A Rating-based Model for Credit Derivatives

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Abstract

We present a model in which a bond issuer subject to possible default is assigned a "continuous" rating $R_t \in [0,1]$ that follows a jump-diffusion process. Default occurs when the rating reaches 0, which is an absorbing state. An issuer that never defaults has rating 1 (unreachable). The value of a bond is the sum of "default-zero-coupon" bonds (DZC), priced as follows:

$$D(t,x,R) = \exp\left(-\ell(t,x) - \psi(t,x,R)\right)$$

The default-free yield $y(t,x,1) = \ell(t,x)\,/\,x$ follows a traditional interest rate model (e.g. HJM, BGM, "string", etc.). The "spread field" $\psi(t,x,R)$ is a positive random function of two variables R and x, decreasing with respect to R and such that $\psi(t,0,R) = 0$. The value $\psi(t,x,0)$ is given by the bond recovery value upon default. The dynamics of ψ is represented as the solution of a finite dimensional SDE. Given ψ such that $\partial\psi/\partial R < 0$ a.s., we compute what should be the drift of the rating process R_t under the risk-neutral probability, assuming its volatility and possible jumps are also given.

For several bonds, ratings are driven by correlated Brownian motions and jumps are produced by a combination of economic events.

Credit derivatives are priced by Monte-Carlo simulation. Hedge ratios are computed with respect to underlying bonds and CDS's.

Most other credit models (Merton, Jarrow-Turnbull, Duffie-Singleton, Hull-White, etc.) can be seen either as particular cases or as limit cases of this model, which has been specially designed to ease calibration.

Long-term statistics on yield spreads in each rating and seniority category provide the diffusion factors of ψ. The rating process is, in a first step, statistically estimated, thanks to agency rating migration statistics from rating agencies (each agency rating is associated with a range for the continuous rating). Then its drift is replaced by the risk-neutral value, while the historical volatility and the jumps are left untouched.

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1. Introduction

Recently, Enron, one of the largest energy brokers in the world, and the leader in energy derivatives, filed for Chapter 11 protection. It could not sustained by itself derivative-linked liabilities on oil and electricity contracts. The cancellation of a buy-out agreement by another energy company, Dynegy, caused the company's failure. This event would have been one among other bankruptcies, hadn't it happened about one year after a famous cascade of electricity company failures in California during the Fall 2000. It would be difficult to find a direct link between these two events, although it is quite obvious that the California defaults had a negative impact on Enron's general financial shape. What can be taken for granted is that both stories happened in a rather turbulent environment for energy markets. This article is an attempt to present a mathematical framework for credit events modelling that is both tractable, in terms of statistics, calibration, credit derivative pricing and hedging, and flexible enough to reproduce the real features of credit events in financial markets, such as the non-causal, though truly existing, link described above.

The issuer of a bond subject to possible default — corporate or else — is assigned a "continuous" rating $R_t \in [0,1]$ that follows a diffusion process, possibly with jumps. Default occurs when the rating reaches 0, which is an absorbing state. Non-defaultable bonds have rating 1, which is unreachable when starting from other ratings. At any time t, the bond is valued as the sum of its scheduled payments, which are proportional to "defaultable discount factors" with rating R_t. The defaultable discount factor with time to maturity x and rating R is denoted $D(t,x,R)$ and decomposed as follows:

$$D(t,x,R) = \exp\left(-\ell(t,x) - \psi(t,x,R)\right)$$

The non-default yield $y(t,x,1) = \ell(t,x)\,/\,x$ follows a traditional interest rate model (e.g. HJM, BGM, etc.). The *spread field* $\psi(t,x,R)$ is a positive random function of the two variables x and R, which is decreasing with respect to R. The usual yield spread over Treasuries is $\psi(t,x,R)/x$, which implies $\psi(t,0,R) \equiv 0$ (unless an actual discount is still observed immediately before payment). The recovery value upon default is $D(t,x,0) = e^{-\chi} D(t,x,1)$, where $\chi = \psi(t,x,0)$. A formal zero recovery rate would correspond to a function ψ that is singular in $R=0$, so that $\chi = +\infty$. Different bonds issued by the same issuer must have the same rating, but could lead to different spread fields ψ, as their recovery rate depends on the bond's seniority level. Let $\varphi = -\partial\psi/\partial R$, so that $\psi(t,x,R) = \int_R^1 \varphi(t,x,u)\,du$. The *derived spread field* φ, or "spread per unit of rating" is a positive random function, represented as the solution of a finite dimensional SDE, as is the case for $\ell(t,x)$ in an HJM-like model, but with one extra variable.

The "continuous" rating of a bond issuer has a rather intuitive meaning: it can be seen as an interpolation of ratings provided by agencies. More precisely, one can specify the model in such a way that a given agency rating corresponds to some sub-interval $[\alpha_i, \alpha_{i+1}] \subset [0,1]$. Rating migrations correspond to crossing one or several (in case of a jump) threshold(s) α_i . Thresholds can be customarily chosen. A market-observed spread jump, due to some negative information on the issuer, is, in our model, linked to some rating jump. This means that the continuous rating is in fact market implied and may anticipate the actual rating provided by agencies. We shall however ignore this possibility when calibrating the diffusion and jump parameters of the rating process.

From the Arbitrage Pricing Theory (A.P.T.), we know that there exists a probability, equivalent to the original "historical" probability, that is risk-neutral for non-defaultable bonds. Under this probability, credit-risk-free contingent claim prices are equal to the expectation of the discounted claim pay-offs. In the defaultable context, φ being given and positive a.s., we show that there exists an equivalent probability that is risk-neutral for both defaultable and non-defaultable bonds, under which credit-risk-dependent contingent claim prices are again equal to the expectation of the discounted credit-event-dependent pay-off, with the recovery value in case of the counterparty default. For this purpose, we compute a "risk-neutral drift" of the rating process R_t, assuming its volatility and possible jumps are given.

Note that the introduction of the rating as a market variable in the model goes beyond the usual A.P.T. framework, in which market variables should be associated with the price of a tradable security. More precisely, the insufficient number of issues from a given issuer on a given seniority level and the impossibility of taking short positions on corporate bonds render the market highly incomplete. In mathematical words, for the model to be Markov, we need to introduce non-tradable market variables. As a consequence, various rating and spread process specifications can lead to exactly the same price and default processes (including recovery rate)[1]. Then, the same original probability distribution may lead to different risk-neutral probabilities and, consequently, different credit derivative prices. In this context, the usual "default intensity" approach is an extreme case, with derivative prices that are, therefore, at the extreme of what is consistent with underlying prices and default probabilities. It is well known that, in A.P.T., the set of acceptable prices for a given derivative is the set of expectations of its discounted future pay-off under the various risk-neutral measures. The different rating and spread process specifications will tend to span the interval of acceptable arbitrage prices, whereas the default intensity approach will in most cases, depending on how one's own default is taken into account, provide one of the interval extremities.

[1]For instance, by changing the rating R with any 1-1 transformation of the interval [0,1] . The multiplicity of prices is a consequence of the market non-completeness. In mathematical words, the filtration spanned by default bond prices is not "weakly Brownian" (see Revuz-Yor [19, p. 219] or Emery [9]), that is, a measurable function of prices cannot always be represented as a stochastic integral. Indeed, if reduced to price processes, the model doesn't, in general, have the Markov property.

Ratings of several issuers are driven by correlated Brownian motions. In the case of pure diffusion processes, joint defaults have zero probability (although a default occurrence increases the intensity of other defaults correlated to the defaulting party). In a jump-diffusion model, jumps are produced by economic events, with a size that depends on the event and on the issuer. Similarly to Duffie-Singleton approach [7, 1997], when an economic event occurs, it may induce the default of each single issuer with a certain probability that depends on its current rating and on the jump size distribution. In this case, a jump-driven default implies a negative (or at least non-positive) impact on other ratings and, consequently, an increase in the default probability.

The pricing of credit derivatives, such as non-standard credit default swaps (CDS), first n to default in a basket, etc., is performed by Monte-Carlo simulation under the credit risk-neutral measure. Hedge ratios are computed with respect to the underlying bonds and standard CDS's, which can be used to obtain negative sensitivities. The prices that we obtain are not necessarily "arbitrage prices", but the risk premia, or rather "reward for risk taking". They are, for every given issuer, consistent with the market price of its bond issues, in the sense that, if default is ignored, then the "theta" (time derivative) of the claim is equal to the instantaneous default-free rate of return of the hedging portfolio, theoretically composed of long or short positions on underlying bonds.

Most famous credit models (Merton [10, 1974], Jarrow-Turnbull [14, 1995], Duffie-Singleton [7, 1997]) can be seen either as particular cases or as limit cases of this model, which is an attempt to encompass in a unique framework the various rating-based models, such as Huge-Lando [12], Hull-White [13] and Crouhy-Im-Nudelman [5]. Possible extensions of this model include string models for the default-free interest rate part, as well as an infinite dimensional random field for the function φ with possible distribution components (i.e. the integral $\int_R^1 \varphi(t, x, u)\, du$ may be discontinuous with respect to R). The rating volatility could also be made stochastic, or subject to regime changes.

This model has been specially designed to ease calibration. Long-term statistics on yield spreads in each rating category and seniority level provide the volatility and factor structure of the random function φ. The rating process is, in a first step, statistically estimated, thanks to rating migration statistics from rating agencies (see above: each agency rating is associated with a range of possible continuous ratings). Then its drift is replaced by the risk-neutral value, while the historical volatility is kept. Jumps are only introduced to model catastrophic events involving several bonds. The rating process being an abstract version of Merton's firm value, we suggest, along with other authors (e.g. [5]), to use issuers' stock correlation for that of their rating processes, although this hypothesis should be tested.

2. Dynamics

2.1. Non-defaultable Bond Pricing

We assume that the dynamics of default-free interest rates are given via a HJM or BGM model (see Heath-Jarrow-Morton [11, 1990] and Brace-Gatarek-Musiela [3, 1997]). More precisely, the default free zero-coupon bond $D(t,x,1)$ with face value 1 and time to maturity x is given by:

$$D(t,x,1) = \exp(-\ell(t,x))$$

The function $\ell(t,x)$,which stands for the opposite logarithm of the discount factor, is a convenient term structure representation for stochastic modeling. The parameter 1 makes precise that we are dealing with non-defaultable bonds (the meaning of this parameter is explained below). One has:

$$\ell(t,x) = \int_t^{t+x} f(t,s)\, ds = y(t,x)\, x$$

where $f(t,s)$ is the forward rate at date s as of t and $y(t,x)$ is the zero-coupon yield. We assume that the interest rate dynamics, under a measure $\mathcal{P}$, is given by:

$$d_t\ell(t,x) = \mu(t,x,\underline{\ell}_t)\, dt + \sum_{i=1}^{m} \nu_i(t,x,\underline{\ell}_t)\, dZ_{i,t}$$

where $Z = (Z_1, \ldots, Z_m)$ is an m-dimensional Brownian motion. The drift μ and the volatility factors ν_i not only depend on the time and maturity, but also on the whole yield curve $\underline{\ell}_t = \ell(t,.)$. Let $r(t) = f(t,t)$ be the short term rate. It is well known that, if $\mathcal{P}$ is a risk-neutral probability, in order to avoid arbitrages, we need that actualised discount factors be martingales, which leads to:

$$\mu(t,x,\underline{\ell}_t) = f(t,t+x) - r(t) + \frac{1}{2}\sum_{i=1}^{m} \nu_{i,t}^2 \qquad \text{where} \quad \nu_{i,t} = \nu_i(t,x,\underline{\ell}_t)$$

2.2. Credit Modelling

Each bond issuer is assessed a rating which can take continuous values $R \in [0,1]$ and is modelled as a random process. The rating $R = 1$ corresponds to issuers that never default, therefore it cannot be reached unless it is the initial value. Default occurs at the first time t where $R_t = 0$, which is an absorbing state.

A bond issued by a company depends on the default-free yield curve and on its yield spread over default-free bonds, which is a function of the company rating and of the recovery rate in case of default. The log-ratio between the actual market price and the "would be" default-free value, which may be different for each bond and, in particular, depends on the bond seniority, is itself modelled as a random function $\psi(t,x,R)$,which we call the *spread field.*

2.2.1. Rating Diffusion Process

The rating process R_t, $t \geq 0$, is, in a first step, modelled as a diffusion process:

$$dR_t = h_t\, dt + \sigma(t, R_t)\, dB_t \tag{2.1}$$

where B is a Brownian motion, as long as $R_t > 0$. Then if $\tau = \min\{t \geq 0,\ R_t = 0\}$, we set $R_t = 0$ for every $t \geq \tau$. The drift h_t and the volatility $\sigma(t, R_t)$ must be chosen so that $R_0 < 1 \Rightarrow \forall t,\ R_t < 1$ a.s., in particular — but this is not sufficient — they must vanish for $R = 1$. We shall assume, in this article, that the volatility $\sigma(t, R)$ is a deterministic, continuous function of t and R, however, the drift can be any integrable process.

We shall see in sect. 2.4 how to add a jump term to the rating process. A stochastic volatility is a possible extension of the model (see comment in sect. 4.1).

2.2.2. Spread Field Process

Let $D(t, x, R)$ be the price of a defaultable zero-coupon bond (DZC in short) with rating R. One can set:

$$D(t, x, R) = \exp(-\ell(t, x) - \psi(t, x, R)) \tag{2.2}$$

where the spread field ψ is defined by:

$$\psi(t, x, R) = \log\left(\frac{D(t, x, 1)}{D(t, x, R)}\right)$$

This is a random function of x and R which, for fixed (t, x), should decrease when R increases and vanish for $R = 1$.

Remark 1. *The usual yield spread is $s(t, x, R) = \psi(t, x, R) / x$, which suggests that the function ψ must satisfy $\psi(t, 0, R) \equiv 0$. However, in certain market conditions, this assumption can be relaxed, for instance when, at the eve of a payment, a default risk still remains and the market significantly underprices the bond with respect to its face value. This justifies why we prefer to work with ψ rather than s.*

Remark 2. *When a given company has several bond issues, the default on one security usually implies a right, for holders of other issues, to ask for immediate reimbursement. Therefore, the default time is the same for all the bonds issued by the same company. However, the yield spreads over Treasuries of the various issues are different, due to different seniority levels and, hence, different recovery rates in case of default. The rating based model is particularly well suited for this situation, where a given company has only one rating process R_t but, for every single bond issue — or, at least, every seniority level — a different spread field, calibrated so as to match the market price of each bond.*

The spread field properties allow us to write it under the form:

$$\psi(t,x,R) = \int_R^1 \varphi(t,x,u)\,du$$

where φ is a non-negative random field, called the *derived spread field.* Following the above remark, φ must satisfy $\varphi(t,0,u) \equiv 0$, with the same comment about payment eve possible discounts.

If the firm defaults at time t, the value of the bond is a percentage of the default-free bond:

$$D(t,x,0) = D(t,x,1)\,e^{-\chi(t,x)}$$

Hence, the spread field value for $R=0$ is linked to the recovery rate by the equation:

$$\psi(t,x,0) = \int_0^1 \varphi(t,x,u)\,du = \chi(t,x) = \log\frac{D(t,x,1)}{D(t,x,0)}$$

A formally zero recovery rate would correspond, in this model, to the fact that $\varphi(t,x,u)$ has a singularity for $u=0$, such that $\displaystyle\int_0^1 \varphi(t,x,u)\,du = +\infty$.

The dynamics of the derived spread field for fixed (x,u) is given by a multi-factor diffusion:

$$d_t\varphi(t,x,u) = \gamma(t,x,u,\underline{\varphi}_t)\,dt + \sum_{i=1}^n \xi_i(t,x,u,\underline{\varphi}_t)\,dW_{i,t}$$

where $W=(W_1,\dots,W_n)$ is an n-dimensional Brownian motion. In this formulation, the drift γ and the volatility factors ξ_i may depend on the whole derived spread field $\underline{\varphi}_t = \varphi(t,.,.)$. We may assume, without loss of generality[2], that the correlations between the different Brownian motions are:

$$\begin{aligned} d\langle Z_i,B\rangle_t &= z_i dt \qquad d\langle W_i,B\rangle_t = w_i dt \\ d\langle W_i,Z_i\rangle_t &= \rho_i dt \qquad d\langle W_i,Z_j\rangle_t = 0 \quad \text{if} \quad i\neq j \end{aligned}$$

Let us make the assumption that the first order partial derivatives $\partial_x\gamma$ and $\partial_R\gamma$, as well as, for every i, $\partial_x\xi_i$ and $\partial_R\xi_i$, exist and are a.s. bounded and continuous with an at most uniform linear growth with respect to φ. If we assume the same property with the initial derived spread field $\varphi_0(x,u) = \varphi(0,x,u)$, then $\partial\varphi/\partial x$ and $\partial\varphi/\partial R$ remain a.s. bounded and continuous for all times.

[2]One can apply a unitary linear transformation to multi-dimensional Brownian motions W and Z.

Lemma 2.1. *For fixed T, the dynamics of the composed spread process Ψ defined by $\Psi_t = \psi(t, T-t, R_t)$ is given by the following formula, in which $x = T - t$:*

$$d\Psi_t = \int_{R_t}^1 d_t\varphi(t,x,u)\,du - \varphi(t,x,R_t)\,dR_t$$

$$-\left(\int_{R_t}^1 \frac{\partial\varphi}{\partial x}(t,x,u)\,du\right)dt - d\langle R,\varphi\rangle_t - \frac{1}{2}\frac{\partial\varphi}{\partial R}(t,x,R_t)\,d\langle R\rangle_t$$

In this formula, $d_t\varphi(t,x,u)$ stands for the time differential of the process $\varphi(t,x,u)$ for fixed (x,u), $\langle R,\varphi\rangle_t$ denotes its bracket with R_t evaluated at $u = R_t$, whereas $\langle R\rangle_t$ is the usual bracket of the process R_t .

Proof The proof of this lemma is given in the Appendix.

Let us now denote:

$$\begin{aligned}
\mu_t &= \mu(t,x,\underline{\ell}_t) \qquad \nu_{i,t} = \nu_i(t,x,\underline{\ell}_t)\\
\Gamma_t &= \Gamma(t,x,R_t) = \int_{R_t}^1 \gamma(t,x,u,\underline{\varphi}_t)\,du\\
\Xi_{i,t} &= \Xi_i(t,x,R_t) = \int_{R_t}^1 \xi_i(t,x,u,\underline{\varphi}_t)\,du\\
\sigma_t &= \sigma(t,R_t) \qquad \xi_{i,t} = \xi_i(t,x,R_t,\underline{\varphi}_t) \qquad \varphi_t = \varphi(t,x,R_t)\\
\varphi'_t &= \frac{\partial\varphi}{\partial R}(t,x,R_t) \qquad \frac{\partial\psi}{\partial x}|_t = \int_{R_t}^1 \frac{\partial\varphi}{\partial x}(t,x,u)\,du
\end{aligned}$$

From the above proposition, we deduce:

$$\begin{aligned}
d_t\ell &= \mu_t\,dt + \sum_{i=1}^m \nu_{i,t}\,dZ_{i,t}\\
d\Psi_t &= \left(\Gamma_t - \varphi_t\,h_t - \frac{\partial\psi}{\partial x}|_t - \sigma_t\sum_{i=1}^n w_i\,\xi_{i,t} - \frac{1}{2}\,\sigma_t^2\varphi'_t\right)dt + \sum_{i=1}^n \Xi_{i,t}\,dW_{i,t} - \sigma_t\,\varphi_t\,dB_t
\end{aligned} \tag{2.3}$$

2.3. Risk-neutral Probability

This calculation, together with Ito lemma applied to formula (2.2), lead to the following proposition.

Proposition 2.2. *For fixed T, the defaultable discount factor dynamics is given by:*

$$\begin{aligned} \frac{dD_t}{D_t} &= -d\ell_t + \frac{1}{2} d\langle \ell \rangle_t \\ &+ \left[-\Gamma_t + \varphi_t\, h_t + \frac{\partial \psi}{\partial x}|_t + \sigma_t \sum_{i=1}^{n} \xi_{i,t}\, w_i + \sum_{i=1}^{m\wedge n} \nu_{i,t}\, \Xi_{i,t}\, \rho_i \right. \\ &+ \left. \frac{1}{2}\left(\sigma_t^2 \varphi_t' + \sum_{i=1}^{n} \Xi_{i,t}^2 + \sigma_t^2 \varphi_t^2 \right) - \varphi_t \sigma_t \left(\sum_{i=1}^{m} \nu_{i,t}\, z_i + \sum_{i=1}^{n} \Xi_{i,t}\, w_i \right) \right] dt \\ &+ martingale \end{aligned}$$

Let us recall that, for $\mathcal{P}$ to be a risk-neutral probability for non-defaultable bonds, one must have, for fixed T :

$$-d\ell_t + \frac{1}{2} d\langle \ell \rangle_t = r_t + martingale$$

Equating the drift of dD_t/D_t to r_t , we get the "risk-neutral drift" $\tilde{h}$ of the rating process, as stated here.

Proposition 2.3. *Assume that $\mathcal{P}$ is a risk-neutral probability for the non-defaultable discount factors $D(t,x,1)$. Assume that $\varphi(t,x,u) > 0$ almost surely and that the process $\tilde{h}_t$ defined by:*

$$\begin{aligned} \varphi_t\, \tilde{h}_t &= \Gamma_t - \frac{\partial \psi}{\partial x}|_t - \sigma_t \sum_{i=1}^{n} \xi_{i,t}\, w_i - \sum_{i=1}^{m\wedge n} \nu_{i,t}\, \rho_i\, \Xi_{i,t} \\ & -\frac{1}{2}\sigma_t^2 \left(\varphi_t' + \varphi_t^2\right) - \frac{1}{2}\sum_{i=1}^{n} \Xi_{i,t}^2 + \varphi_t \sigma_t \left(\sum_{i=1}^{m} \nu_{i,t}\, z_i + \sum_{i=1}^{n} \Xi_{i,t}\, w_i \right) \end{aligned}$$

is such that:

$$\forall T > 0, \qquad \mathbf{E}_{\mathcal{P}} \left[\exp\left(\frac{1}{2} \int_0^T \frac{(\tilde{h}_t - h_t)^2}{\sigma_t^2}\, dt \right) \right] < +\infty \tag{2.4}$$

Then the probability $\widetilde{\mathcal{P}}$ defined by:

$$\frac{d\widetilde{\mathcal{P}}}{d\mathcal{P}} = \exp\left(\int_0^T \frac{\tilde{h}_t - h_t}{\sigma_t}\, dB_t - \frac{1}{2} \int_0^T \frac{(\tilde{h}_t - h_t)^2}{\sigma_t^2}\, dt \right)$$

is a risk-neutral probability for defaultable zero-coupon bonds.

Proof Girsanov theorem shows that, under condition (2.4), $\widetilde{\mathcal{P}}$ is a probability measure equivalent to $\mathcal{P}$. The above calculation shows that discounted prices of defaultable zero-coupon bonds are martingales with respect to $\widetilde{\mathcal{P}}$. □

2.4. Rating Jump-diffusion Process

In this section, we add a jump term to the rating process. More precisely, we model the rating process R_t as the following jump-diffusion process:

$$dR_t = h_t dt + \sigma(t, R_{t-})\, dB_t + \theta(t, R_{t-})\, dM_t \tag{2.5}$$

where B is a Brownian motion and M is the compensated martingale associated to a Poisson process N with deterministic intensity $\lambda(t)$, that is:

$$dM_t = dN_t - \lambda(t)dt$$

The drift term h and the functions σ and θ must be chosen such that if $0 < R_0 < 1$ then, for any $t > 0$, one has $0 \leq R_t < 1$. Taking care of jumps in the dynamics of Ψ_t, we obtain:

$$\begin{aligned}
d\Psi_t &= \int_{R_t}^{1} d_t\varphi(t,x,u)\, du - \left(\int_{R_t}^{t} \frac{\partial \varphi}{\partial x}(t,x,u)du\right) dt \\
&- \varphi(t,x,R_{t-})\, dR_t - d\langle R^c, \varphi\rangle_t - \frac{1}{2}\frac{\partial \varphi}{\partial R}(t,x,R_t)\, d\langle R^c\rangle_t \\
&+ \left[\psi(t,x,R_{t-}+\theta_t) - \psi(t,x,R_{t-}) + \varphi(t,x,R_{t-})\,\theta_t\right] dN_t
\end{aligned}$$

where $\theta_t = \theta(t, R_{t-})$ and R^c_t is the continuous part of the process R_t. Then, replacing stochastic differential terms by their value:

$$\begin{aligned}
d\Psi_t &= \left[\Gamma_t - \varphi_t\, h_t - \frac{\partial \psi}{\partial x}|_t - \sigma_t \sum_{i=1}^{n} w_i \xi_{i,t} - \frac{1}{2}\,\sigma_t^2 \varphi_t' + \lambda(t)\left(\Delta\Psi_t + \varphi_{t-}\theta_t\right)\right] dt \\
&+ \sum_{i=1}^{n} \Xi_{i,t}\, dW_{i,t} - \sigma_t\, \varphi_{t-}\, dB_t + \Delta\Psi_t\, dM_t
\end{aligned}$$

where

$$\Delta\Psi_t = \Psi_t - \Psi_{t-} = \int_{R_{t-}+\theta_t}^{R_{t-}} \varphi(t,x,u)\, du$$

Finally, the risk neutral drift $\tilde{h}_t$ satisfies:

$$\begin{aligned}
\varphi_t\, \tilde{h}_t &= \Gamma_t + \lambda(t)\left(1 - \exp\left(-\Delta\Psi_t\right) + \varphi_{t-}\theta_t\right) \\
&- \frac{\partial \psi}{\partial x}|_t - \sigma_t \sum_{i=1}^{n} \xi_{i,t}\, w_i - \sum_{i=1}^{m\wedge n} \nu_{i,t}\, \Xi_{i,t}\, \rho_i \\
&- \frac{1}{2}\sigma_t^2\left(\varphi_t' + \varphi_t^2\right) - \frac{1}{2}\sum_{i=1}^{n} \Xi_{i,t}^2 + \varphi_t \sigma_t \left(\sum_{i=1}^{m} \nu_{i,t}\, z_i + \sum_{i=1}^{n} \Xi_{i,t}\, w_i\right)
\end{aligned}$$

3. Link with Other Models

3.1. Structural Models

Structural models, such as that described in Merton's famous 1974 article [10, 1974], can be seen as particular cases of our model. The rating is the firm value, scaled in a non-linear way in order to remain in the interval $[0, 1)$. Bonds value, which depends on the investor's risk aversion, is a deterministic function of the yield curve, the probability of default and the maturity. Therefore, so are the spread field ψ and its rating derivative φ.

3.2. Default-intensity-based Models

The relation between rating-based and default-intensity-based models is less straightforward. The seminal articles of this family of models are Jarrow-Turnbull [14, 1995] and Lando [17, 1998]. Only the possible default of bond issues is observed, their rating is ignored. The time τ at which a bond issuer defaults is modelled as a Poisson process. In Jarrow-Turnbull model, the yield spread of zero-coupon bonds over the equivalent non-defaultable bonds is a deterministic function of time and maturity, and so is the hazard rate, i.e. the intensity of the Poisson process. In Lando model, the hazard rate is stochastic and, consequently, so are yield spreads. These models could be seen as a limit case of a rating-based model. The apparently obvious formulation, where the rating R takes only two possible values, $\bar{R}$ and 0, and jumps from $\bar{R}$ to 0 according to a Poisson process is incorrect, because the "risk-neutralization" of the original probability $\mathcal{P}$ is different[3]. A better, but still incorrect parallel is to assume that φ is for instance a constant and that R_t is set so as to match the bond value.

The correct approach is, conversely, to start from a *risk-neutral* Jarrow-Turnbull (or Lando) model, then, thanks to a classical result[4], represent, at a given origin of time t_0, the default time τ as the first hitting time of a process R_t to some time dependent barrier $H(t)$. The change of variable that moves this barrier to the axis $R = 0$ is usually singular, in the sense that R_t has an infinite negative drift at $t = t_0$ in order to reach a default probability of the order of $t - t_0$ for t close to t_0 (one has $H(t_0) = 0$ and $H(t) \approx -a\sqrt{t - t_0}$ for small $t - t_0$). This is the reason why we speak here of a "limit case". Then the derived spread field φ is any smooth function of (x, R) such that bond prices match in both models. Now, a change of intensity of τ — which is thus not anymore risk-neutral — can be translated into a change of barrier $H(t)$ to some barrier $H'(t)$. If we keep the same function φ, the new, non risk-neutral, rating process $R'_t = R_t + H(t) - H'(t)$ has the same volatility, but a different drift. Our "risk-neutralization" procedure will lead us to the original Jarrow-Turnbull (or Lando) model.

[3] If the rating is ignored as a market variable, credit models are incomplete and several risk-neutral probability measures may exist.

[4] Although this result is often referred to in the literature, it hasn't been possible to identify the original reference.

In fact the two models are still different because, in the rating-based model, the change of rating process necessarily implies a change in the spread process Ψ_t . This feature is inherent to the global idea of a rating, because, when defaulting in a non-jumping manner, a bond has a price that continuously tends to the recovery value. Pure Poisson models, such as Jarrow-Turnbull and Lando, would correspond to a derived spread field φ that is a distribution concentrated on the axis $\{R = 0\}$ and 0 elsewhere. This difference has a small impact on the price of CDS's with respect of that of bonds and will be fully justified later on, when speaking of default correlation. Another reason for preferring the rating-based approach is that, in general, credit derivative prices and hedge ratios depend more gently on model parameters.

The real behaviour of bonds about to default seems to be in between: a price that consistently decreases before default, but still incurs some true jump when default is actually observed, which would correspond to a non-zero φ with some Dirac component along the axis $\{R = 0\}$.

3.3. Rating-based Models

Crouhy & Al. [5] model the rating as a Markov chain with finitely many states, in order to mimic agency ratings. They directly input migration statistics. With several issuers, ratings are correlated with the same correlation as issuers stocks. They build a risk-neutral probability that is inconsistent with an interpolation of discontinuous ratings by continuous ones. Nevertheless, it is close enough to ours to lead to similar derivative prices.

Hull-White model [13] can be viewed as a particular case our model. They define a rating process R_t which is a pure Brownian motion, but the "default barrier" is adapted so as to match the default probability. In particular, it is not necessarily a straight line. In order to get a risk-neutral probability, they hence modify the location of the barrier. This model can be identified with ours by changing the rating with some non-linear transformation that fits it into $[0, 1)$. The change to a "risk-neutral" barrier becomes a change in the in drift of the process, exactly like in our model. Notwithstanding the elegance of Hull-White's approach, we prefer our framework, in which the rating has a real practical meaning and is thereby easier to calibrate.

Avellaneda-Zhu [2] introduce the idea of a "risk-neutral distance-to-default process" of a firm. They characterize risk-neutrality by the fact the default index satisfies a Fokker-Planck-like parabolic PDE. Although their study only concerns one issuer, they show the easiness of calibration and the "square root" shape of barriers mentioned above.

4. Correlation of defaults

This aspect of credit models is probably the most sensitive and has lead to a thick literature on the topic. One advantage of the rating based approach is to make the joint default "mechanism" for several issuers very transparent. Moreover, although we increased the dimensions of the model, the number of parameters to calibrate remains tractable and the methodology for performing statistics is rather straightforward. Generally speaking, this approach is much more practical than, say, copula distributions (see sect. 4.3).

4.1. Correlated Diffusion Parts

We consider here a set of q companies with ratings $R_{i,t}$ at time t, $i = 1 \ldots q$, such that:

$$dR_{i,t} = h_{i,t}dt + \sigma_i(t, R_{i,t})\, dB_{i,t}$$

Brownian motions B_i and B_j are correlated with correlation ρ_{ij}. Such Brownian motions can be deduced from a series of q independent ones thanks to a Cholesky decomposition of the correlation matrix $[\rho_{ij}]_{1 \leq i,j \leq q}$.

Assume that the rating correlation of two companies is positive. If the first company defaults, then its rating went to 0. The positive correlation indicates that the second company rating is more likely to be low, thus it has a high probability to default. Suppose now that we know nothing about ratings and only observe defaults. Before the first company defaults, we don't know about its bad shape (except the possible high spread, but this could be hidden by a high recovery rate). Its rating could be anywhere, as well as that of the second company. As soon as the default of the first company is observed, then we have an implicit information on the rating probability distribution of the second one, which significantly increases its default probability. This is an essential feature of rating-based models that the default of a company suddenly increases the default probability, over a given period of time, of correlated companies.

This feature is actually observed in reality, though, under turbulent conditions in a given industrial sector, surviving companies tend to strengthen because competition vanishes. In other words, a default increases the rating volatility. Again we could reverse the causality by extending the model to ratings with stochastic volatility. When a default is observed, then the rating volatility, which is not an observed variable, is more likely to be high, inducing, on the one hand, more defaults but, on the other hand, faster rating improvements.

It is not easy to perform joint statistics on defaults, and even on rating migrations, because these are rather seldom events. This is the reason why we recommend here, in the absence of any better assumption and along with other authors (see Crouhy-Im-Nudelman [5]), to use for ρ_{ij} the correlation between the stocks of company i and company j (provided these are listed stocks). However, due to the custom scaling, the volatility should still be estimated statistically.

4.2. Correlated Jumps

In order to model the rating jumps of several companies, we consider k various independent Poisson processes $dN_{1,t}, \ldots, dN_{k,t}$. Then, the rating $R_{i,t}$ of company j

follows a jump-diffusion process of the type:

$$dR_{i,t} = h_{i,t}\,dt + \sigma_i(t, R_{i,t})\,dB_{i,t} + \sum_{i=1}^{k} \theta_{ij}(t, R_{i,t-})\,dM_{j,t}$$

where

$$dM_{j,t} = dN_{j,t} - \lambda_j(t)\,dt \qquad j = 1 \dots k$$

Along with Duffie-Singleton approach, one can see the Poisson processes $dN_{j,t}$ as "economic events" that impact each rating, but with a different coefficient θ_{ij} on each company. In particular, the probability, when an event j occurs that a given company i defaults depends on the company rating at that time and on the size of the coefficient θ_{ij} .

4.3. Link with Copula Distributions

Copula distributions can, again, be seen as a particular case of a rating based model. Indeed, consider, for instance, a pair of issuers with default times τ_1 and τ_2 . A copula model (see for instance Schönbucher-Schubert [20] and ref. cit.) explicitly provides the joint distribution of these two random times. In a rating-based model, we need to find a pair of processes $R_{1,t}$ and $R_{2,t}$ such that τ_1 and τ_2 are respectively the first hitting time of $R_{1,t}$ and $R_{2,t}$ at the 0 level. It is not obvious, and perhaps not even true, that any pair of stopping times can be seen at first hitting times of the level 0 by correlated diffusion processes. However, the richness of the class of rating-based models allows at least to reasonably approach any copula model.

The full superiority of the rating approach appears when modelling more than two issuers. Indeed, the specification of copula distributions and, in particular, of the pairwise, triplet-wise, etc., joint default probabilities becomes completely intractable as soon as the number of issuers is above 5 or 6. The simulation of ratings only needs to specify pairwise issuer correlations.

5. Credit Derivative Pricing

5.1. General Pricing Method

Up to now, we only focused on modelling the stochastic behaviour of defaultable bonds that underlie credit derivatives. In fact, once this is done, derivative pricing straightforwardly follows from the general arbitrage theory. Let C be a credit derivative that delivers a pay-off $P(\tau; X_{1,\tau}, \dots, X_{q,\tau})$ depending on the price of bonds X_i issued by companies i with rating $R_{i,t}$ and spread fields $\psi_i(t, x, u)$, at a time τ that depends on default times τ_i , $i = 1 \dots q$. This pay-off shape is even not the most general one: it could as well contain payments prior to τ, depend on defaults prior to τ, or even on price or spread moves, official rating changes, etc. We must include in the global simulation the rating $R_{w,t}$ of the derivative writer and its default time

τ_w. If the latter defaults prior to τ, then the derivative pay-off is simply cancelled (or multiplied by a damping factor β if a non-zero recovery rate is assumed). The price of C at time t for the derivative buyer is given by the following formula, where r is the short term refinancing rate of the buyer:

$$C(t) = \mathbf{E}_{\widetilde{\mathcal{P}}}\left[\exp\left(-\int_t^\tau r(u)\,du\right) P(\tau; X_{1,\tau}, \ldots, X_{q,\tau})\ (\mathbf{1}_{\tau_w > \tau} + \beta\,\mathbf{1}_{\tau_w \leq \tau})\right]$$

If the pay-off contains payments prior to τ, say a value $P(T; X_{1,T}, \ldots, X_{q,T})$ at time T, provided $\tau > T$, then we should add the following term:

$$\mathbf{E}_{\widetilde{\mathcal{P}}}\left[\exp\left(-\int_t^T r(u)\,du\right) P(T; X_{1,T}, \ldots, X_{q,T})\,\mathbf{1}_{\tau > T}\right]$$

Hedge ratios are obtained as the partial derivative of the price with respect to hedging instruments..

Due to the large number of variables, we recommend implementing the model with Monte-Carlo simulations, for which above formula is well suited. Following is a list of examples of credit derivatives that can be priced with this model.

5.1.1. One Issuer

- **Credit default swaps:** The swap buyer holds a defaultable bond X delivering coupons at dates $T_1, \ldots, T_n$ and wishes protection against default. He will pay, on every coupon payment date, provided he is paid himself, a fixed amount to the swap writer. Conversely, upon default, the writer will buy the bond at a given price K, e.g. face value. In this case, $P(\tau; X_\tau)$ has two parts. The positive part is $(K - X_\tau)_+$ and the negative part is the sum of payments at dates $T_k < \tau$.

- **Brady bond options:** In these emerging market government issues, as long as no default is observed, the first two coupons and the principal are guaranteed by the United States, so that the default risk on these payments can be neglected. If a coupon payment is missed — i.e. payed by the US. — then only one coupon and the principal are guaranteed in the sequel, and only the principal if two payments are missed (see Avellaneda-Wu [1, 2001]). In each Monte-Carlo path, we simulate the issuer rating and the history of payments, which provides the value of the bond and the option pay-off. The option value is obtained as the expectation of the discounted pay-off.

- **Convertible bonds:** They require to introduce in the model the underlying stock process S_t as a log-normal process, possibly with jumps. This process should be correlated with other market variables of the model: rating, interest rates and spread field. Probably, the most important correlation is the rating, as the stock is likely to drop drastically in case of default, whereas the rating should strengthen if the company value increases (see Davis-Lischka [6]). Note that Merton's structural model [10, 1974] assumes 100% correlation between these two variables.

5.1.2. Several Issuers

- **Basket protection:** This is a default insurance on a basket of bonds from several issuers, usually capped at a certain level, so that it cannot be decomposed into the sum of protections on each single component of the basket. The ratings and defaults of issuers are jointly simulated and, in each path, the pay-off is computed, including capping, and discounted back to the current date.
- **Tranche insurance:** This option is similar to the previous one, but only guarantees the part of losses that exceeds a certain amount, or is between two levels. The pricing method is the same.
- **First n to default:** The underlying of this option is again a basket of bonds. It provides protection against the first n defaults observed in the basket, possibly over a given period of time. The pricing method is still the same as above. This type of option is very sensitive to the joint behaviour of defaults and, in particular, to the impact of a default on the default probability of other assets. We believe that the rating-based approach provides prices that are more in line with market practice than other models.

6. Model Calibration

6.1. Rating Process Calibration

Rating agencies provide annual statistics of rating migrations and defaults. In a first step, we see the rating process as a Markov chain with finitely many states, the lowest one being default, which is absorbing. As defaults and migrations are rather rare events that, moreover, highly depend on the economical context, in the absence of specific information, it is preferable to assume that the Markov chain transition matrix is stationary. In the case of newly issued bonds and/or low rating (e.g. CCC) which have a smaller probability of keeping the same rating, one may relax the stationary hypothesis. In order to avoid biases, these statistics should be performed on bonds that kept their rating. Reasons for losing a rating could be the issuer buy-out by some other company, the bond call-back (for callable bonds), the bond conversion (for convertible bonds), or even when the issue is fully bought by one bond holder or by the issuer itself.

Then, one must "Interpolate" the Markov chain by a jump diffusion of the form (2.1) with constant thresholds α_i. Parameters h, σ, θ and λ (preferably stationary) are estimated by maximum likelihood. It is recommended to manually identify jumps and estimate their frequency and size. Then, θ and λ being given, estimate h and σ, otherwise the maximum likelihood method may provide unstable results. In a last step, the "risk-neutral" drift $\tilde{h}_t$ is computed in order to price credit derivative.

6.2. Spread Field Calibration

For this purpose, bonds are sorted by country, industrial sector, agency rating and seniority. For each class of Country / Industrial Sector / Rating / Seniority, the average spread curve over government bonds is computed every day. In a second step, we identify agency ratings with a range $R \in [\alpha_i, \alpha_{i+1}]$, e.g. $\alpha_i = i/p$ where p is the number of ratings and compute, every day, the spread field $\psi(t,x,R)$ and its rating derivative $\varphi(t,x,R)$. Finally, for each class of Country / Industrial Sector / Seniority, we perform a PCA (or Karhunen-Loeve analysis) of the function $\delta\varphi(t,x,R) = \varphi(t+1,x,R) - \varphi(t,x,R)$ in order to get the diffusion factors ξ_i . See Duffie-Singleton [8, 1997] for a study of yield spreads behaviour with respect to the rating.

7. Appendix

Proof of lemma 2.1: Let δt be a finite time interval, which will eventually tend to 0. Calculations made here are valid up to a term of the order $|\delta t|^{\frac{3}{2}}$:

$$\begin{aligned}\delta\Psi_t &= \Psi_{t+\delta t} - \Psi_t = \int_{R_{t+\delta t}}^{1} \varphi(t+\delta t, T-t-\delta t, u)\,du - \int_{R_t}^{1} \varphi(t, T-t, u)\,du \\ &= \int_{R_{t+\delta t}}^{R_t} \varphi(t+\delta t, T-t-\delta t, u)\,du + \int_{R_t}^{1} \left(\varphi(t+\delta t, T-t-\delta t, u) - \varphi(t, T-t, u)\right) du\end{aligned}$$

Let us define:

$$\begin{aligned}I_1 &= \int_{R_{t+\delta t}}^{R_t} \varphi(t+\delta t, T-t-\delta t, u)\,du \\ I_2 &= \int_{R_t}^{1} \left(\varphi(t+\delta t, T-t-\delta t, u) - \varphi(t, T-t, u)\right) du\end{aligned}$$

Let us set $\delta R_t = R_{t+\delta t} - R_t$. In the following calculations, $\mathcal{O}\left(|\delta t|^{\frac{3}{2}}\right)$ represents a random function of time $X(t)$ such that, for every t, $\mathbf{E}\left(X(t)^2\right) \leq C\,\delta t^3$ where C is a constant independent of t :

$$\begin{aligned}I_1 &= -\frac{1}{2}\left(\varphi(t+\delta t, T-t-\delta t, R_t) + \varphi(t+\delta t, T-t-\delta t, R_{t+\delta t})\right)\delta R_t + \mathcal{O}\left(|\delta t|^{\frac{3}{2}}\right) \\ &= -\varphi(t+\delta t, T-t-\delta t, R_t)\,\delta R_t - \frac{1}{2}\frac{\partial\varphi}{\partial R}(t+\delta t, T-t-\delta t, R_t)\,\delta\langle R\rangle_t + \mathcal{O}\left(|\delta t|^{\frac{3}{2}}\right) \\ &= -\varphi(t+\delta t, T-t, R_t)\,\delta R_t - \frac{1}{2}\frac{\partial\varphi}{\partial R}(t, T-t, R_t)\,\delta\langle R\rangle_t + \mathcal{O}\left(|\delta t|^{\frac{3}{2}}\right) \\ &= -\varphi(t, T-t, R_t)\,\delta R_t - \delta\langle\varphi, R\rangle_t - \frac{1}{2}\frac{\partial\varphi}{\partial R}(t, T-t, R_t)\,\delta\langle R\rangle_t + \mathcal{O}\left(|\delta t|^{\frac{3}{2}}\right)\end{aligned}$$

where $\delta\langle R\rangle_t = \langle R\rangle_{t+\delta t} - \langle R\rangle_t$ and $\delta\langle\varphi, R\rangle_t = \langle\varphi, R\rangle_{t+\delta t} - \langle\varphi, R\rangle_t$. This can be easily seen by writing $\varphi(t, x, R)$ as a stochastic integral. For I_2, we have:

$$\begin{aligned} I_2 &= \int_{R_t}^1 (\varphi(t+\delta t, T-t-\delta t, u) - \varphi(t, T-t-\delta t, u))\, du \\ &\quad + \int_{R_t}^1 (\varphi(t, T-t-\delta t, u) - \varphi(t, T-t, u))\, du \\ &= \int_{R_t}^1 \delta\varphi(t, T-t-\delta t, u)\, du \\ &\quad - \int_{R_t}^1 \frac{\partial\varphi}{\partial x}(t, T-t, u)\, du\, \delta t + \mathcal{O}\left(|\delta t|^2\right) \end{aligned}$$

where $\delta\varphi(t, x, u) = \varphi(t+\delta t, x, u) - \varphi(t, x, u)$. Because, for fixed x and u, $\partial\varphi/\partial x(t, x, u)$ is an Ito process, we have:

$$\begin{aligned} \int_{R_t}^1 \delta\varphi(t, T-t-\delta t, u)\, du &= \int_{R_t}^1 \delta\varphi(t, T-t, u)\, du + \int_{R_t}^1 \int_{T-t}^T \delta\frac{\partial\varphi}{\partial x}(t, s, u)\, ds du \\ &= \int_{R_t}^1 \delta\varphi(t, T-t, u)\, du + \mathcal{O}\left(|\delta t|^{\frac{3}{2}}\right) \end{aligned}$$

where $\delta\partial\varphi/\partial x(t, x, u) = \partial\varphi/\partial x(t+\delta t, x, u) - \partial\varphi/\partial x(t, x, u)$. Therefore:

$$I_2 = \int_{R_t}^1 \delta\varphi(t, T-t, u)\, du - \int_{R_t}^1 \frac{\partial\varphi}{\partial x}(t, T-t, u)\, du\, \delta t + \mathcal{O}\left(|\delta t|^{\frac{3}{2}}\right)$$

We get the lemma by putting back together the two terms of $\delta\Psi_t$. □

References

[1] Avellaneda, Marco and Li-Xin Wu *"Credit Contagion: Pricing Cross Country Risk in Brady Debt Markets"*, Int. Jour. of Th. and Appl. Finance, 2001.

[2] Avellaneda, Marco and Jing-Yi Zhu *"Modeling the Distance-to-default Process of a Firm"*, Preprint, New York University – 14 pages – July 2001, to appear in Risk.

[3] Brace, Alan, Darusz Gatarek and Marek Musiela *"The Market Model of Interest Rate Dynamics"*, Math. Finance, Vol. 7, 1997, p. 127-154.

[4] Crouhy, Michel, Dan Galai and Robert Mark *"A Comparative Analysis of Current Credit Risk Models"*, Journal of banking and finance, Vol. 24, 2000, pp. 49–117.

[5] Crouhy, Michel, John Im and Gregory Nudelman *"Measuring Credit Risk: The Credit Migration Approach Extended for Credit Derivatives"*, Preprint, Canadian Imperial Bank of Commerce, 2001.

[6] Davis, Mark and Fabian R. Lischka *"Convertible Bonds with Market Risk and Credit Risk"*, Preprint, Imperial College – 16 pages – October 10, 1999.

[7] Duffie, Darrell and Kenneth J. Singleton *"An Economic Model of the Term Structure of Interest Rate Swap Yields"*, Journal of Finance, Vol. LII, No. 4, (September 1997), pp. 1287-1321.

[8] Duffie, Darrell and Kenneth J. Singleton *"Ratings-based Term Structures of Credit Spreads"*, Preprint, Stanford University, 1997.

[9] Emery, Michel *"Espaces probabilisés filtrés : de la théorie de Vershik au mouvement Brownien via des idées de Tsirelson"*, Séminaire Bourbaki, No. 882, 2000-2001.

[10] Merton, Robert C. *"On the Pricing of Corporate Debt: The Risk Structure of Interest Rates"*, Journal of Finance, Vol. 29, MIT (1974), pp. 449-470.

[11] Heath, David, Robert Jarrow, and Andrew Morton, *"Bond Pricing and the Term Structure of Interest Rates: A Discrete Time Approximation"*, Journal of Financial and Quantitative Analysis, Vol. 25, No. 4, Cornell University, University of Illinois at Chicago, (December 1990), pp. 419-440.

[12] Huge, Brian and David Lando *"Swap Pricing with Two-sided Default Risk in a Rating-based Model"*, Preprint Dept. of Operations Research, University of Copenhagen – 33 pages – August 4, 1998.

[13] Hull, John and Alan White *"Valuing Risk Default Swaps, I: No Counterparty Default, II: Modelling Default Correlation"* Preprint, Rotman School, University of Toronto, 2000.

[14] Jarrow, Robert and Stuart M. Turnbull. *"Pricing Derivatives on Financial Securities Subject to Credit Risk"*, Journal of Finance, Vol. L, No. 1, Cornell University, and Queen's University (Canada) (Mar-1995), pp. 53-85.

[15] Jarrow, Robert and Fan Yu *"Counterparty Risk and the Pricing of Defaultable Securities"*, Journal of Finance, Vol. LVI, No. 5, (October 2001), pp. 1765-1800.

[16] Jeanblanc, Monique and Marek Rutkowski *"Modelling of Default Risk: Mathematical Tools"*, Preprint, Evry University – 67 pages – March 2

[17] Lando, David *"On Cox Processes and Credit Risky Securities"*, Review of Derivatives Research, Vol. 2, 1998, pp. 99–120.

[18] Lando, David *"Some Elements of Rating-based Credit Risk Modeling"*, Preprint, University of Copenhagen – 22 pages – February, 1999.

[19] Revuz, Daniel and Marc Yor *"Continuous Martingales and Brownian Motions"*, 3rd ed., Springer, 1999.

[20] Schönbucher, Philipp J., and Dirk Schubert *"Copula-dependent Default Risk in Intensity Models"*, Preprint, Bonn University – 23 pages – April 2001.

See www.defaultrisk.com for a very complete list of references on credit modelling.