DEFAULT IMPLIED VOLATILITY FOR CREDIT SPREAD

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This paper presents a simple reduce-form approach to pricing credit derivatives. The definition of default is purely based on the market value of a risky bond and its potential recovery value. A risky bond is treated as a riskless bond with an embedded short position on a barrier option. The risky bond market implicitly prices this barrier option. The default implied volatility (DIV) curve for credit spread is derived from the values of barrier options. The DIV curve is useful for pricing volatility-sensitive credit derivatives. In this paper, we show how the DIV curve is used to consistently price a credit spread put option and a first-to-default swap. The correlation matrix of reference entities' credit spreads (*not* default arrival times) is used for pricing a first-to-default swap.

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Recently there has been much interest in the pricing of credit derivatives, and the demand for credit derivatives is booming. It was spurred by Russia's default on its sovereign debts and the global credit crisis in the summer of 1998. The latest survey by the *Risk* magazine (*Risk*, April 1999) puts the global credit derivatives market at $477 billion. One popular approach for the credit derivatives is the "reduced form" based approach which models the default event as a random surprise[¶]. This paper presents a new simple approach to price credit derivatives along this line. The basic logic behind this paper is based upon the following observation: the wider the credit spread, other things being equal, the higher the probability of default, thus the higher the future potential credit spread volatility (since in a default event, the credit spread may move drastically). We call this future potential credit spread volatility, implied by the credit spread, the default implied volatility (DIV) for the credit spread. Our focus on the DIV is mainly motivated by the fact that many credit derivatives, like other derivatives, depend heavily on volatility.

To derive the DIV, we use a continuous credit spread model to describe the default characteristics, rather than treating a default event as a random surprise or a Poisson process. We assume that a risky bond issuer defaults, if and only if its shortest-maturity bond price drops to R percent of the otherwise equivalent riskless bond with the same coupon and maturity. When a bond issuer

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¶There are mainly two approaches to model the credit spreads. Examples of firm value based approach include Black and Cox (1976), Chance (1990), Hull and White (1995), Longstaff and Schwartz (1992), Merton (1974), and Nielson, Saa-Requejo and Santa-Clara (1993). Examples of "reduced form" based approach include Artzner and Delbaen (1994), Cooper and Mello (1996), Duffie and Singleton (1998), Fons (1994), Jarrow, Lando, and Turnbull (1997), Lando (1998), Litterman and Iben (1988), Madan and Unal (1993), Nielsen and Ronn (1995), Ramaswamy and Sundaresan (1986), and Schonbucher (1997).

defaults, each of its outstanding bonds is priced at R percent of the otherwise equivalent riskless bond, where R is the recovery rate (Jarrow, Lando, and Turnbull, 1997, use a similar recovery rate). This threshold-based definition of default is analogous to Black and Cox (1976)'s definition of default which is based on a threshold of the firm value. In this paper, a default event is defined by the shortest-maturity risky bond price relative to the riskless bond price. We call the shortest-maturity bond the trigger bond since default is triggered by its price relative to the riskless bond price. We use the shortest-maturity bond as the trigger bond mainly because: a) this bond is generally the most liquid, and b) when the issuer defaults on this bond, it must default on all other outstanding bonds. Our definition of default is reasonable if one is comfortable with the recovery rate. If the market is certain that when the trigger bond defaults, the recovery rate is R, then when the trigger bond price hits the recovery value, the bond must default.

From this definition, a risky bond can be decomposed into two parts: a riskless bond and a barrier option. To long a risky bond is same as to long the equivalent riskless bond and to short a barrier option. The barrier option pays (1-R) percent of the equivalent riskless bond if the trigger bond price reaches R percent of its equivalent riskless bond price (i.e. the bond issuer defaults), zero if otherwise. The value of the barrier option is equal to the riskless bond price minus the risky bond price. Therefore, the bond market implicitly prices this barrier option through the risky and riskless bond prices. From this option price, we can derive the default implied volatility for credit spread via a term structure model (we use the Cox, Ingersoll, and Ross, 1985, model in this paper). The DIV is purely derived from the underlying bond prices, but fully accounts for the credit spread changes due to economic conditions, possibility of credit changes, and possibility of default. This default implied volatility can then be used to price other credit derivatives consistently. In this paper, we price the credit spread put option and the first-to-default swap as examples.

The primary motivation of using a continuous credit spread model and defining default by the trigger bond price is to make the credit derivatives more understandable. Although the credit spread does jump occasionally, the default implied volatility fully incorporates the possibility of these occasions. This observation is analogous to stock option traders' rationale. It is well known that the stock prices often jump, and there are numerous stock option pricing models which explicitly try to explain these movements. However, the option traders are still using the Black-Scholes (1973) model which is purely based on a continuous setting. The possibility of jumps is simply factored into higher implied volatilities. In our setting, we treat the credit derivatives in a way that is very similar to how we treat the usual (i.e. riskless) interest rate based derivative products. For example, we can make a direct comparison between the DIV and the historical credit spread volatility. We can use the credit spread correlation matrix of the basket of reference entities to price the first-to-default swap. This correlation matrix, which is based on credit spreads, can be measured by historical data. It is far different from the correlation matrix of default arrival times, which is used in Poisson based models. The correlation matrix of default arrival times is very subjective, and can not be measured by historical data.

THE CIR MODEL FOR THE DEFAULTABLE BOND

Let $B(t,T)$ and $B_d(t,T)$ be the zero coupon bond prices at time t with maturity T for the riskless government bond and a risky defaultable corporate or sovereign bond correspondingly. Let $r(t)$ and *s(t)* be the short term interest rate and the short term credit spread. We assume each of *r(t)* and *s(t)* follow a one-factor CIR model, i.e.

(1) $r(t) = c_1(t) + X_1(t)$

(2) $s(t) = c_2(t) + X_2(t)$

and the state variable $X_i(t)$ follows a square-root process,

(3) $dX_i(t) = [\kappa_i \theta_i - (\kappa_i + \lambda_i) X_i(t)]\, dt + \sigma_i \sqrt{X_i(t)}\, dZ_i(t)$ for i = 1, 2.

here $Z_i(t)$, i = 1, 2, are independent standard Brownian motions in the risk-neutral measure. κ_i, θ_i, and σ_i are the long-term mean, the mean-reversion speed, and the volatility. λ_i are the risk parameters. And $c_i(t)$ are deterministic functions which make the model fit the initial interest rate curve and the credit spread curve exactly. Then CIR shows that the bonds prices

(4) $$B(t,T) = E_t\{\exp(-\int_t^T r(t)\, dt)\}$$

$$= \exp\{-\int_t^T c_1(t)\, dt + ln\, A_1(t,T) - B_1(t,T)\, X_1(t)\}$$

(5) $$B_d(t,T) = E_t\{\exp(-\int_t^T [r(t) + s(t)]\, dt)\}$$

$$= B(t,T) \exp\{-\int_t^T c_2(t)\, dt + ln\, A_2(t,T) - B_2(t,T)\, X_2(t)\}$$

where

(6) $$ln\, A_i(t,T) = \frac{2k_i q_i}{s_i^2}\, ln\left[\frac{2g_i \exp((k_i + l_i - g_i)(T-t)/2)}{2g_i e^{-g_i(T-t)} + (k_i + l_i + g_i)(1-e^{-g_i(T-t)})}\right]$$

(7) $$B_i(t,T) = \frac{2(1-e^{-g_i(T-t)})}{2g_i e^{-g_i(T-t)} + (k_i + l_i + g_i)(1-e^{-g_i(T-t)})}$$

(8) $\gamma_i = \sqrt{(k_i + l_i)^2 + 2s_i^2}$ for i = 1, 2

We assume, for the simplicity of discussion, that the bond issuer issues zero coupon bonds only. Furthermore, at time t, the shortest-maturity bond has maturity t^+. From our definition, $B_d(t, t^+)$ is the trigger bond price at time t. When the trigger bond price, at time τ, touches the recovery value $RB(\tau,\tau^+)$, where R is the recovery rate, the bond issuer defaults on all the outstanding bonds at time τ. At that time, each of the outstanding bonds is valued at R percent of its equivalent riskless bond price, i.e. $B_d(\tau, T) = R\,B(\tau, T)$, for all $T > \tau$.

Now let us consider zero coupon risky bond with maturity T at time t. Let D(t,T) be the value of a barrier option embedded in $B_d(t,T)$ at time t. This barrier option pays:

(9) $= (1–R)B(\tau,T)$ at time τ, if $B_d(\tau,\tau^+)$ reaches $RB(\tau,\tau^+)$ for the first time at τ ($t< \tau <T$);

0, otherwise.

Then long the defaultable bond is equivalent to long the riskless bond with a short position on the barrier option, i.e.

(10) $B_d(t,T) = B(t,T) – D(t,T)$

or $D(t,T) = B(t,T) – B_d(t,T)$,

so, whenever the market prices the riskless bond and the defaultable bond, it implicitly prices the barrier option. Now let us take a closer look on this barrier option. The barrier on the trigger bond price is $B_d^*(\tau,\tau^+) = RB(\tau,\tau^+)$, for $t< \tau <T$. Now let us transfer this barrier on the bond price to the barrier on the credit spread factor $X_2(\tau)$ by using equation (5). The barrier on the credit spread factor is

(11) $$X_2^*(\tau) = [-\,ln R - \int_t^{t^+} c_2(u)\,du + ln A_2(\tau,\tau^+)\,]/B_2(\tau,\tau^+)$$

This barrier is plotted in Figure 1. Here we assume at time 0, the outstanding risky bonds are 10 zero coupon bonds maturing at year i, i = 1, 2, …, 10. The parameter set is

(12) $\kappa_2 = 0.01$, $\theta_2 = 0.02$, $\lambda_2 = 0.001$, $X_2(0) = 0.01$, R = 50%, *r(t)* = 5.00%, for all t

and $\sigma_2 = 150\%$, $c_2(t) = 0$, for all t. We can see that this barrier is not constant, but time-varying. As time t gets closer to the maturity t^+ of the trigger bond, the barrier moves further away. This is because the shorter the trigger bond's maturity, the wider the credit spread to make the trigger bond price equal to a fixed R percent of the equivalent riskless bond price.

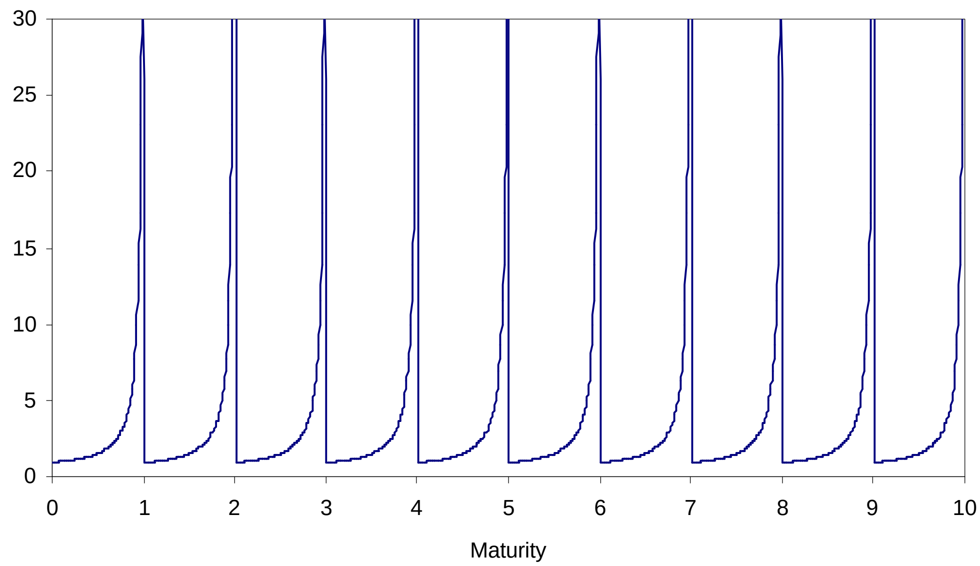

Figure 1. The Barrier for the credit spread state variable $X_2(t)$.
This barrier is based on θ_2=0.02, λ_2=0.001, κ_2=0.01, σ_2=150%, $c_2(t)$=0, $X_2(0)$=0.01, R=50%, and r(t)=5%. The outstanding bonds at time 0 are zero coupon bonds with maturity i=1,2,…,10.

PRICE THE BARRIER OPTION

Given the parameters κ_2, λ_2, θ_2, $X_2(0)$, and volatility σ_2 (from hereon we assume the reader can estimate the parameters κ_1, λ_1, θ_1, σ_1, $X_1(0)$, and $c_1(t)$ for the riskless interest rate curve, see CIR), as well as the deterministic function $c_2(t)$, we can price the barrier option by Monte Carlo simulation. Let f(t) be the first-passage density function for $X_2(t)$ to reach the barrier (11) at time t. Since the riskless interest rate and the credit spread are independent,

$$(13) \qquad D(t,T) = E_t[\int_t^T B(t,\tau)\,[(1 - R)\,B(\tau,T)]\,f(\tau)\,d\tau]$$

$$= (1 - R)\,B(t,T) \int_t^T f(\tau)\,d\tau = (1 - R)\,B(t,T)\,p(t,T)$$

where p(t,T) is the probability of $X_2(t)$ hitting the barrier (11) between [t, T]. To calculate p(t,T), we can discretize the diffusion process (3) by

$$(14) \qquad \Delta X_2(t_i) = (\kappa_2\theta_2 - (\kappa_2+\lambda_2)\,X_2(t_{i-1}))\,\Delta t_i + \sigma_2\sqrt{X_2(t_{i-1})\Delta t_i}\;\xi_i, \quad \text{for } i = 1, 2, \ldots, M$$

where $t = t_0 < t_1 < t_2 < ... < t_M = T$, and ξ_i are i.i.d. standard normal random variables. For each simulation run, we check at each t_i if $X_2(t_i)$ has reached the barrier (11). If it reaches at any point for the j-th simulation run, assign $p_j(t,T) = 1$; otherwise $p_j(t,T) = 0$. The probability p(t,T) is the average of these $p_j(t,T)$ over the total number of simulation runs.

Given the CIR parameters κ_2, λ_2, θ_2, $X_2(0)$, and σ_2, the deterministic function $c_2(t)$ totally specifies the initial credit spread curve. But $c_2(t)$ cannot be chosen arbitrarily, since $c_2(t)$ must be chosen to satisfy

$$(15) \qquad B_d(0,t) = B(0,t) - D(0,t)$$

here $B_d(0, t)$ depends on $c_2(\tau)$, $\tau < t$, through equation (5), and $D(0, t)$ depends on $c_2(\tau)$, $\tau < t$, through equation (11). If we assume $c_2(t)$ a piece-wise constant $c_2(t) = c_{2,j}$, for $T_{j-1} < t < T_j$, $T_0 = 0 < T_1 < T_2 < \ldots < T_N = T$, we can recursively solve the nonlinear equation (15) for all $c_{2,j}$. More specifically, if we have already solved $c_{2,i}$, i = 1, 2, …, j, then $c_{2,j+1}$ is the solution to the nonlinear equation (15) (with $t = T_{j+1}$) with the constrains (5) (with t = 0, $T = T_{j+1}$) and (11). With all these $c_{2,j}$, j = 1, 2,…, N,

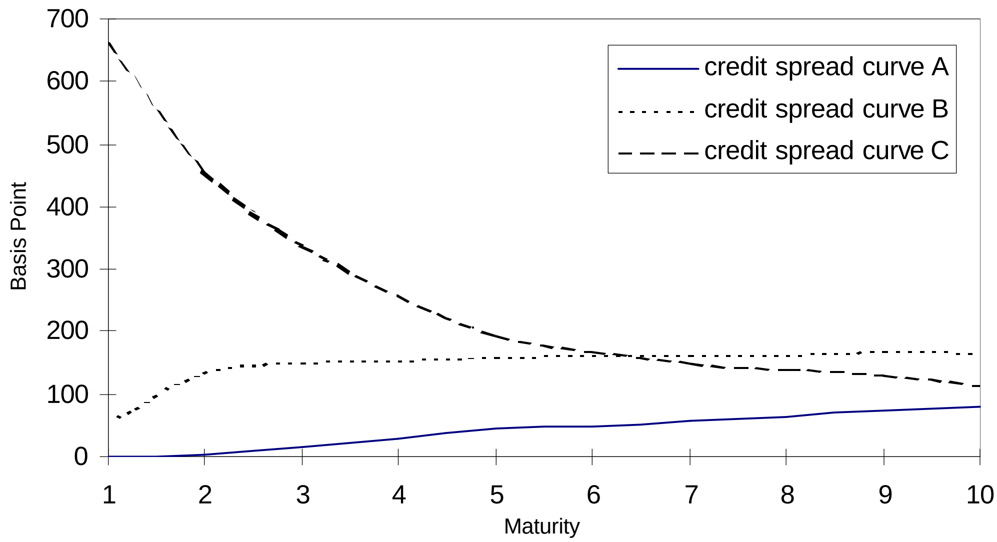

Figure 2. The derived spot credit spread curve based on the inputted volatility curve in Figure 3. The parameters are κ_2=0.01, θ_2=0.02, λ_2=0.001, $X_2(0)$=0.01, R=50%, r(t)=5%. The outstanding bonds at time 0 are zero coupon bonds with maturity i=1,2,…,10.

we can calculate the initial credit spread curve. Figure 2 and 3 show the calculation results. Here we use the parameter set (12). The credit spread curve B in Figure 2 is derived from the credit spread volatility curve B in Figure 3.

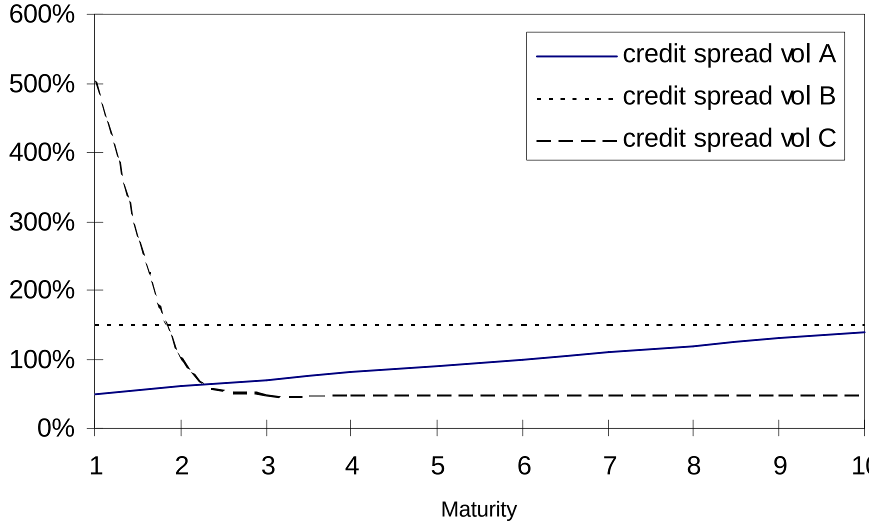

Figure 3. The inputted CIR credit spread volatility curves used to generate the spot credit spread curves in Figure 2.

CIR MODEL WITH PIECEWISE CONSTANT VOL

To make the model more flexible, we assume the credit spread volatility is piecewise constant $\sigma_2 = \sigma_2(\tau) = \sigma_{2,j}$, for $T_{j-1} < \tau < T_j$, $t = T_0 < T_1 < ... < T_N = T$. It can be shown (see Pitman and Yor, 1982) that

(16) $$B_d(t,T) = B(t,T)\exp\{-\int_t^T c_2(\tau)\,d\tau + \sum_{j=0}^{N-1} ln A_2(T_j, T_{j+1}, \alpha_{j+1}) - B_2(t, T_1, \alpha_1)\, X_2(t)\}$$

Where

$ln A_2(T_j, T_{j+1}, \alpha_{j+1}) =$

$$\frac{2k_2 q_2}{s_{2,j+1}^{\;2}}\, ln\left[\frac{2g_{2,j+1}\exp((k_2 + l_2 - g_{2,j+1})(T_{j+1} - T_j)/2)}{2g_{2,j+1}e^{-g_{2,j+1}(T_{j+1}-T_j)} + (k_2 + l_2 + g_{2,j+1} + s_{2,j+1}^{2}a_{j+1})(1 - e^{-g_{2,j+1}(T_{j+1}-T_j)})}\right]$$

$$B_2(T_j,T_{j+1},\alpha_{j+1}) = \frac{(2-(k_2+l_2)a_{j+1})(1-e^{-g_{2,j+1}(T_{j+1}-T_j)}) + g_{2,j+1}a_{j+1}(1+e^{-g_{2,j+1}(T_{j+1}-T_j)})}{2g_{2,j+1}e^{-g_{2,j+1}(T_{j+1}-T_j)} + (k_2+l_2+g_{2,j+1}+s_{2,j+1}^2 a_{j+1})(1-e^{-g_{2,j+1}(T_{j+1}-T_j)})}$$

$$\gamma_{2,j} = \sqrt{(k_2+l_2)^2 + 2s_{2,j}^2}$$

here $\alpha_N = 0$, $\alpha_j = B(T_j, T_{j+1}, \alpha_{j+1})$, for j = 1, ...N-1.

The barrier (11) can be easily replaced by

$$(17)\qquad X_2^*(\tau) = [-\ln R - \int_t^{t^+} c_2(u)\,du + \ln A_2(\tau, T_{m_1}, a_{m_1}) + \sum_{j=m_1}^{m_2-1} \ln A_2(T_j,T_{j+1},\alpha_{j+1})$$

$$+ \ln A_2(T_{m_2}, \tau^+, a_{m_2+1})\,]/B_2(\tau, T_{m_1}, a_{m_1}),$$

if $\tau \in (T_{m_1-1}, T_{m_1}]$, and $\tau^+ \in (T_{m_2}, T_{m_2+1}]$, here $a_{m_2+1} = 0$, $a_{m_2} = B_2(T_{m_2}, \tau^+, a_{m_2+1})$, and $\alpha_j =$

$B_2(T_j,T_{j+1},\alpha_{j+1})$, for j = m_1, …, m_2-1.

Now given the credit spread volatility term structure $\sigma_{2,j}$, j = 1, 2, …, N, we can use the same recursive procedure described above to calculate $c_{2,j}$ by solving the nonlinear equation (15). With $c_{2,j}$, we get the initial credit spread curve. Figure 2 and 3 show the impacts of different credit spread volatility curves on the initial credit spread curves. Here we use the parameter set (12). The credit spread curves A, B, and C in Figure 2 are derived from the credit spread volatility curves A, B, and C in Figure 3 correspondingly.

The model generates an upward-sloping credit spread curve A by assuming an increasing credit spread volatility curve A. This shape of the credit spread curve is applicable to an AAA-rated issuer. For this issuer, the market has more confidence in nearer term than in longer term (i.e. if something is to go wrong, it must be years away). Translating this confidence into volatility, it means the market expects less volatility in nearer term than in longer term. So the upward-sloping credit spread volatility curve is logical. The model can also generate a humped credit spread curve B by assuming a constant volatility term structure B, which is a typical credit spread curve for a BBB-rated issuer. For this issuer, the market has almost equal confidence in nearer term vs. longer term. If the issuer experiences any financial difficulty, it could be in either the nearer term or longer term. Hence a flat volatility curve makes sense. Finally, the model generates a downward-sloping credit spread curve C by assuming huge volatility in the near term and much smaller volatility in medium and long terms. This is generally true for a CCC-rated issuer. This issuer generally has a very difficult time in near term, however if it survives in near term, it will more likely improve in the long run. Therefore, the credit spread volatility is much higher in the near term than in the long term.

Now we can obviously reverse the procedure above, given the initial credit spread curve, to calculate the volatility term structure. More specifically, assume we have already arrived at the solutions up to i, $\sigma_{2,j}$, for j = 1, 2, ..., i, we want to calculate $\sigma_{2,i+1}$. With the initial credit spread curve, we know $B_d(0, T_j)$, j = 1, ..., i. From equation (16), we can calculate $c_{2,j}$, j = 1, ..., i. Now $\sigma_{2,i+1}$ is the solution to the following nonlinear equation

(18) $$B_d(0, T_{i+1}) = B(0, T_{i+1}) - D(0, T_{i+1})$$

with $c_{2,i+1}$ satisfying equation (16) (with t = 0, T = T_{i+1}), and the barrier condition (17) with $\tau \in [0,T_{i+1}]$.

We call this volatility term structure the Default Implied Volatility (DIV) term structure of credit spread. Figure 4 shows the default implied volatility curve based on the initial credit spread curve. Here we use the parameter set (12). One must notice that the DIV for credit spread is much higher than usual (riskless) interest rate vol. The DIV is in the order of 50% to 150%, while the usual interest rate vol is somewhere around 5% to 10% in the CIR world. It is not surprising that the DIV is much higher. The DIV compensates for the possibility of default (in this case, the credit spread

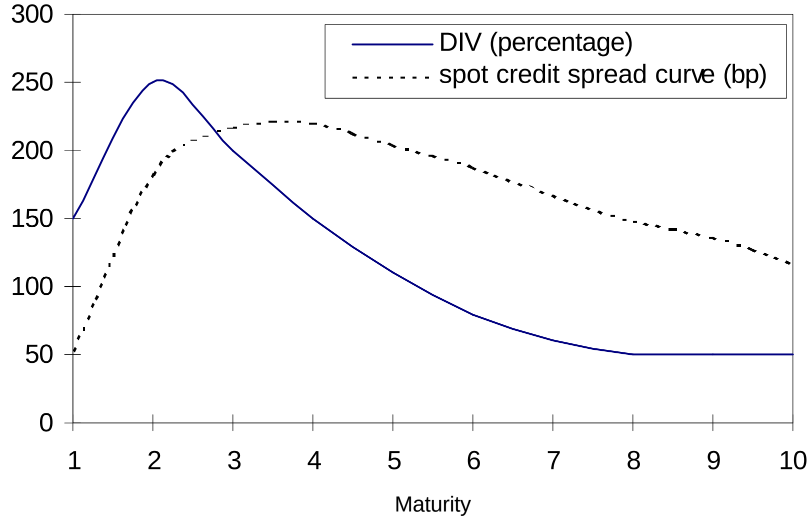

Figure 4. The Default Implied Volatility (DIV) curve for credit spread. The parameters are κ_2=0.01, θ_2=0.02, λ_2=0.001, $X_2(0)$=0.01, R=50%, r(t)=5%. The outstanding bonds at time 0 are zero coupon bonds with maturity i=1,2,…,10.

may have a big jump), the possibility of credit-rating change (in this case, the credit spread may have a smaller jump), and the general credit spread swing due to the economic condition changes.

The DIV is the volatility for the "instantaneous" short-term credit spread. To get a broader picture, let us link the DIV to the credit yield spread standard deviation. The credit yield spread Y(t, T) for the zero coupon bond Bd(t, T) is

$$Y(t, T) = -\frac{\ln(B_d(t,T)/B(t,T))}{(T-t)}$$

$$= -[-\int_t^T c_2(\tau)\, d\tau + \sum_{j=0}^{N-1} \ln A_2(T_j, T_{j+1}, \alpha_{j+1}) - B_2(t, T_1, \alpha_1)\, X_2(t)\,] / (T-t)$$

and the standard deviation is

$$\text{std}(Y(t, T)) = B_2(t, T_1, \alpha_1)\, \sigma_{2,1} \sqrt{X_2(t)} \;/ (T-t)$$

This standard deviation is plotted in Figure 5 in the case of the flat short-term credit spread volatility curve $\sigma_2 = 150\%$. Notice that the shorter the maturity of the zero coupon bond, the larger the standard deviation of the credit yield spread. This is because the shorter the maturity of the bond, given a fixed recovery rate R, the wider the credit yield spread moves if the bond defaults. For example, given the recovery rate R=50%, and constant 5% riskless interest rate, if a 10-year zero coupon bond defaults, its credit yield spread will jump to Y(0, 10) = 6.93% (from exp(-(Y(0,10)+5%)x10) = 0.5 exp(- 5% x10)). But if a 1-year zero coupon bond defaults, its credit yield spread will jump to 69.3%.

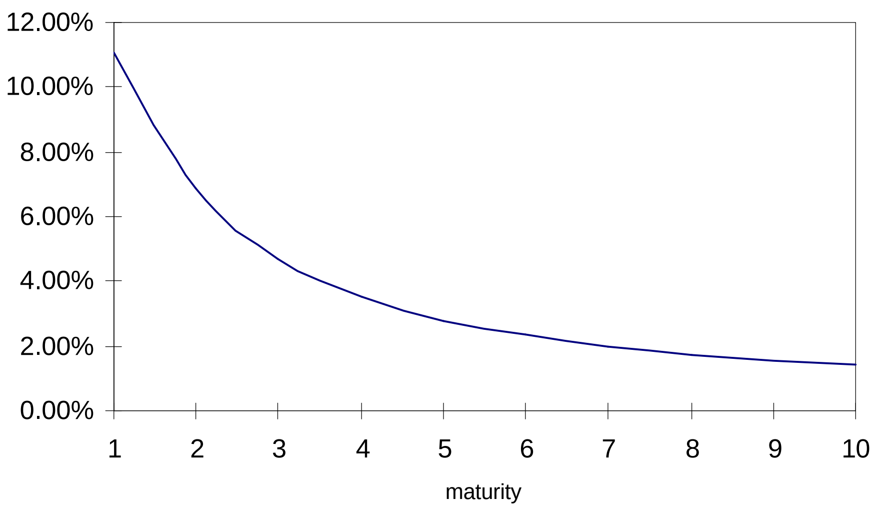

Figure 5. The credit yield spread standard deviation for zero coupon bonds. The parameters are κ_2=0.01, θ_2=0.02, λ_2=0.001, σ_2=150%, $X_2(0)$=0.01.

PRICE CREDIT DERIVATIVES

The model can easily be used to price credit derivatives consistently with the underlying defaultable bond market. In this section, we price a credit spread put option, and a first-to-default swap.

A credit spread put option on the zero coupon defaultable bond $B_d(t,T)$ is an option which pays the difference between a reference price (here we use the defaultable bond price $B_d(\tau,T; S_K)$ with its yield at the option strike) and the default value of the bond if it defaults at any time τ ($0<\tau<T^*$) before the option maturity T*; or $\max[B_d(T^*,T;S_K) - B_d(T^*,T; S(T^*)), 0]$ at the option maturity date T* if it doesn't default at [0, T*]. Here S_K is the option strike, and $S(T^*)$ is the credit spread at the option maturity date T*. We can value this credit spread put option as a barrier option. When the state variable $X_2(t)$ hits the barrier (17) (i.e. the bond defaults) at time $\tau<T^*$, it pays $B_d(\tau,T; S_K) - RB(\tau,T)$; otherwise it pays $\max[B_d(T^*,T;S_K) - B_d(T^*,T;S(T^*)), 0]$. Since the interest rate and the credit spread are assumed independent, we can take the expectation of the option payoffs over the interest rate. Thus if the option hits the barrier (17) at time τ, the present value of the expected payoff is

(19) $$E_0\{ B(0,\tau) [B_d(\tau,T; S_K) - RB(\tau,T)] \}$$

$$= E_0[B(0,\tau)B(\tau,T)] (\exp(- S_K (T-\tau)) - R)$$

$$= B(0,T) (\exp(- S_K (T-\tau)) - R)$$

otherwise, the present value of the expected payoff is

(20) $$E_0\{B(0,T^*) [B_d(T^*,T;S_K) - B_d(T^*,T;S(T^*))]^+\}$$

$$= E_0[B(0,T^*)B(T^*,T)]\,[\exp(-\,S_K\,(T-T^*))$$

$$-\exp(-\int_{T^*}^{T} c_2(u)\,du + \sum_{j=n_1}^{N-1} \ln A_2(T_j,T_{j+1},\alpha_{j+1}) - B_2(T^*,T_{n_1+1},a_{n_1+1})X_2(T^*))]^+$$

$$= B(0,T)\,[\exp(-S_K(T-T^*))$$

$$-\exp(-\int_{T^*}^{T} c_2(u)\,du + \sum_{j=n_1}^{N-1} \ln A_2(T_j,T_{j+1},\alpha_{j+1}) - B_2(T^*,T_{n_1+1},a_{n_1+1})X_2(T^*))]^+$$

here we assume $T^* = T_{n_1}$, and $T = T_N$.

We can price the barrier option with payoffs (19)-(20) by discretizing the state variable $X_2(t)$ as (14) and using Monte Carlo simulation. Table 1 shows the value of the credit spread put option.

Table 1. The value of the credit spread put options on 10-year zero coupon defaultable bond

Maturity	Strike				
	50bp OTM	25bp OTM	ATM	25bp ITM	50bp ITM
1	1.74	1.91	2.14	3.24	4.44
2	2.42	2.63	2.90	3.95	5.05
3	3.28	3.49	3.77	4.73	5.71
4	3.95	4.19	4.48	5.32	6.20
5	4.54	4.78	5.07	5.82	6.58

Here we use the parameter set (12), the credit spread vol curve B in Figure 3, and the credit spread curve B in Figure 2. The At-The-Money option is at-the-money forward on the credit spread. The option values are presented as a percentage of the notional. The maturity is in years.

Now let us look at the first-to-default swap. In a first-to-default swap (also called basket swap), there is more than one reference entity. If any of the reference entities defaults before the swap maturity date, the protection buyer will receive the difference between a reference price (here we use the corresponding riskless bond price) and the defaulted bond price. The protection buyer makes periodic (e.g. annually) payments to the protection seller up to the default date (here we assume the payments at arrears, and if default occurs, the buyer pays only the accrual for the last payment).

Let us assume there are J reference entities in the first-to-default swap. For the j-th entity, the short credit spread $s_j(t) = c_2^j(t) + X_{2j}(t)$, and $X_{2j}(t)$ follows

$$dX_{2j}(t) = (\kappa_{2j}\theta_{2j} - (\kappa_{2j}+\lambda_{2j})\, X_{2j}(t))\, dt + \sigma_{2j}(t)\sqrt{X_{2j}(t)}\, dZ_{2j}(t), \qquad j = 1, 2, ..., J \tag{21}$$

where $Z_{2j}(t)$, j = 1, 2, ..., J are standard Brownian motions, and they are independent from $Z_1(t)$. The correlation between $dZ_{2i}(t)$ and $dZ_{2j}(t)$ (with $i \neq j$) is ρ_{ij}. The recovery rate for the j-th entity is R_j. To price the first-to-default, we first use each reference entity's initial credit spread curve to derive the default implied volatility. Now assume Y to be the annualized periodic payment the buyer must pay for a first-to-default swap on T-year zero coupon bonds (assuming each entity has a T-year zero coupon bond). Let T* be the maturity of the swap, and t_i, i = 1, 2, …, P, the periodic payment dates with $T^* = t_P$. Then the cash flows of this swap deal to the buyer are $[\, B(\tau, T) - R_j\, B(\tau, T)\,] - Y(\tau - t_q)$ at time τ, and $-Y(t_i - t_{i-1})$ at time t_i, for i = 1, 2, …, q, if j-th entity first defaults at time $\tau \in (t_q, t_{q+1}]$, q<P; otherwise if there is no entity defaults at [0, T*], the cash flows are $-Y(t_i - t_{i-1})$ at time t_i, for i = 1, 2, …, P (here $t_0 = 0$). Now we can value the first-to-default swap as a barrier option, with a total of J barrier equations (17) with one for each of the reference entity. If one of the entities (e.g. the j-th) hits the barrier (17) (with j-th entity's parameters in the equation) at time $\tau \in (t_q, t_{q+1}]$, q<P, the present value of the expected cash flows of the deal over the interest rate is

$$E_0\{B(0, \tau)\, [\, B(\tau, T) - R_j\, B(\tau, T)\,] - \sum_{i=1}^{q} Y(t_i - t_{i-1})\, B(0, t_i) - Y(\tau - t_q)\, B(0, \tau)\} \tag{22}$$

$$= B(0, T)\, (1 - R_j) - \sum_{i=1}^{q} Y(t_i - t_{i-1})\, B(0, t_i) - Y(\tau - t_q)\, B(0, \tau)$$

otherwise if there is no entity hitting the barrier (17) at [0, T*], the present value of the cash flows is

(23) $$-\sum_{i=1}^{P} Y(t_i - t_{i-1})\, B(0, t_i)$$

We can price this barrier option via Monte Carlo simulation. Y is the solution which makes the value of the swap deal worth zero. Figure 6 shows the annualized premium for a two-year first-to-default swap on 10-year zero coupon bonds with two reference entities. Here we use the parameter set (12), the volatility curve B in Figure 3, and the credit spread curve B in Figure 2 for each of the entities. This graph confirms the intuition that the lower the correlation on credit spreads, the higher the probability of default on at least one of the entities; thus, the higher the protection premium the buyer must pay.

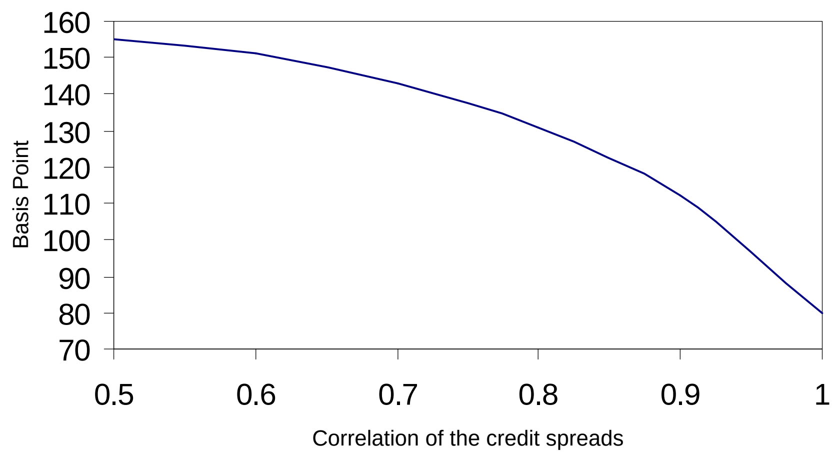

Figure 6. The annual premiums of the first-to-default swap with two entities. It is a two-year swap on 10-year zero coupon bonds. The parameters for each entity are κ_2=0.01, θ_2=0.02, λ_2=0.001, $X_2(0)$=0.01, R=50%, r(t)=5%.

CONCLUSIONS

In practice, one needs to generate a defaultable zero coupon bond curve from the market prices of the outstanding coupon bonds (Jarrow, Lando, Turnbull, 1997, give some details for this procedure). Although our discussion is focused on the zero coupon bonds, derivatives on coupon bonds can be dealt with in the same fashion. One can also easily generalize our model to correlate credit spreads with interest rates, and to make the recovery rate stochastic.

In this paper, we present a simple approach to model the term structure of credit spreads. From our framework, the following three positions:

P_1) Long the riskless bond $B(t, T)$ and short the defaultable bond $B_d(t, T)$;

P_2) Long a credit spread put option on $B_d(t, T)$;

P_3) Long a default swap on $B_d(t, T)$;

are all barrier options (a default swap is a first-to-default swap with one reference entity). These barrier options have a common barrier (17), although the payoffs are different. We believe the market values of P_1-type positions contain enough information to price the P_2- and P_3-type positions. After deriving the default implied volatility (DIV) curve for the credit spread based on the market values of P_1-type positions, the DIV is applied to price P_2 and P_3 positions consistently.

Although our model assumes the credit spread follows a continuous process, the DIV curve fully captures the possibility of spread jumps due to default or credit-rating changes. Through our volatility-based paradigm, one can consistently hedge P_2- or P_3-type positions by P_1-type positions, i.e. consistently hedge the credit derivatives by the underlying defaultable bonds. Our model simply treats the credit derivatives as some usual (i.e. riskless) exotic interest rate derivatives. The DIV is directly comparable to the historical credit spread volatility. The first-to-default swap can be priced by using the correlation matrix of the credit spreads of the reference entities. This correlation matrix, which is based on credit spreads, can be measured by historical data. It is far different from the correlation matrix of default arrival times, which is used in hazard rate based paradigms. The correlation matrix of default arrival times is very subjective, and can not be measured by historical data.

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