the effective date (the date at which the protection begins), and $T_N > T_0$ the maturity date of the forward CDS. The protection buyer is denoted with $\mathbf{A}$, the protection seller

[1] Of course, if the forward start date equals the trade date we recover the plain vanilla CDS.

is **B**, and the reference entity is **C**. The model is set up in a filtered probability space[2] $(\Omega, (\mathcal{F}_t)_{(t\geq 0)}, P)$. Absence of arbitrage implies the existence of a spot-martingale measure $Q \sim P$, and we also assume the existence of $b(t) = e^{\int_0^t r(s)ds}$, the continuously-compounded bank account. The time of the **C**-credit event is modelled with a stopping time τ.

1.1. **The Fee Leg.** At the payment dates $T_n, 1 \leq n \leq N$, the protection buyer **A** pays the regular fee payments of

$$\overline{s}\delta_n \mathbf{1}_{\{T_n \leq \tau\}} \qquad \text{at } T_n \tag{1}$$

to the protection seller **B**. Here $\overline{s}$ is the *forward CDS rate* for the forward CDS, δ_n is the daycount fraction for the interval $[T_{n-1}, T_n]$ and $\mathbf{1}_{\{T_n \leq \tau\}}$ is the indicator function that the default has not occurred before the payment date T_n.

Furthermore, at the time of default the protection buyer makes a final payment covering the time between the last payment date and the default: Let $n^* = \max\{n | T_n \leq \tau\}$ be the last payment date before default. Then **A** pays to **B**

$$\overline{s}\delta^* \mathbf{1}_{\{T_0 \leq \tau \leq T_N\}} \qquad \text{at } \tau, \tag{2}$$

where δ^* is the daycount fraction for the time interval $[T^*, \tau]$. If we denote with $V^{\text{fee}}(t)$ the value at time $t < T_1$ of receiving 1bp of fee payments

$$V^{\text{fee}}(t) = \mathbf{E}^Q \left[\sum_{n=1}^{N} \frac{1}{b(T_n)} \delta_n \mathbf{1}_{\{T_n \leq \tau\}} + \frac{1}{b(\tau)} \delta^* \mathbf{1}_{\{T_0 \leq \tau \leq T_N\}} \;\middle|\; \mathcal{F}_t \right], \tag{3}$$

then the value of the fee leg is $\overline{s}V^{\text{fee}}(t)$. Note that after default no fees are paid, i.e. $V^{\text{fee}}(t) = 0$ for $\tau \leq t$.

1.2. **The Protection Leg.** If the default occurs before T_N, i.e. if $\tau \leq T_N$, then **B** pays to **A** the amount of $(1 - R)$, where R is the recovery rate on bonds of **C**. (More precisely, R is the value at time τ of the cheapest bond that is deliverable under the terms of the contract.) Thus, the payment of **B** to **A** is

$$(1 - R)\mathbf{1}_{\{T_0 \leq \tau \leq T_N\}} \qquad \text{at } \tau. \tag{4}$$

We denote the value of the protection leg at time $t < T_0$ with

$$V^{\text{prot}}(t) = \mathbf{E}^Q \left[\frac{1}{b(\tau)} (1 - R)\mathbf{1}_{\{T_0 \leq \tau \leq T_N\}} \;\middle|\; \mathcal{F}_t \right]. \tag{5}$$

Note that the delivery option is explicitly recognized (R may be stochastic), and we also model the value of accrued fee payments in (2).

[2]The usual regularity conditions on $(\Omega, (\mathcal{F}_t)_{(t\geq 0)}, P)$ apply. We do not assume that Q is unique, markets may be incomplete.

1.3. **Fair CDS Rates, Marking-to-Market and Option Payoff Functions.** Given (3) and (5), the *fair forward CDS rate* at time t of a forward CDS over the interval $[T_0, T_N]$ is the rate at which the fee leg has the same value as the protection leg, i.e.

$$\overline{s}(t) = \frac{V^{\text{prot}}(t)}{V^{\text{fee}}(t)}, \qquad \text{for } t < \tau. \tag{6}$$

As $V^{\text{fee}}(t) = 0$ for $\tau < t$, the fair forward CDS rate is *not defined* after default.

If the forward CDS over $[T_0, T_N]$ was entered (as protection buyer) before t at a rate $\overline{s}_0 \neq \overline{s}(t)$, then the markt-to-market value of this position at time t is

$$(\overline{s}(t) - \overline{s}_0) V^{\text{fee}}(t), \tag{7}$$

because one can lock into a fee stream of $(\overline{s}_0 - \overline{s}(t))$ by selling the forward CDS protection at the current market rate $\overline{s}(t)$.

If instead one just has the *right* and not the *obligation* to enter the forward CDS at time t at the forward CDS rate $\overline{s}^*$, then the value of this right is

$$(\overline{s}(t) - \overline{s}^*)^+ V^{\text{fee}}(t). \tag{8}$$

This is the payoff of an *option on a forward-starting CDS* with maturity t.

2. The Change of Numeraire Technique

It is well-known[3] that one can use every asset with a positive cash price process $A(t)$ to express the prices of all other assets in terms of units of $A(t)$. E.g. if another asset has cash-price $X(t)$, then

$$\widehat{X}(t) := X(t)/A(t) \tag{9}$$

is the price of that same asset expressed in A-units. If this is done for all traded assets, $A(t)$ is called the *numeraire* of this new price system. The numeraire of the cash price system is the continuously compounded bank account[4] $b(t)$, other commonly used numeraires are T-maturity zero coupon bonds (yielding a system of T-forward prices), foreign currencies or – in the case of interest-rate swaps – the value of a fee stream of 1bp at a set of payment dates. Furthermore it is also well-known that for any given spot-martingale-masure Q and numeraire $A(t)$, one can define an equivalent pricing measure Q^A using the Radon-Nikodym density process

$$\left.\frac{dQ^A}{dQ}\right|_t = L^A(t) := \frac{A(t)}{b(t)} \frac{b(0)}{A(0)}. \tag{10}$$

[3]See Schroder (1999) for an introduction to and references on the classical (default-free) change-of-numeraire technique.

[4]The bank account is used instead of cash because it is dividend-free.

The measure Q^A has the very useful property that *prices in the A-numeraire are Q^A-martingales.*

A look at equations (8) and (6) suggests the use of the value of the fee stream V^{fee} as a numeraire in order to price the CDS option: The CDS-rate is the V^{fee}-relative price of protection, and the option's payoff function in (8) is also easily expressed in this numeraire. Unfortunately a direct application of the previously cited results is not possible because V^{fee} *can be zero.* Thus a price system in terms of V^{fee} is *not always defined:* It is undefined after defaults, i.e. for all times t and states of nature $\omega \in \Omega$ with $\tau(\omega) \leq t$.

Fortunately, while equation (9) defining the new price system can break down if $A(t) = V^{\text{fee}}(t) = 0$, equation (10) will still remain valid as long as $A(0) > 0$ and it will define a mathematically admissible Radon-Nikodym density and a mathematically sound new probability measure $\overline{P}$. Furthermore, if $X(t)V^{\text{fee}}(t)$ is the cash-price of a traded asset, then it also follows from (10) that $X(t)$ is a $\overline{P}$-martingale.

Assume $\overline{A}(t)$ is the price process of a defaultable asset with zero recovery. For a given $T > t$ we can then define a "promised" payoff $A'(T)$ via $A'(T)\mathbf{1}_{\{T<\tau\}} = \overline{A}(T)$, and we can define a corresponding default-free asset which will have the price process $A(t) := \mathbf{E}^Q\left[\frac{b(t)A'(T)}{b(T)}\right]$. This default-free asset pays $A'(T)$ at T for sure. We can use both, defaultable and default-free assets in (10) to define two new measures: Q^A for $A(t)$ and $Q^{\overline{A}}$ for $\overline{A}(t)$ respectively.

Often we can choose the promised payoff such that it is positive: $A'(T) > 0$. In this case it can be shown that $Q^{\overline{A}}$ is the measure that is reached *when Q^A is conditioned on survival until T (i.e. $\tau > T$).* For this reason, probability measures that are reached by using defaultable numeraires in equation (10) are termed *survival measures* in Schönbucher (1999). For example, the T-forward-measure uses a T-maturity default-free zero coupon bond $B(t,T)$ as numeraire. If this measure is conditioned on survival of an obligor until T, we reach the T-*surival measure* which could also be reached by using a T-maturity defaultable zero coupon bond with zero recovery $\overline{B}(t,T)$.

It should be noted that survival measures are *not equivalent* to the spot martingale measure Q any more, they are only absolutely continuous with respect to Q. The new probability measure $Q^{\overline{A}}$ attaches zero probability to all events which involve a default before T, i.e. if $E \in \mathcal{F}_T$ and $E \subset \{\tau \leq T\}$, then $L^{\overline{A}} = 0$ on E and $\mathbf{Q}^{\overline{A}}[\,E\,] = 0$. This may seem an inconvenient property, but it follows naturally from the interpretation as conditioned measure, and with a little creativity the practical restrictions imposed by this can be circumvened:

Value of $XA'(T)$ at T in survival:

$$\mathbf{E}^Q\left[\frac{b(t)}{b(T)}\mathbf{1}_{\{\tau>T\}}XA'(T)\;\middle|\;\mathcal{F}_t\right]=\overline{A}(t)\mathbf{E}^{Q^{\overline{A}}}\left[\,X\mid\mathcal{F}_t\,\right]. \tag{11}$$

Value of $YA'(T)$ at T if $\tau\le T$: (using $\mathbf{1}_{\{\tau\le T\}}=1-\mathbf{1}_{\{\tau>T\}}$)

$$\mathbf{E}^Q\left[\frac{b(t)}{b(T)}\mathbf{1}_{\{\tau\le T\}}XA'(T)\;\middle|\;\mathcal{F}_t\right]=A(t)\mathbf{E}^{Q^A}\left[\,Y\mid\mathcal{F}_t\,\right]-\overline{A}(t)\mathbf{E}^{Q^{\overline{A}}}\left[\,Y\mid\mathcal{F}_t\,\right]. \tag{12}$$

Similarly, the value of receiving $Z(T_i)$ at T_i if $\tau\in]T_{i-1},T_i]$ is evaluated. If the time points $T_0<T_1<\ldots<T_I$ are spaced densely enough this can be used to price recovery payoffs.

3. Pricing and Hedging Options on Credit Default Swaps

Armed with the survival-based pricing measures we can further analyse the CDS option by using the defaultable fee stream V^{fee} for a change of measure to a survival measure. The payoff of the CDS option is given in equation (8), using (11) yields the price of the CDS option

$$\begin{aligned}V^{\text{CDSO}}(t)&=\mathbf{1}_{\{\tau>t\}}\mathbf{E}^Q\left[\frac{b(t)}{b(T_0)}(\overline{s}(T_0)-\overline{s}^*)^+V^{\text{fee}}(T_0)\mathbf{1}_{\{\tau>T_0\}}\;\middle|\;\mathcal{F}_t\right]\\&=\mathbf{1}_{\{\tau>t\}}V^{\text{fee}}(t)\mathbf{E}^{P^{V^{\text{fee}}}}\left[\,(\overline{s}(T_0)-\overline{s}^*)^+\;\middle|\;\mathcal{F}_t\,\right]. \end{aligned}\tag{13}$$

This is the furthest we can get without making any modelling assumptions apart from the absence of arbitrage. Equation (13) tells us that all we need to know (or to model) is the distribution of the forward CDS-rate $\overline{s}(T_0)$ at time T_0 under the $P^{V^{\text{fee}}}$-measure. But we already know one more thing about $\overline{s}(t)$: according to (6) it is a relative price under the numeraire V^{fee}, so it must be a *martingale* under $P^{V^{\text{fee}}}$. Here are some possible choices for martingale dynamics for $\overline{s}(t)$:

Lognormal Brownian Motion:

$$d\overline{s}(t)=\overline{s}(t)\sigma dW^{\text{fee}}(t) \tag{14}$$

where σ is a constant and $W^{\text{fee}}(t)$ is a $P^{V^{\text{fee}}}$-Brownian Motion. In this case we can evaluate (13) in closed-form and reach the famous Black (1976) formula

$$V^{\text{CDSO}}(t)=\mathbf{1}_{\{\tau>t\}}V^{\text{fee}}(t)\big[\overline{s}(t)N(d_1)-s^*N(d2)\big], \tag{15}$$

$$\text{where}\qquad d_{1,2}=\frac{\ln(\overline{s}(t)/s^*)\pm\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}. \tag{16}$$

Rating Transitions:
Let $\lambda_{ij},\ i,j\le K$ be the $P^{V^{\text{fee}}}$-transition intensity from rating class i to rating class j. Let $p_i(T)$ be the $P^{V^{\text{fee}}}$-probability of reaching rating class i at time T, and let $\overline{s}_i$ be the obligor's CDS-rate if he is in rating class i at time T. In principle the transition probabilities can be calculated from the transition intensities λ_{ij},

but it is sufficient to directly specify the transition probabilities until the time-of-maturity T, as long as a few basic requirements are observed: $\overline{s}$ must be a $P^{V^{\text{fee}}}$-martingale, therefore $\overline{s}(0) = \sum_{i=1}^{K} p_i(T)\overline{s}_i$ is necessary, and we must also require that the $P^{V^{\text{fee}}}$-transition probability to the default rating class is zero. Then the price of the CDS option is given by

$$V^{\text{CDSO}}(t) = \mathbf{1}_{\{\tau>t\}} V^{\text{fee}}(t) \sum_{i=1}^{K} p_i(T)(\overline{s}_i - s^*)^+. \tag{17}$$

Of course, many other specifications of the $P^{V^{\text{fee}}}$-dynamics of the forward cd-srate $\overline{s}(t)$ are feasible, in the light of empirical CDS-rate dynamics, jump-diffusion-dynamics seem to be a reasonable choice.

The pricing formulae (15) or (17) have some properties that may surprise at first. For example, the *recovery rate* does not appear explicitly (although it does appear implicitly if one prices the fee stream V^{fee}). The reason for this is twofold: First, the recovery rate is already implicitly priced into the current forward-CDS rate itself. Second, it does not enter the volatility either because it only enters as a factor in front of the CDS rate (see e.g. (5)). Another puzzling property is that even for a *put* on a CDS, its value increases for higher volatility of the forward CDS spread. Intuitively one would assume that a very high volatility would make an early default (and thus a knockout) more likely. Again the resolution lies in the fact that the forward CDS rate $\overline{s}$ is already given, and the market has already priced that risk into it. All this is similar to the usual Black-Scholes pricing of equity options where again the fundamental *share valuation* problem is shown to be irrelevant for the pricing of the option, here the underlying *credit risk assessment* is not necessary, because it has already been done implicitly by the market when the price $\overline{s}(t)$ was formed.

3.1. **Hedging and Implementation.** Despite the large freedom in the choice of the dynamics of the forward CDS rate, lognormal dynamics do have their advantages. First, these dynamics can be linked to a more fundamental Defaultable Libor-Market-Model setup as it is described in Schönbucher (1999). Here, the implied volatility of the forward CDS rate can be approximately equated to the average volatility of the discrete default hazard rates.

Secondly, lognormal dynamics lead to *complete markets* for derivatives on the forward CDS. Thus, using a continuous trading strategy in the underlying forward CDS, a dynamic hedge can be established which (at least in theory) perfectly replicates the CDS-option's payoff. The hedge strategy is: At time $t < T$, $(t < \tau)$, and for a given value of $\overline{s}(t)$, hold

$$\alpha_1(t) := N(d_1) \tag{18}$$

units of forward CDS protection, and

$$\alpha_2(t) := \frac{V^{\mathrm{CDSO}}(t)}{V^{\mathrm{fee}(t)}} \tag{19}$$

in the fee stream asset $V^{\mathrm{fee}}(t)$. This hedge is the classical delta-hedge using the *forward CDS* to offset the risk of changes in the forward CDS-rate. The forward CDS protection is the ideal hedge instrument, because even at an early default of the reference credit the hedge works out perfectly: Both hedge and option are knocked out.

Of course, implementation of such a hedge strategy runs into many obstacles in real markets. First, pure forward CDS are usually not traded, the practical solution is to trade a long position in a T_2-maturity CDS and a short position in a $T_1 < T_2$-maturity CDS in order to approximatively replicate the $[T_1, T_2]$ forward CDS. This approximation is perfect if both CDS have the same fees, otherwise there will be a continuous cash-flow of the difference of the two CDS fee streams $-\overline{s}_1 + \overline{s}_2$ until T_1 (or τ if $\tau < T_1$). Usually, this difference is an order of magnitude smaller than the originally spreads and it can be ignored.

More problems are posed by the lack of liquidity and the large bid-ask-spreads in CDS markets. In practice, an adjustment of the hedge cannot be done too often. Nevertheless it is well-known from classical Black-Scholes theory that even discrete hedges can reduce the risk of the position significantly, if the hedge-volatility is chosen large enough it may even be possible to construct a super-hedge in discrete hedging (see e.g. El Karoui et al. (1998)). All these results carry over – mutatis mutandis – to this situation.

The levels of CDS-rate volatilities that are reported by Hull and White (2003) (between 60% and 120% p.a.) may seem very high, in particular in comparison to equity or interest-rate volatilities, but they are indeed typical values for the volatilities of spot CDS rates. Here, we are talking about the volatility of *forward* CDS rates. As in the interest-rate world, the volatilities of these forward CDS rates can be significantly lower than the volatility of spot CDS rates if there is a term structure of spread volatilities with strong mean-reversion. Modelling this would require a full model of the term structure of interest-rates and credit spreads as it is described in Schönbucher (1999).

4. Conclusion

In this note we have shown that the pricing of derivatives on CDS can be reduced to "classical" pricing problems if a suitable numeraire asset and pricing measure is chosen. These pricing measures have some peculiarities as they are based upon *defaultable* numeraire assets which are worthless at default. We showed that these *survival measures* can be viewed as versions of default-free pricing measures, conditioned upon survival of the obligor. The pricing of options on CDS is not the only case in which this technique can be applied quite successfully, other applications are the setup of a Libor-market model with default risk as in Schönbucher (1999), or the analysis and pricing of *counterparty risk* in OTC derivatives transactions. Either way, the *survival measure* should not be missing in any credit derivatives modeller's toolkit.

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