PRICING CREDIT RISK DERIVATIVES

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Abstract. In this paper the pricing of several credit risk derivatives is discussed in an intensity-based framework with both risk-free and defaultable interest rates stochastic and possibly correlated. Credit spread puts and exchange options serve as main examples. The differences to standard interest rate derivatives are analysed. Where possible closed-form solutions are given.

1. Introduction

Credit derivatives are derivative securities whose payoff depends on the credit quality of a certain issuer. This credit quality can be measured by the credit rating of the issuer or by the yield spread of his bonds over the yield of a comparable default-free bond. In this paper we will concentrate on the latter case.

Credit Risk Derivatives (CRD) have enjoyed much attention and publicity in the derivatives world in recent years. They have been hailed as the major new risk management tool for managing credit exposures, as the most important new market in fixed income derivatives and as a breakthrough in loan risk management. This praise is at least to some part justified: Credit risk derivatives can make large and important risks tradeable. They form an important step towards market completion and efficient risk allocation, they can help bridge the traditional market segmentation between corporate loan and bond markets.

Despite their large potential practical importance, there have been very few works specifically on the pricing and hedging of credit risk *derivatives*. The majority of papers is concerned with the pricing of defaultable *bonds* which is a necessary prerequisite for credit derivatives pricing. The leading recovery model in this paper is the fractional recovery / multiple defaults model (see Duffie and Singelton [7, 8] and also Schönbucher [17, 18]). Applying this model to credit derivatives will yield important insights into the subtler points of the model used. The choice of model is not essential to the main point of the paper which is to show the methodology that has to be used in an

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intensity-based credit spread model. It is easy to transfer most of the results to other intensity-based models like e.g. the models by Jarrow and Turnbull [12], Lando [14, 13], Madan and Unal [16] or Artzner and Delbaen [1], in fact it is demonstrated that under some circumstances it may be more appropriate to model the recoveries following one of these models.

In the following section this model will be recapitulated, followed by a presentation of the credit derivatives that will be priced. As the vast majority of credit derivatives are OTC derivatives, no standard specification has evolved for second generation derivatives yet. Therefore we chose an array of typical and/or natural specifications to exemplify typical cases, including *default swaps*, straight *put options* on defaultable bonds, *credit spread puts* with different specifications for the treatment of default events, and *default digital puts*.

Then we define the specification of the stochastic processes. We chose a multifactor Cox Ingersoll Ross (CIR) [4] square-root diffusion model both for the credit spread and for the default-free short rate. This setup can allow for arbitrary correlation between credit spreads and the risk-free term structure of interest rates. To keep the results as flexible as possible we used this specification only in the last step of the calculation of the prices, and we also give the expectations that have to be calculated if another model is used.

This specification is used in the next section to price the credit risk derivatives that were introduced before. The pricing formulae are expressed in terms of the multivariate noncentral chi-squared distribution function, using results from Jamshidian [11]. The conclusion sums up the main results of the paper and points out the consequences that these results have for the further development of credit spread models.

2. Defaultable Bond Pricing with Cox Processes

The model used here is an extension of the Duffie-Singleton [7] model to multiple defaults. More details to the model can be found in Schönbucher [17, 18]. The new feature of this model is that a default does not lead to a liquidation but a reorganisation of the issuer: defaulted bonds lose a fraction q of their face value and continue to trade. This feature enables us to consider European-type payoffs in our derivatives without necessarily needing to specify a payoff of the derivative at default (although we will consider this case, too).

All dynamics that we introduce are understood to be specified with respect to a common martingale measure. First we will have to introduce some notation:

2.0.1. *The default-free term structure.* The term structure of default-free bond prices is given by the time t prices $P(t,T)$ of the default-free zero coupon bonds with maturity $T \geq t$. It is well-known that under the martingale measure

$$P(t,T) = \mathrm{E}_t \left[e^{-\int_t^T r(s)ds} \right], \tag{1}$$

where r denotes the default-free short rate. We will specify the dynamics of r later on. The discount factor $\beta_{t,T}$ over the interval $[t,T]$ is defined as

$$\beta_{t,T} = e^{-\int_t^T r(s)ds}. \tag{2}$$

2.0.2. *The defaultable bond prices.* Defaultable bonds are the underlying securities to most credit risk derivatives. The time t price of a defaultable zero coupon bond with maturity T is denoted [1] with

$$P'(t,T).$$

The defaults themselves are modeled as follows:
There is an increasing sequence of stopping times $\{\tau_i\}_{i\in\mathbb{N}}$ that define the times of default. At each default the defaultable bond's face value is reduced by a factor q, where q may be a random variable itself. This means that a defaultable zero coupon bond's final payoff is the product

$$Q(T) := \prod_{\tau_i \le T} (1 - q_i) \tag{3}$$

of the face value reductions after all defaults until the maturity T of the defaultable bond. The loss quotas q_i can be random variables drawn from a distribution $K(dq)$ at time τ_i, but for the first calculations we will assume $q_i = q$ to be constant. To simplify notation the time of the *first* default will often be referred to with $\tau := \tau_1$.

We assume that the times of default τ_i are generated by a Cox process. Intuitively, a Cox Process is defined as a Poisson process with stochastic intensity h (see Lando [13]). Formally the definition is:

Definition 1. *The default counting process*

$$N(t) := \max\{i | \tau_i \le t\} = \sum_{i=1}^{\infty} \mathbf{1}_{\{\tau_i \le t\}} \tag{4}$$

is called a Cox process, if there is a nonnegative stochastic process $h(t)$ (called the intensity of the Cox process), such that given the realization $\{h(t)\}_{\{t>0\}}$ of the intensity, $N(t)$ is a time-inhomogeneous Poisson process with intensity $h(t)$.

This means that the probability of having exactly n jumps, given the realisation of h, is

$$\Pr\left[\, N(T) - N(t) = n \mid \{h(s)\}_{\{T\ge s\ge t\}} \,\right] = \frac{1}{n!}\left(\int_t^T h(s)ds\right)^n \exp\{-\int_t^T h(s)ds\}, \tag{5}$$

and the probability of having n jumps (without knowledge of the realisation of h) is found by conditioning on the realisation of h *within* an outer expectation operator:

$$\begin{aligned} \Pr\left[\, N(T) - N(t) = n \,\right] &= \mathrm{E}_t\left[\; \Pr\left[\, N(T) - N(t) = n \mid \{h(s)\}_{\{T\ge s\ge t\}} \,\right] \;\right] \\ &= \mathrm{E}_t\left[\; \frac{1}{n!}\left(\int_t^T h(s)ds\right)^n \exp\{-\int_t^T h(s)ds\} \;\right], \end{aligned} \tag{6}$$

[1] Defaultable quantities usually carry a dash (').

Specifically, the *survival probability* $P_h(t,T)$ of survival until time T as viewed from time t is

$$P_h(t,T) = \mathrm{E}\left[\, \mathbf{1}_{\{\tau>T\}} \,\right] = \mathrm{E}\left[\, e^{-\int_t^T h(s)ds} \,\right]. \tag{7}$$

This application of the law of iterated expectations will prove extremely useful later on, it was first used in a credit risk context by Lando [13].

The specification of the default trigger process as a Cox process precludes a dependence of the default intensity on previous defaults and also ensures totally inaccessible stopping times τ_i as times of default. Apart from this it allows rich dynamics of the intensity process, specifically, we can reach stochastic credit spreads. Equations (5) and (6) will be used frequently later on.

It is now easily seen (using the iterated expectations, see also Duffie and Singleton and Schönbucher [7, 17, 18]) that in this setup the price of a defaultable zero coupon bond is given by

$$P'(t,T) = Q(t)\,\mathrm{E}_t\left[\, e^{-\int_t^T r'(s)ds} \,\right]. \tag{8}$$

The process r' is called the defaultable short rate r' and it is defined by

$$r' = r + hq. \tag{9}$$

Here r is the default-free short rate, h the hazard rate of the defaults and q is the loss quota in default. If q is stochastic then q has to replaced by its (local) expectation $q_t^e = \int qK_t(dq)$ in equation (9). The similarity of equation (8) to the default-free bond pricing equation (1) will be used later on.

It is convenient to decompose the defaultable bond price P' as follows:

$$P'(t,T) = Q(t)P(t,T)\hat{P}(t,T). \tag{10}$$

Here $Q(t)$ denotes the number of defaults until now (time t). For all applications we will be able to set $t=0$ and thus $Q(t)=1$, but for the analysis at intermediate times it is important to be clear about the notation[2]. The defaultable bond price $P'(t,T)$ is thus the product of $Q(t)$, the influence of previous defaults, and the product of the default-free bond price $P(t,T)$ and the third factor $\hat{P}(t,T)$ which is uniquely defined by equation (10), or equivalently:

$$\hat{P}(t,T) = \frac{1}{Q(t)}\frac{P'(t,T)}{P(t,T)}. \tag{11}$$

$\hat{P}(t,T)$ is related to the *survival probability* of the defaultable bond: If r and h are independent and there is a total loss $q=1$ at default then $\hat{P}(t,T)$ is the probability (under the martingale measure) that there is no default in $[t,T]$. If r and h are not independent, $\hat{P}(t,T)$ is the survival probability under the T-forward measure

$$\hat{P}(t,T) = \mathrm{E}_t^T\left[\, e^{-\int_t^T qh(s)ds} \,\right]. \tag{12}$$

[2] For example, at the expiry date of an option we would like to separate previous defaults and credit spreads in the price of the underlying.

(Under independence $\mathrm{E}_t^T\left[e^{-\int_t^T qh(s)ds}\right]$ and $\mathrm{E}_t\left[e^{-\int_t^T qh(s)ds}\right]$ coincide.)

3. The Credit Derivatives

This section contains a short overview of the specification and uses of the credit derivatives we will concentrate on later. The credit derivatives can be broadly classified into first generation products like default swaps and puts and default digital puts, and second generation products like Put options on defaultable bonds and Credit *Spread* Put options.

3.1. **First Generation Products.** The first generation CRDs that we will consider here are: default swap (also known as credit swap), default put, and default digital put.

A *default swap* on a given defaultable bond (the reference security) is a swap contract where

- counterparty A pays the swap rate s
  until default or maturity T of the default swap, and
- conterparty B pays *at default* the difference between the recovery value of the defaulted bond (measured by its post-default market price) and its notional value.

If there is no default of the reference security until T, counterparty B pays nothing.

The reference security to a default swap is usually not a defaultable zero coupon bond but a defaultable coupon bond which trades close to par. Apart from the scarcity of defaultable zero coupon bonds in the markets this also takes care of another problem: the prepayment effect at early defaults. If there was an early default, a large fraction of counterparty B's payment would not compensate for the default loss but it would compensate for the (risk-free) discounting of the final payment of the bond, whereas the default swap was only meant to cover the loss quota in default.

The price of a default swap is closely related to the price of a *defaultable floating rate note (FRN).* Although the defaultable FRN is not a derivative, its pricing is nevertheless nontrivial because it does not always have to trade at par like default-free FRNs.

A *default put* is similar to a default swap, with the difference, that counterparty A pays a lump sum payment up front instead of a regular rate until default. Again the same remarks with respect to the reference security of the default swap applies.

The *default digital put* differs from the default put in that the payment of counterparty B at default is fixed in advance, i.e. B pays at default \$ 1 to A. This mitigates potential problems in identifying the defaulted bond's recovery value. For constant and known recovery rates the default digital put is equivalent to the default put. The default digital put is also an important building block of more complex credit derivatives which often imply a constant rebate at default. These derivatives can be viewed as derivatives that are knocked out at default plus a default digital put that pays off the rebate.

There is obviously also a swap version of the default digital put which is called a *default digital swap*. Here A pays the swap rate until default or until maturity of the contract. The valuation of a periodic payment until a fixed date in the future is standard and will not be treated here but we will consider the case when the fixed side of the swap is terminated at default and give the value of this payoff stream.

3.2. **Second Generation Credit Risk Derivatives.** The two types of second generation instruments that we will discuss here are plain put options on defaultable bonds and credit spread put options.

As with all credit risk derivatives it is very important how a default event is handled, i.e. whether the derivative covers the defaults or whether the contract is killed at default. Lump sum payments at default can be modelled as additional default digital put options, therefore we will not consider them here any more.

A plain *put option* on a defaultable bond is different from classical bond options in that the price of the underlying (the defaultable bond) may jump downwards at defaults. It offers protection against rising default-free interest rates, rising credit spreads and (if the option survives defaults) against defaults. Concerning the treatment of defaults we will consider both, (a) the option is not modified at intermediate defaults and (b) the option is knocked out at default.

A *credit spread put option* on a defaultable bond only conditions on the credit spread of the defaultable bond price over the spread of an otherwise identical default-free bond price. Credit spread puts can be viewed as exchange options, giving the bearer the right to exchange one defaultable bond for a certain number (< 1) of default free bonds. Again we can have survival and knockout at intermediate defaults.

Obviously one can imagine similar contracts with American early-exercise features or with barriers on the spread etc., and with different underlying securities. These will have to be valued with a suitable numerical algorithm. Here we shall concentrate on the analytic valuation of the derivatives specified above.

4. Valuation of First Generation Credit Risk Derivatives

In this section we begin the discussion of the pricing of CRDs with the first-generation products that were introduced before. The simpler structure of these products allows us to keep our analysis at a more general level, we do not need to specify the stochastic differential equations satisfied by interest rates and credit spreads yet.

The default digital put option and the default digital swap will be analysed first, as these products are the simplest and the pricing of the other products can often be reduced to the pricing of default digital puts. Next are defaultable FRNs and the default swap.

The following data is needed:

- the default-free term structure of bond prices
  $P(0,T) = \mathrm{E}\left[\, e^{-\int_0^T r(s)ds} \,\right]$
- the defaultable bond prices under zero recovery
  $P'(0,T) = \mathrm{E}\left[\, e^{-\int_0^T r(s)+h(s)ds} \,\right]$
- the survival probabilities
  $P_h(0,T) = \mathrm{E}\left[\, e^{-\int_0^T h(s)ds} \,\right]$.

We also need some information about the correlation between the default-free term structure of interest rates and the term structure of credit spreads. In this section we will assume independence of the two (which will yield $P_h = P'/P$), the case of correlation requires a more precise specification and will be treated in the following section.

We assumed above (to simplify the notation) that the data is given for a recovery rate of zero (i.e. a loss quota $q = 1$ of one). This is especially easy to recover from the defaultable bond prices if the recovery is modeled in terms of equivalent default-free securities[3] and the expected recovery rate is known:

In this model, 1 defaultable security F' becomes $1 - c$ otherwise equivalent default-free securities F at default, where c can be stochastic. Thus, the price of a defaultable zero coupon bond is given by

$$\begin{aligned} P'(0,T) &= \mathrm{E}\left[\, \beta_{0,T}\mathbf{1}_{\{\tau>T\}} + c\beta_{0,\tau}P(\tau,T)\mathbf{1}_{\{\tau\le T\}} \,\right] \\ &= \mathrm{E}\left[\, \beta_{0,T}\mathbf{1}_{\{\tau>T\}} \,\right] + c\mathrm{E}\left[\, \beta_{0,T}\mathbf{1}_{\{\tau\le T\}} \,\right] \\ &= \mathrm{E}\left[\, \beta_{0,T}\mathbf{1}_{\{\tau>T\}} \,\right] + c\mathrm{E}\left[\, \beta_{0,T} \,\right] - c\mathrm{E}\left[\, \beta_{0,T}\mathbf{1}_{\{\tau>T\}} \,\right] \\ &= (1-c)\mathrm{E}\left[\, \beta_{0,T}\mathbf{1}_{\{\tau>T\}} \,\right] + cP(0,T), \end{aligned}$$

the defaultable zero coupon bond price can be decomposed into c default-free bonds and $(1-c)$ defaultable bonds with zero recovery $\mathrm{E}\left[\, \beta_{0,T}\mathbf{1}_{\{\tau>T\}} \,\right]$. Given information about c, the prices of zero-recovery defaultable bonds are therefore easily implied from $P(0,T)$ and $P'(0,T)$.

In the fractional recovery model it is not so easy to derive the value of a zero-recovery defaultable bond $\mathrm{E}\left[\, e^{-\int_0^T r(s)+h(s)ds} \,\right]$ just from knowledge of the recovery rate q, the defaultable bond price $\mathrm{E}\left[\, e^{-\int_0^T r(s)+qh(s)ds} \,\right]$ and the default-free bond prices. Nevertheless, with a suitable choice of (time dependent) parameters both recovery models, fractional recovery (with q as loss *quota*) and recovery in terms of default-free securities (with c as default *loss*) can be transformed into each other: The value of the security in default is only expressed in different units, once in terms of defaultable bonds and once in terms of default-free bonds. Both approaches are therefore equivalent and one should use the specification that is best suited for the issue at hand. Here the recovery in terms of default-free securities has an advantage over the fractional recovery model.

[3]E.g. this model of recovery has been proposed by Lando [13] and Madan and Unal [16].

The assumptions about the available data are open to criticism. One of the largest difficulties in credit risk modelling is the scarcity of useful data, and only rarely (maybe for some souvereign issuers) a full term structure of defaultable bond prices is available, let alone survival probabilities or recovery rates. These data problems will have to be addressed separately, here we just point out that all of the defaultable inputs (recovery rates, $P'(0,T)$ or $P_h(0,T)$) can also stem from other sources than market prices, e.g. fundamental analysis or historical data of the same rating class or industry. The only requirement is that this information is given under risk-adjusted (pricing) probabilities, we are working under the martingale measure already.

As shown above, the zero recovery assumption is mainly a tool to simplify the notation if defaultable bond prices, default-free bond prices and expected default losses c are known. It can be ignored if the survival probabilities are given directly. The assumption of independence on the other hand is much more restrictive. This assumption is needed to evaluate some expectations and can only be dropped in the full multifactor specification of interest-rate and spread dynamics in the following section.

4.1. **Default Digital Payoffs.** The default digital put option has the payoff 1 *at the time of default.* The timing is important because it determines until when the payoff has to be discounted, but we will consider a simpler specification in a first step:

As a first example consider an European-style default digital put which pays off 1 *at* T iff there has been a default at some time before (or including) T. Its value is easily found (τ denotes the time of the first default):

$$\begin{aligned} D =& \mathrm{E}\left[e^{-\int_0^T r(s)ds} \mathbf{1}_{\{\tau<T\}} \right] \\ =& \mathrm{E}\left[\mathrm{E}\left[e^{-\int_0^T r(s)ds} \mathbf{1}_{\{\tau<T\}} \,\middle|\, \{h(s)\}_{s\le T} \right] \right] \\ =& \mathrm{E}\left[e^{-\int_0^T r(s)ds} (1 - e^{-\int_0^T h(s)ds}) \right] \\ =& P(0,T) - \mathrm{E}\left[e^{-\int_0^T r(s)+h(s)ds} \right] = P(0,T) - P'(0,T). \end{aligned} \tag{13}$$

This follows also direclty from the fact that – under zero recovery – this default digital put replicates the payoff of a portfolio that is long one default-free bond and short one defaultable bond.

In the case when the payoff takes place *at* default, the expectation to calculate is

$$D = \mathrm{E}\left[e^{-\int_0^\tau r(s)ds} \mathbf{1}_{\{\tau<T\}} \right].$$

Conditioning on the realization of h yields

$$\begin{aligned} D =& \mathrm{E}\left[e^{-\int_0^\tau r(s)ds} \mathbf{1}_{\{\tau<T\}} \right] \\ =& \mathrm{E}\left[\mathrm{E}\left[e^{-\int_0^\tau r(s)ds} \mathbf{1}_{\{\tau<T\}} \,\middle|\, h \right] \right]. \end{aligned} \tag{14}$$

Now note that the probability distribution of τ (given h) is

$$\Pr[\,\tau \le T\,] = 1 - \Pr[\,\tau > T\,] = 1 - \exp\{-\int_0^T h(s)ds\},$$

so the density of τ is

$$h(t)\exp\{-\int_0^t h(s)ds\}. \tag{15}$$

Substituting equation (15) into (14) yields for the price of the default digital put:

$$\begin{aligned}
D &= \mathrm{E}\left[\,\mathrm{E}\left[\,e^{-\int_0^\tau r(s)ds}\mathbf{1}_{\{\tau<T\}}\;\middle|\;h\,\right]\,\right] \\
&= \mathrm{E}\left[\,\int_0^T h(t)e^{-\int_0^t h(s)ds}e^{-\int_0^t r(s)ds}dt\,\right] \\
&= \int_0^T \mathrm{E}\left[\,h(t)e^{-\int_0^t h(s)ds}e^{-\int_0^t r(s)ds}\,\right]dt, \qquad (16)
\end{aligned}$$

where we assume sufficient regularity to allow the interchange of expectation and integration. Under independence the expectation in the integral in equation (16) can be further simplified to

$$\begin{aligned}
&\mathrm{E}\left[\,h(t)e^{-\int_0^t h(s)ds}e^{-\int_0^t r(s)ds}\,\right] \\
=&P(0,t)\mathrm{E}\left[\,h(t)e^{-\int_0^t h(s)ds}\,\right]
\end{aligned}$$

changing the measure

$$\begin{aligned}
=&P(0,t)\hat{P}(0,t)\,\mathrm{E}_t^{\hat{P}_t}\left[\,h(t)\,\right] \\
=&P(0,t)\hat{P}(0,t)h(0,t) = P'(0,t)h(0,t).
\end{aligned}$$

Here $\hat{P}(0,t) = \mathrm{E}\left[\,e^{-\int_0^t h(s)ds}\,\right]$ is the survival probability, and

$$h(0,t) = -\frac{\partial}{\partial T}\ln \hat{P}(0,T)\bigg|_{T=t}$$

is the associated 'forward rate' of the spreads (for zero recovery $q = 1$). The survival probability $\hat{P}(0,t)$ was used as numeraire in the change of measure from the martingale measure to $\hat{P}^t$ to remove $\exp\{-\int_0^t h(s)ds\}$ from the expectation. The final line follows from the well-known fact in risk-free interest rate theory that under the T-forward measure the expectation of the short rate $r(T)$ at time T is exactly equal to the T-forward rate $f(0,T)$. Here we have a mathematically equivalent situation, just changing the notation such that $\hat{P}(0,T)$ are our bond prices, $\hat{P}_T$ is the forward measure, $h(T)$ is the short rate at T and $h(0,T)$ is the T-forward rate.

Putting all together we gain the following representation for the value of a default digital put with maturity T under independence and zero recovery:

$$D = \int_0^T P'(0,t)h(0,t)dt. \tag{17}$$

The restrictive assumption of *independence* of interest rates and spreads will be relaxed later on.

There is another simple representation of the default digital put if it is specified in a *swap*-like form. The swap specification is as follows:

Party A pays 1 at default if there is a default. Discounted this can be written as

$$\beta_{0,\tau}\mathbf{1}_{\{\tau\leq T\}} = \int_0^T \beta_{0,t} d\mathbf{1}_{\{\tau\leq T\}}.$$

Party B pays a regular fee $x(t)dt$ until default, i.e. he pays until maturity of the swap (also already in discounted form)

$$\int_0^T \beta_{0,t} x(t)\mathbf{1}_{\{t\leq\tau\}}dt$$

What should be the fair fee $x(t)$? If it can be chosen variable it should be the local credit spread $h(t)$. The intuitive reason is that by

$$\mathrm{E}_t\left[\ \mathbf{1}_{\{\tau\in[t,t+dt]\}}\ \right] = h(t)dt$$

$h(t)$ it is the local default probability at time t. Suppose that the default protection is re-negotiated for every small time-interval $[t, t+dt]$, then the fee should be $h(t)dt$ for this interval.

Mathematically this follows from the fact that $\int_0^t h(s)\mathbf{1}_{\{s\leq\tau\}}ds$ is the predictable compensator of $\mathbf{1}_{\{\tau\leq t\}}$, thus the expected values of stochastic integrals w.r.t. either of them are equal:

$$\mathrm{E}_t\left[\ \beta_{0,\tau}\ \right] = \mathrm{E}_t\left[\int_0^T \beta_{0,t} d\mathbf{1}_{\{\tau\leq t\}}\right] = \mathrm{E}_t\left[\int_0^T \beta_{0,t} h(t)\mathbf{1}_{\{t\leq\tau\}}dt\right].$$

The expected discounted values of both sides of the swap are equal. This result is independent of the maturity of the swap and also valid for correlation between interest rates and defaults.

The previous result required a continuous adaptation of the swap rate. If the swap rate $x(t)$ is to be determined at the inception of the swap at $t = 0$ for all future times $t \leq T$, then the value of the fixed leg is (again assuming the necessary regularity for the interchange of integration and expectation)

$$\begin{aligned}\mathrm{E}\left[\ \int_0^T \beta_{0,t}x(t)\mathbf{1}_{\{\tau\geq t\}}dt\ \right] &= \int_0^T x(t)\ \mathrm{E}\left[\ \beta_{0,t}\mathbf{1}_{\{\tau\geq t\}}\ \right]\ dt \\ &= \int_0^T x(t)P'(0,t)dt. \qquad (18)\end{aligned}$$

Equation (18) gives the value of a constant or time-dependent income stream (i.e. a fixed swap leg) that *is terminated at default* or T. Equation (18) remains valid under correlation.

The value of the fixed leg in equation (18) has to be set equal to the value of the variable (i.e. the default-protection) side of the swap, which is given (under independence) in equation (17):

$$\int_0^T P'(0,t)h(0,t)dt.$$

For a constant fixed rate $x(t) = x$ this yields the fair swap rate for a default digital swap:

$$x = \frac{\int_0^T P'(0,t)h(0,t)dt}{\int_0^T P'(0,t)dt}. \tag{19}$$

Compare this to the plain vanilla fixed-for-floating swap rate in default-free interest rate model

$$s_r = \frac{1 - P(0,T)}{\int_0^T P(0,t)dt} = \frac{\int_0^T P(0,t)f(0,t)dt}{\int_0^T P(0,t)dt}.$$

(The equality follows from $\frac{\partial}{\partial t}P(0,t) = -f(0,t)P(0,t)$ and integrating by parts.) We see that the equation for the default digital swap rate has a structure that is very similar to the structure of the equation for plain fixed-for-floating swap rates. The forward rate $f(0,t)$ in the risk-free case has been replaced by the credit spread $h(0,t)$ and the default-free bond prices $P(0,t)$ are replaced by their defaultable counterparts $P'(0,t)$. In both cases the numerator denotes the value of the floating leg of the swap, and the denominator the fixed leg of the swap, where the floating leg for the default digital swap is the default contingent payment.

The fact that the default swap is terminated at default introduces another difference to default-free swaps: The fixed leg has to be discounted using *defaultable* bond prices (P', the defaultable bond prices for zero recovery). If the swap was not killed at default, the fixed side would have to be discounted with default-free rates.

4.2. **Defaultable FRNs and Default Swaps.** There is a link between default digital puts and the prices of defaultable floating rate notes. Consider a defaultable floating rate note $F'(t,T)$ with maturity T that pays a continuous coupon of $r(t)dt$ until T and the principal of 1 at T. [4] Assume furthermore zero recovery in default. Then a portfolio of a defaultable floating rate note and a default digital put with the same maturity is fully hedged against default risk, its value is always one. Thus

$$F'(t,T) = 1 - D(t,T),$$

where $D(t,T)$ is the value of the default digital put with maturity T.

If there is a fractional recovery of $(1-q)$ at default then this hedge becomes only approximately valid: A portfolio of one $F'(t,T)$ and q default digital puts $D(t,T)$ is protected against *one* default, but not against any more.

Again, this can be resolved if the recovery is modeled in terms of equivalent default-free securities[5]: At default, 1 defaultable security F' becomes $1-c$ otherwise equivalent default-free securities F, where c can be stochastic.

[4] Usually a defaultable FRN pays a coupon of $r + \bar{s}$ with a fixed spread $\bar{s}$ over the risk-free rate. We ignore these spread payments for now. They can be decomposed into a portfolio of $\bar{s}dt$ zero coupon bonds of each maturity.

[5] See e.g. Lando [13] and Madan and Unal [16].

For the floating rate notes this means that (if c is constant) they can be hedged with c default digital puts:

$$F'(t,T) + cD(t,T) = 1.$$

If c is stochastic, this equality still holds for the *prices* under the martingale measure, but perfect hedging is obviously impossible, then.

The pricing of the *default swap* can also be reduced to the pricing of a default digital put, it only differs from the default digital put in that the default swap pays off the loss quota c at default. Therefore it can be used to set up a perfect hedge of a defaultable FRN even for stochastic recovery rates. Instead of reiterating the analysis of the default digital put, we therefore use this property to give a characterisation of the default swap rate in terms of the credit spread of defaultable FRNs.

Assume the following payoffs: The defaultable FRN pays the floating rate $r(t)$ plus a constant spread $\bar{s}$ per time. The default swap rate is s, and the default-free FRN pays the floating rate r. Both floaters have a final payoff of 1. The payoffs are shown in table 1.

	defaultable FRN	default swap	default-free FRN
Price at $t = 0$	$F'(0)$	0	1
periodic payments	$(r(t) + \bar{s})dt$	$-sdt$	$r(t)dt$
final payoff at $t = T$	$1 + (r(t) + \bar{s})dt$	$-sdt$	$1 + r(t)dt$
value at default	$1 - c$	c	1

Table 1. The Payoffs of defaultable FRN, default swap and default-free FRN.

If the defaultable FRN trades at par at $t = 0$, i.e. $F'(0) = 1$ then initially (at $t = 0$), at defaults and at $t = T$ we have

defaultable FRN + default swap = default-free FRN.

As both the spread $\bar{s}$ of the defaultable FRN and the default swap rate s are constant they must coincide. Therefore

$$\bar{s} = s,$$

Find a defaultable FRN that trades at par and the spread $\bar{s}$ of its coupons over r. Then $s = \bar{s}$ is the fair swap rate for a default swap of the same maturity.

This argument only uses a simple comparison of payoff schedules, it does not use any assumptions about the dynamics and distribution of risk-free interest rates or credit spreads or about the recovery rates.

Finally we can use these results to price a defaultable FRN F' with a coupon of $(r+\bar{s})dt$ and principal of 1 (under zero recovery and independence):

$$\begin{aligned} F'(0,T) &= \mathrm{E}\left[\, \int_0^T r(t)\beta_{0,t}\mathbf{1}_{\{\tau>t\}}dt \,\right] + \mathrm{E}\left[\, \int_0^T \bar{s}\beta_{0,t}\mathbf{1}_{\{\tau>t\}}dt \,\right] + \mathrm{E}\left[\, \beta_{0,T}\mathbf{1}_{\{\tau>T\}} \,\right] \\ &= \int_0^T \mathrm{E}\left[\, r(t)\beta_{0,t} \,\right]\mathrm{E}\left[\, \mathbf{1}_{\{\tau>t\}} \,\right]dt + \bar{s}\int_0^T P'(0,t)dt + P'(0,T) \\ &= \int_0^T f(0,t)P(0,t)\hat{P}(0,t)dt + \bar{s}\int_0^T P'(0,t)dt + P'(0,T) \\ &= \int_0^T (f(0,t)+\bar{s})P'(0,t)dt + P'(0,T). \end{aligned}$$

Again we see a strong similarity between the values of defaultable and default-free FRNs. The extension to positive recovery rates is not difficult now, and independence was only used in the evaluation of $\mathrm{E}\left[\, r(t)\beta_{0,t}\mathbf{1}_{\{\tau>t\}} \,\right]$. Setting $F'=1$, solving for $\bar{s}$ and integrating by parts will transform this result into the swap rate (19) of the default digital swap.

5. The Multifactor Model

5.1. **Specification.** In the previous section we could derive some pricing formulae for the first-generation CRDs without having to specify the dynamics we assume for credit spreads and interest rates, but we had to assume independence of the realisations of r and h in the final step of the analysis.

For the pricing of the second generation credit risk derivatives and for correlation between r and h we need a full specification of the dynamics of the risk-free interest rate r and the intensity of the default process h. A suitable specification should have the following properties:

- Both r and h are stochastic. Stochastic risk-free interest rates are a must anyway, and a stochastic default intensity is required to reach a stochastic credit spread which is necessary for meaningful prices for the spread options.
- The dynamics of r and h are rich enough to allow for a realistic description of the real-world prices. Duffie and Singleton [8] and Duffee [5] come to the conclusion that in many cases a multifactor model for the credit spreads is necessary.
- There should be scope to include correlation between credit spreads and risk-free interest rates.
- Interest rates and credit spreads should remain positive at all times. Negative credit spreads would represent an arbitrage opportunity but maybe relaxing this requirement in favour of a Gaussian specification is still acceptable because of the analytical tractability that is gained.

Here we chose a *multifactor Cox Ingersoll Ross* [4] setup, following mainly Jamshidian [11]. (Related models can also be found in Jamshidian [9, 10], Duffie and Kan [6], Chen and Scott [2] and Longstaff and Schwartz [15].) This model setup gives us the required properties while still retaining a large degree of analytical tractability. Furthermore, models of credit spreads of the CIR square-root type have been estimated by Duffie and Singleton [8] and Duffee [5]. Nevertheless, the derivation of the pricing formulae will be kept as general as possible to ensure that a transfer of the results to a different model specification (e.g. Gaussian) will be straightforward.

Interest rates and default intensities are driven by n independent factors x_i, $i = 1, \dots, n$ with dynamics of the CIR square-root type:

$$dx_i = (\alpha_i - \beta_i x_i)dt + \sigma_i \sqrt{x_i} dW_i. \tag{20}$$

We require $\alpha_i > \frac{1}{2}\sigma_i^2$ to ensure strict positivity of the factors.

The risk-free short rate is defined as a positive linear combination of these factors:

$$r(t) = \sum_{i=1}^{n} a_i x_i(t), \tag{21}$$

and the credit spread is another positive linear combination of these factors:

$$h(t) = \sum_{i=1}^{n} a'_i x_i(t). \tag{22}$$

With all coefficients $a_i \geq 0$ and $a'_i \geq 0$ nonnegative we have ensured that $r > 0$ and $h > 0$ a.s. Unfortunately this specification can only generate *positive* correlation between r and h, if negative correlation is needed one could define modified factors x'_i that are negatively correlated to the x_i by $dx'_i = (\alpha_i - \beta_i x_i)dt - \sigma_i \sqrt{x_i} dW_i$ (note the minus in front of the Brownian motion). The analysis would not be changed significantly, only the changes of measure will be slightly different. Therefore we stick with the empirically more relevant case of positive correlation.

Typically one would reserve the first $m < n$ factors for the risk-free term structure (i.e. $a_i = 0$ for $i > m$, $m = 3$ would usually be sufficient), and use the full set of state variables for the spreads (maybe $n = 4$ or $n = 5$, this will have to be determined empirically). The additional $n - m$ factors for the spreads ensure that the dynamics of the credit spreads have components that are independent of the risk-free interest rate dynamics. The simplest example would be independence between r and h, where $r = x_1$ and $h = x_2$: one factor driving each rate,

$$\begin{aligned} n &= 2 & m &= 1 \\ a_1 &= 1 & a_2 &= 0 \\ a'_1 &= 0 & a'_2 &= 1. \end{aligned}$$

In the general specification we have for a linear multiple cx_i (c is a positive constant) of the factor x_i (see CIR [4]) the following representation for a 'bond price:

$$\mathrm{E}_t\left[\exp\{-\int_t^T cx_i(s)ds\}\right] = H_{1i}(T-t,c)e^{-H_{2i}(T-t,c)cx_i} \tag{23}$$

where

$$H_{1i}(T-t,c) = \left[\frac{2\gamma e^{\frac{1}{2}(\gamma+\beta_i)(T-t)}}{(\gamma+\beta_i)(e^{\gamma(T-t)}-1)+2\gamma}\right]^{2\alpha_i/\sigma_i^2} \tag{24}$$

$$H_{2i}(T-t,c) = \frac{2(e^{\gamma(T-t)}-1)}{(\gamma+\beta_i)(e^{\gamma(T-t)}-1)+2\gamma} \tag{25}$$

$$\gamma = \sqrt{\beta_i^2+2c\sigma_i^2}. \tag{26}$$

The default-free bond prices are given by

$$\begin{aligned} P(t,T) &= \mathrm{E}_t\left[e^{-\int_t^T r(s)ds}\right] = \mathrm{E}_t\left[e^{-\sum_i \int_t^T a_i x_i(s)ds}\right] \\ &= \mathrm{E}_t\left[\prod_i e^{-\int_t^T a_i x_i(s)ds}\right] = \prod_i \mathrm{E}_t\left[e^{-\int_t^T a_i x_i(s)ds}\right] \end{aligned}$$

because the factors are independent. The bond price is thus a product of one-factor bond prices

$$P(t,T) = \prod_{i=1}^n H_{1i}(T,t,a_i)e^{-H_{2i}(T,t,a_i)a_i x_i(t)}. \tag{27}$$

Similarly, the defaultable bond prices are given by:

$$P'(t,T) = Q(t)\prod_{i=1}^n H_{1i}(T,t,a_i+qa_i')e^{-H_{2i}(T,t,a_i+qa_i')(a_i+qa_i')x_i}, \tag{28}$$

because of $P'(t,T) = \mathrm{E}_t\left[\exp\{-\int_t^T r(s)+qh(s)ds\}\right]$.

5.2. **Affine Combinations of independent non-central chi-squared distributions.**

The mathematical tools for the analysis of the model have been provided by Jamshidian [11], who used them to price interest-rate derivatives in a risk-free interest rate environment. Of these tools we need (a) the expressions for the evaluation of the expectations that will arise in the pricing equations, and (b) the methodology of a change of measure to remove discount factors from these expectations.

First the distribution function of an affine combination of noncentral chi-squared random variables is presented. Most of the following expressions are based upon this distribution function. Then we consider the expressions for the expectations of noncentral chi-squared random variables, and finally the change-of-measure technique that has to be applied in this context.

5.2.1. *The distribution of noncentral chi-square random variables.*

The factors x_i have square-root dynamics which give rise to noncentral chi-square distributed final values[6].

The dynamics of a factor x under the martingale measure are

$$dx = (\alpha - \beta x)dt + \sigma\sqrt{x}dW. \tag{29}$$

The density function of the factor $x(s)$ at time s given its value $x(t)$ for an earlier time $t < s$ is

$$f(x(s), s; x(t), t) = ce^{-u-v}\left(\frac{v}{u}\right)^{q/2} I_q(2(uv)^{1/2}), \tag{30}$$

where $I_q(\cdot)$ is the modified Bessel function of the first kind of order q and

$$c = \frac{2\beta}{\sigma^2(1 - e^{-\beta(s-t)})} \qquad q = \frac{2\alpha}{\sigma^2} - 1 \tag{31}$$

$$u = cx(t)e^{-\beta(s-t)} \qquad v = cx(s). \tag{32}$$

The distribution function of $2cx(s)$ given $x(t)$ is

$$\chi^2(\quad 2cx(s) \quad ; 2q + 2, 2u) \tag{33}$$

the noncentral chi-squared distribution function with $2q + 2$ degrees of freedom and a parameter of noncentrality of $2u$.

As interest rates and spreads are affine combinations of these factors we need to analyse their distribution.

Let Y be an affine function of n noncentral chi-square distributed random variables z_i:

$$Y = \epsilon + \sum_{i=1}^{n} \eta_i z_i, \tag{34}$$

where the z_i are one-dimensional noncentral chi-squared distributed with

$$z_i \sim \chi^2(\,\cdot\,;\ \nu_i, \lambda_i)$$

ν_i degrees of freedom and noncentrality parameter λ_i. Call the distribution function of Y

$$Y \sim \chi^2(\,\cdot\,;\ \nu,\ \lambda,\ \eta,\ \epsilon),$$

where η, ν and λ are vectors, and for $\epsilon = 0$ we will usually omit the last argument.

The evaluation of this distribution function can be efficiently implemented via a fast Fourier transform of its characteristic function (see e.g. Chen and Scott (1992)). Alternatively, Jamshidian gives the following formula (which is basically the transform integral of the characteristic function):

$$\chi^2(y;\ \nu, \lambda, \eta) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \Psi(\frac{\nu}{2}, \lambda, 2\xi^2\eta^2, 0)\ \sin(\xi y - \Phi(\nu, \lambda, \xi\eta))\frac{d\xi}{\xi}, \tag{35}$$

[6] See e.g. CIR [3, 4].

where

$$\Psi(\nu,\lambda,\eta,\epsilon)=\mathrm{E}\left[\,e^{-Y}\,\right]$$

is given by equation (39) and

$$\Phi(\nu,\lambda,\gamma)=\sum_{i=1}^{n}\left(\frac{\nu_i}{2}\arctan(2\gamma_i)+\frac{\gamma_i\lambda_i}{1+4\gamma_i^2}\right).$$

The integral in (35) is numerically well-behaved in that the limit of the integrand as $\xi\to 0$ is finite and the integral is absolutely integrable. We will see later on that most valuation problems in the multifactor setup can be reduced to this one-dimensional integral. The evaluation of the noncentral chi-squared distribution function which one encounters in the one-factor independence case, also requires a numerical approximation, the implementation effort of the multifactor case therefore does not seem to be very much higher than the effort required for a model with one independent factor each for interest rates and spreads.

5.2.2. *The values of some expectations.*

Let Y' be similarly distributed to Y with η' and ϵ' respectively but *the same* z_i, i.e.

$$Y\sim\chi^2(\,\cdot\,;\ \nu,\ \lambda,\ \eta,\ \epsilon),\qquad Y'\sim\chi^2(\,\cdot\,;\ \nu,\ \lambda,\ \eta',\ \epsilon'),\tag{36}$$

and let y be a constant. Then the following expressions give:
… the expectation of Y:

$$\mathrm{E}[\,Y\,]=\epsilon+\sum_{i=1}^{n}\eta_i(\lambda_i+\nu_i),\tag{37}$$

… the distribution of Y:

$$\Pr[\,Y\le y\,]=\chi^2(y-\epsilon;\ \nu,\lambda,\eta),\tag{38}$$

… the expectation of e^{-Y}:

$$\mathrm{E}\left[\,e^{-Y}\,\right]=e^{-\epsilon}\prod_{i=1}^{n}\frac{1}{(1+2\eta_i)^{\nu_i/2}}\exp\left\{\frac{\eta_i\lambda_i}{1+2\eta_i}\right\}\tag{39}$$

… a call option on e^{-Y} with strike e^{-y}:

$$\begin{aligned}\mathrm{E}\left[\,(e^{-Y}-e^{-y})^+\,\right]&=\mathrm{E}\left[\,e^{-Y}\,\right]\chi^2\left(y-\epsilon;\nu,\frac{\lambda}{1+2\eta},\frac{\eta}{1+2\eta}\right)\\&\quad-e^{-y}\,\chi^2(y-\epsilon;\nu,\lambda,\eta)\end{aligned}\tag{40}$$

… an exchange option on e^{-Y} and $e^{-Y'}$:

$$\begin{aligned}\mathrm{E}\left[\,(e^{-Y}-e^{-Y'})^+\,\right]&=\mathrm{E}\left[\,e^{-Y}\,\right]\chi^2\left(\epsilon'-\epsilon;\nu,\frac{\lambda}{1+2\eta},\frac{\eta-\eta'}{1+2\eta}\right)\\&\quad-\mathrm{E}\left[\,e^{-Y'}\,\right]\chi^2\left(\epsilon'-\epsilon;\nu,\frac{\lambda}{1+2\eta'},\frac{\eta-\eta'}{1+2\eta'}\right).\end{aligned}\tag{41}$$

5.3. **The change of measure technique.**

The change of measure necessary to eliminate discounting with one of the factors is defined in the following Lemma which is due to Jamshidian [9]. The proof has been slightly adapted and can be found in the appendix:

Lemma 1. *Let x follow a CIR-type square-root process under the measure P:*

$$dx = (\alpha - \beta x)dt + \sigma\sqrt{x}\, dW, \tag{42}$$

so that the model admits an affine term structure:

$$G(x,t,T) = \mathrm{E}_t\left[e^{-\int_t^T x(s)ds} \right] = H_{1x}(T,t)e^{-H_{2x}(T,t)x}. \tag{43}$$

and such that $x = 0$ is an unattainable boundary ($\alpha > \frac{1}{2}\sigma^2$). Then

$$\mathrm{E}_t^P\left[e^{-\int_t^T x(s)ds} F(x(T)) \right] = G(x(t),t,T)\,\mathrm{E}_t^{P^Z}\left[F(x(T)) \right] \tag{44}$$

where under the measure P^Z the process x has the dynamics

$$dx = [\alpha - (\beta + H_{2x}(T,t)\sigma^2)x]dt + \sigma\sqrt{x}\, dW^Z, \tag{45}$$

with W^Z a P^Z-Brownian motion, and

$$dW = dW^Z - H_{2x}(T-t)\sigma\sqrt{x}dt.$$

The measure P^Z is defined via the Radon-Nikodym density

$$Z(t) = \mathcal{E}\left(- \int_0^t \sigma(s)\sqrt{x(s)}H_{2x}(T,s)dW(s) \right).$$

The dynamics of the other factors are unaffected.

Remark: The lemma also holds for time-dependent parameters with $\alpha(t)/\sigma^2(t) > 1/2$, but here we only need the time-independent case. The distribution of $x(T)$ under the new measure P^Z is given in the following lemma which is due to Jamshidian [9, 11]:

Lemma 2. *The time T-value $x(T)/\delta_T$ of the process x has a noncentral chi-squared distribution under P^Z (i.e. given the x dynamics (45)), with ν degrees of freedom and noncentrality parameter $\lambda_T x(t)$, where:*

$$\Pr[\, x(T)/\delta_T \le X \,] = \chi^2(X\,;\,\nu, \lambda_T x(t)) \tag{46}$$

where

$$\delta_T = \frac{\sigma^2}{4} H_{2x}(T-t) \tag{47}$$

$$\nu = \frac{4\alpha}{\sigma^2} \tag{48}$$

$$\lambda_T x(t) = \frac{4}{\sigma^2} \frac{\frac{\partial}{\partial T} H_{2x}(T-t)}{H_{2x}(T-t)} x(t). \tag{49}$$

Proof: See Jamshidian [9, 11].

We will also need the same change of measure with a constant multiple $y = cx$ of x. Lemmata 1 and 2 now take the following form:

Lemma 3. *The dynamics of x are given by* (42) *and The model admits an affine term structure:*

$$G(x, t, T, c) = \mathrm{E}_t \left[e^{-\int_t^T cx(s)ds} \right] = H_{1x}(T, t, c) e^{-H_{2x}(T,t,c)cx}. \tag{50}$$

and $x = 0$ is still an unattainable boundary ($\alpha > \frac{1}{2}\sigma^2$). Then

$$\mathrm{E}_t^P \left[e^{-\int_t^T cx(s)ds} F(x(T)) \right] = G(x(t), t, T, c)\, \mathrm{E}_t^{P_c^Z} [\ F(x(T))\ ] \tag{51}$$

where under the measure P_c^Z the process x has the dynamics

$$dx = [\alpha - (\beta + H_{2x}(T, t)c\sigma^2)x]dt + \sigma\sqrt{x}\, dW^Z, \tag{52}$$

with W^Z a P_c^Z-Brownian motion, and

$$dW = dW^Z - H_{2x}(T - t, c)c\sigma\sqrt{x}dt.$$

The measure P_c^Z is defined via the Radon-Nikodym density

$$Z(t) = \mathcal{E}\left(- \int_0^t \sigma(s)c\sqrt{x(s)}H_{2x}(T, s, c)dW(s) \right).$$

And lemma 2 becomes now

Lemma 4. *The time T-value $x(T)/\hat{\delta}_T$ of the process x has a noncentral chi-squared distribution under P_c^Z (i.e. given the x dynamics* (52)*), with $\hat{\nu}$ degrees of freedom and noncentrality parameter $\hat{\lambda}_T x(t)$, where:*

$$\Pr\left[x(T)/\hat{\delta}_T \le X \right] = \chi^2(X\ ;\ \hat{\nu}, \hat{\lambda}_T x(t)) \tag{53}$$

where

$$\hat{\delta}_T = \frac{\sigma^2}{4} H_{2x}(T - t, c) \tag{54}$$

$$\hat{\nu} = \frac{4\alpha}{\sigma^2} \tag{55}$$

$$\hat{\lambda}_T x(t) = \frac{4}{c\sigma^2} \frac{\frac{\partial}{\partial T} H_{2x}(T - t, c)}{H_{2x}(T - t, c)} x(t). \tag{56}$$

Proof: Apply lemmata 1 and 2 to $y = cx$. Note that y has the dynamics

$$dy = d(cx) = (c\alpha - \beta y)dt + \sigma\sqrt{c}\,\sqrt{y}\, dW \tag{57}$$

$$=: (\hat{\alpha} - \hat{\beta} y)dt + \hat{\sigma}\sqrt{y}\, dW, \tag{58}$$

which is still of the CIR square-root form.

The change of measure to P_c^Z removes the discounting with $e^{-\int_0^T cx(t)dt}$, while changing the distribution of $\hat{\delta}_T x(T)$ to a noncentral chi-squared distribution with parameters given in equations (54) to (56).

6. Valuation of Second Generation Credit Risk Derivatives

6.1. **Credit Spread Put.** A *Credit Spread Put* on a defaultable bond $P'(t,T_2)$ with maturity $T_1 < T_2$ gives the holder the right to sell defaultable bonds at time T_1 at a price that corresponds to a yield spread of $\bar{s}$ above the yield of an (otherwise identical) default-free bond $P(T_1,T_2)$. Define the exchange ratio $\bar{S} := e^{-\bar{s}(T_2-T_1)}$. These credit derivatives entitle the holder to *exchange one defaultable bond $P'(T_1,T_2)$ for $\bar{S}$ default-free bonds $P(T_1,T_2)$ at time T_1*. The payoff function is:

$$(\bar{S}P(T_1,T_2) - P'(T_1,T_2))^+ \tag{59}$$

Here the default is a jump in the price of the underlying security, we will first consider the case when the contract is not modified at default, then the case when the option is knocked out by a default. In the first case the payoff is conditioned on the full defaultable bond price $P'(T_1,T_2)$, and the default risk is borne by the writer of the security. The holder of the security is protected by the contract against any losses worse than a final credit spread of $\bar{s}$ and against all defaults, but not against interest rate risk. The exchange option is therefore well suited to lay off most of the credit exposure in the underlying instrument while keeping some limited exposure to the credit spread of the underlying.

For the valuation of the exchange option we find

$$\begin{aligned} D_t &= \mathrm{E}_t\left[\ \beta_{t,T_1}(\bar{S}P(T_1,T_2) - P'(T_1,T_2))^+\ \right] \\ &= \mathrm{E}_t\left[\ e^{-\int_t^{T_1} r(s)+h(s)ds} P(T_1,T_2)(\bar{S} - \hat{P}(T_1,T_2))^+\ \right] \\ &\quad + \bar{S}P(t,T_2) - P'(t,T_2) \\ &\quad - \mathrm{E}_t\left[\ e^{-\int_t^{T_1} r(s)+h(s)ds} P(T_1,T_2)(\bar{S} - \hat{P}(T_1,T_2))\ \right]. \end{aligned} \tag{60}$$

This is reached as follows: The expectation to evaluate is (assuming no defaults so far $Q(t) = 1$)

$$\begin{aligned} D_t &= \mathrm{E}_t\left[\ \beta_{t,T_1}(\bar{S}P(T_1,T_2) - P'(T_1,T_2))^+\ \right] \\ &= \mathrm{E}_t\left[\ \beta_{t,T_1}(\bar{S}P(T_1,T_2) - P(T_1,T_2)Q(T_1)\hat{P}(T_1,T_2))^+\ \right] \\ &= \mathrm{E}_t\left[\ \beta_{t,T_1}P(T_1,T_2)(\bar{S} - Q(T_1)\hat{P}(T_1,T_2))^+\ \right] \\ &= \mathrm{E}_t\left[\ \mathbf{1}_{\{\tau>T_1\}}\beta_{t,T_1}P(T_1,T_2)(\bar{S} - \hat{P}(T_1,T_2))^+\ \right] \\ &\quad + \mathrm{E}_t\left[\ \mathbf{1}_{\{\tau\le T_1\}}\beta_{t,T_1}P(T_1,T_2)(\bar{S} - Q(T_1)\hat{P}(T_1,T_2))^+\ \right]. \end{aligned}$$

where we split up the payoff in the case without intermediate default ($\mathbf{1}_{\{\tau>T_1\}}$) and the case with intermediate default ($\mathbf{1}_{\{\tau\le T_1\}}$). If there has been an intermediate default, we assume that the option is in the money and we omit the maximum sign for that event:

$$= \mathrm{E}_t\left[\ \mathbf{1}_{\{\tau>T_1\}}\beta_{t,T_1}P(T_1,T_2)(\bar{S} - \hat{P}(T_1,T_2))^+\ \right]$$

$$+ \ \mathrm{E}_t \left[\ \mathbf{1}_{\{\tau \le T_1\}} \beta_{t,T_1} P(T_1,T_2)(\bar{S} - Q(T_1)\hat{P}(T_1,T_2)) \ \right],$$

and then substitute $\mathbf{1}_{\{\tau \le T_1\}} = 1 - \mathbf{1}_{\{\tau > T_1\}}$ to reach

$$\begin{aligned}
&= \mathrm{E}_t \left[\ \mathbf{1}_{\{\tau > T_1\}} \beta_{t,T_1} P(T_1,T_2)(\bar{S} - \hat{P}(T_1,T_2))^+ \ \right] \\
&\quad + \ \mathrm{E}_t \left[\ \beta_{t,T_1} P(T_1,T_2)(\bar{S} - Q(T_1)\hat{P}(T_1,T_2)) \ \right] \\
&\quad - \ \mathrm{E}_t \left[\ \mathbf{1}_{\{\tau > T_1\}} \beta_{t,T_1} P(T_1,T_2)(\bar{S} - Q(T_1)\hat{P}(T_1,T_2)) \ \right] \\
&= \mathrm{E}_t \left[\ \mathbf{1}_{\{\tau > T_1\}} \beta_{t,T_1} P(T_1,T_2)(\bar{S} - \hat{P}(T_1,T_2))^+ \ \right] \\
&\quad + \ \mathrm{E}_t \left[\ \beta_{t,T_1} P(T_1,T_2)\bar{S} \ \right] - \ \mathrm{E}_t \left[\ \beta_{t,T_1} Q(T_1)\hat{P}(T_1,T_2)) \ \right] \\
&\quad - \ \mathrm{E}_t \left[\ \mathbf{1}_{\{\tau > T_1\}} \beta_{t,T_1} P(T_1,T_2)(\bar{S} - Q(T_1)\hat{P}(T_1,T_2)) \ \right] \\
&= \mathrm{E}_t \left[\ \mathbf{1}_{\{\tau > T_1\}} \beta_{t,T_1} P(T_1,T_2)(\bar{S} - \hat{P}(T_1,T_2))^+ \ \right] \\
&\quad + \bar{S}P(t,T_2) - P'(t,T_2)) \\
&\quad - \ \mathrm{E}_t \left[\ \mathbf{1}_{\{\tau > T_1\}} \beta_{t,T_1} P(T_1,T_2)(\bar{S} - \hat{P}(T_1,T_2)) \ \right].
\end{aligned}$$

In the last line we used the fact that $\mathbf{1}_{\{\tau > T_1\}} Q(T_1) = \mathbf{1}_{\{\tau > T_1\}}$, the indicator function of 'no default until T_1' switches off the 'influences $Q(T_1)$ of default until T_1'.

From the decomposition of the payoff in a 'no default' part and a 'default part' we can also directly see the expectation for the price of a credit spread option that is killed at default: It is

$$\tilde{D}_t = \ \mathrm{E}_t \left[\ \beta_{t,T_1}(\bar{S}P(T_1,T_2) - P'(T_1,T_2))^+ \ \right]. \tag{61}$$

Next we have to eliminate the indicator functions from the expectations. This is done by conditioning on the realisation of h and using the Cox process properties of the default triggering process:

$$\begin{aligned}
&\mathrm{E}_t \left[\ \mathbf{1}_{\{\tau > T_1\}} \beta_{t,T_1} P(T_1,T_2)(\bar{S} - \hat{P}(T_1,T_2))^+ \ \right] \\
&= \mathrm{E}_t \left[\ \mathrm{E} \left[\ \mathbf{1}_{\{\tau > T_1\}} \beta_{t,T_1} P(T_1,T_2)(\bar{S} - \hat{P}(T_1,T_2))^+ \ \middle| \ h \right] \ \right] \\
&= \mathrm{E}_t \left[\ e^{-\int_t^{T_1} h(s)ds} \beta_{t,T_1} P(T_1,T_2)(\bar{S} - \hat{P}(T_1,T_2))^+ \ \right].
\end{aligned}$$

The second expectation is treated similarly to reach equation (60). Until now we have not used the multifactor CIR specification yet, the evaluation of the expectations (60) and (61) could be implemented with any other specification, too.

To eliminate the integrals with the exponentials of $\int r + h \, ds$ we will have to perform a change of measure for each factor involved. Recalling that

$$-\int_t^{T_1} r(s) + h(s)ds = -\sum_{i=1}^{n} \int_t^{T_1} (a_i + a_i')x_i(s)ds$$

we have to eliminate each factor individually, using Lemma 1. Because all factors are independent the change of measure procedure is not difficult and follows directly: For every function $F(x(T_1))$ of the factors at T_1 we have

$$\begin{aligned}(62)\quad & \mathrm{E}_t\left[\, e^{-\int_t^{T_1} r(s)+h(s)ds} F(x(T_1))\,\right] \\ & = \mathrm{E}_t^{P^{rh}}\,[\, F(X(T_1)\,]\prod_{i=1}^{n} H_1(T_1,t,a_i+a_i')\ \exp\{-H_2(T_1,t,a_i+a_i')(a_i+a_i')x_i(t)\},\end{aligned}$$

where under the measure P^{rh} the factor $x_i(T)/\delta_i^{rh}$ is noncentral chi-square distributed with ν_i^{rh} degrees of freedom and parameter of noncentrality λ_i^{rh}:

$$\delta_i^{rh} = \frac{(a_i+a_i')\sigma_i^2}{4} H_2(T_1-t,a_i+a_i') \tag{63}$$

$$\nu_i^{rh} = \frac{4\alpha_i}{\sigma_i^2} \tag{64}$$

$$\lambda_i^{rh} = \frac{x_i(t)}{\delta_i^{rh}}\frac{\partial}{\partial T} H_2(T_1-t,a_i+a_i'). \tag{65}$$

To evaluate the remaining expectations we define the following random variables:

$$\begin{aligned} Y :=& \ln(\bar{S}P(T_1,T_2)) \\ =&\bar{s}(T_2-T_1)+\sum_{i=1}^{n}\ln H_{1i}(T_2-T_1,\,a_i)-\sum_{i=1}^{n}\delta_i^{rh}H_{2i}(T_2-T_1,\,a_i)\left(\frac{x_i}{\delta_i^{rh}}\right) \quad (66)\\ Y' :=& \ln(P(T_1,T_2)\hat{P}(T_1,T_2)) \\ =&\sum_{i=1}^{n}\ln H_{1i}(T_2-T_1,\,a_i+qa_i')-\sum_{i=1}^{n}\delta_i^{rh}H_{2i}(T_2-T_1,\,a_i+qa_i')\left(\frac{x_i}{\delta_i^{rh}}\right). \quad (67)\end{aligned}$$

Note that both Y and Y' have a representation as an linear affine combination of (under P^{rh}) noncentral chi-square distributed random variables $x_i(T_1)/\delta_i^{rh}$. Therefore their distributions (using the notation of Jamshidian [11]) are

$$Y \equiv\chi^2\left(\nu^{rh};\ \lambda^{rh};\ -\delta^{rh}H_2(T_2-T_1,\,a);\ \bar{s}(T_2-T_1)+\sum_{i=1}^{n}\ln H_{1i}(T_2-T_1;a_i)\right) \tag{68}$$

$$Y' \equiv\chi^2\left(\nu^{rh};\ \lambda^{rh};\ -\delta^{rh}H_2(T_2-T_1,\,a+qa');\ \sum_{i=1}^{n}\ln H_{1i}(T_2-T_1;a_i+qa_i')\right) \tag{69}$$

The closed form solutions for the expectations $\mathrm{E}_t^{P^{rh}}\left[\, e^{Y}\,\right]$, $\mathrm{E}_t^{P^{rh}}\left[\, e^{Y'}\,\right]$ and for the option $\mathrm{E}_t^{P^{rh}}\left[\,(e^{Y}-e^{Y'})^{+}\,\right]$ can be read from equations (37) to (41). Substitution of these solutions and (62) into (60) yields the closed-form solution for the price of the exchange option.

6.2. **Put Options on Defaultable Bonds.** The credit risk derivatives considered so far conditioned on the *spread* of the defaultable bond over a default-free bond, but the reference to a default-free bond is not necessary, we can also consider derivatives on the defaultable bond alone, e.g. a Put or call on a defaultable bond. Again we can consider the question whether the Put covers default losses, or whether it is killed at default.

A European Put with maturity T_1 and strike K on a defaultable bond $P'(t, T_2)$ pays off

$$(K - P'(T_1, T_2))^+. \tag{70}$$

This derivative protects the buyer against all risks: Interest rates, credit spreads or defaults that could move the defaultable bond price below K.

To price this derivative we have to evaluate

$$\mathrm{E}_t \left[e^{-\int_t^{T_1} r(s)ds} (K - P'(T_1, T_2))^+ \right].$$

This can be expanded by the now familiar technique of first conditioning on the realisation of h and then substituting the event probabilities:

$$\begin{aligned} & \mathrm{E}_t \left[e^{-\int_t^{T_1} r(s)ds} (K - P'(T_1, T_2))^+ \right] \\ = & \mathrm{E}_t \left[e^{-\int_t^{T_1} r(s)+h(s)ds} (K - \bar{P}(T_1, T_2))^+ \right] \\ & + P(t, T_1)K - P'(t, T_2) \\ & - \mathrm{E}_t \left[e^{-\int_t^{T_1} r(s)+h(s)ds} (K - \bar{P}(T_1, T_2)) \right]. \end{aligned} \tag{71}$$

Equation (71) is very similar to the pricing equation (60) of the exchange option. The only difference is that $\bar{S}P(T_1, T_2)$ in (60) has been replaced with K in (71). The valuation is therefore very similar: The same change of measure removes the discounting, the only difference is that equation (39) (the plain option) instead of (40) (the exchange option) is to be used in the final substitution.

It is important to note that the value of this option is *not* the value of an European Put on a bond in a world where the short rate is $r' = r + qh$. (One might be lead to this view by the simple representation (8) of the defaultable bond prices.) This view would ignore the large jumps the bond price makes at defaults and only price the spread component. To illustrate this point we introduce a new security: the put option on the yield with *security protection.* This option is the one that is similar to a default-free put option in a world with interest rate $r' = r + qh$, but its specification conditions on the default model that is used, and therefore will not be viable in real-world applications.

6.2.1. *Put option on the yield of a defaultable bond: Security protection.* A European Put with security protection, maturity T_1 and strike K on the yield of a defaultable bond $P'(t, T_2)$ pays off

$$Q(T_1)(K - \bar{P}(T_1, T_2))^+. \tag{72}$$

The term 'security protection' has been chosen to point out the special feature in this specification: At each default the face value of the defaultable bond is reduced by a factor $1 - q$. This payoff function is also reduced by the same factor, such that this option always covers exactly one defaultable bond, the size of the protection is adjusted to the 'size' (=face value) of the security. Therefore we called it Put with *security protection.*

This security can be used to artificially shorten the maturity of a defaultable bond $P'(t,T_2)$: With $K = 1$ a portfolio of this credit risk derivative and the underlying defaultable bond has the same payoff as a defaultable bond with maturity T_1: the payoff is

$$P'(T_1,T_2) + Q(T_1)(1 - \bar{P}(T_1,T_2)) = P'(T_1,T_2) + Q(T_1) - Q(T_1)\bar{P}(T_1,T_2)) = Q(T_1).$$

Again very similarly to (60) we reach the expectation

$$\mathrm{E}_t\left[\, e^{-\int_t^{T_1} r(s)+qh(s)ds}(K - \bar{P}(T_1,T_2))^+ \,\right],$$

which is exactly identical to the expectation that has to be evaluated to price a put option on a zero coupon bond in a world where $r' = r + qh$ is the riskless short rate. To evaluate the expectation in the multifactor CIR-framework we have to perform the same change of measure as for the credit spread put with security protection, again only $\bar{S}P(T_1,T_2)$ has been replaced by K.

7. Conclusion

In this paper we demonstrated the Cox process approach to the pricing of credit risk derivatives, and some closed form solutions were given for the case of a multifactor CIR-specification of the credit spread and interest rate dynamics.

The Cox process modelling approach derives its tractability from the fact that the pricing of derivatives can be done in stages:

First, by conditioning on the realisation of the intensity process h the pricing problem can be reduced to the fairly straightforward case of inhomogeneous Poisson processes, for which most expected values are easily calculated. Thus the explicit reference to default events can be eliminated and replaced by expressions in the intensity of the default process. After this stage the methods of risk-free interest rate theory can be applied, because the reference to jump processes has disappeared.

We used specifically the change of measure technique in several to remove discount factors of the form $\exp - \int_0^T g(s)ds$ where g is an 'interest-rate like' process, and then some well-known properties of the final distributions of the factor processes. In most cases we needed to change the measure to remove the discount factor

$$e^{-\int_0^T r(t)+h(t)dt}$$

which typically arose from expectations of the form

$$\mathrm{E}\left[\, e^{-\int_0^T r(t)dt}\mathbf{1}_{\{\tau>T\}}X \,\right] = \mathrm{E}\left[\, e^{-\int_0^T r(t)+h(t)dt}X \,\right],$$

i.e. survival contingent payoffs. The resulting measure can be termed a *default measure.*

The change of measure *after* the elimination of the default indicator function makes sure that the new measure remains equivalent to the original martingale measure. Simply taking a defaultable bond as numeraire would mean that this property is lost, and the change of measure would become much more complicated because the price path of a defaultable bond can be discontinuous.

Even when no closed form solutions are available the conditioning technique is still very useful as pre-processing procedure for a subsequent numerical solution of the pricing equation. Especially Monte Carlo methods are notoriously slow to converge for low-probability events (like defaults), the results can be speeded up significantly if the default events have been removed. P.d.e. methods (like finite differences solvers) are usually based on a Feynman-Kac representation of the price-expectation as an expectation of a stochastic integral and a final payoff in terms of *diffusion processes*. Therefore they cannot handle discrete default events directly[7], a pre-processing to remove the jumps is necessary here, too. The only case when this does not make sense is if one would like to recover the full distribution of the payoffs from the numerical scheme, including default events.

It was seen furthermore that the analogy to the risk-free interest rate world does not carry through completely. There is the simple representation (8) of a defaultable bond price as 'bond price expectation' with defaultable interest rates given by $r' = r + qh$ but this does *not* mean that options on defaultable bonds can also be treated as if there were no defaults but just a new interest rate r'. In the last section this was shown to be quite wrong.

Most of the results of this paper are independent of the specification used for the interest-rate and credit spread processes, the point of the paper was to present the techniques that have to be used in all specifications and the transfer to another specification should not be too hard.

Another point to mention is the amount of data and information that is needed to price credit risk derivatives. One of the advantages of the Duffie-Singelton setup appeared to be the fact, that default intensities h and default loss rates q only appeared linked as a product in the bond pricing equation, an qh could therefore be estimated from the credit spreads – without any further need for separate default and recovery information. This breaks down when the credit risk derivatives are considered, the default intensity h *does* appear separate from the default losses q, both pieces of information are needed separately.

In a practical implementation much of this data will not be available and the financial engineer will have to resort to historical data, market data from other issuers of the same industry, region and rating class, or to numbers from fundamental analysis. On the other hand, every derivative needs an underlying, and this underlying should be fairly immune to market manipulation. This means that second generation products ar only viable iff there is some degree of liquidity in the market for the underlying defaultable bond. Liquidity can usually only be expected for large issuers and this means better quality on historical price data on the one hand and probably the existence of further

[7]It is possible to incorporate the jump processes in a finite-difference scheme, but at the cost of computing speed and having full matrices.

defaultable bonds by the same issuer that trade in liquid markets and that can be used to build a credit spread curve. Therefore CRDs will only be viable in markets and for issuers where the data problems are less severe anyway. Nevertheless, data availability and quality is the most pressing problem in the field at the moment.

Appendix A. Appendix

Proof of Lemma 1. The proof of Lemma 1 is an application of the Girsanov theorem and the change of measure technique: Define

$$g(0,t) = e^{-\int_0^t x(s)ds}. \tag{73}$$

Then

$$Z(t) := \frac{1}{G(x(0),0,T)} g(0,t) G(x(t),t,T) \tag{74}$$

is a positive martingale with initial value $Z(0) = 1$ and can therefore be used as a Radon-Nikodym density for a change of measure from P to P^Z defined by $dP/dP^Z = Z$, and

$$Z(t) = \mathcal{E}\left(-\int_0^t \sigma\sqrt{x(s)} H_{2x}(T-s) dW(s) \right).$$

Then

$$Z(t)\, \mathrm{E}_t^{P^Z}\left[\, F(x(T))\,\right] = \mathrm{E}_t^P\left[\, Z(T)F(x(T))\,\right] = \mathrm{E}_t^P\left[\, e^{-\int_t^T x(s)ds} F(x(T))\,\right]$$

It remains to show that x has indeed the postulated dynamics under P^Z. The P dynamics of x are

$$dx = \kappa(\alpha - x)dt + \sigma\sqrt{x}\, dW$$

where W is a P-Brownian motion. By Girsanov's theorem

$$dW = dW^Z - H_{2x}(T-t)\sigma\sqrt{x} dt$$

where W^Z is a P^Z-Brownian motion. Thus

$$\begin{aligned} dx &= \kappa(\alpha - x)dt - H_{2x}(T-t)\sigma^2 x dt + \sigma\sqrt{x}\, dW^Z \\ &= [\kappa\alpha - \sigma^2(\kappa + H_{2x}(T-t))x]dt + \sigma\sqrt{x}\, dW^Z. \end{aligned} \tag{75}$$

The invariance of the dynamics of the other factors follows from the independence of the individual factors. □

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