Optimal Shortfall Hedging of Credit Risk *

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Abstract

In this paper we examine the problem of partially hedging a given credit risk exposure. We derive hedges which satisfy certain optimality criteria: For a given investment into the hedge they minimize the remaining risk, or vice versa. This is motivated by the fact that it is a core business of financial intermediaries to carry risks, and that therefore they do not want to hedge their risks completely. In contrast to the usual mean-variance criterion, our hedging strategies try to minimize either the shortfall probability (SP) or the expected shortfall (ES). In complete markets this allows the investor to save money by hedging only part of the claim, while taking a certain (minimal) risk that the hedge does not cover the claim completely. In incomplete markets, a perfect hedge is not always available, and this methodology introduces a new way to find a hedging strategy which minimizes the shortfall risk. We apply this to a credit risk model, where default occurs at the first jump time of a Poisson process. The write-down after default is stochastic and independent of the time of default. In this stylized model we compare hedging strategies for defaultable bonds and credit default swaps which minimize either the SP (Quantile Hedging) or the ES. We consider first a complete market where the martingale measure is unique and derive explicit results. Hedging strategies for both objectives are compared. In the incomplete markets setting, we consider two situations: In the first, we assume that the default risk premium is unknown from the beginning, and therefore we have to select the worst-case martingale measure from the set of possible martingale measures. In the second, the market is complete at the beginning, but at a future time point the default risk parameter will change randomly, for example because of a rating change, and this makes the market incomplete. Strategies for both situations are developped.

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1 Introduction

The last years have seen a tremendous increase in theoretical models for the valuation of credit risky securities. These were applied with some success to the valuation of defaultable bonds as well as options on credit risk, such as credit default swaps. Compared to this progress the analysis of questions related to the measurement of credit risk itself and hedging strategies for credit risky securities has fallen behind. In this paper we want to focus on new ways to measure credit risk, such as the Shortfall Probability (SP) related to Value-at-Risk and the Expected Shortfall (ES).

Our paper is based on existing credit risk models, of which we will now give a short account. The theory of the valuation of credit risky securities is divided into two branches. While the classical approach uses the value of the firm to determine the time of default, newer papers model the intensity of a jump-process, and the time of its first jump is the time of default.

The basis for all classical models can be found in the papers of Black and Scholes [1973] and Merton [1974]. They view a corporate bond as a derivative on the underlying firm value, so that classical option pricing formulae can be used to value those bonds. Merton-based models may be criticized on two points: First, due to the structure of the model, the time of default is predictable. Second, with few exceptions, these models describe complete markets. This implies that it is theoretically possible to hedge all risks by using a dynamic trading strategy. In reality this is not the case. Even newer models of this class, such as Shimko et al. [1993], Longstaff and Schwartz [1994] and Duffie and Lando [1998] do not recognize this problem.

In contrast to classical models, the time of default in intensity models is not determined via the value of the firm, but it is the first jump of a point process (for example, a Poisson process). Important examples of this way of modelling default risk are the works of Jarrow and Turnbull [1995] and Duffie and Singleton [1997]. In these models, the time of default is totally inaccessible and markets are typically incomplete. The incompleteness leads to the non-uniqueness of the martingale (pricing-) measure, and therefore prices of securities and options are no longer uniquely defined. However, this problem is not discussed in these models, but the existence of a certain measure is assumed which is used to value all cash-flows.

Finally, none of the above-mentioned credit risk models considers the problem of partially hedging the risk from default, while bearing the other part. However, risk bearing is an important business for the banking sector. In our paper, we apply two concepts of measuring risk to a simple intensity model of credit risk, where the write-down after default is stochastic and independent of the time of default. We will determine the optimal hedging strategies for both concepts and compare the results.

The classical way of measuring risk is through the mean and variance of a given portfolio or contingent claim. In recent years, a new approach was developed as an alternative to the measurement of risk via mean and variance: The so-called Value-at-Risk of a contingent claim is the worst possible loss which can occur with a given probability (for example 99%). This measure is widely used in official capital adequacy requirements for banks, but also for internal risk-management purposes. Tha VaR is a static concept which does not take into account that it is possible to use a dynamically rebalanced trading strategy to hedge against the risk. Föllmer and Leukert [1997] generalize the concept of VaR to a dynamic setting and characterize in a continuous time setup the associated hedging strategy: For a given initial investment, the payoff from this strategy will have the maximum probability of being equal to or greater than a predetermined payoff obligation. With other words, this strategy minimizes the Shortfall Probability (SP) of the hedge.

The latest development in this area are risk measures which are defined via certain desirable axioms, the so-called coherent risk measures (cf. Artzner et al. [1998]). One of these axioms says, for example, that the total risk of two contingent claims should not be greater than the sum of their individual risks, which is intuitive when one considers the case that the one contingent claim is used to hedge the risk of the other. However, the VaR does not possess this property, so that the VaR is not a coherent risk measure.

Alternative coherent risk measures are proposed by Artzner et al. [1998]. Although the Expected Shortfall (ES) is not a coherent risk measure, it possesses similar properties to those required for this class and is superior to the SP. Föllmer and Leukert [1998] show how to calculate a hedging strategy which minimizes the ES for a given initial investment.

In a complete market, these hedging strategies have the following interpretation: They allow the calculation of how much risk (in the sense of SP or ES) one has to take if one does not want to invest the full price of the contingent claim into the hedging strategy. Alternatively, the investor can fix the risk he wants to bear (in terms of SP or ES) and find the cheapest hedging strategy which leaves him with only so much risk.

Föllmer and Leukert [1997,1998] apply their methodology to the problem of hedging a call option on an underlying stock. The results obtained there can be applied directly to a credit risk model of the Merton [1974] type: here the firm is financed by debt and equity, the equity corresponds to a call option on the underlying firm value and the value of debt is obtained as the difference between firm value and value of equity. Without going into a full discussion of advantages and disadvantages of firm value versus intensity models, it seems to us that the latter class is more accepted in literature and industry, because in these models the time of default is unpredictable and they can be fitted quite easily to an observed term structure of credit risky bonds.

The same methods can be applied in an incomplete markets setup, where the martingale measure is not unique and a perfect hedge does not exist for every contingent claim.

The question of risk-minimization in an incomplete market was first discussed by Föllmer and Sondermann [1986]. They introduced the notions of local and global variance minimization. Further progress in this direction was made, for example, by Föllmer and Schweizer [1990] and Schweizer [1990,1991,1993] and, recently, by Gourieroux, Laurent and Pham [1998]. These concepts try to find a trading strategy which has the same expected payoff as the contingent claim and which minimizes the variance of the difference (the so-called hedge error). The concept of local risk-minimization was applied to credit risky securities by Lotz [1997]. In this setup, the strategies minimizing the SP or ES offer an alternative which makes use of more recently developped risk measures.

In the present paper we will apply the mathematical results from Föllmer and Leukert [1997,1998] to a credit risk setting of the intensity class. We start in a complete market to explain the methodology, calculate some introductory examples and compare this with other hedging strategies. Our standard example will be the optimal hedge of the (random) loss from a defaultable bond. This is important for the investor who has invested in this bond or for a bank which has sold credit protection to its customers (for example in the form of a credit default swap). Then we go on to the incomplete market case. Here we consider two situations: In the first case, the default risk premium is unknown from the beginning, and the criterion of minimizing shortfall probability or expected shortfall leads to the so-called worst-case martingale measure under which the hedging strategy is calculated. In the second case the risk-premium is known, but at some intermediate time the default risk will change randomly, for example because of a rating change. We will derive optimal hedging strategies and show the relationship between initial investment and shortfall probability or expected shortfall.

The next section explains our basic mathematical setup and summarizes some key results from Föllmer and Leukert [1997, 1998]. In section 3, we apply these results to the hedging of credit risky securities in a complete market and give general results as well as some explicit examples. In section 4 we consider the problem of shortfall hedging in an incomplete market. Section 5 concludes.

2 Mathematical Setup

This section outlines the general principles of our model. First, the basic process generating the risks in our market is described. Then, the approach of Föllmer and Leukert [1997,1998] is explained.

2.1 The Market

Instead of constructing a credit risk model of utmost generality, we want to focus on the issue of partial hedging. For this reason, we intentionally restrict ourselves to a rather simple credit risk model and also set interest rates to zero. It is straightforward to combine the model with an interest rate term structure of ones choice by simply going over to the appropriate forward measure.

We consider a continuous trading economy with trading interval $[0, T]$ for a fixed $T > 0$. The uncertainty in our model is specified by a probability space $(\Omega, \mathcal{F}, P)$. We assume that this probability space is big enough to hold a marked one-jump Poisson process defined as follows:

2.1.1 Definition *A marked one-jump Poisson process on a mark space $\mathbb{E}$ is a pair $(N(t), Z)$, where*

- *$N(t)$ is a Poisson process with intensity λ, and*
- *Z is a $\mathbb{E}$-valued random variable which is drawn at the first jump time τ of the Poisson process.*

For further information on marked Poisson or point processes, see Brémaud [1981] or Last and Brand [1997]. In our case, we will take $\mathbb{E} \subset \mathbb{R}^+$. The random variable Z has a density f on $\mathbb{E}$ under P, which may depend on the time of the jump, $f = f(\tau, \cdot)$. Generated by the marked Poisson process is a complete, right-continuous filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$. The probability measure P is called the objective or historical probability.

The interpretation of this setup is the following: The first jump of the point process determines the time of default τ. The mark variable Z will influence the payoff of the credit-risky security after default. For illustration purposes, we want to present some special cases which can be treated with the above specification of the marked Poisson process driving default:

1) The mark space consists of n discrete points, $\mathbb{E} = \{z_1, \ldots, z_n\}$. Each point corresponds to a payout pattern of bonds of different seniority classes. For example, we could have $n = 2$, a high payout Δ_H and a low payout Δ_L as well as a senior bond B_S and a junior bond B_J, and the following payout pattern:

State	B_S	B_J	Probability
z_1	Δ_H	Δ_L	$f(z_1)$
z_2	Δ_L	Δ_L	$f(z_2)$

In that model, the senior bond may recover the higher or lower payout after default, while the junior bond will always recover only the low payout.

2) Another possibility is to take $\mathbb{E} = [0,1]$ and use the Beta distribution with parameters c and d as the distribution of payout ratios. The parameters c and d could also depend on the default time τ, thus introducing a correlation between default time and payout ratio.

The assumption captures most important features of a typical credit risk model, such as unpredictable default times and random payoff after default. We will introduce more specific assumptions later on which allow us to compute most results explicitly. The next step of generalization would be to use a jump process with time-inhomogeneous, deterministic intensity. All results from this paper can be easily rewritten for this additional complexity, but results can then only be obtained by numerical methods.

To each set $A \subset \mathbb{E}$ corresponds a jump process $N(t, A)$ via

$$N(t, A) = N(t) 1_{\{Z \in A\}}$$

The processes $\bar{N}(t, A)$ and $\bar{N}(t)$, defined by

$$\bar{N}(t, A) := N(t, A) - \lambda \int_0^t \int_A f(s, z)\, dz\, ds; \qquad \bar{N}(t) := N(t) - \lambda t$$

are martingales (for a proof see Brémaud [1981]).

2.1.2 Assumption *The set of possible equivalent martingale measures is given by*

$$\mathbb{P} = \left\{ P^* : \mathcal{G} := \frac{dP^*}{dP} = \mathcal{E}\left\{ \int_0^T \int_{\mathbb{E}} (\mu h(s,z) - 1) f(s,z)\, dz\, d\bar{N}(s) \right\}; \mu \in \mathbb{R}^+ \backslash \{0\}; \right.$$
$$\left. h(s, \cdot) > 0, \mathbb{E} - \textit{measurable}, \int_{\mathbb{E}} f(s,z) h(s,z)\, dz = 1 \quad \forall s \in [0, T] \right\}$$

Therefore, under the equivalent martingale measure P^* the Poisson process has the intensity $\lambda^* = \lambda\mu$, and the mark variable Z has the distribution $f^* = fh$.

The transition density h corresponds to the risk premium for the payout ratio after default and may depend on the time of default. For the two specifications of the mark space $\mathbb{E}$ given above, h looks as follows:

1) When the mark space $\mathbb{E}$ consists of 2 discrete points as above, h can be described by a pair of real numbers $(h(z_1), h(z_2))$ modifying the probabilities of both events, so that under the equivalent measure the probabilities are $(f(z_1)h(z_1), f(z_2)h(z_2))$. For a risk averse investor, we would assume $h(z_1) < 1$ and $h(z_2) > 1$, so that claims are valued as if the high payoff is less likely than under the historical or objective measure.

2) On the other hand, we could assume that $\mathbb{E} = [0,1]$ and f and f^* are Beta distributions with parameters c, d and c^*, d^* respectively. Then for a risk averse investor, the transition density h would be such that $d^* > d$ and/or $c^* < c$, because both lowers the expected value of the payout.

The parameter μ in the change of measure corresponds to the change of default intensity or probability of a default. Because the new default intensity under the martingale measure is given by $\lambda^* = \lambda\mu$, we will have $\mu > 1$ for a risk-averse investor. We assume μ to be constant over time. This imposes implicit structure on the risk premia for default in the economy, which must then also be constant over time. This assumption can easily be relaxed in favor of time-dependent risk premia, but this would only make notation more complicated without adding significant new results.

2.1.3 Lemma *The density for the change of measure can be written as*

$$\mathcal{G} = \mu h(\tau, \cdot) e^{-\lambda(\mu-1)\tau} 1_{\{\tau \leq T\}} + e^{-\lambda(\mu-1)T} 1_{\{\tau > T\}}$$

Proof Follows directly from the definition of $\mathcal{G}$.

2.2 Shortfall Hedging with the Neyman-Pearson Lemma

The mathematical techniques used in sections 3 and 4 come from Föllmer and Leukert [1997,1998]. Both papers deal with the problems of shortfall hedging and use the Neyman-Pearson Lemma to determine the optimal hedging strategies. While Föllmer and Leukert [1997] focus on the Shortfall Probability (SP) and show how to minimize this probability (so-called quantile hedging), the later paper Föllmer and Leukert [1998] is concerned with more general risk measures, from which we will use the Expected Shortfall (ES). We start with an explanation of these risk measures. After that, we review the methods of Föllmer and Leukert [1997,1998] which allow us to find the optimal solution to the hedging problems.

The technique of quantile hedging was introduced in Föllmer and Leukert [1997]. It is closely related to the concept of Value-at-Risk (VaR). We assume here that the market is complete so that a perfect hedge is available for every contingent claim at its arbitrage-free, uniquely defined price and that the martingale measure is denoted by P^*. Consider a contingent claim G with initial price G_0^*, which should be viewed as a payment obligation due at time T. Recall the definition of Value-at-Risk as given in Artzner et al. [1998]:

$$VaR_\epsilon(G) = \inf\{c : P[G \geq c] \leq \epsilon\}$$

where x is a constant real number. In other words, the VaR is that boundary so that the final payout obligation G is greater than that number only with probability

ϵ. The concept of Quantile Hedging can be seen as a generalization of the VaR to a stochastic boundary. Here we consider the problem: Find a contingent claim H, the hedge against G, with initial price H_0^* so that the SP is smaller than or equal to ϵ,

$$\min H_0^* \text{ so that } P[G \geq H] \leq \epsilon \tag{1}$$

The Value-at-Risk for a certain probability ϵ is widely used in the banking industry for a variety of purposes. However, it has several shortcomings. For example, the investor above is only concerned with the probability of a loss, but not with the size of the loss. Furthermore, as shown by Albanese [1997], it is possible that the VaR increases after diversification of risks, which is counterintuitive. Consider an investment opportunity which yields 1 in 99% of all cases, but with 1% probability the investor loses 10. Because G above is a payment obligation, we have $G = -1$ or $G = 10$. Consequently, the 1% VaR of G is -1, which means that there is no risk at the 1% level. Now, instead ten independent projects are undertaken, which yield a gain of 0.1 or or a loss of -1 each, with the same probabilities as above. Again, this translates into values for G_i, $i = 1, \ldots, 10$ of $G_i = -0.1$ or $G_i = 1$. In this situation, the 1% VaR is 0.1. Thus, in spite of the diversification of risk, the VaR has increased.

Driven by this analysis, we also look at hedging strategies which minimize the ES. Here we follow Föllmer and Leukert [1998]. The ES of the obligation G over a constant amount c is defined as

$$E[(G - c)^+]$$

It is easy to see that this risk measure can be reduced through diversification of risks and is therefore superior to the SP (or VaR): For this purpose, consider one large investment G on the one hand, and n small investments of size $1/n$, $G_1, \ldots, G_n$, on the other hand. The $G_i, i = 1, \ldots, n$ are distributed as G/n. Now, we have

$$\begin{aligned} E\left[\left(\sum_{i=1}^{n} G_i - c\right)^+\right] &\leq E\left[\sum_{i=1}^{n}\left(G_i - \frac{c}{n}\right)^+\right] \\ &= \sum_{i=1}^{n} E\left[\left(G_i - \frac{c}{n}\right)^+\right] \\ &= nE\left[\left(\frac{G}{n} - \frac{c}{n}\right)^+\right] \\ &= E\left[(G - c)^+\right] \end{aligned}$$

So, the ES of investing into the diversified projects $G_1, \ldots, G_n$ is smaller than or equal to the risk of investing into the big project G.

The generalization of this risk measure to a stochastic boundary is straightforward, and here we consider the problem: Find a contingent claim H with initial price H_0^*, so that the ES is smaller than or equal to η, formally

$$\min H_0^* \text{ so that } E[(G-H)^+] \leq \eta \tag{2}$$

In both cases (SP and ES), it is also useful to consider the equivalent dual problem: Given an initial amount of money V_0 for the hedge, find a contingent claim H with initial price $H_0^* \leq V_0 < G_0^*$ so that the SP or the ES is minimized, formally

$$\min P[G \geq H] \text{ so that } H_0^* \leq V_0 \tag{3}$$
$$\text{or } \min E[(G-H)^+] \text{ so that } H_0^* \leq V_0 \tag{4}$$

We will mostly consider the problems in this form, because then it is much easier to compare the results.

The problems described above are solved in two steps: First, the optimal contingent claim (hedge) H has to be found. Second, a trading strategy has to be constructed which generates the hedge.

The optimal hedges H are described in Föllmer and Leukert [1997,1998]. In general, H is given by a modification of the payment obligation G, $H = \varphi G$, where φ is a random variable from the set

$$\mathcal{R} = \{\varphi : \Omega \to [0,1], \varphi \quad \mathcal{F}_T - \text{measurable}\}$$

φ is called the success ratio and its shape depends on the problem at hand. The optimal success ratio is known explicitly in a complete market, where the set of equivalent martingale measures consists of only one element, $\mathbb{P} = P^*$. In the case where we want to minimize the SP with a given investment V_0, the optimal success ratio $\hat{\varphi}^Q$ is given by

$$\hat{\varphi}^Q = 1_{\mathcal{A}^Q} + \gamma 1_{\{\frac{dP}{dP^*} = \tilde{a}G\}}$$

with

$$\mathcal{A}^Q = \left\{\frac{dP}{dP^*} > \tilde{a}G\right\}; \quad \tilde{a} = \inf\left\{a : E^*[G1_{\{\frac{dP}{dP^*} > aG\}}] \leq V_0\right\}; \quad \gamma = \frac{V_0 - E^*[G1_{\{\frac{dP}{dP^*} > \tilde{a}G\}}]}{E^*[G1_{\{\frac{dP}{dP^*} = \tilde{a}G\}}]}$$

$\mathcal{A}^Q$ is called the success set of the hedge.

On the other hand, if we want to minimize the ES with a given investment V_0, the optimal success ratio $\hat{\varphi}^{ES}$ is given by

$$\hat{\varphi}^{ES} = 1_{\mathcal{A}^{ES}} + \gamma 1_{\{\frac{dP}{dP^*} = \tilde{a}\}}$$

with

$$\mathcal{A}^{ES} = \left\{ \frac{dP}{dP^*} > \tilde{a} \right\}; \quad \tilde{a} = \inf \left\{ a : E^*[G 1_{\{\frac{dP}{dP^*} > a\}}] \le V_0 \right\}; \quad \gamma = \frac{V_0 - E^*[G 1_{\{\frac{dP}{dP^*} > \tilde{a}\}}]}{E^*[G 1_{\{\frac{dP}{dP^*} = \tilde{a}\}}]}$$

In section 3, we will use these results to compute the optimal hedge for credit risky securities in a complete market. In an incomplete market, the situation is different and more complicated. The problems (3) and (4) translate into

$$\min P[G \ge H] \text{ so that } E^*[H] \le V_0 \quad \forall P^* \in \mathbb{P} \tag{5}$$

$$\text{or } \min E[(G - H)^+] \text{ so that } E^*[H] \le V_0 \quad \forall P^* \in \mathbb{P} \tag{6}$$

Here, under certain conditions, the following form of $\hat{\varphi}$ is necessary for optimality (see Föllmer and Leukert [1997]). There exist measures $\hat{P}^Q \in \mathbb{P}$ and $\hat{P}^{ES} \in \mathbb{P}$ such that the optimal success ratios fulfill

$$\varphi^Q = \begin{cases} 1 & \text{if } dP > \hat{a}^Q H d\hat{P}^Q \\ 0 & \text{if } dP < \hat{a}^Q H d\hat{P}^Q \end{cases}; \quad \hat{E}^Q[H\hat{\varphi}^Q] = V_0$$

$$\varphi^{ES} = \begin{cases} 1 & \text{if } dP > \hat{a}^{ES} d\hat{P}^{ES} \\ 0 & \text{if } dP < \hat{a}^{ES} d\hat{P}^{ES} \end{cases}; \quad \hat{E}^Q[H\hat{\varphi}^{ES}] = V_0$$

In section 4, we use this to find the optimal success set for a credit risk security in an incomplete markets setup.

After the construction of the optimal claim H used to hedge against the payment obligation G, one can proceed to generate the contingent claim H by a dynamic trading strategy. In a complete market this is straightforward, because here by definition for every contingent claim there exists a duplicating dynamic trading strategy. In an incomplete market, the results of Kramkov [1996] and Föllmer and Kabanov [1998] can be used. Define

$$H(t) = H_0 \text{ ess.sup }_{\tilde{P} \in \mathbb{P}} \tilde{E}[H | \mathcal{F}_t]$$

This process admits an optional decomposition

$$H(t) = H(0) + \int_0^t \xi(s)\, dX(s) - C(t),$$

where C is an increasing optional process and ξ is an admissible strategy. C can be interpreted as amount which can be withdrawn from the trading strategy while more and more information becomes available.

In credit risk markets, such a dynamic hedging strategy is unlikely to be successful because of the illiquidity of the market. On the other hand, most credit risk derivatives are traded in an over-the-counter fashion, so that the exact parameters needed by the counterparties can be entered into the contracts. For these two reasons, we do not consider step 2 (replication of the optimal hedge) in this paper, but are satisfied with the construction of the optimal hedge (step 1).

3 Complete Markets

In this section we will apply the results from Föllmer and Leukert [1997,1998] to a credit risk setting with a complete market. Therefore, regarding the risk-neutral probability measure P^*, we make the following

3.0.1 Assumption *There exists a unique martingale measure P^* equivalent to P, and its density is given by*

$$\frac{dP^*}{dP} = \mathcal{E}\left\{\int_0^T \int_E (\mu h(s,z) - 1) f(s,z)\, dz\, d\bar{N}(s)\right\}$$

For the two specifications of our mark space, we discuss the requirements on traded assets which make the market complete:

1) For the case when the mark space $\mathbb{E}$ consists of two points, three assets have to be traded in order to guarantee completeness of the market: These are a non-defaultable bond, a senior bond and a junior bond.
2) On the other hand, if the mark space is a continuum, for example $\mathbb{E} = [0,1]$, then infinitely many securities are needed for the market to be complete. This is analogous to the situation in Björk et al. [1997]. Although this is a strong assumption on the market, it is nevertheless often used in the credit risk literature, see for example Schönbucher [1996,1997].

In this probabilistic setting, we consider a contingent claim G, which we view as a payment obligation due at date T, of the following form:

3.0.2 Assumption

$$G := C1_{\{\tau > T\}} + D1_{\{\tau \leq T\}}$$

where

- *$C \geq 0$ is a constant and*

- *$D \geq 0$ is a random variable with distribution $f^* = fh$ under P^* on the mark space $\mathbb{E}$.*

3.0.3 Remark The specification is quite general and allows G to be interpreted as one of several securities associated to credit risk:

- For $C = 1$ and $D = \Delta$ this is a *credit risky bond* of face value 1, where Δ is the payoff ratio of the bond in the case of default. For example, a corporation which wants to sell its bonds to an insurance company could be required to package the bonds with some kind of default insurance by a third party so that the overall shortfall risk meets regulatory requirements for the insurance company.

- for $C = 0$ and $D = (1 - \Delta)$, this is a *credit default swap* which protects against the default of the credit risky bond above. Here, we could imagine a bank which has sold credit protection against default to its customers and now wants to reinsure itself.

3.1 The Optimal Shortfall Hedge for a given Initial Investment

Föllmer and Leukert [1997,1998] deal with the hedging of a call option in a Black-Scholes world with Brownian motions. In contrast, in our setting we have much less price dynamics: The only time there is a random price change is at the time of default. Therefore, a dynamic hedging strategy loses much of its appeal in our setting. Instead, we focus on the optimal contingent claim H for the hedging problem, which will be a contract specifying a certain payout depending on the time of default τ and payout ratio Δ. Another argument for this treatment is that credit derivatives are traded in an over the counter market and standardized contracts have not yet emerged, so that there is the possibility to specify the contracts according to the needs of the counterparties. Therefore, we consider the following

3.1.1 Problem *Given an initial amount of capital V_0, find a contingent claim with payout H^Q at time T and price $H_0^Q \leq V_0$ so that the probability of a successful hedge, $P[H^Q \geq G]$ is maximal. Following Föllmer and Leukert, we call this the problem of quantile hedging.*
Alternatively, we can search for a contingent H^{ES} so that the expected shortfall $E[(G - H^{ES})^+]$ is minimal.

It turns out that the optimal contingent claims H are given by a modification of the payment obligation φG, where φ is the so-called success ratio which can be determined explicitly. The following theorem provides the optimal success ratio for both problems:

3.1.2 Theorem *For $C \neq 0$, there are two critical values of initial investment V_1, V_2 for the Quantile and Expected Shortfall case with*

$$V_1^Q = \int D1_{\{Da_K^Q \mu h e^{-\tau(\mu-1)\lambda} < 1\}}\, dP^* \qquad V_1^{ES} = \int D1_{\{a_K^{ES} \mu h e^{-\tau(\mu-1)\lambda} < 1\}}\, dP^*$$
$$V_2^Q = V_1^Q + CP^*[\tau > T] \qquad V_2^{ES} = V_1^{ES} + CP^*[\tau > T]$$

where the constants a_K^Q and a_K^{ES} are given by

$$a_K^Q = \frac{1}{C} e^{T(\mu-1)\lambda} \qquad a_K^{ES} = e^{T(\mu-1)\lambda}$$

Depending on the initial investment V_0, we have the following three cases:

1) $V_0 < V_1$: Here the optimal success ratio reduces to a success set given by

$\mathcal{A}_1^Q = \{Da_1^Q \mu h e^{-\tau(\mu-1)\lambda} < 1\} \cap \{\tau \leq T\}$ *for the Quantile case and*
$\mathcal{A}_1^{ES} = \{a_1^{ES} \mu h e^{-\tau(\mu-1)\lambda} < 1\} \cap \{\tau \leq T\}$ *for the Expected Shortfall case.*

In both cases, a_1 is the solution to

$$V_0 = \int D1_{\mathcal{A}_1}\, dP^*$$

and we have $a_1 > a_K$.

2) $V_0 \in [V_1; V_2]$: Define

$$\mathcal{A}_2^Q := \{a_2^Q D \mu h e^{-\tau(\mu-1)\lambda} < 1\} \cap \{\tau \leq T\}$$
$$\mathcal{A}_2^{ES} := \{a_2^{ES} \mu h e^{-\tau(\mu-1)\lambda} < 1\} \cap \{\tau \leq T\}$$

In both cases, the optimal success ratio is given by

$$\varphi = 1_{\mathcal{A}_2} + \frac{V_0 - V_1}{CP^*[\tau > T]} 1_{\{\tau > T\}}$$

and $a_2 = a_K$.

3) $V_0 > V_2$: Again, the optimal success ratio reduces to a success set given by

$$\mathcal{A}_3^Q = \{\tau > T\} \cup (\{a_3^Q D\mu h e^{-\tau(\mu-1)\lambda} < 1\} \cap \{\tau \leq T\})$$
$$\mathcal{A}_3^{ES} = \{\tau > T\} \cup (\{a_3^{ES} \mu h e^{-\tau(\mu-1)\lambda} < 1\} \cap \{\tau \leq T\})$$

and in both cases a_3 is the solution to

$$V_0 = \int H 1_{\mathcal{A}_3} \, dP^*$$

For $C = 0$, the critical values of investment do not exist and the solution reduces to case 3) above.

Proof: See appendix.

3.1.3 Remark The randomization which occurs in the theorem above if the capital invested into the hedge lies in the interval $[V_1, V_2]$ is difficult to interpret in economic terms: It corresponds to making the payout of the hedge dependent on a probabilistic event which is completely independent from the payout obligation G, such as a throwing of dice. It is extremely unlikely that a contract which involves such a randomization will be traded in reality, so typically investors will use a hedge H with a price below V_1 or above V_2.

3.2 The Optimal Hedge for a given Shortfall Probability or Expected Shortfall

Conversely, it is also possible to start with a given shortfall probability ϵ and find the minimal required initial amount of capital. In other words, we have the

3.2.1 Problem *Given a probability of a successful hedge $1 - \epsilon$, find a contingent claim with payout H^Q at time T and with minimal price H_0^Q at time zero so that the probability of a successful hedge,*

$$P[H^Q \geq G] = 1 - \epsilon$$

In the following, we define by G_0^* the unique price of the payment obligation G at time 0. The theorem which provides a solution to this problem is similar to theorem 3.1.2:

3.2.2 Theorem *For $C \neq 0$, there are two critical values ϵ_1, ϵ_2 for the shortfall probability ϵ,*

$$\epsilon_1 = P\left[\frac{D\mu h}{b_K^Q G_0^*} e^{-\tau(\mu-1)\lambda} > 1\right]; \qquad \epsilon_2 = \epsilon_1 + P[\tau > T]$$

with

$$b_K^Q = \frac{C}{G_0^*} e^{-\lambda(\mu-1)T} = \frac{1}{G_0^*}\frac{1}{a_K^Q}$$

We have the following three cases:

1) $\epsilon < \epsilon_1$: Here the optimal success ratio reduces to a success set given by

$$\mathcal{B}_1^Q = \{\tau > T\} \cup (\{\frac{D\mu h}{b_1^Q G_0^*} e^{-\tau(\mu-1)\lambda} < 1\} \cap \{\tau \leq T\})$$

where b_1^Q is the solution to

$$1 - \epsilon = P[\mathcal{B}_1^Q]$$

2) $\epsilon \in [\epsilon_1; \epsilon_2]$: The optimal success ratio is given by

$$\mathcal{B}_2^Q = \left\{\frac{D\mu h}{b_2^Q G_0^*} e^{-\lambda(\mu-1)\tau} < 1\right\} \cap \{\tau \leq T\}$$

$$\hat{\psi}^Q = 1_{\mathcal{B}_2^Q} + \frac{\epsilon_2 - \epsilon}{P[\tau > T]} 1_{\{\tau > T\}}$$

We have $b_2^Q = b_K^Q$.

3) $\epsilon > \epsilon_2$: Again, the optimal success ratio reduces to a success set given by

$$\mathcal{B}_3^Q = (\{\frac{D\mu h}{b_3^Q G_0^*} e^{-\lambda(\mu-1)\tau} < 1\} \cap \{\tau \leq T\})$$

and b_3^Q is the solution to

$$1 - \epsilon = P[\mathcal{B}_3^Q]$$

For $C = 0$, the critical values for the shortfall probability do not exist, and the solution reduces to case 1) above.

Proof: See appendix.

Instead of a shortfall probability ϵ we can also use a given expected shortfall η and find the minimal required initial amount of capital. In other words, we consider the

3.2.3 Problem *Given an expected shortfall of a successful hedge η, find a contingent claim with payout H^{ES} at time T and minimal initial price H_0^{ES} at time 0 so that*

$$E[(G - H^{ES})^+] = \eta$$

Again, following Föllmer and Leukert, the solution in our case is given by a success ratio ψ^{ES} with

$$\psi^{ES} = 1_{\{\frac{dP}{dP^*} > \frac{1}{b^{ES}}\}} + \gamma 1_{\{\frac{dP}{dP^*} = \frac{1}{b^{ES}}\}}$$

and again there exists a critical value for b^{ES}, namely

$$b_K^{ES} = \frac{C}{G_0^*} e^{-\lambda(\mu-1)T} = \frac{1}{G_0^*} \frac{1}{a_K^{ES}}$$

The theorem which provides a solution to this problem is similar to theorem 3.1.2:

3.2.4 Theorem *For $C \neq 0$, there are two critical values η_1, η_2 for the expected shortfall η,*

$$\eta_1 = \int D 1_{\{\frac{1}{b_K^{ES}} \mu h e^{-\lambda(\mu-1)\tau} < 1\}} \, dP^*; \qquad \eta_2 = \eta_1 + C P^*[\tau > T]$$

where

$$b_K^{ES} = \frac{C}{G_0^*} e^{-\lambda(\mu-1)T} = \frac{1}{G_0^*} \frac{1}{a_K^{ES}}$$

We have the following three cases:

1) $\eta < \eta_1$: Here the optimal success ratio reduces to a success set given by

$$\mathcal{B}_1^{ES} = \{\tau > T\} \cup \left(\left\{ \frac{\mu h}{b_1^{ES}} e^{-\lambda(\mu-1)\tau} < 1 \right\} \cap \{\tau \leq T\} \right)$$

where b_1^{ES} is the solution to

$$\eta = E[H(1 - 1_{\mathcal{B}_1^{ES}})]$$

2) $\eta \in [\eta_1; \eta_2]$: The optimal success ratio is given by

$$\mathcal{B}_2^{ES} = \left\{ \frac{\mu h}{b_2^{ES}} e^{-\tau(\mu-1)\lambda} < 1 \right\} \cap \{\tau \le T\}$$
$$\hat{\psi}^{ES} = 1_{\mathcal{B}_2^{ES}} + \frac{\eta_2 - \eta}{CP^*[\tau > T]} 1_{\{\tau > T\}}$$

We have $b_2^{ES} = b_K^{ES}$.

3) $\eta > \eta_2$: Again, the optimal success ratio reduces to a success set given by

$$\mathcal{B}_3^{ES} = (\{\frac{\mu h}{b_3^{ES}} e^{-\lambda(\mu-1)\tau} < 1\} \cap \{\tau \le T\})$$

and b_3^{ES} is the solution to

$$\eta = E[H(1 - 1_{\mathcal{B}_3^{ES}})]$$

For $C = 0$, the critical values for the expected shortfall do not exist, and the solution reduces to case 1) above.

Proof: See appendix.

3.3 Example

In order to provide some explicit results, in this section we consider a specific example and make the following additional assumptions:

3.3.1 Assumption

- *$\mu > 1$: This means that under the risk-neutral measure P^*, the jump intensity is higher ($\mu\lambda > \lambda$) than under the objective probability, and therefore corresponds to a risk-averse investor.*
- *$f = f^* = 1_{[0,1]}$; $h \equiv 1$: In the case of a jump, the payoff of the contingent claim is uniformly distributed between zero and one, under the objective as well as under the risk-neutral measure. This assumption implies that there is a zero risk premium attached to the payout ratio. It simplifies the calculations, while the results can be easily adjusted for the case of a non-zero risk premium.*
- *$C = 1$: In the event of no jump, the contingent claim G pays out 1 at maturity.*

With these parameter settings, the contingent claim can be interpreted as a defaultable bond. Default occurs when the Poisson process jumps, in which case the bond pays out something between zero and one. If no default occurs, the bond is worth one at maturity T. Its price at time zero is

$$E^*[C1_{\{\tau > T\}} + D1_{\{\tau \leq T\}}] = \frac{1}{2}(1 + e^{-\lambda\mu T})$$

The result will be a hedge for a defaultable bond. This is potentially useful for the enhancement of the risk structure of the defaultable bond which might be required by the investors due to regulatory rules (see remark 3.0.3). Moreover, preliminary analysis shows that the following theorems will be important for the incomplete case, and they also provide some interesting insight into the structure of shortfall hedging. Finally, once the results for the credit risky bond are established, minor modifications yield the results for the credit default swap.

Let us consider first the strategy which minimizes the shortfall probability.

3.3.2 Theorem *The critical values are given by*

$$a_K^Q = e^{\lambda(\mu-1)T}; \quad \mu \neq 2 : \bar{V}_1^Q = \frac{e^{-2\lambda(\mu-1)T} - e^{-3\lambda\mu T}}{2\mu(2-\mu)}; \quad \mu = 2 : \bar{V}_1^Q = \frac{\lambda T}{4} e^{-2\lambda T};$$
$$\bar{V}_2^Q = \bar{V}_1^Q + e^{-\lambda\mu T}$$

As above for $\bar{V}_1^Q$, in the following formula there will occur apparent poles which can be closed by using l'Hôspital's rule. Depending on the initial investment V_0, the following list contains the success set as well as probability and expected size of the shortfall:

1) $V_0 < \bar{V}_1^Q$:

$$\textit{Success Set: } \left\{ \tau \in]0,T], \Delta \in [0,1] : \Delta < \frac{1}{\mu}\sqrt{\frac{2V_0\mu(2-\mu)}{(1-e^{-\lambda(2-\mu)T})}} e^{\lambda(\mu-1)\tau} \right\}$$

$$\textit{Prob. of Shortfall: } 1 - \sqrt{\frac{2V_0(1-e^{\lambda(\mu-2)T})}{\mu(2-\mu)}}$$

$$\textit{Expected Shortfall: } \frac{1}{2}(1+e^{-\lambda T}) - \frac{V_0(2-\mu)(1-e^{\lambda(2\mu-3)T})}{\mu(3-2\mu)(1-e^{\lambda(\mu-2)T})}$$

2) $V_0 \in [\bar{V}_1^Q, \bar{V}_2^Q]$*:*

$$\text{Success Set: } \left\{ \Delta < \frac{1}{\mu} e^{-(T-\tau)(\mu-1)\lambda} \right\} \text{ and randomized}$$
$$\text{with probability } (V_0 - \bar{V}_1^Q) e^{\lambda\mu T} \text{ on } \{\tau > T\}$$

$$\text{Prob. of Shortfall: } 1 - \frac{e^{\lambda(1-\mu)T} - e^{-\lambda T}}{\mu(2-\mu)} - (V_0 - \bar{V}_1^Q) e^{\lambda(\mu-1)T}$$
$$\text{Expected Shortfall: } \frac{1}{2}\left(1 + e^{-\lambda T} - \frac{e^{2\lambda(1-\mu)T} - e^{-\lambda T}}{\mu^2(3-2\mu)}\right) - (V_0 - \bar{V}_1^Q) e^{\lambda(\mu-1)T}$$

3) $V_0 > \bar{V}_2^Q$*: No closed formulae for the shortfall probability or for the expected shortfall exist. In this segment, both have to be determined numerically.*

Proof: See appendix.

Note that the success set depends not only on the time of default τ, but also on the size of the payout ratio in the case of default, Δ. It can be seen that the later the default happens, the bigger is the payout ratio Δ which is hedged by the claim, up to the total amount of 1. The optimal hedge contract is of the knock-out type: If the write-down is below a certain boundary, it is covered by the contract, but if it is above the boundary, the hedge pays out nothing. The knock-out form of the hedge is typical for shortfall hedging, compare Föllmer and Leukert [1997,1998].

Figure 2 shows several boundaries depending on the initial investment into the hedging strategy (in percent of the bond's price). Here, parameters are $\mu = 2, \lambda = 0.1, T = 10$. The area below the boundary is the area secured by the hedging strategy. The double dependence of the success set on the time of default as well as the size of default is the main difference between the strategy minimizing the shortfall probability and the strategy minimizing the expected shortfall. We see that at later possible default dates, higher loss rates are insured by the optimal hedging strategy. There is an intuitive reason for this shape of the success set: Figure 1 shows that the density of default times under the martingale measure is falling faster than the density under the historical measure. This implies that it costs less to insure against the possibility of default at later dates, and therefore a higher amount of the investment into the hedge can be used to insure against a higher loss rate in the case of default.

Let us consider now the strategy which minimizes the expected shortfall.

3.3.3 Theorem *The critical values are given by*

$$a_K^{ES} = e^{\lambda(\mu-1)T}; \quad \bar{V}_1^{ES} = 0; \quad \bar{V}_2^{ES} = e^{-\lambda\mu T}$$

Depending on the initial investment, the success set, probability and expected size of the shortfall are given by

	$V_0 \in [\bar{V}_1^{ES}, \bar{V}_2^{ES}]$	$V_0 > \bar{V}_2^{ES}$
Success Set	*Randomized on* $\tau > T$ *with probability* $V_0 e^{\lambda\mu T}$	$T \geq \tau > -\frac{\log(2V_0 - e^{-\lambda\mu T})}{\lambda\mu}$
Shortfall Prob.	$1 - V_0 e^{-\lambda(1-\mu)T}$	$(2V_0 - e^{-\lambda\mu T})^{-1/\mu} - e^{-\lambda T}$
Expected Shortfall	$\frac{1}{2}(1 + e^{-\lambda T}) - V_0 e^{-\lambda(1-\mu)T}$	$\frac{1}{2}((2V_0 - e^{-\lambda\mu T})^{-1/\mu} - e^{-\lambda T})$

Proof: See appendix.

3.3.4 Remark

- First of all, note the difference in success sets between the strategy minimizing the SP and the strategy minimizing the ES: In the latter, the success set does not depend on the size of the default, but only on the time of default. For a low investment into the hedging strategy, the risk of the contingent claim is hedged first at the long end, and with rising investment the success set covers dates nearer to the present. This is due to the same reasons which influence the shape of the success set in the case of the strategy which minimizes the shortfall probability: The density of default times is falling faster under the martingale measure than under the historical measure, and therefore it is cheaper to hedge late default times first.
- Note that the initial investment enters in two different ways depending whether it is in the interval $[0, \bar{V}_2^{ES}]$ or greater than $\bar{V}_2^{ES}$: In the first case it enters linearly, in the second case as the reciprocal of some root. It can be shown that the derivative with respect to V_0 is greater in absolute value in the first case than in the second case. This means that a marginal increase in V_0 for $V_0 \in [0, \bar{V}_2^{ES}]$ reduces shortfall probability and the expected size of the shortfall more than for $V_0 > \bar{V}_2^{ES}$.

The figures 3 and 4 compare the strategies minimizing the probability of a shortfall and the expected shortfall with respect to the resulting shortfall probability and expected shortfall for a given initial investment in percent of the bond's price. Parameters are $\mu = 2, \lambda = 0.1, T = 10$. We can see that, for example, investing 50% of the bond's price into the strategy minimizing the SP results in a 10% smaller SP, but 2.5% higher ES than when the same amount is invested into the strategy minimizing the ES. Of course, the strategy minimizing the SP has a lower SP than the strategy minimizing the ES, and vice versa.

With the results for the credit risky bond, it is relatively easy to treat also the case of a credit default swap. For this purpose, we modify assumption 3.3.1 in the following way:

3.3.5 Assumption *Let μ, f, f^* and h as before, and $C = 0$, i.e. if there is no default, the claim G pays nothing.*

The price of this instrument is

$$E^*[D1_{\{\tau \leq T\}}] = \frac{1}{2}(1 - e^{-\lambda\mu T})$$

The following theorem provides some results on the optimal shortfall hedging strategies of the credit default swap.

3.3.6 Theorem *As in the last case of theorem 3.3.2, the strategy which minimizes the SP has to be determined numerically. For the strategy which minimizes the ES, we have the following explicit results:*

$$\begin{aligned} \textit{Success Set: } & \tau > -\frac{1}{\lambda\mu}\log(2V_0 + e^{-\lambda\mu T}) \\ \textit{Prob. of Shortfall: } & 1 - (2V_0 + e^{-\lambda\mu T})^{1/\mu} \\ \textit{Expected Shortfall: } & \frac{1}{2}\left(1 - (2V_0 + e^{-\lambda\mu T})^{1/\mu}\right) \end{aligned}$$

Proof: See appendix.

The success sets in this situation look very similar to the case of a defaultable bond, and this is also apparent in the figures 5 and 7. Consequently, an investor who wants to save money on the hedging of the loss from holding a defaultable bond and who is interested in minimizing the ES should take out an insurance against default at the later dates and bear the risk of default at the earlier dates. On the other hand,if the investor is interested in minimizing his SP, then his optimal hedge should cover him against larger and larger losses from default, the later default happens.

3.4 Discussion

In the following, we want to show the advantages of our hedging strategies over competing contracts. The discussion is based on the last specification of the contract in the preceding section. Thus, we consider a claim which pays a random amount D uniformly distributed on $[0, 1]$ if default τ occurs before maturity $T = 1$ year, where τ is exponentially distributed with intensity $\lambda = 0.1$, and the risk premium for the jump is $\mu = 2$. To explain our standard example, think about an investor who owns a credit risky bond which pays 1 if default τ occurs after maturity T of the bond ($\tau > T$) and which pays a random amount $\Delta \in [0, 1]$ if default occurs before maturity ($\tau \leq T$). This investor faces a random loss of $D = (1 - \Delta)$ in the case of default. We can interpret this as a payment obligation of random size, which also fits the situation where a bank has sold credit protection for this bond to a customer and might have to pay this amount in the case of default. Note that the price of this payment obligation is 0.091, while its 5% VaR is 0.475. In other words, the bank would have to hold this amount in order to be sure that in only 5% of all possible cases, they have to pay more than that to their customer. Apart from diversification, which is a possibility we do not consider in this paper, the investor or the bank could try to reduce the risk of the random loss by entering into a (re-)insurance contract. Three possible types of contracts come to mind, which we will explain below and compare the results with the optimal hedging strategies derived before:

1) PER: Imagine first a contract which pays a certain percentage of the loss in the case of default, $V(T) = cD1_{\{\tau \leq T\}}$. Because this contract pays always less than the actual loss, the shortfall probability (SP) is 100% for all $c < 1$.

2) CON: Another possibility would be a contract which pays a fixed amount in the case of default, $V(T) = c1_{\{\tau \leq T\}}$. For a SP of 5%, we calculate $c = 0.475$ and the price of this contract is 0.086.

3) MIN: Finally, consider a contract which pays, in the event of default, the minimum of a constant c and the actual loss D, $V(T) = (c \wedge D)1_{\{\tau \leq T\}}$. For a SP of 5%, this contract costs 0.066 and is therefore cheaper than contract number 2).

Although the last contract seems to be quite good, it is far outperformed by the optimal strategies calculated by the application of the Neyman-Pearson lemma. The strategy which minimizes the SP for this contract only costs 0.021 (about one third of the best strategy in our list above) in order to guarantee a 5% SP.

We can also compare all strategies with respect to their expected shortfall (ES). Here the differences are less pronounced, but it is still possible to save some percent in hedging costs by using a strategy which minimizes the ES versus, for example, the last contract in the list above. For example, to guarantee an ES of 0.01, the contract number 3) above costs 0.072, while the strategy minimizing the ES costs 0.071, a saving of about 1.4%. The results (initial investment for required SP or ES) are summarized in the table below.

Strategy	5% Shortfall Probability	0.01 Expected Shortfall
VaR	0.475	0.542
PER	0.091	0.072
CON	0.087	0.099
MIN	0.066	0.072
MinShortProb	0.021	0.072
MinExpShort	0.042	0.071

4 Incomplete Markets

In this section, we analyze hedging strategies for a credit default swap ($C = 0$) in incomplete markets. Thus, we consider a payment obligation of the form

$$G = D1_{\{\tau \leq T\}}$$

with D uniformly distributed on $[0, 1]$. We look at the strategies which minimize, for a given initial investment, the Sp or the ES, respectively. We do this in two different incomplete market setups:

- First, we assume that the default risk premium is unknown from the beginning. This approach leads to an optimization over all possible martingale measures, which results in the so-called "worst-case" martingale measure.
- In our second setup, we assume that at a predetermined time t_0 between 0 and T a rating change will occur, so that after time t_0 the default intensity λ is distributed with the density g on $\mathbb{R}^+$. This setup is similar to the one used in the papers by Föllmer and Leukert [1997,1998], and the optimal hedging strategies can be found in an analogous way.

In both cases, we can use results from the complete markets case, especially theorem 3.1.2. We will compare the results from both strategies.

4.1 Unknown Default Risk Premium

The markets where credit risk is traded, such as the corporate bond market, are often very illiquid, and a large part of credit derivatives are traded over the counter, so that it may often be difficult to obtain accurate information on current prices for credit risky securities. These prices are often used to back out the risk premium from a credit risk model, so that in the absence of price information this risk premium is not known. However, the optimal strategies as well as the minimal shortfall probability and expected shortfall may differ quite a lot depending on the value of μ, as we can see in figure 8.

We substitute assumption 3.0.1 with the following:

4.1.1 Assumption • *The constant μ, which determines the default intensity under the risk-neutral measure, is not unique, but lies somewhere in the interval $[\mu_L, \mu_H]$ with $\mu_L > 1$.*

- *As in our example in section 3.3, we assume that $f = f^* = 1_{[0,1]}$ and $h \equiv 1$.*

The assumption implies that the risk premium for the default intensity is unknown. Regarding the payout ratio after default, Δ, we keep the assumption from the previous section that it has the same distribution (uniform on $[0, 1]$) under the historical and under the martingale measure, which implies that the risk premium for the payout ratio is zero. The results can be generalized to the case where there is a positive risk premium for the payout ratio which may also be unknown. However, the assumptions can be justified by the fact that the risk premium for the default intensity plays a much bigger role in practice than the risk premium for the payout ratio, as most models used in the industry still use a constant payout ratio. Therefore, we consider the case of the unknown default risk premium.

Under this assumption, we face again the problem 3.1.1. The optimal hedge can be written as before by $H = \varphi G$, where $\varphi \in \mathcal{R}$ (see section 2.2). To find the optimal function φ, we introduce a family of probability measures $\{Q^* | P^* \in \mathbb{P}\}$, where Q^* is defined by

$$\frac{dQ^*}{dP^*} = \frac{G}{E^*[G]}$$

Because the P^* are indexed by μ, we also write $Q^* = Q^*(\mu)$. The problem can now be interpreted as testing the compound hypothesis $\{Q^* | P^* \in \mathbb{P}\}$ against the simple alternative P in the SP case and against Q in the ES case, where

$$\frac{dQ}{dP} = \frac{H}{E[H]}$$

Witting [1985] gives a sufficient condition for the optimal function φ: There exists a finite measure L on $[\mu_L, \mu_H]$ so that

$$\varphi^Q = \begin{cases} 1 & \text{if } dP > \int_{\mu_L}^{\mu_H} dQ^*(\mu)\, dL^Q(\mu) \\ 0 & \text{if } dP < \int_{\mu_L}^{\mu_H} dQ^*(\mu)\, dL^Q(\mu) \end{cases}; \quad \tilde{E}^Q[H\hat{\varphi}^Q] = V_0$$

$$\text{or } \varphi^{ES} = \begin{cases} 1 & \text{if } dQ > \int_{\mu_L}^{\mu_H} dQ^*(\mu)\, dL^{ES}(\mu) \\ 0 & \text{if } dQ < \int_{\mu_L}^{\mu_H} dQ^*(\mu)\, dL^{ES}(\mu) \end{cases}; \quad \tilde{E}^{ES}[H\hat{\varphi}^{ES}] = V_0$$

where $\tilde{E}$ denotes the expectation under the optimal mixture of measures for the SP or the ES case, respectively. Practically, it is nearly impossible to find the optimal measure $\hat{L}$ on the parameter interval $[\mu_L, \mu_H]$. Only in the case that the mixture of measures Q^* above is again a member of the family $\{Q^*|P^* \in \mathbb{P}\}$, as in Föllmer and Leukert [1997], there is some hope of a solution of this problem. Under these circumstances, we know as in the complete case for $C = 0$, that the optimal success ratio reduces to a success set

$$\mathcal{A}^Q(\tilde{\mu}) = \{\tau > T\} \cup (\{a^Q D\tilde{\mu}e^{-\lambda(\tilde{\mu}-1)\tau} < 1\} \cap \{\tau \leq T\}) \quad (7)$$

$$\mathcal{A}^{ES}(\tilde{\mu}) = \{\tau > T\} \cup (\{a^{ES}\tilde{\mu}e^{-\lambda(\tilde{\mu}-1)\tau} < 1\} \cap \{\tau \leq T\}) \quad (8)$$

Although in our setup the family $\{Q^* : P^* \in \mathbb{P}\}$ is not measure convex, we believe that it is possible to find a nearly optimal solution by restricting our attention to success sets of the form above.

Note that inside the success set, the parameter $\tilde{\mu}$ is not fixed, as in the complete case, but may vary in the interval $[\mu_L, \mu_H]$. Similarly, due to incompleteness of the market, there is a whole range of possible measures P^*, which are associated to the risk premium $\mu^* \in [\mu_L, \mu_H]$. In contrast to the complete case, we have to determine $\mathcal{A}$ in such a way that under all possible martingale measures, the price of the hedge is less than or equal to our fixed initial investment V_0,

$$\sup_{P^*} E^*[1_{\mathcal{A}}G] \leq V_0$$

4.1.2 Theorem *In the expected shortfall case, the worst case martingale measure $\hat{\mu}$ is given by*

$$\hat{\mu} = \arg\min_{\mu}(2V_0 + e^{-\lambda\mu T})^{(1/\mu)}$$

In the shortfall probability case, the worst case martingale measure can only be found numerically.
In both cases, success set, shortfall probability and expected shortfall of the optimal hedge can be obtained as in the complete case by using the measure associated to $\hat{\mu}$ as the unique martingale measure P^*.

Proof. We treat the strategy minimizing the expected shortfall first. Following Föllmer and Leukert [1997], we know that for the worst case martingale measure $\hat{\mu}$ we have

$$E^{\hat{\mu}}[1_{\mathcal{A}^{ES}(\hat{\mu})}G] = V_0$$

We use this condition together with our results from the complete case to find possilbe combinations of worst case martingale measures $\hat{\mu}$ and associated critical constants $\hat{a}^{ES}$ for a given initial investment V_0. These are connected by

$$\hat{a}^{ES}(\hat{\mu}) = \frac{1}{\hat{\mu}}(2V_0 + e^{-\lambda\hat{\mu}T})^{(1-\hat{\mu})/\hat{\mu}}$$

(see the proof of theorem 3.3.3). Together with equation (8) it is clear that the success set $\mathcal{A}^{ES}$ depends on the parameter $\hat{\mu}$. Now, $\hat{\mu}$ has to be determined so that under all alternative measures $\tilde{\mu} \in [\mu_L, \mu_H]$, the price of our hedge is less than the initial investment, $E^{\tilde{\mu}}[1_{\mathcal{A}^{ES}(\hat{\mu})}G] \leq V_0$. Computing the expectation (same calculations as in section 3), we find

$$E^{\tilde{\mu}}[1_{\mathcal{A}^{ES}(\hat{\mu})}G] = \frac{1}{2}[e^{-\frac{\tilde{\mu}}{\hat{\mu}-1}\log \hat{a}^{ES}\hat{\mu}} - e^{-\lambda\tilde{\mu}T}] \stackrel{!}{\leq} V_0 \quad \forall\tilde{\mu} \in [\mu_L, \mu_H]$$

Substituting the expression for $\hat{a}^{ES}$ from above, we see that this condition is equivalent to

$$(2V_0 + e^{-\lambda\hat{\mu}T})^{(1/\hat{\mu})} \leq (2V_0 + e^{-\lambda\tilde{\mu}T})^{(1/\tilde{\mu})} \quad \forall\tilde{\mu}$$

Therefore, we have to minimize the function

$$(2V_0 + e^{-\lambda\mu T})^{(1/\mu)}$$

with respect to μ, and the solution will give us the worst-case martingale measure $\hat{\mu}$.

In the case where the investor wants to minimize the shortfall probability, such an explicit solution for the worst-case martingale measure is not available. This is mainly due to the fact that the optimal parameter $\hat{a}^Q$ can not be computed explicitly as a function of $\hat{\mu}$. This problem was already outlined in the proof of theorem 3.3.2, case 3). It is still possible to determine the worst case martingale measure numerically. As in the expected shortfall case, the optimal success set depends on the parameter $\hat{\mu}$, while the expectation has to be taken under the measure which corresponds to $\tilde{\mu}$. Thus,

$$E^{\tilde{\mu}}[1_{\mathcal{A}^{Q}(\hat{\mu})}G] = F(\tilde{\mu}, \hat{\mu}) \stackrel{!}{\leq} V_0 \quad \forall\tilde{\mu} \in [\mu_L, \mu_H]$$

We solve this problem by a grid search method: We discretize the interval $[\mu_L, \mu_H]$ and on the resulting grid for $\tilde{\mu}, \hat{\mu}$ we find the value of $\hat{\mu}$ which fulfills the condition above. Of course, this method is not very sophisticated, and improvements are possible. □

Figures 10, 11 and 12 show the worst-case martingale measure, shortfall probability and expected shortfall for various investments V_0 into both hedging strategies. Here, the interval $[\mu_L, \mu_H]$ was set to $[1, 3]$. We see that the parameter $\hat{\mu}$ is a rising function of the initial investment V_0. This behaviour of the worst-case martingale measure can be explained in the following way: Remember that from all possible martingale measures, the optimal hedge is most expensive under the worst case martingale measure. We know from the complete markets case that the optimal success sets for the expected shortfall case always start at the long end and hedge against late default times first. The more money is invested into the hedge, the earlier default times are insured. In the shortfall probability case, the loss from default which is hedged is biggest at the long end. However, an increase in the risk premium parameter $\hat{\mu}$ prices default risk as if an earlier default was more probable, i.e. it shifts more mass of the distribution of the default time τ to earlier dates (compare figure 1), and this makes the hedge which covers the earlier dates more expensive.

The curves showing shortfall probability and expected shortfall of the optimal hedge have the same form as in the complete case, although the difference in numbers can be quite significant (as we showed in figure 8).

4.2 Rating Change

Rating changes are a common occurence in credit risk markets and have a strong impact on the value of securities traded in these markets. In this section we consider the case where a rating change takes place at a predetermined point in time, t_0. For this purpose, we introduce the

4.2.1 Assumption *The intensity λ governing the risk of default is known from time 0 up to time t_0. At time t_0, a rating change occurs, and the new default intensity is distributed with density g on $\mathbb{R}^+$. Contracts on the new default intensity (except the one under consideration) do not exist, and therefore the market after date t_0 is incomplete.*

This assumption is analogous to the assumptions in Föllmer and Leukert [1997,1998], where they consider a volatility jump of the driving Brownian motion at time t_0. Our setup differs in two respects from theirs:

1) In Föllmer and Leukert, it turns out only at the maturity T of the claim G whether the hedge was successful or not, whereas in the present paper this may be known before, namely at the time of default τ.

2) Although the volatility σ jumps, they keep the drift α of the stock price constant. Therefore, the risk premium α/σ jumps together with the volatility. In our setup, the situation is different: The default intensity λ jumps, but the risk premium parameter μ stays constant. This seems to be a plausible assumption for our credit risk setting.

As before, we focus on a credit default swap, where the random payoff D at the default time τ is uniformly distributed on $[0,1]$. We consider the shortfall probability case first, the expected shortfall case can then be treated in an analogous fashion. Before we present the optimal strategy, we introduce the following

4.2.2 Definition *For a given realization λ_2 of the default intensity λ and an investment of Y into the hedging strategy at time t_0, we denote by $\alpha^{\lambda_2}(Y)$ its minimal shortfall probability after time t_0 if default has not happened before time t_0.*

Because for a given realization of λ_2 from time t_0 onwards the market is complete, we can use the methods developed in section 3 to calculate $\alpha^{\lambda_2}(Y)$.

4.2.3 Theorem *With this definition, the optimal hedging strategy which minimizes the probability of a shortfall for a given initial investment V_0 at time 0 can be obtained in two steps:*

1) *For a given amount Y at t_0 and using the methods from section 3, compute the average shortfall probability for times $t > t_0$ given by*

$$\bar{\alpha}_2(Y) := \int \alpha^{\lambda_2}(Y) g(\lambda_2)\, d\lambda_2$$

2) *In a second step, compute the optimal hedge with initial investment V_0 which minimizes the shortfall probability $\alpha_1(Y)$ for the claim $Y1_{\{\tau > t_0\}} + D1_{\{\tau \leq t_0\}}$ at time t_0.*

The total shortfall probability for a given Y is then the (weighted) sum of the shortfall probabilities from 1) and 2):

$$\alpha(Y) = \alpha_1(Y) + \bar{\alpha}_2(Y) * P[\tau > t_0]$$

The total shortfall probability depends on the intermediate investment Y. The intermediate investment Y which yields the lowest shortfall probability for a given initial investment V_0 can be found numerically.

The strategy which minimizes the expected shortfall can be obtained in a similar way.

Proof Consider a hedge H with initial investment H_0 and $\mathcal{F}_t$-measurable value process H_t.

At time t_0, due to the structure of the underlying probability space, the value of the hedge can be written as

$$H_{t_0} = \bar{H}_1 1_{\{\tau > t_0\}} + \bar{H}_2(\tau, D) 1_{\{\tau \leq t_0\}},$$

where $\bar{H}_1$ is a constant and $\bar{H}_2$ is a function of the time of default, τ, and the mark space variable D. Given that no default has happened up to time t_0, it is optimal to use the value $\bar{H}_1$ to implement a hedging strategy which minimizes the shortfall probability from time t_0 onwards. This can be done as in section 3, because the new value λ_2 of the default intensity is now known.

In the case that default happens before time t_0, the shortfall probability is surely minimized by implementing a shortfall probability minimizing hedge for the claim

$$\tilde{H}_{t_0} = \bar{H}_1 1_{\{\tau > t_0\}} + (1 - \Delta) 1_{\{\tau \leq t_0\}}$$

Note that for a given initial investment H_0, the optimal shortfall hedge in this first segment is fully determined by the payoff H_1 at time t_0. Finally, $\bar{H}_1$ has to be chosen such that the overall shortfall probability is minimized. □

5 Conclusion

We show how the results of Föllmer and Leukert [1997,1998] can be applied to credit risky securities. We show how an investor can save money by using hedging strategies which minimize the shortfall probability or the expected shortfall instead of a perfect hedge in complete markets. General theorems for credit risky securities are derived, and we provide explicit results for the optimal hedges of a defaultable bond and a credit default swap. The hedges are compared with respect to their shortfall probability and expected shortfall. We show that in the case of an investor who wants to minimize the shortfall probability, it is optimal to hedge only losses below a certain, time dependent boundary, while for the strategy which minimizes the expected shortfall it is optimal to hedge the whole possible loss, starting at a certain point in time which is determined by the initial investment into the hedging strategy.

In an incomplete markets setup, we compute optimal hedges for both objectives in the cases where the risk premium for default is unknown or a future rating event will change the default intensity randomly.

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Proofs

Proof of theorem 3.1.2 We treat the Quantile case (minimization of shortfall probability) first. According to the results of Föllmer and Leukert [1997], the optimal success ratio has the form

$$\hat{\varphi}^Q = 1_{\mathcal{A}^Q(\hat{a}^Q)} + \gamma^Q 1_{\{\frac{dP}{dP^*}=\hat{a}^Q H\}} \tag{9}$$

where the set $\mathcal{A}^Q(a)$ and the two constants $\hat{a}^Q$ and γ^Q are given by

$$\mathcal{A}^Q(a) = \{\frac{dP}{dP^*} > aG\} \tag{10}$$

$$\hat{a}^Q = \inf\{a : \int 1_{\mathcal{A}^Q(a)} G dP^* \leq V_0\} \tag{11}$$

$$\gamma^Q = \frac{V_0 - \int 1_{\{\mathcal{A}^Q(\hat{a}^Q)\}} G\, dP^*}{\int 1_{\{\frac{dP}{dP^*}=\hat{a}^Q G\}} G\, dP^*}$$

For a given initial investment V_0, the success set is fully described through the constants $\hat{a}^Q$ and γ^Q, which in turn depend on the initial investment into the hedging strategy, V_0. Note that through equation 10 the function $P^*[\mathcal{A}^Q(a)]$ is decreasing in a. In return, through equation 11 the optimal parameter $\hat{a}^Q$ is weakly decreasing in the initial investment V_0: Increasing V_0 means that $\mathcal{A}^Q(a)$ increases weakly, so that $\hat{a}^Q$ decreases weakly.

In a first step, we determine the values of $\hat{a}^Q$ for which the success ratio reduces to a success set. Thus, we study the condition

$$P[\frac{dP}{dP^*} = aH] = 0$$

To analyze this condition, we decompose the set into

$$\left\{\frac{dP}{dP^*} = aH\right\} = \left\{aH\frac{dP^*}{dP} = 1\right\}$$
$$= (\{aCe^{-\lambda(\mu^*-1)T} = 1\} \cap \{\tau > T\}) \cup (\{aD\mu^* h^* e^{-\lambda(\mu-1)\tau} = 1\} \cap \{\tau \leq T\})$$

While the second partition (for $\tau \leq T$) has measure zero for any a, the first partition has positive mass for the parameter

$$a_K^Q = \frac{1}{C} e^{\lambda(\mu^*-1)T}$$

Consequently, for $\hat{a}^Q \neq a_K^Q$, the optimal success ratio reduces to the success set $\mathcal{A}^Q(\hat{a}^Q)$. For $\hat{a}^Q = a_K^Q$, randomization occurs on the set $\{\tau > T\}$ (and on a subset of $\{\tau \leq T\}$, which has measure zero and is therefore irrelevant).

As we argued above, $\hat{a}^Q$ is a decreasing function of V_0. Thus, for small values of V_0, we have $\hat{a}^Q > a_K^Q$, while for large values of V_0, we have $\hat{a}^Q < a_K^Q$. In these cases, the success set is solely determined through the value of the parameter $\hat{a}^Q$, the parameter γ^Q does not occur. As we can see from the decomposition of the success set $\mathcal{A}^Q$,

$$\begin{aligned}\mathcal{A}^Q &= \left\{aH\frac{dP^*}{dP} < 1\right\} \\ &= (\{aCe^{-\lambda(\mu^*-1)T} < 1\} \cap \{\tau > T\}) \cup (\{aD\mu^* h^* e^{-\lambda(\mu-1)\tau} < 1\} \cap \{\tau \leq T\})\end{aligned}$$

for $\hat{a}^Q > a_K^Q$, $\mathcal{A}^Q$ consists only of a subset of $\{\tau \leq T\}$, while for $\hat{a}^Q < a_K^Q$ the success set also comprises the set $\{\tau > T\}$. Of course, in the case $C = 0$ the set $\{\tau > T\}$ is always part of the success set, and the following computations simplify accordingly. The relationship between initial investment V_0 and the parameter $\hat{a}^Q$ in the case $\hat{a}^Q > a_K^Q$ is given by

$$V_1^Q(\hat{a}^Q) = \int D1_{\{D\hat{a}^Q \mu h e^{-\lambda(\mu-1)\tau} < 1\}}\, dP^*$$

and in the case $\hat{a}^Q > a_K^Q$ by

$$V_2^Q(\hat{a}^Q) = \int D1_{\{D\hat{a}^Q \mu h e^{-\lambda(\mu-1)\tau} < 1\}}\, dP^* + CP^*[\tau > T]$$

For $\hat{a}^Q = a_K^Q$, the optimal solution is given by the success ratio $\hat{\varphi}^Q$ as defined in equation 9. Consequently, this parameter value is obtained in a range of initial investments $[V_1^Q, V_2^Q]$, which correspond to the interval $\gamma^Q \in [0, 1]$.

The optimal success ratio in the Expected Shortfall case is

$$\hat{\varphi}^{ES} = 1_{\mathcal{A}^{ES}(\hat{a}^{ES})} + \gamma 1_{\{\frac{dP}{dP^*} = \hat{a}^{ES}\}}$$

where $\mathcal{A}^{ES}(a), \hat{a}^{ES}$ and γ^{ES} are given by

$$\begin{aligned}\mathcal{A}^{ES}(a) &= \{\frac{dP}{dP^*} > a\} \\ \hat{a}^{ES} &= \inf\{a : \int 1_{\mathcal{A}^{ES}(\hat{a}^{ES})} H dP^* \leq V_0\} \\ \gamma^{ES} &= \frac{V_0 - \int 1_{\{\mathcal{A}^{ES}(\hat{a}^{ES})\}} H\, dP^*}{\int 1_{\{\frac{dP}{dP^*} = \hat{a}^{ES}\}} H\, dP^*}\end{aligned}$$

The success ratio reduces to a success set under the condition

$$P[\frac{dP}{dP^*} = a] = 0$$

so that the critical parameter for the Expected Shortfall case is given by

$$a_K^{ES} = e^{\lambda(\mu-1)T}$$

Consequently, for $\hat{a}^{ES} \neq a_K^{ES}$, the optimal success ratio reduces to the success set $\mathcal{A}^{ES}(\hat{a}^{ES})$. For $\hat{a}^{ES} > a_K^{ES}$, the success sets consist only of a subset of $\{\tau \leq T\}$, while for $\hat{a} < a_K$ the success sets also comprise the set $\{\tau > T\}$. The initial investment for a hedging strategy of the first kind is given by

$$V_1^{ES}(\hat{a}^{ES}) = \int D 1_{\{\hat{a}^{ES}\mu h e^{-\lambda(\mu-1)\tau} < 1\}}\, dP^*$$

and for the second kind by

$$V_2^{ES}(\hat{a}^{ES}) = \int D 1_{\{\hat{a}^{ES}\mu h e^{-\lambda(\mu-1)\tau} < 1\}}\, dP^* + CP^*[\tau > T]$$

For $\hat{a}^{ES} = a_K^{ES}$, the critical values V_1^{ES} and V_2^{ES} can be found as in the Quantile case. □

Proof of theorem 3.2.2 Define the set

$$\mathcal{B}^Q(b) = \left\{ \frac{dP}{dP^*} > \frac{G}{bE^*[G]} \right\}$$

According to Föllmer and Leukert [1997], the optimal success ratio has the form

$$\hat{\psi}^Q = 1_{\mathcal{B}^Q(\hat{b}^Q)} + \delta^Q 1_{\{\frac{dP}{dP^*} = \frac{G}{\hat{b}^Q E^*[G]}\}}$$

where

$$\hat{b}^Q = \inf\left\{ b : P\left[\frac{dP}{dP^*} < \frac{G}{bE^*[G]} \right] \leq \epsilon \right\}$$
$$\delta^Q = \frac{(1-\epsilon) - P[\mathcal{B}(\hat{b}^Q)]}{P[\frac{dP}{dP^*} = \frac{G}{bE^*[G]}]}$$

The rest of the proof is along the same lines as the proof to theorem 3.1.2. □

Proof of theorem 3.2.4 Define the set

$$\mathcal{B}^{ES}(b) = \left\{ \frac{dP^*}{dP} < b \right\}$$

According to Föllmer and Leukert [1998], the optimal success ratio has the form

$$\hat{\psi}^{ES} = 1_{\mathcal{B}^{ES}(\hat{b}^{ES})} + \delta^{ES} 1_{\{\frac{dP^*}{dP} = \hat{b}^{ES}\}}$$

where

$$\hat{b}^{ES} = \inf\{b : \int (1 - 1_{\mathcal{B}^{ES}(\hat{b}^{ES})}) G\, dP \le \eta\}$$
$$\delta^{ES} = 1 - \frac{\eta - \int (1 - 1_{\mathcal{B}^{ES}(\hat{b}^{ES})}) G\, dP}{\int 1_{\{\frac{dP}{dP^*} = \hat{b}^{ES}\}} G\, dP}$$

The rest of the proof is along the same lines as the proof to theorem 3.1.2. □

Proof of theorem 3.3.2 We start with the observation that

$$a_K^Q z \mu e^{-\lambda(\mu-1)\tau} < 1 \Leftrightarrow z < \frac{1}{\mu} e^{-(T-\tau)(\mu-1)\lambda}$$

where $a_K^Q = \exp\{\lambda(\mu - 1)T\}$ from theorem 3.1.2 and the far right side is smaller than one for $\tau < T + \frac{\log \mu}{\lambda(\mu-1)}$. With this, the double integral reduces to

$$\bar{V}_1^Q = \int_0^T \int_0^1 z 1_{\{z\mu e^{\lambda(\mu-1)(T-\tau)}\}}\, dz \lambda \mu e^{-\lambda\mu\tau}\, d\tau = \frac{1}{2} \int_0^T \left(\frac{1}{\mu} e^{-\lambda(\mu-1)(T-\tau)} \right)^2 \lambda \mu e^{-\lambda\mu\tau}\, d\tau$$

Integration gives the value of $\bar{V}_1^Q$, while we know from theorem 3.1.2 that

$$\bar{V}_2^Q = \bar{V}_1^Q + CP^*[\tau > T] = \bar{V}_1^Q + e^{-\lambda\mu T}$$

The optimal solution for the three cases in the theorem is obtained as follows:

1) $V_0 < \bar{V}_1^Q$: Here we have $\hat{a}^Q > a_K^Q$ and therefore

$$\int z 1_{\{\hat{a}^Q z\mu e^{-\lambda(\mu-1)\tau} < 1\}}\, dP^* = \frac{1}{2} \int_0^T \left(\frac{1}{a^Q \mu} e^{\lambda(\mu-1)\tau} \right)^2 \lambda \mu e^{-\lambda\mu\tau}\, d\tau \stackrel{!}{=} V_0$$

This equation determines the optimal value $\hat{a}^Q$. The success set is then given by

$$\{\hat{a}^Q z \mu \exp(-\lambda(\mu - 1)\tau) < 1\}$$

where τ must lie between zero and T, and z between zero and one. The probability of a shortfall is one minus the probability of the success set under the measure P. The expected shortfall can be computed as

$$\int z 1_{\{\hat{a}^Q z\mu \exp(-\lambda(\mu-1)\tau) < \ge 1\}}\, dP + P[\tau > T]$$

2) $V_0 \in [\bar{V}_1^Q, \bar{V}_2^Q]$: The optimal value $\hat{a}^Q$ is equal to a_K^Q. In this segment, randomization occurs. On the set $\{\tau \leq T\}$, the success set is given by

$$\{z < \exp\{-\lambda(\mu-1)(T-\tau)\}/\mu\}$$

For $\{\tau > T\}$, success is randomized with $(V_0 - \bar{V}_1^Q)/P^*[\tau > T]$. On $\{\tau \leq T\}$, shortfall probability and expected shortfall have to be determined by integration as in the first segment. On $\{\tau > T\}$, both are given by $P[\tau > T] - (V_0 - \bar{V}_1^Q)P[\tau > T]/P^*[\tau > T]$.

3) $V_0 > \bar{V}_2^Q$: Here we have $\hat{a}^Q < a_K^Q$ and the credit risky bond is completely hedged on the set $\{\tau > T\}$.

On $\{\tau \leq T\}$, the situation is more complicated: We have to determine $\hat{a}^Q$ such that

$$\int_0^T \int_0^1 z 1_{\{\hat{a}^Q z\mu e^{-\lambda(\mu-1)\tau} < 1\}} \, dz \lambda\mu e^{-\lambda\mu\tau} \, d\tau = V_0 - P^*[\tau > T]$$

We solve the inequality in the indicator set for z,

$$z < \frac{1}{\hat{a}^Q \mu} e^{\lambda(\mu-1)\tau}$$

and this is smaller than one only for

$$\tau < \frac{\log(\hat{a}^Q \mu)}{(\mu-1)\lambda} =: \tilde{T}$$

Hence, for $\tilde{T} \geq T$, the integral is equal to

$$\frac{1}{2} \int_0^T \left(\frac{1}{\hat{a}^Q \mu} e^{\lambda(\mu-1)\tau} \right)^2 \lambda\mu e^{-\lambda\mu\tau} \, d\tau$$

On the other hand, if $\tilde{T} < T$, we split up the integral into

$$\frac{1}{2} \int_0^{\tilde{T}} \left(\frac{1}{\hat{a}^Q \mu} e^{\lambda(\mu-1)\tau} \right)^2 \lambda\mu e^{-\lambda\mu\tau} \, d\tau + \frac{1}{2} \int_{\tilde{T}}^T \lambda\mu e^{-\lambda\mu\tau} \, d\tau$$

These integrals can be computed explicitly, but result in complicated functions of $\hat{a}^Q$ (note that $\hat{a}^Q$ enters not only inside the integral, but also through the boundaries of the integral). The optimal value of $\hat{a}^Q$ in this situation for given V_0 can only be found numerically.

□

Proof of theorem 3.3.3 First observe that

$$\mu > 1 \Leftrightarrow \mu - 1 > 0$$

and $a_K^{ES} = \exp\{\lambda(\mu-1)T\}$, which comes directly from theorem 3.1.2. The success set is given by

$$\mathcal{A}_K^{ES} := \{a_K^{ES}\mu e^{-\lambda(\mu-1)\tau} < 1\} \cap \{\tau \le T\}$$

Substituting a_K^{ES}, we see

$$\{\mu e^{\lambda(\mu-1)(T-\tau)} < 1\} = \{\tau > T + \frac{\log\mu}{(\mu-1)\lambda}\}$$

Because of $\log\mu/(\mu-1)\lambda > 0$, the success set $\mathcal{A}_K^{ES}$ is empty. Consequently, we have $\bar{V}_1^{ES} = 0$ and $\bar{V}_2^{ES} = \exp\{-\lambda\mu T\}$.
For an investment of V_0 in the interval $[0, \exp\{-\mu\lambda T\}]$, the probability of a shortfall is given by

$$P[\{a_K^{ES}\mu e^{-\tau(\mu-1)\lambda} \ge 1\} \cap \{\tau \le T\}] + (1-\gamma)P[\tau > T]$$

The set inside the first probability reduces to $\{\tau \le T\}$ because

$$\begin{aligned} &a_K^{ES}\mu e^{-\tau(\mu-1)\lambda} \ge 1 \\ \Leftrightarrow &\mu e^{\lambda(\mu-1)(T-\tau)} \ge 1 \end{aligned} \tag{12}$$

and this is true for all $\tau \le T$ because $\mu > 1$. The second term of the sum is equal to $P[\tau > T] + V_0 P[\tau > T]/P^*[\tau > T]$ according to theorem 3.1.2, and this gives us the result in the table. The expected shortfall is given by

$$\int_0^T \int_0^1 z 1_{\{a_K^{ES}\mu e^{-\lambda(\mu-1)\tau} \ge 1\}}\, dz \lambda e^{-\lambda\tau}\, d\tau + (1-\gamma)P[\tau > T]$$

Due to the same argument as in (12), the double integral reduces to $(1-\exp\{-\lambda T\})/2$, and together with the second part this is equal to the formula in the table.
If the initial investment V_0 is greater than $\bar{V}_2^{ES}$, we first have to determine the optimal success set via its parameter $\hat{a}^{ES}$. The success set is given by

$$\mathcal{A}_3^{ES} = \{\tau > T\} \cup (\{\hat{a}^{ES}\mu e^{-\lambda(\mu-1)\tau} < 1\} \cap \{\tau \le T\})$$

Therefore, we calculate

$$\begin{aligned} &\int_0^T \int_0^1 z 1_{\{\hat{a}^{ES}\mu e^{-\tau(\mu-1)\lambda} < 1\}}\, dz \mu\lambda e^{-\mu\lambda\tau}\, d\tau = \frac{1}{2}P^*[\tau \in [\frac{\log(\hat{a}^{ES}\mu)}{(\mu-1)\lambda}; T]] \\ =&V_0 - e^{-\lambda\mu T} \end{aligned}$$

and solve this equation for $\hat{a}^{ES}$

$$\Leftrightarrow \hat{a}^{ES} = \frac{1}{\mu}(2V_0 - e^{-\mu\lambda T})^{\frac{1}{\mu}-1}$$

Therefore, the probability of a shortfall for this investment is

$$\begin{aligned} &P[\{\hat{a}^{ES}\mu e^{-\tau(\mu-1)\lambda} \geq 1\} \cap \{\tau \leq T\}] \\ =&(2V_0 - e^{-\lambda\mu T})^{-1/\mu} - e^{-\lambda T} \end{aligned}$$

and the expected shortfall is exactly one half that number. □

Proof of theorem 3.3.6 As we saw already in theorem 3.1.2, in the case $C = 0$ the critical values do not exist and the success sets have the form

$$\begin{aligned} \mathcal{A}^Q &= \{\tau > T\} \cup (\{\hat{a}^Q D\mu e^{-\tau(\mu-1)\lambda} < 1\} \cap \{\tau \leq T\}) \\ \mathcal{A}^{ES} &= \{\tau > T\} \cup (\{\hat{a}^{ES}\mu e^{-\tau(\mu-1)\lambda} < 1\} \cap \{\tau \leq T\}) \end{aligned}$$

Minimizing the shortfall probability, we have to solve

$$\int_0^T \int_0^1 z 1_{\{\hat{a}^Q z\mu e^{-\lambda(\mu-1)\tau} < 1\}} \, dz \lambda\mu e^{-\lambda\mu\tau} \, d\tau = V_0$$

for $\hat{a}^Q$. This has to be done numerically, compare the proof for theorem 3.3.2. For the expected shortfall, we have to solve

$$\int_0^T \int_0^1 z 1_{\{\hat{a}^{ES}\mu e^{-\tau(\mu-1)\lambda} < 1\}} \, dz \mu\lambda e^{-\mu\lambda\tau} \, d\tau = \frac{1}{2} P^*[\tau \in [\frac{\log(\hat{a}^{ES}\mu)}{(\mu-1)\lambda}; T]] \stackrel{!}{=} V_0$$

for $\hat{a}^{ES}$:

$$\Leftrightarrow \hat{a}^{ES} = \frac{1}{\mu}(2V_0 + e^{-\lambda\mu T})^{1-\frac{1}{\mu}}$$

Therefore, the probability of a shortfall for this investment is

$$\begin{aligned} &P[\{\hat{a}^{ES}\mu e^{-\tau(\mu-1)\lambda} \geq 1\} \cap \{\tau \leq T\}] \\ =&1 - (2V_0 + e^{-\lambda\mu T})^{1/\mu} \end{aligned}$$

and the expected shortfall is again one half that number. □

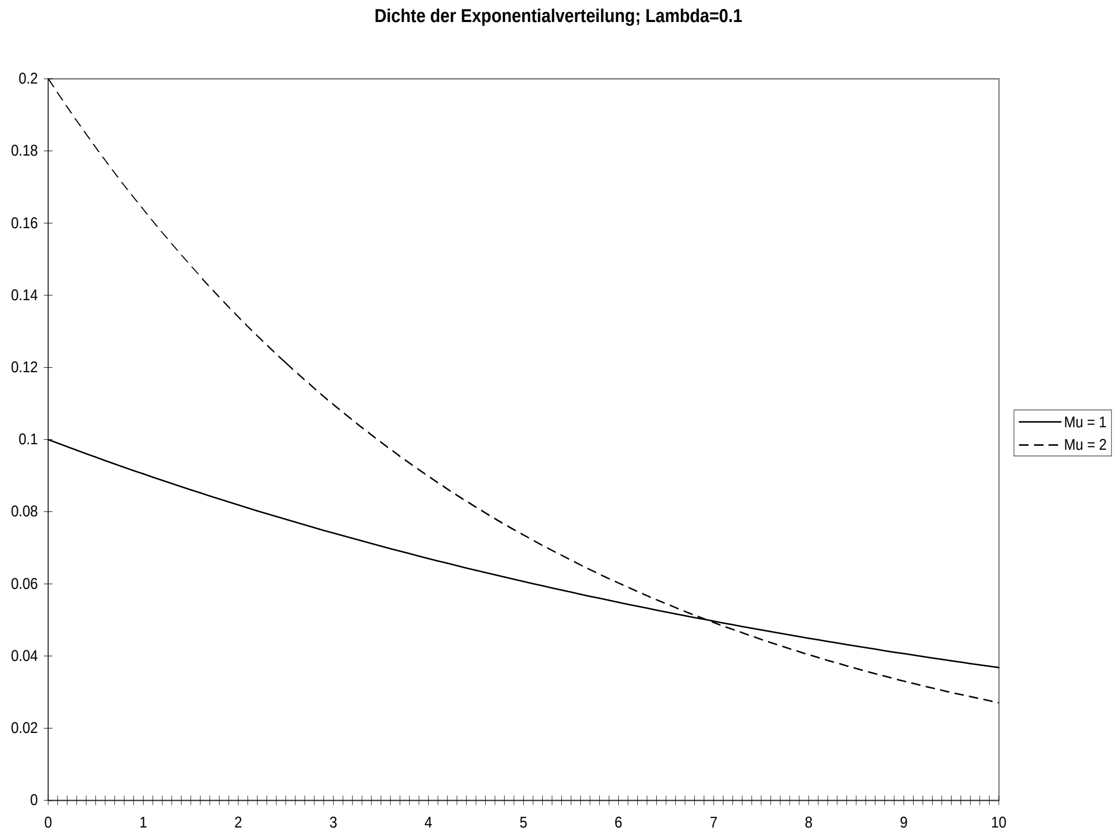

Figure 1: Density of Default Times

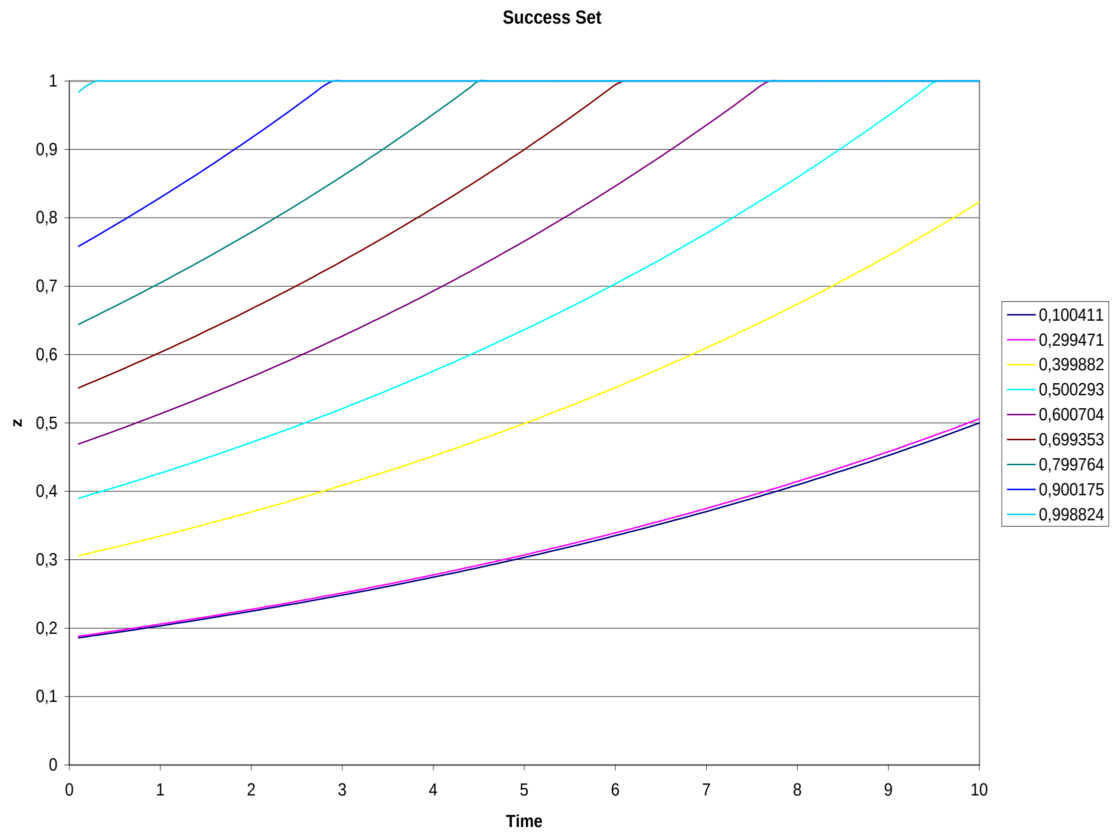

Figure 2: Credit Risky Bond: Success Set

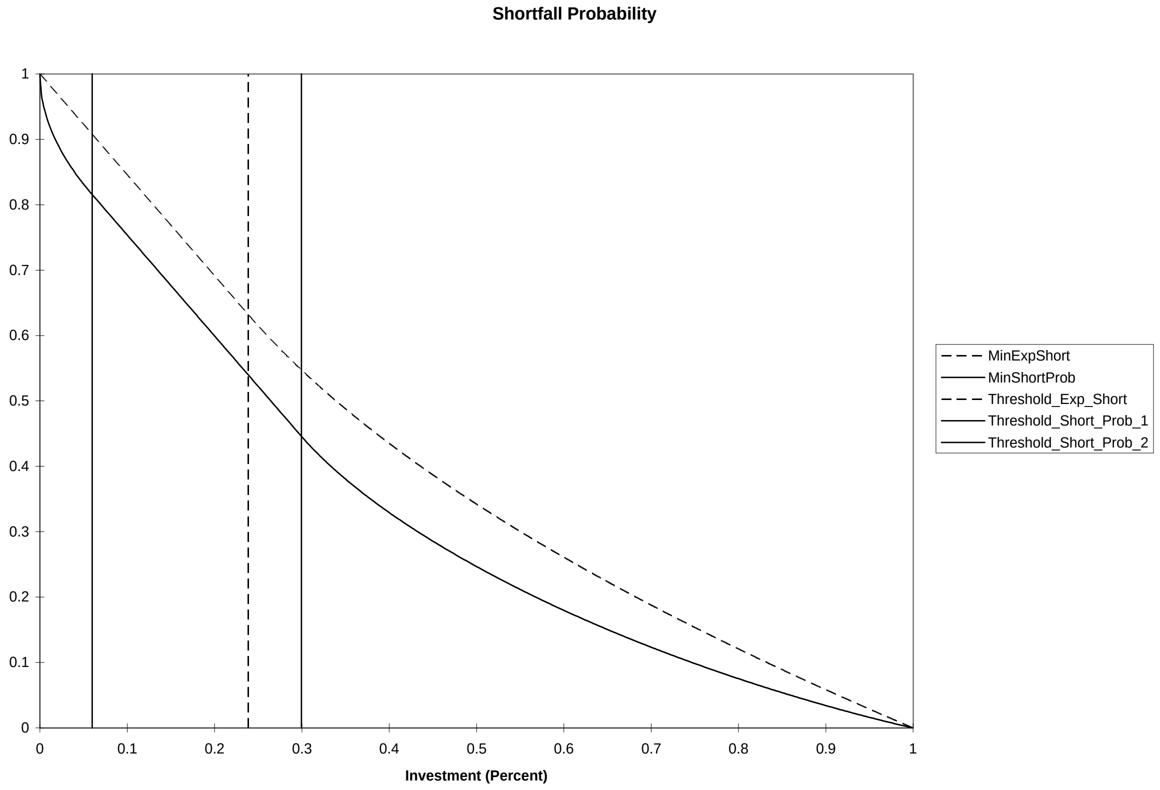

Figure 3: Credit Risky Bond: Shortfall Probability

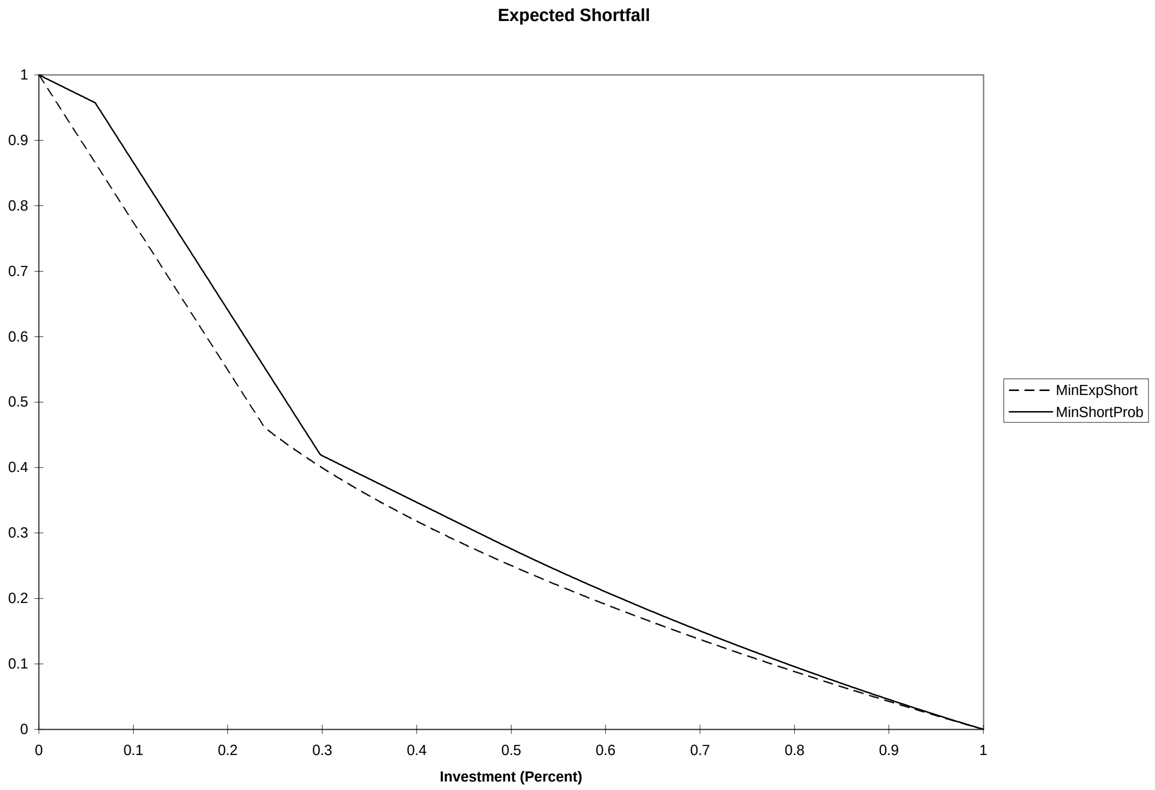

Figure 4: Credit Risky Bond: Expected Shortfall

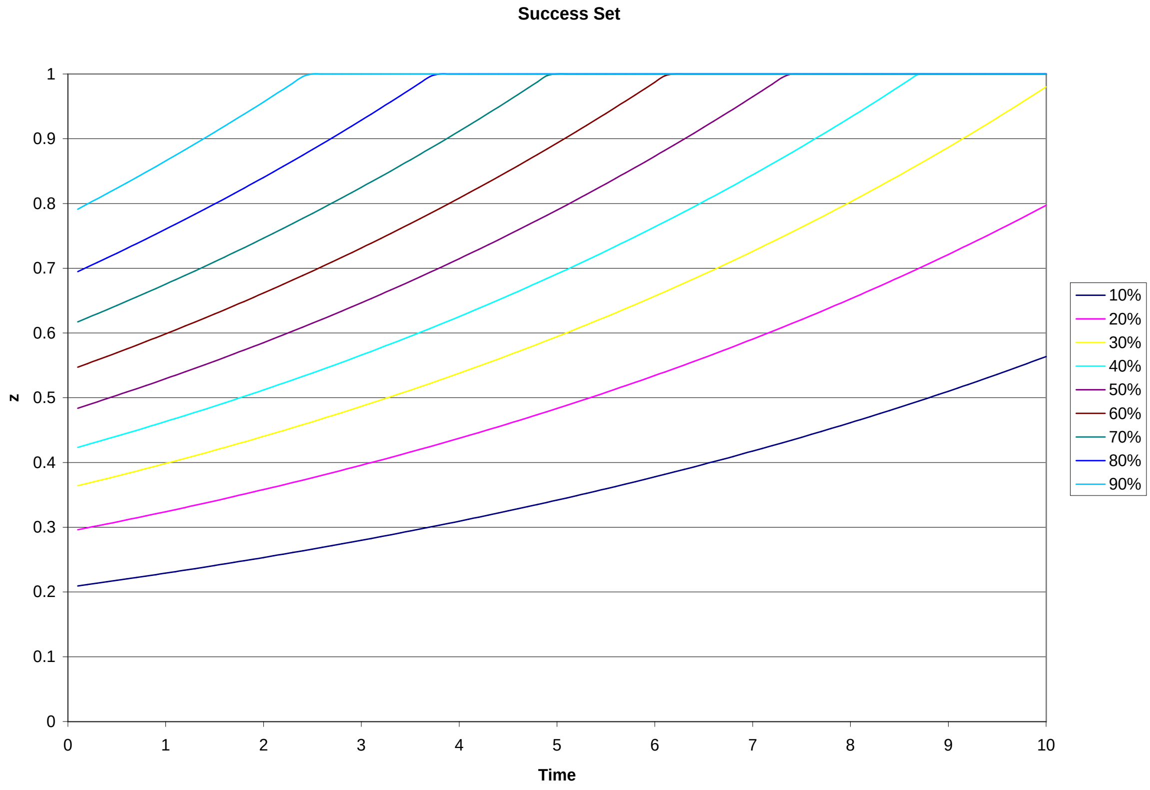

Figure 5: Credit Default Swap: Success Set

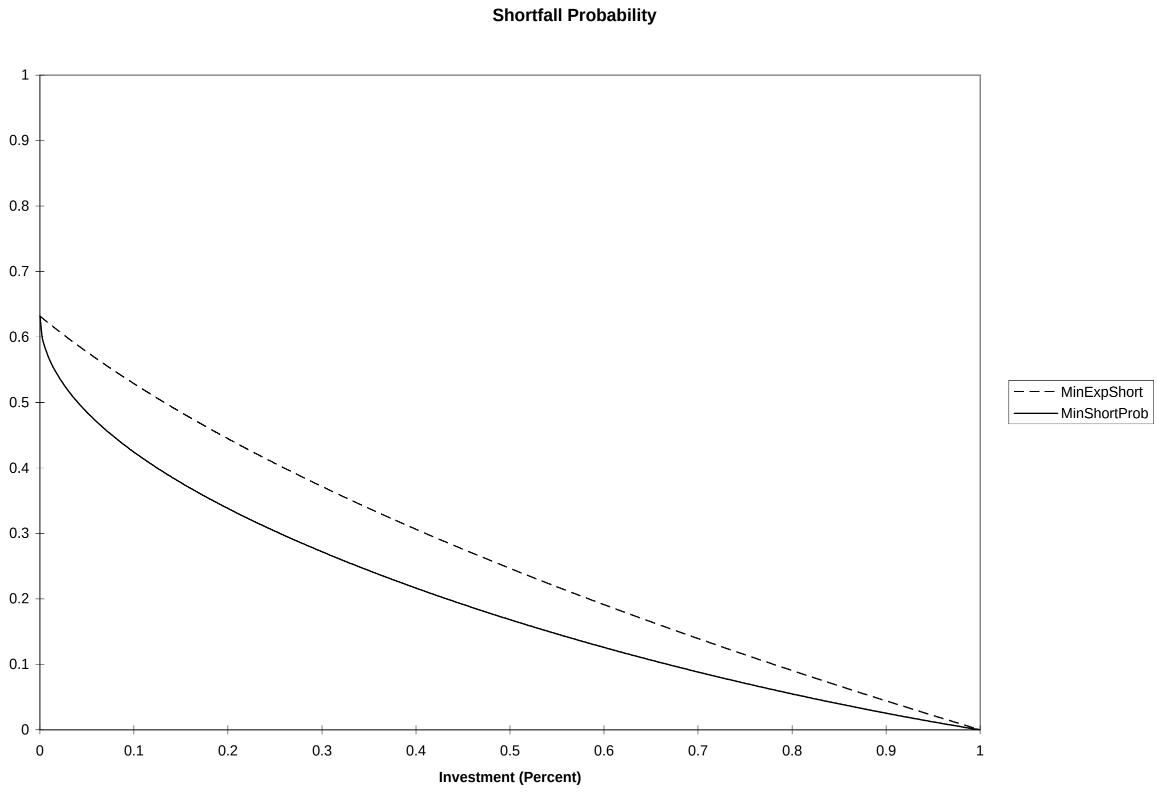

Figure 6: Credit Default Swap: Shortfall Probability

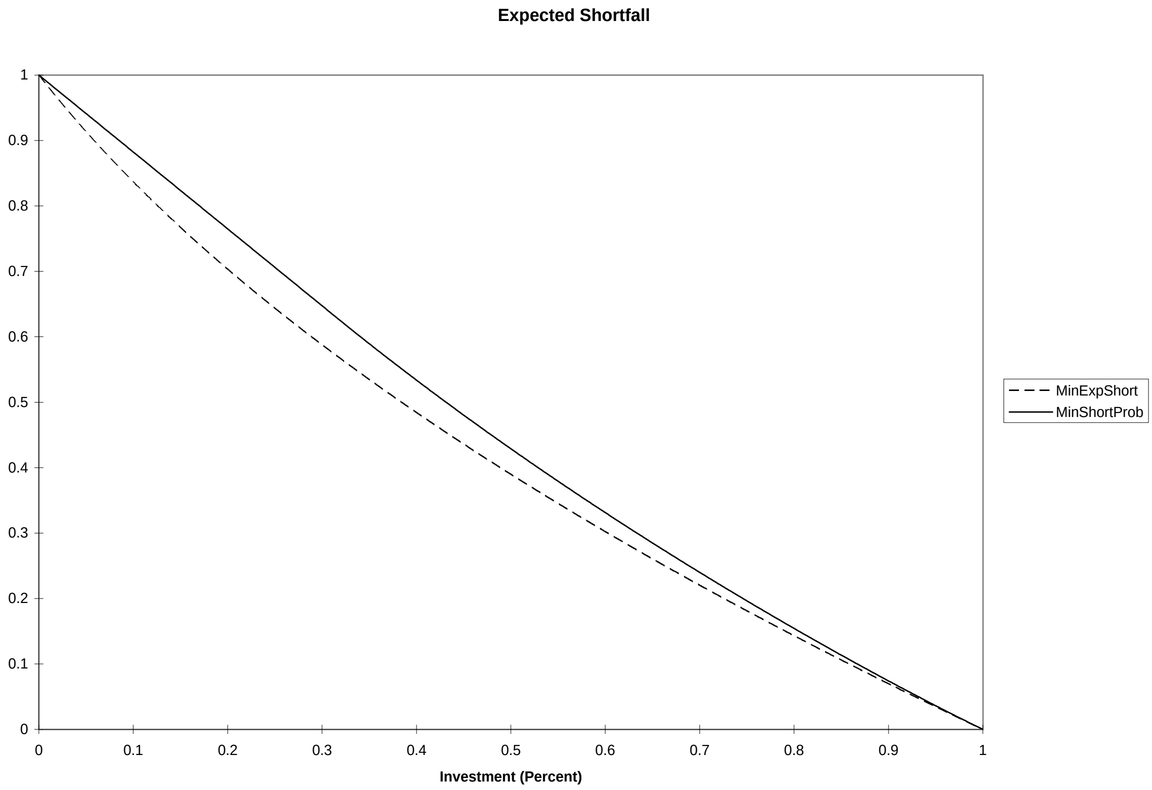

Figure 7: Credit Default Swap: Expected Shortfall

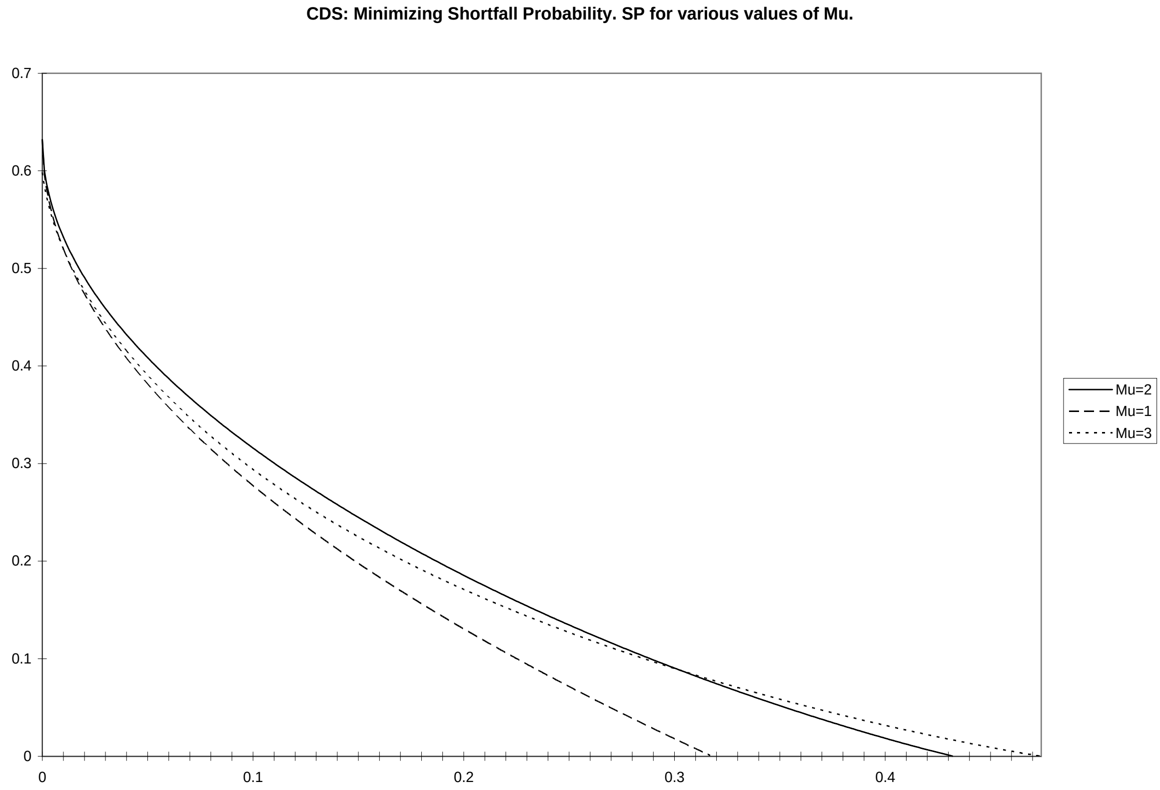

Figure 8: Credit Default Swap, Minimizing Shortfall Probability: SP for various values of Mu

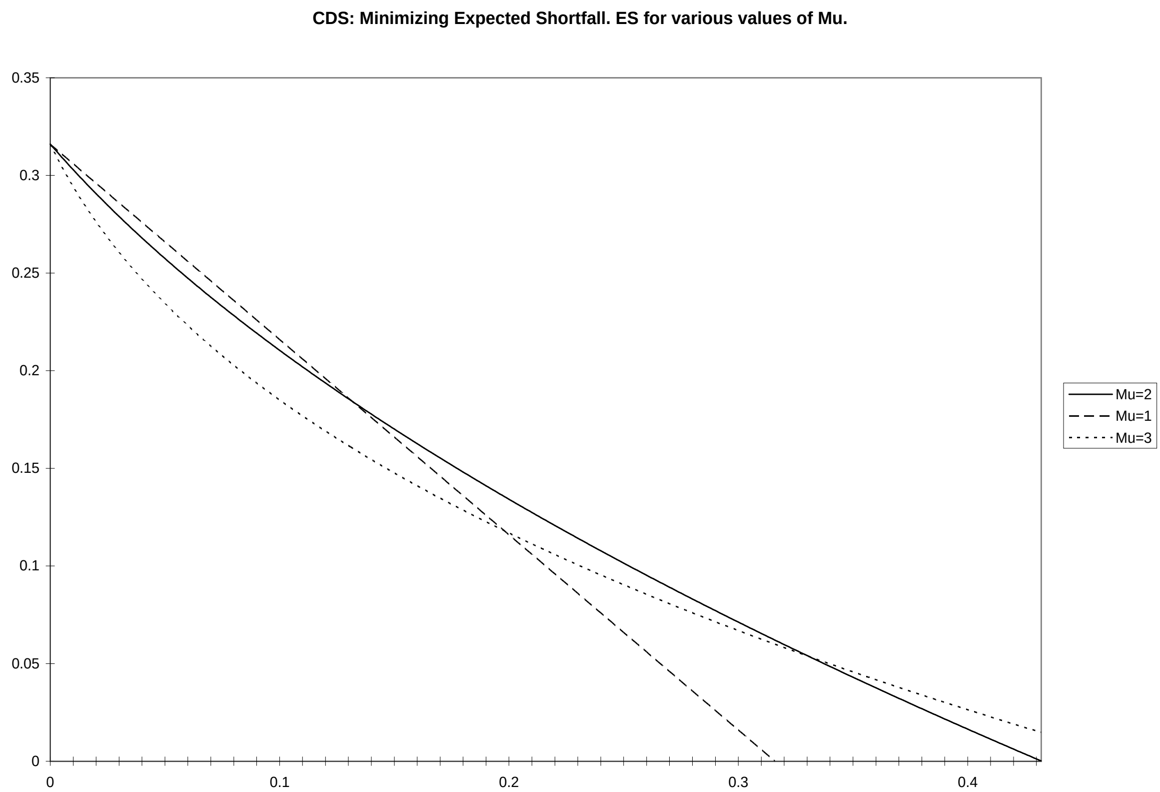

Figure 9: Credit Default Swap, Minimizing Expected Shortfall: ES for various values of Mu

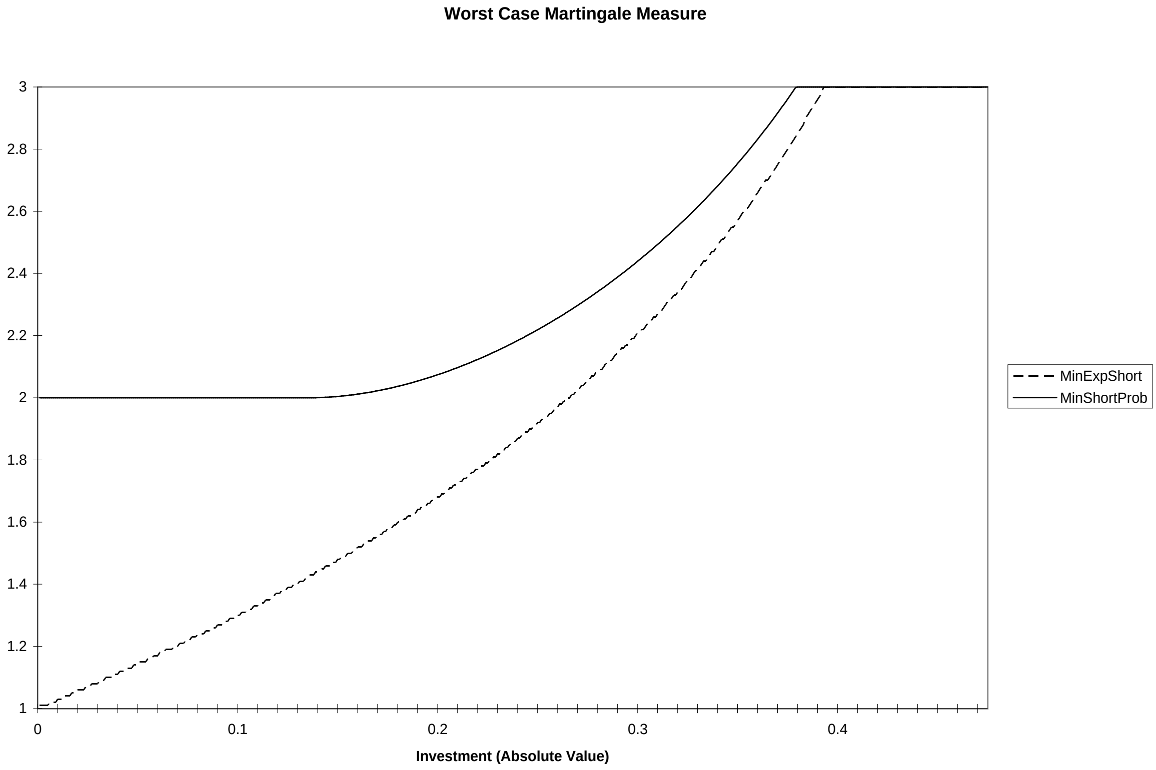

Figure 10: Incomplete Market: Worst Case Martingale Measure

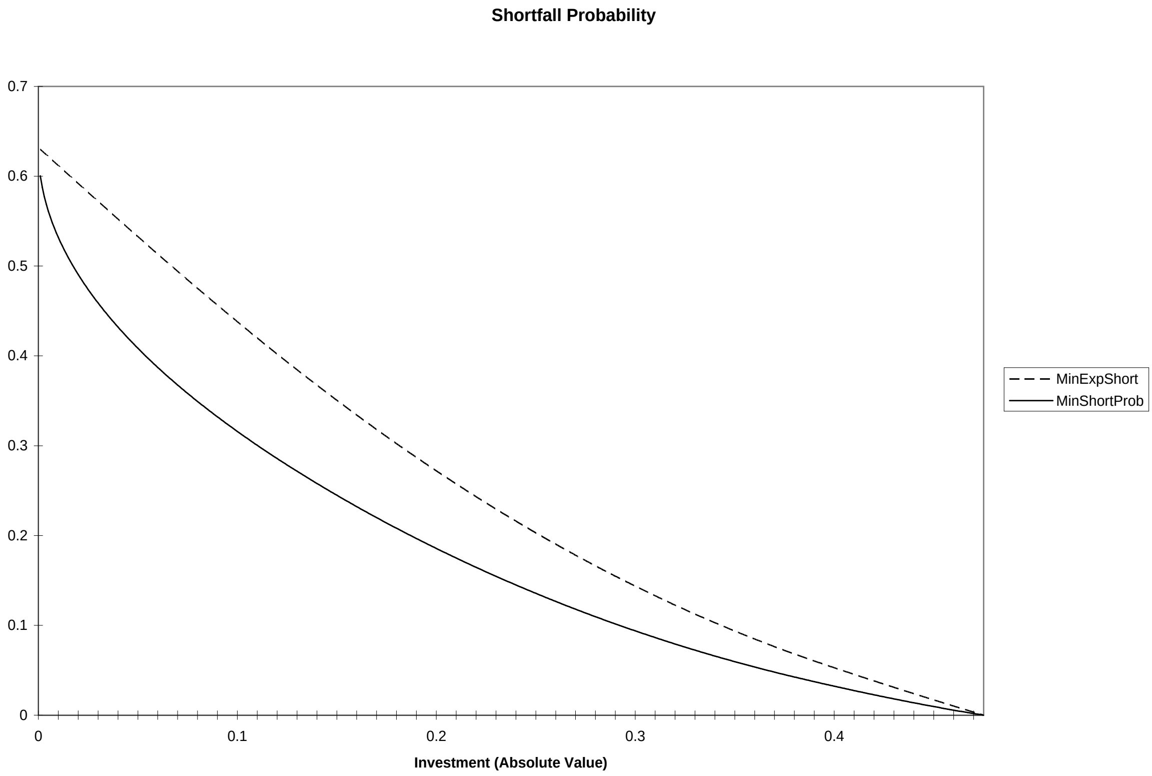

Figure 11: Incomplete Market, Credit Default Swap: Shortfall Probability

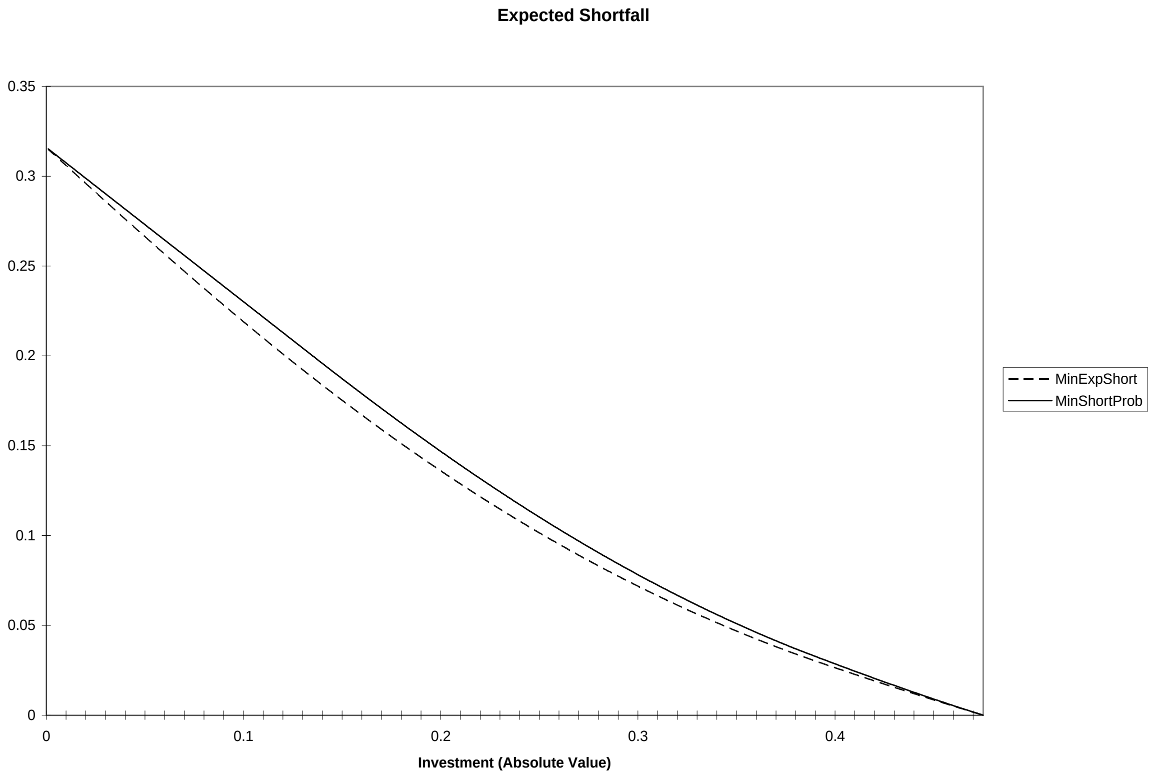

Figure 12: Incomplete Market, Credit Default Swap: Expected Shortfall

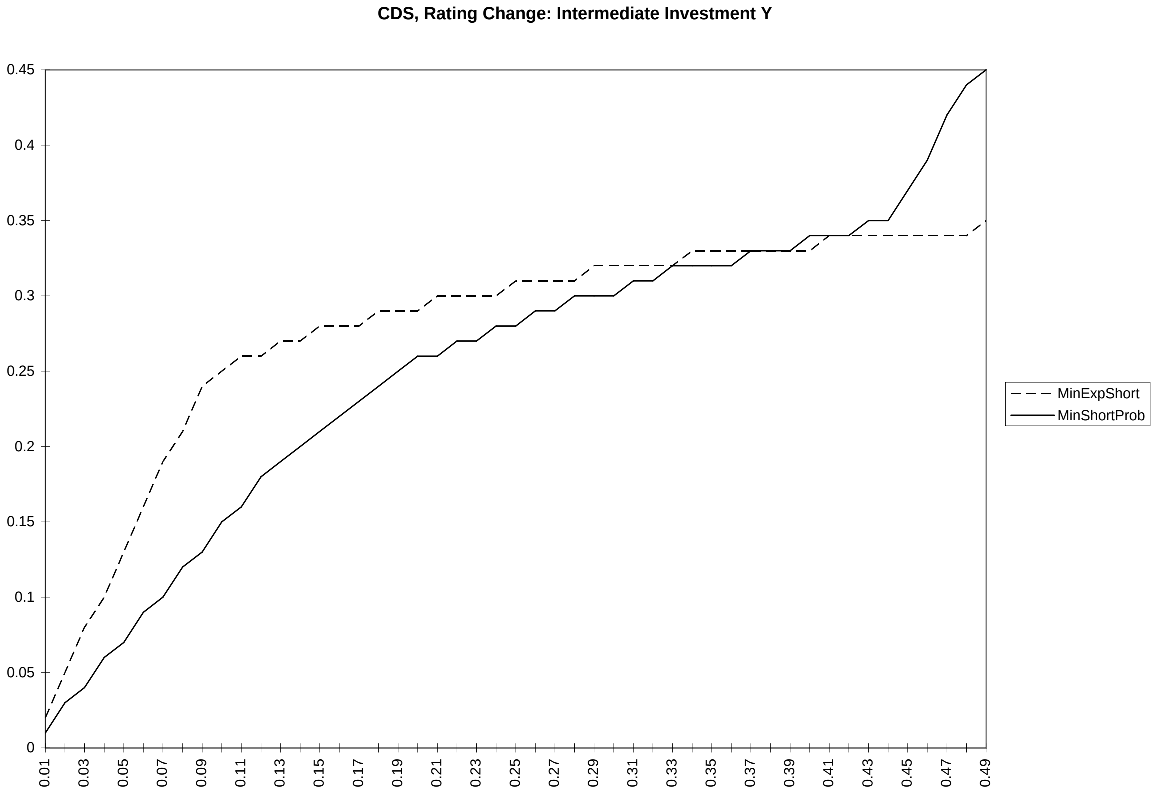

Figure 13: Incomplete Market, Rating Change: Intermediate Investment

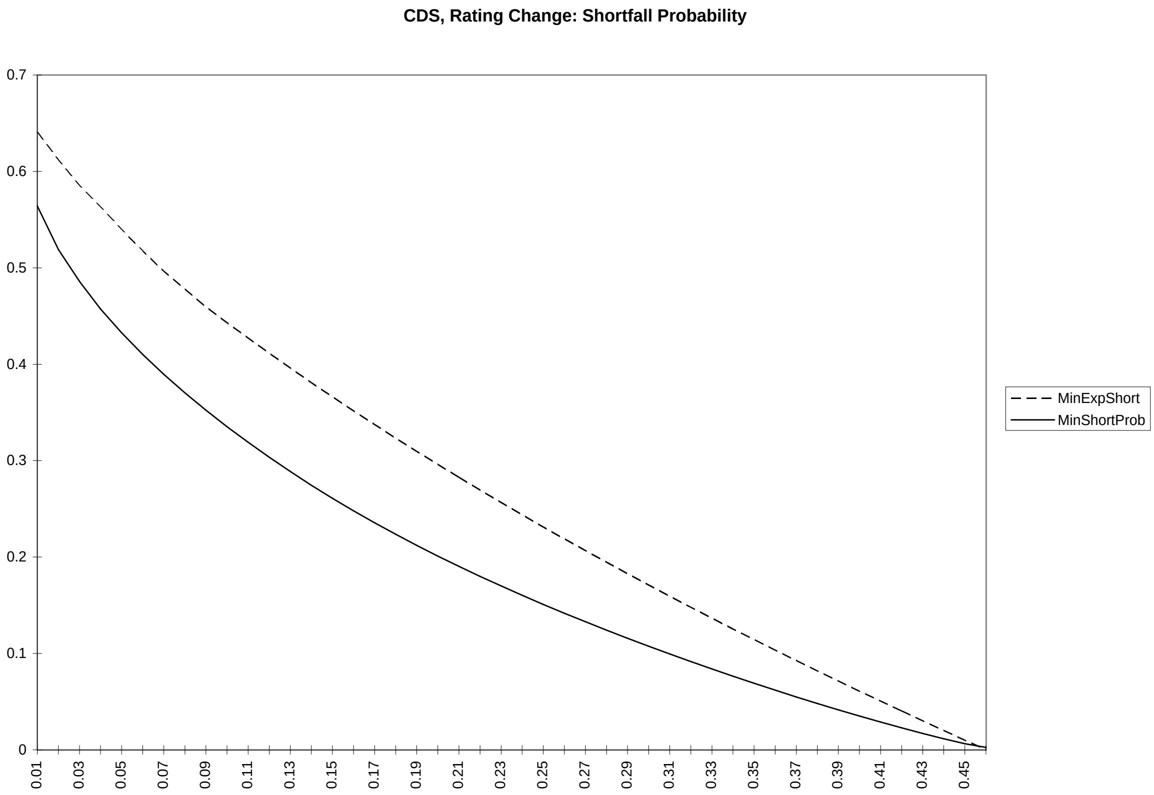

Figure 14: Incomplete Market, Rating Change: Shortfall Probability