THE VALUATION OF DEFAULT-TRIGGERED CREDIT DERIVATIVES

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ABSTRACT

Credit derivatives are among the fastest growing contracts in the derivatives market. We present a simple, easily implementable model to study the pricing and hedging of two widely traded default-triggered claims: default swaps and default baskets. In particular, we demonstrate how default correlation (the correlation between two default processes) impacts the prices of these claims. When we extend our model to continuous time, we find that, once default correlation has been taken into consideration, the spread dynamics have very little explanatory power.

THE VALUATION OF DEFAULT-TRIGGERED CREDIT DERIVATIVES

Credit derivatives are among the fastest growing contracts in the derivatives market. They are exercised either upon the occurrence of a default event or when spreads exceed certain pre-defined boundaries. Among default-triggered credit derivatives, default swaps (the most popular) and default baskets are the most widely traded. Spurred by the recent Asian financial crises, the trade volume in default swaps has increased dramatically. According to Reyfman and Toft (2001), "Globally, more than $1 billion in default swap notional exposures to more than several dozen names are traded daily." Over the years, many default swap contracts have been standardized in the over-the-counter market and there have emerged specialized broker-dealers (e.g., Tullet and GFI Inc.) who provide two-way quotes.

Default-triggered credit derivatives, such as default swaps and default baskets, are regarded as "cash product" which can be priced off the default probability curve. Like the risk-free forward rate curve, the default probability curve describes default probabilities for various future points in time. While deriving the single default probability is simple, the derivation of the *joint* default probability curve is very complex. Correlations among various default processes play a dominant role in deriving the joint default probability curve. Unfortunately default swaps and default baskets both involve more than one default process. Default swaps involve two default processes: counterparty default and issuer default. Default baskets involve multiple default processes: counterparty default and defaults of multiple issuers.

Default can be modeled in a variety of ways: one popular way is through the use of a Poisson distribution. However, correlation is not clearly defined between two Poisson processes. In this paper, we present a simple model for the default correlation. The proposed correlation model not only preserves the desired features for default-triggered contracts but also is very easy to implement. With our model, we are able to demonstrate, in a single-period setting, how the default correlation plays a dominant role in the pricing of default products. We then extend the model to multiple periods and incorporate random spreads and random interest rates. In this setting, we demonstrate that correlations among spreads have literally no impact on prices, instead we demonstrate that default correlation is the relevant (dominant) consideration.

The paper is organized as follows. Section I provides a description and some institutional details related to default swaps and default baskets. Section II discusses the relevant literature. Section III models default as an exogenous process. Section IV models the default correlation and demonstrates how one can easily correlate two Poisson processes. After we have built the model for the correlation, Section V then prices default swaps and default baskets and demonstrates that the default correlation plays a dominant role in the pricing of these contingent claims. Section VI introduces stochastic interest rates and spreads and demonstrates that spread correlations have little impact on the price of a default product. Section VII concludes.

Section I. A description of default swaps and default baskets.

A. Default swaps

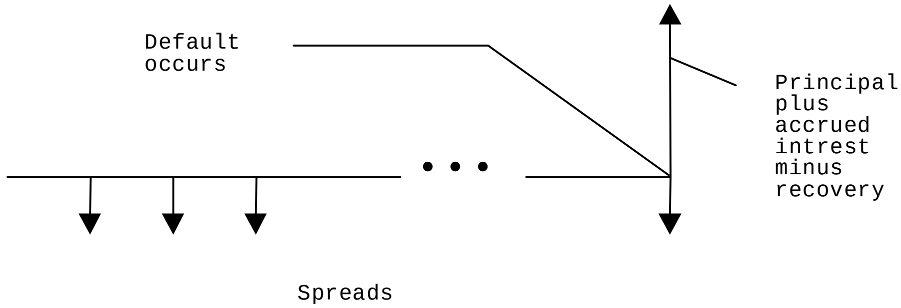
A default swap contract offers protection against default of a pre-specified corporate bond issue. In the event of default, a full recovery default swap will pay the principal and accrued interest in exchange for the defaulted bond.[1,2] While there are many different types of default swaps, the most popular is one that is physically delivered, has full recovery, and carries no embedded options.[3] These swaps, like all swaps, have two "legs" associated with them. The "premium" leg contains a stream of payments, called spreads, paid by the buyer of the default swap to the seller until either default or maturity, whichever is earlier. The other leg, the "protection" leg, contains a lump-sum payment from the seller to the buyer if default occurs, and nothing otherwise.

The following picture describes the cash flows of the default swap in the event that default occurs:

[1] See Hull (1999) for a similar definition of default swaps.

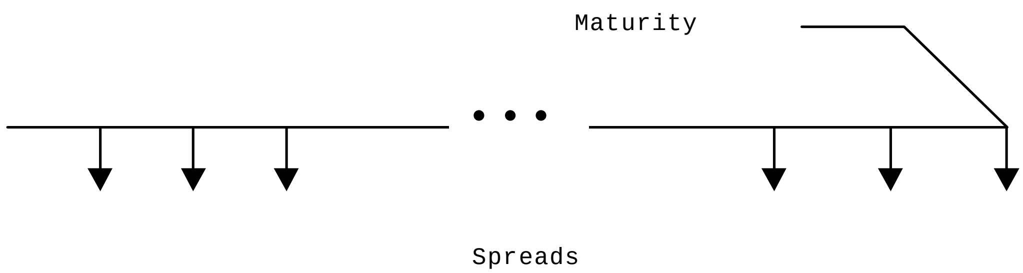
[2] Default swaps can also be designed to protect a corporate name. These default swaps are usually digital default swaps.

[3] Some default swaps do not pay accrued interest or pay only partial accrued interest. These default swaps obviously do not provide full protection of the underlying bond since the accrued interest is lost when default occurs. A cash-settled default swap contract does not require the physical delivery of the defaulted bond. Instead, the payoff upon default is principal plus accrued interest minus recovery. One example of cash-settled default swaps is digital default swap. This type of default swaps gives the buyer all-or-nothing payoff (all if default occurs and nothing if default does not occur). Foreign currency denominated default swaps have emerged recently due to Asian crises. Many investors were seeking hedges of their Asian credit exposures from US defaults swap market.



In the event of default, the buyer will stop making spread payments and will deliver the defaulted bond to the seller. In return, the buyer receives from the seller the bond's principal and any accrued interest. If the defaulted bond has a recovery value, then on the default date, the net payment received from the seller will be the principal plus any accrued interest minus the recovery value.

If default does not occur, then the buyer will continue to make spread payments until maturity. The non-default cash flows are depicted by the following picture:



If we assume there is no counterparty risk, then default swaps are generally regarded as perfect hedges for their corresponding floaters. If we further assume that defaults occur only at coupon dates which are also floating rate reset dates, then the spread on a par floater should be identical to the spread on a default swap purchased to protect against default of the floater or an arbitrage opportunity will occur. For example, suppose an investor purchases a par floater that pays the risk-free rate plus a spread and finances it by borrowing at the risk-free rate. At the same time, this investor also purchases a default swap which requires the payment of the same spread as on the floater. If default does not occur, then the investor uses the spread earned from the floater to make the spread payments for the default swap and uses the risk-free interest to pay for the borrowing and ends up with a net zero position. If default occurs, then the investor delivers the defaulted floater and in return receives par which is used to pay for borrowing. Since default takes place at the coupon time that is also the reset time, the borrowing is paid in full. Hence, once again, the investor has a net zero position. Since the investor has a zero net position whether or not the bond defaults, the default swap and the par floater must have the same spread.

However, in reality, there is always a discrepancy between the spread on a floater and the spread on its corresponding default swap. The first source of discrepancy is counterparty risk.[4] Default swaps are issued by investment banks while floaters are issued by companies. Given the fact that most investment banks are single-A rated, for a default swap on a triple-A issuer, the counterparty risk of the contract is higher than the underlying bond. However, recently banks have attempted to circumvent this concern by establishing triple-A rated subsidiaries that handle their swap business. Secondly, floating rates are not necessarily reset at the same time as coupon dates. For example, it is not uncommon to use a 3-month LIBOR (London InterBank Offer Rate) to index the semi-annual interest payment of a floater. Finally, default can occur at any time. If default occurs in between two coupon dates, then the recovery value from the default swap may or may not be enough to cover the borrowing (due to differences attributed to accrued interest calculations). Note, however, that even though *technical* default, in the sense that the market value of the firm's debt exceeds the market value of its assets, can occur between two coupon dates, default in the sense of a missed coupon payment is unlikely since the firm is not required to make actual cash interest payments between any two coupon dates.

Default swap contracts are highly flexible instruments. Buyers and sellers are free to design contracts of any maturity, any seniority, any form of payoff, and any recovery structure that may or may not tie into the recovery structure of the underlying bond. It should be noted that the value of a default swap changes over time: it is a function of the bond's credit quality. Just like any other swap contract, default swap traders can unwind the contract whenever the spread changes favorably. Hence, default swaps can be viewed as spread instruments and investors can trade them just like spread options.

Another way to view a default swap contract is as a put option where the underlying asset is the corporate bond. Thus, if default occurs, the bond can be put back to the seller of the swap at a strike price, which equals the principal plus any accrued interest. Consequently, buying a default swap provides an efficient hedge of the underlying bond against default.

[4] Collin-Dufresne and Solnik (2001) demonstrate that counterparty default risk is a dominant factor in the determination of the spread between the swap rate and the LIBOR rate.

It is not unusual for some corporate bonds to trade only twice in their lifetimes: at issuance and redemption. Thus, shorting these thinly traded bonds may not be possible. Default swaps provide an indirect way for investors to short corporate bonds, even when the bonds themselves are not frequently traded. Since default swaps can be regarded as put options, buying a default swap is equivalent to shorting the underlying bond.

B. Default baskets

Unlike default swaps, which provide protection for one bond, default baskets provide protection against a basket (collection) of bonds. When any single bond in the basket defaults, the seller of the basket will pay principal plus accrued interest minus recovery of the defaulted bond. Like with default swaps, the buyer of a default basket makes a stream of spread payments until either maturity or default. In the event of default, the buyer receives a single lump-sum payment. Default baskets have become popular because purchasing individual default swaps for a collection of bonds can be very expensive, especially considering how unlikely it is that all the bonds in a given basket will default simultaneously. Buying a basket, instead, provides a much cheaper solution. The most popular default basket contract is the *first-to-default* basket.[5] In this contract, the seller pays (the default event occurs) when the first default is observed among the bonds in the basket. Presumably, there can also be protection against higher order default, however, as of yet, these contracts have not yet received much attention.

There is a relationship between default swap and default basket prices. This can be seen clearly in a simple discrete-time model where there are two bonds in a basket and there is no counterparty risk. Default dependency is directional. For example, consider a large automobile manufacturer and a small supplier of decals for whom the large manufactured comprises 90% of its total business. The small supplier has near perfect default dependency on the manufacturer (if the manufacturer defaults so will the supplier), but the manufacturer has near zero dependency on the small supplier (if the supplier defaults, the manufacture will simply find another supplier with minimal losses). Take an extreme case of perfect positive dependency between one corporate name and another: then buying the first-to-default basket is equivalent to buying a single default swap written on the riskier bond since the default that triggers the default swap also triggers the basket. In the other extreme case where the one-sided dependency is zero (negative default correlation), then buying the first-to-default basket is equivalent to buying two single default swaps: one for each of the two bonds in the basket. Hence the value of the default basket varies between that of a single default swap and the sum of

[5] The number of names in a typical basket varies from 3 to 5.

two default swaps, depending upon the default correlation between the two bonds.[6] As a result, it is natural to use default swaps as a hedge against default baskets. The default correlation is used for computing the correct hedge ratio.

When the default correlation is low, the first-to-default basket would be especially expensive and any higher order baskets would be relatively cheap. When the default correlation is high, the bonds in the basket will tend to default together, so all orders of basket contracts will be roughly equally priced (and equal to the price of the default swap). It should be understood that buying all orders of the basket contracts is similar to buying default swaps of all the bonds in the basket because both provide full protection of all the bonds in the basket.

Section II. Literature Review

Duffie (1998b) provides an excellent overview and valuation of the default swap contract using the Duffie-Singleton model (1999). However, his valuation does not include counter-party risk and hence does not consider default correlation. Duffie (1998a) models correlated spreads but not correlated defaults. This distinction is especially important, since we find that, when we introduce default correlation, spread correlation becomes an unimportant factor in the pricing of default swaps and baskets.

There are two primary types of models that attempt to describe default processes in the literature: "structural form" models and "reduced form" models. In structural form models, default occurs endogenously when the debt value of the firm exceeds the total value of the firm. In reduced form models, default occurs unexpectedly and follows some random jump processes. There are pros and cons to models of either form. Structural form models, pioneered by Merton (1974), assume that recovery processes are endogenously determined: no additional assumption is necessary. However, structural form models have difficulty incorporating complex debt structures.[7] In addition, structural form models are harder to calibrate to the existing market prices. On the other hand, reduced form models, proposed by Jarrow and Turnbull (1995), Duffie and Singleton (1999), and others, are easy to calibrate, but have difficulty with the incorporation of realistic recovery assumptions.[8]

[6] This result is demonstrated more formally in Section V.
[7] For example, different definitions of default for different debt classes.
[8] See also Madan and Unal (1993), Jarrow, Lando, and Turnbull (1997), Longstaff and Schwartz (1995) and Neilson and Ronn (1995), among others.

Section III. Modeling default as an exogenous process

In this paper, we adopt the reduced form approach and assume that defaults occur unexpectedly and follow a Poisson process; that is, defaults are defined by observing jumps in the process. Furthermore, we define the Poisson processes under the risk-neutralized measure. Thus, we can identify a series of risk neutral default probabilities so that the weighted average of the default and no-default payoffs can be discounted at the risk-free rate.

We follow Jarrow-Turnbull (1995) and Duffie-Singleton (1999) and assume $Q(t)$ to be the risk neutral survival probability from now until some future time t. Then

Eq 1 $$Q(t) = E[1_{\{u>t\}}]$$

where 1 is the indicator function and u is the default time. From Lando (1998), we know that if the spreads are stochastic, then this survival probability should also be stochastic. But here, throughout the paper (except for Section VI), we assume the survival probability to be deterministic. The instantaneous forward default probability is given by

Eq 2 $$\text{Instantaneous default probability} = -dQ(t)$$

The total probability of default over the life of the default swap is the integral of all the instantaneous default probabilities:

Eq 3 $$\int_0^T -dQ(t)dt = 1 - Q(T)$$

As one would expect, the total default probability is equal to one minus the total survival probability.

The survival probability has a very useful application. Assuming deterministic risk-free discounting and deterministic spreads (so that the survival probability is deterministic), a \$1 risky cash flow received at time t has a risk neutral expected value of $Q(t)$ and a present value of $P(t)Q(t)$, where P is the risk-free discount factor. Therefore, the present value of a risky annuity of \$1 (i.e., \$1 received until default) can be written as:[9]

$$\text{PV(risky \$1 annuity)} = \sum_{j=1}^{n} P(j\tau)Q(j\tau)$$

[9] The valuation under stochastic interest rates and spreads can be found in Section VI.

where τ is the time to default. Thus, the price of a risky bond with no recovery upon default can be written as:

$$\text{PV(risky bond with no recovery)} = c\sum_{j=1}^{n} P(j\tau)Q(j\tau) + P(n\tau)Q(n\tau)$$

where *c* is the coupon rate of the risky bond. This result is similar to the risk-free coupon bond where only risk-free discount factors are used. Due to this similarity, we can regard *PQ* as the risky discount factor. The logarithmic difference between the risk-free discount factor and the risky discount factor is often referred as a yield spread:

$$\text{yield spread} = \ln(P(t)) - \ln(P(t)Q(t)) = -\ln Q(t)$$

which is simply the natural logarithm of the survival probability.

The conditional forward default probability is a conditional default probability for a forward time interval conditional on surviving until the beginning of the interval. Naturally this probability can be expressed as:

Eq 4 $$p(t) = \frac{-dQ(t)}{Q(t)}$$

Consider a simple contract that pays $\$X$ upon default and 0 otherwise. Then the value of this contract should be the recovery value upon default multiplied by the default probability:

Eq 5 $$V = E\left[P(u)1_{u<T}\,X(u)\right] = \int_0^T -P(u)dQ(u)X(u)du$$

Default baskets usually contain 3 to 5 credit names (individual bonds). The first-to-default basket pays principal and accrued interest minus the recovery value of the first defaulted bond in the basket. Define u_j to be the time to default of the first bond that defaults in basket: $u_j = \min\{u_k\}$ for any arbitrary bond *k* in the basket. Hence, for pricing the default basket, we can generalize Eq 5 as follows:

Eq 6 $$V = E\left[P(u_j)1_{u_j<T}(1 - R_j(u_j) + A_j(u_j))N_j\right]$$

where 1 is the indicator function, u_j is the default time of the *j*-th bond, R_j is recovery rate of the *j*-th bond, A_j is accrued interest of the *j*-th bond, and N_j is the notional amount of the *j*-th bond. The basket pays when it experiences the first default, i.e. $\min\{u_j\}$.[10]

Obviously, the above equation has no easy solution when the default events (or default times, u_j) are correlated. For the sake of exposition, we assume two default processes and label the survival probabilities of the two credit names as $Q_1(t)$ and $Q_2(t)$. In the case of independence, the default probabilities at some future time *t* are $-dQ_1(t)$ and $-dQ_2(t)$ respectively. The default probability of either bond defaulting at time *t* is:

Eq 7 $$-d[Q_1(t)Q_2(t)]$$

The above equation represents a situation wherein both credit names jointly survive until *t*, but not until the next instant of time; hence one of the bonds must have defaulted instantaneously at time *t*. Subtracting the default probability of the first credit name from the probability of either defaulting gives rise to the probability that only the second name (but not the first) defaults:

Eq 8 $$\begin{aligned}\int_0^T -d[Q_1(t)Q_2(t)] + dQ_1(t)dt &= [1-Q_1(T)Q_2(T)] - [1-Q_1(T)] \\ &= Q_1(T)[1-Q_2(T)]\end{aligned}$$

This probability is equal to the probability of survival of the first name and default of the second name; thus, it is with this probability that the payoff to the second name is paid. By the same token, the default probability of the first name is $1-Q_1(T)$, and it is with this probability that the payoff regarding to the first name is paid.

In a basket model specified in Eq 6, the final formula for the price an *N* bond basket under independence is:

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[10] In either the default swap or default basket market, premium is usually paid in a form of spreads. The spread is paid until either the default or maturity, whichever is earlier. From the total value of the default swap, we can convert it to a spread that is paid until default or maturity:

$$s = \frac{V}{\sum_{j=1}^{n} P(j\tau)Q(j\tau)}$$

Eq 9 $V = \int_0^T \sum_{i=1}^{N} P(t)\left[-d\Pi_{j=1}^{i} Q_j(t) + d\Pi_{j=0}^{i-1} Q_j(t)\right](1 - R_i(t) + A_i(t))$

where $Q_0(t) = 1$ and hence $dQ_0(t) = 0$. The above formula assumes that the last bond (i.e. bond *N*) has the highest priority in compensation, i.e. if the last bond jointly defaults with any other bond, the payoff is determined by the last bond. The second to last bond has the next highest priority in a sense that if it jointly defaults with any other bond *but the last*, the payoff is determined by the second to last bond. This priority prevails recursively to the first bond in the basket.

Investment banks that sell or underwrite default baskets are themselves subject to default risks. If a basket's underlying issuers have a higher credit quality than their underwriting investment bank, then it is possible that the bank may default before any of the issuers. In this case, the buyer of the default basket is subject to not only the default risk of the issuers of the bonds in the basket, but also to that of the bank as well. This is called the "counterparty risk." If the counterparty defaults before any of the issuers in the basket does, the buyer suffers a total loss of the whole protection (and the spreads that had been paid up to that point in time). We modify Eq 9 to incorporate the counterparty risk, by adding a new asset with 0 payoff to the equation:

Eq 10 $V = \int_0^T \sum_{i=1}^{N+1} P(t)\left[-d\Pi_{j=1}^{i} Q_j(t) + d\Pi_{j=0}^{i-1} Q_j(t)\right](1 - R_i(t) + A_i(t))$

where the first asset represents the counterparty whose payoff is 0, i.e.:

Eq 11 $1 - R_1(t) + A_1(t) = 0$ for all *t*.

Note that the counterparty payoff has the lowest priority because the buyer will be paid if the counterparty jointly defaults with any issuer.

The default swap is a special case of the default basket with $N = 1$. However, with a default swap, the counterparty risk is more pronounced than that with a basket deal. With only one issuer, Eq 10 can be simplified to:

Eq 12

$$V = \int_0^T P(t)\left[-dQ_1(t)(1 - R_1(t) + A_1(t)) + (-dQ_1(t)Q_2(t) + dQ_1(t))(1 - R_2(t) + A_2(t))\right]$$

$$= \int_0^T P(t)\left[-dQ_1(t)Q_2(t) + dQ_1(t))(1 - R_2(t) + A_2(t))\right]$$

Eq 12 implies that the investor who buys a default swap on the underlying issuer effectively sells a default swap of joint default back to the counterparty.

When the defaults of the issuers and the counterparty are correlated, the solution to Eq 10 becomes very complex. When the correlations are high, issuers in the basket tend to default together. In this case, the riskiest bond will dominate the default of the basket. Hence, the basket default probability will approach the default probability of the riskiest bond. On the other hand, when the correlations are low, individual bonds in the basket shall default in different situations. No bond will dominate the default in this case. Hence, the basket default probability will be closer to the sum of individual default probabilities.

To see more clearly how correlation can impact the basket value, we take a two-bond basket as an example. For the sake of exposition, we also assume no counterparty risk. In the extreme case where the default correlation is 1, the two bonds in the basket should default together. In this case, the basket should behave like a single bond. On the other extreme, if the correlation is –1 (the bonds are perfect compliments of one another), default of one bond implies the survival of the other and vice versa. In this case, the basket should reach the maximum default probability: 100%.

Section IV. Modeling default correlation

Although it is possible to describe the default correlation, it is not easy to actually model it. The concept of correlation is well defined only for normal random variables. That is, the correlation between two normal random variables uniquely defines the joint distribution of the two normal variables. But what if the random variables are not normally distributed, as would be the case with default probabilities? Unfortunately, dependency between any two non-normally distributed random variables cannot be characterized by correlation.[11]

There has not been much discussion on the default correlation. Duffie and Singleton (1998) simulate correlated spreads under the Merton model but defaults in their model remain independent. In another article, Duffie and Singleton (1996) model the correlation between default processes by combining (summing or subtracting) independent Poisson processes. The limitation of this approach is that one has to pre-specify the sign of the correlation. Furthermore, when there are many issuers, the correlation structure under this model will be quite limited.

[11] See, for example, Schweizer and Wolff (1981).

More recently, Das, Fong, and Geng (2001) find significant correlations in default rates in credit portfolios and document significant differences in credit losses when default correlation is ignored. Zhou (2001) derived the joint default probability for multiple assets under the structural form framework. For a special specification of the Merton model, Zhou is able to present the result in closed form for two assets. Another correlation model derived by Hull and White (2001) is also under the structural form framework. Their model, similar to Zhou's, is closed form for two assets. Both models have difficulty extending the closed form result to more than two assets.[12] To compute the joint default probability in the structural framework, one has to assume an artificial default barrier crossing which (by the underlying firm value) triggers the event of default. This approach does not permit changes of default correlation without changing the level of spread.

The reduced form models, on the contrary, allow spread levels (which are characterized by the hazard rate) and default correlations (which are random draws from the Poisson distribution) to be determined independently. Hence, it allows us to analyze the correlation effect of defaults while holding spreads constant. So far there has been little development of default correlations under the reduced form framework. The only study is Li (2000) who uses a Copula function to parametize the default correlation. In this section, we model defaults non-parametrically using simple Bernoulli events and conditional default probabilities for any given time period. This general approach does not require any distributional assumptions.[13] Furthermore, this approach allows as many assets as desired. Finally, this approach is very easy to implement with Monte Carlo simulations.

A. A Two-Asset Example

In order to properly correlate default events, maintain the flexibility of freely specifying individual default probabilities, and be able to uniquely define a joint distribution, we use conditional default probabilities to describe the dependency of two

[12] Hull and White acknowledge that under multiple assets, their model cannot be carried out without Monte Carlo simulation.

[13] Strictly speaking, our approach to model defaults is not restricted to reduced form models and can be made consistent with structural form models as well.

default events (rather than specifying the correlation). In other words, we specify the default probability of the second bond given that the first bond has already defaulted.

For a single period, the default event should follow a Bernoulli distribution where 0 represents no default and 1 represents default. Under the Bernoulli distribution, the specification of the conditional probability (given the marginal probabilities) uniquely defines the joint default probability. The joint default probability can be formally delineated as:

Eq 13 $$p(A \cap B) = \begin{cases} p(A|B)\,p(B) & \text{or} \\ p(B|A)\,p(A) \end{cases}$$

where A and B represent default events, i.e. a short hand notation for $A = 1$ and $B = 1$.

This specification of the joint distribution is desirable in that it is realistic in defining the default relationship between two bonds. If two bonds have different default probabilities (marginals) but they are extremely highly correlated, it means (in our model) that if the less risky one defaults, the riskier one will surely default but not the other way around. Recall the example of the small auto part supplier whose sole client is a large automobile manufacturer. If the manufacturer defaults, it is highly likely that the small part supplier will eventually default as well. The small part supplier may also default on its own, independent of what happens to the automobile manufacturer.

For example, we let A be default on the part of the manufacturer and B be the default on the part of the small supplier. Each has 10% and 20% default probability for the period, respectively. The part supplier is completely dependent upon the manufacturer, i.e. the conditional default probability of the supplier on the manufacturer is one. Then the joint probability of both firms defaulting is:

Eq 14 $$\begin{aligned} p(A \cap B) &= p(B|A)\,p(A) \\ &= p(A) \\ &= 10\% \end{aligned}$$

Hence, we obtain the following joint distribution for the two companies:

B \ A	0	1	
0	80%	0	80%
1	10%	10%	20%
	90%	10%	100%

The dependency of the manufacturer on the supplier can be calculated to be:

$$
\begin{aligned}
p(A|B) &= \frac{p(A \cap B)}{p(B)} \\
&= \frac{10\%}{20\%} \\
&= 50\%
\end{aligned}
\qquad \text{Eq 15}
$$

which is equal to the ratio of the two individual default probabilities. The unconditional correlation of the two companies is 0.6667.

The two companies can also have perfectly negative dependency: default of one company implies the survival of the other. In this case, the conditional probability is:

Eq 16 $p(B^C \mid A) = 1$ or $p(B \mid A) = 0$

Using the numbers in the previous example yields the following joint distribution:

B \ A	0	1	
0	70%	10%	80%
1	20%	0	20%
	90%	10%	100%

Notice that, even though there is perfect negative dependency, the unconditional correlation is only –0.1667.

B. Extension to Multiple Assets

To generalize our two-asset analysis and build a model for the dependency for multiple variables, we need more than conditional probabilities between two assets. We also need the joint default probabilities. As a result, we need a model that can sufficiently define a joint distribution for the default events with no more than conditional probabilities as inputs. We achieve the objective by using multivariate normal distributions.

We define the conditional default probability of any two assets as follows:

Eq 17
$$\begin{aligned} p(X_i \mid X_j) &= p(X_i \cap X_j)\, p(X_j) \\ &= M(K_i, K_j, \rho_{ij})\, p(X_j) \end{aligned}$$

where $M(K_i, K_j, \rho_{ij})$ is a bivariate normal probability function in which ρ_{ij} is set so that the bivariate normal probability is equal to the joint default probability and

Eq 18
$$\begin{aligned} K_i &= N^{-1}(p(X_i)) \\ K_j &= N^{-1}(p(X_j)) \end{aligned}$$

where $N^{-1}(\cdot)$ is the inverse normal probability function. This way, the conditional probabilities of any two assets translate into a set of correlations among normal variables. Given that correlations are sufficient to uniquely determine the joint distribution, all the joint default probabilities of multiple assets can be determined.

Obviously, our model requires numerical searches for the correlations from joint default probabilities, however, quick and efficient algorithms exist for the calculation of bivariate normal probabilities.

Section V. Valuation of default swaps and default baskets

In this section, we employ the model developed in Section IV to investigate the price behavior of default swaps and baskets as default interdependency changes. In order to obtain a closed form solution for the contract value, we examine a two-asset basket with no counterparty risk.[14] The model can easily be extended to a multi-asset basket, but the value must be solved numerically. From Eq 6, the closed form solution for the basket's value is:

Eq 19
$$\begin{aligned} V &= P(T)\, p(A \cup B) \\ &= P(T)[\, p(A) + p(B) - p(B \mid A)\, p(A)] \end{aligned}$$

[14] In a single period, default can occur only at the end of the period and hence there is no accrued interest. We also assume no recovery.

where P is the risk-free discount factor. As can be seen, the basket value is linear in the conditional probability. We should note that the unconditional default correlation, which is calculated as follows:

Eq 20 $$\rho(A,B)=\frac{p(B\,|\,A)\,p(A)-p(A)\,p(B)}{\sqrt{p(A)(1-p(A))\,p(B)(1-p(B))}},$$

is also linear in the conditional probability. It is easy to demonstrate that the basket value is a (negative) linear function of the default correlation. When the default correlation is small (or even negative), the issuers tend to default randomly: this increases the basket risk. When the correlation is large, the issuers tend to default together: this decreases the basket risk. Using the above numerical example of the automobile manufacturer and the parts suppler, we show the relationship between the default correlation and the basket value in Figure 1. As will be demonstrated below, the value of the default basket will vary between that of a single default swap (when there is perfect dependency) and the sum of the two individual default swaps (when there is zero dependency).

Note that in the event of perfect dependency (if B defaults then necessarily A will default as well), $p(B\,|\,A)=1$, the basket valuation formula in Eq 19 becomes:

Eq 21 $$\begin{aligned} V &= P(T)[\,p(A)+p(B)-p(B\,|\,A)\,p(A)] \\ &= P(T)\,p(B) \end{aligned}$$

which implies that the value of the basket is equal to the value of a single swap. Note, if there is perfect dependency and the two companies were to have identical default probabilities, then the default correlation will be one. Therefore, our dependency definition is general enough to contain the usual concept of perfect positive correlation as a special case. This can be seen in the following.

Eq 22 $$\begin{aligned} \rho(A,B)=1 &= \frac{p(A)(1-p(B))}{\sqrt{p(A)(1-p(A))\,p(B)(1-p(B))}} \\ 1 &= \frac{\sqrt{p(A)(1-p(B))}}{\sqrt{(1-p(A))\,p(B)}} \end{aligned}$$

in which the only solution is $p(A)=p(B)$. On the other hand, if the dependency of B on A is zero ($p(B\,|\,A)=0$) then:

Eq 23 $\rho(A,B) = \dfrac{-p(A)p(B)}{\sqrt{p(A)(1-p(A))p(B)(1-p(B))}}$

which is always negative. In this case, the basket valuation becomes:

Eq 24
$$\begin{aligned} V &= P(T)[p(A) + p(B) - p(B \mid A)p(A)] \\ &= P(T)[p(A) + p(B)] \end{aligned}$$

which implies that the value of the basket is equal to the sum of the two individual swaps. Note that if there is no dependency and the two companies were to have default probabilities that sum to exactly 100%, then the joint distribution would degenerate and result in an unconditional correlation of –1. Again, our model of dependency covers the usual definition of perfect negative correlation as a special case. This can be seen in the following:

Eq 25
$$\begin{aligned} \rho(A,B) = -1 &= \frac{-p(A)p(B)}{\sqrt{p(A)(1-p(A))p(B)(1-p(B))}} \\ 1 &= \frac{\sqrt{p(A)p(B)}}{\sqrt{(1-p(A))(1-p(B))}} \end{aligned}$$

for which the only solution is $p(A) + p(B) = 1$.

Default swaps have a very similar default probability profile. Those with counterparty risk will be paid only in the situation where the counterparty survives AND the issuer defaults. Thus, the relevant default probability for default swaps is:

Eq 26 $p(A \cap B^C) = p(A) - p(A \cap B)$

and the valuation formula is:

Eq 27 $V = P(T)p(A)(1 - p(B \mid A))$

Again, note that the relationship is linear in the default probability. Using the same numerical values, we plot the default swap value in Figure 2. Note that if there were no counterparty risk, then Eq 27 would degenerate to

Eq 28 $V = P(T)p(A)$.

The results can be extended to multiple assets, though the calculations of the probabilities become multi-dimensional. We can use Monte-Carlo methods to calculate the joint normal probabilities in high dimensions. In the next section, we shall extend the model to multiple periods, and in addition, include random spreads. As will be seen, our initial choice of a deterministic survival likelihood has very little impact on the result: spread correlation seems to have no impact on basket value.

Section VI. Valuation in continuous time with random spreads

The Bernoulli distribution for a single period can be extended to a Poisson process in continuous time. In a continuous time model, the above default correlation should be interpreted as the instantaneous default correlation. In a Poisson process, defaults occur unexpectedly with intensity λ (the hazard rate). Hence, the survival probability can be written as:

Eq 29 $Q(t) = \exp\left(-\int_0^t \lambda(u)du\right)$

Given Eq 29, the instantaneous conditional forward default probability (the continuous time counterpart to Eq 4) should become:

Eq 30 $p(t) = \dfrac{-dQ(t)}{Q(t)} = \lambda(t)dt$

The Poisson process intensity parameter can thus be interpreted as an annualized conditional forward default probability. The yield spread can also be represented by the intensity parameter:

Eq 31 $s(t) = -\ln Q(t) = \int_0^t \lambda(u)du$

If default is governed by a Poisson process, then for any infinitesimally small period, the default event is a Bernoulli distribution. Thus, our single period analysis in the previous section can readily be generalized to continuous time. In multiple periods (continuous time), spreads change randomly but are correlated with one another; moreover, they are also correlated with changing interest rates. To accommodate this innovation, we randomize the hazard rate following the mathematical foundation established by Lando (1998).

For any issuer, the risky discount factor is written as:

Eq 32
$$\begin{aligned} D(0,t) &= E^{r,\lambda,\tau}\left[1_{\{\tau>t\}}\exp\left(-\int_0^t r_u du\right)\right] \\ &= E^{r,\lambda}\left[E[1_{\{\tau>t\}} \mid r,\lambda]\exp\left(-\int_0^t r_u du\right)\right] \end{aligned}$$

where the interest rate, r, hazard rate, λ, and the default time, τ, are all random variables. In order to investigate the isolated impact of spread correlations on the value of default swaps and baskets, we assume that the default time, τ (conditional on λ), is uncorrelated with the interest rate, r. Thus the first term inside the expectation simplifies to:

Eq 33 $$E[1_{\{\tau>t\}} \mid r,\lambda] = E[1_{\{\tau>t\}} \mid \lambda] = \exp\left(-\int_0^t \lambda_u du\right)$$

As a result, we can write the risky discount factor as:

Eq 34
$$\begin{aligned} D(0,t) &= E^{r,\lambda,\tau}\left[1_{\{\tau>t\}}\exp\left(-\int_0^t r_u du\right)\right] \\ &= E^{r,\lambda}\left[\exp\left(-\int_0^t r_u + \lambda_u du\right)\right] \end{aligned}$$

Note that the interest rate and the hazard rate are correlated. Hence, the closed form solution does not exist except for the case where both variables are normally distributed. Unfortunately, a normally distributed hazard rate is infeasible since it may cause the survival probability to be greater than one (or the default probability to be negative).

Although the value of the risky discount factor in Eq 32 is easy to obtain, the value of the default swap (and basket) is considerably more complex, since it requires the evaluation of a more complex integral where the risk-free discount factor is a function of the default time. See Eq 6.

In order to show the impact of the spread correlation, we present an explicit example where the hazard rate follows a log-normal process:

Eq 35 $$d\ln\lambda_j = \alpha(t)dt + \sigma dz_j$$

where $j = 1, \ldots, N$ representing various issuers in the basket, dz is normal(0, dt) and $\alpha(t)$ is a time dependent parameter used for calibrating the model to the default probability curve (see an Appendix for details). The correlations are defined as $\rho_{ij} dt = dz_i dz_j$.[15]

We continue with our previous example of two issuers (the auto manufacturer and the supplier):

	Auto manufacturer	Supplier
First period default probability	10%	20%
Second period default probability	10%	20%
Second period hazard rate volatility	100% (σ_1)	100% (σ_2)

We start by simulating the defaults in the first period assuming 0 correlation. We label this the "base case." Then we simulate Eq 35 with the correlation value from –1 to 1.[16] The simulated hazard rates are used to compute the default probabilities for the second period. Then the second period defaults are simulated. A payoff of $1 is recorded if there is a default in a simulation run. We assume 5% risk-free interest rate for each period.

Since there is no closed-form solution for the value of the default basket, closed-form comparative static results are unattainable. However, the comparative static impact of the interaction between default correlation and other factors on basket value can easily be teased out of the model using numerical methods. We begin by examining the interaction between spread correlation and default correlation on basket value. Our expectations are that the basket value will be decreasing in the default correlation, but unaffected by changes in the spread correlation.[17] Figure 3 presents a three-dimensional graphical depiction of the impact of spread correlation and default correlation on default basket value. As evidenced by the figure, the relationship is indeed planar: the spread

[15] We follow Duffie (1998a) and set the default correlations to 0.

[16] See Appendix B for details.

[17] Default swaps and baskets are "cash products." Their prices are derived exclusively from the "default" probability curve; thus, only correlations among the default processes should be important. Another example of a "cash product" is interest rate swaps, which are "cash products" of the yield curve: the yield curve alone is sufficient to compute the swap rate, which is the weighted average of forward yields. This is in contrast to a non-cash product like a spread option, whose price is derived from not only the default curve, but also the volatility and correlations among the spreads.

correlation has no substantial impact on basket value regardless of the choice of default correlation, but as the default correlation increases, the value of the default basket decreases. Since the spread correlation has no impact on default probability in even a simple a two-period setting, it is trivial to extend the results to the case of multiple periods.

Our intuition is that volatility should have no effect on the value of the default basket. Changes in volatility are captured by the drift term (α) of the hazard rate (λ) process, which is calibrated to today's spread curve. Figure 4 presents the impact of the interaction between volatility and default correlation on basket value. As expected, this relationship is also planar. Basket value is again decreasing in the default correlation, but is hardly affected by changes in volatility.

As the number of assets in the basket increases, we expect that the basket value should increase because the greater the number of assets increases the likelihood of basket default. This effect should be more pronounced when the default correlations among the assets in the basket is low. Figure 5 presents the relationship among the number of assets in the basket, default correlation, and basket value. The first two bonds in the basket correspond to the two bonds in our base case. The additional bonds each have default probabilities of 15% in both period one and period two and spread volatilities of 100%. Once again, the basket value is decreasing in default correlation, however the effect is more pronounced as the number of assets in the basket increases. Also, not surprisingly, as the number of assets in the basket increases, the basket value increases as well; however, this effect is more prominent at lower levels of default correlation.

We also examine the interaction between volatility and the number of assets in the basket. Once again, our expectation is that volatility should not have much of an impact on the value of the basket, but that the basket value will be increasing in the number of assets. Figure 6 presents the relationship between volatility, the number of assets, and basket value. The basket value is indeed increasing in the number of assets. Also, as expected, the relationship between the number of bonds in the basket and basket value is hardly affected by changes in volatility.

Lastly, Figure 7 examines the interaction between the default probability curve and the default correlation on the values of a two-asset basket. We model the default probability curve as being either upward sloping, flat, or downward sloping. The upward sloping forward default probability curve for the first bond is 5% in period one and 15% in period two and for the second bond is 15% and 25%, respectively. The downward sloping curve is 15% and 5% for the first bond and 25% and 15% for the second bond.

The flat curve is the base case of 10% and 10% for the first bond and 20% and 20% for the second bond. As can be seen in Figure 7, the basket value seems to be unaffected by the slope of the forward default probability curve.

In summation, basket value is affected by the default correlation and the number of bonds in the basket, but is not affected much by spread correlation, spread volatility, or the slope of the forward default probability curve.

Section VII. Conclusion

In this paper, we analyze the pricing behavior of two default-triggered credit derivative contracts: default swaps and default baskets. In particular, we study how correlated default events can impact prices. Given that there is no unique way to define two correlated Poisson processes, we present a simple model for the correlation which not only preserves the desired properties but is very simple to implement. Using our model, we demonstrate that the default correlation plays a dominant role in characterizing the risk of default-triggered credit derivatives. When we extend our model to continuous time, we find that, once the default correlation is taken into consideration, the spread dynamics have very little explanatory power of the risk.

I. APPENDIX

A. MODELING DEFAULT CORRELATION

In this section we show the relationship between the correlation coefficient of two normal random variables and the conditional default probability of one Bernoulli variable on the other. In one extreme case where $\rho = 1$, the joint default probability generated by the joint normal distribution is equal to the default probability of the less risky bond, i.e.:

A 1 $$M(K_1, K_2, \rho) = N(K_1) = p(X_1)$$

Hence, the conditional probability of the more risky bond on the less risky bond is one, i.e.:

A 2 $$p(X_2 \mid X_1) = \frac{p(X_1)}{p(X_1)} = 1$$

On the other hand, the conditional probability of the less risky bond on the more risky bond is the ratio of the two marginal probabilities:

A 3 $$p(X_1 \mid X_2) = \frac{p(X_1)}{p(X_2)}$$

This is precisely the result we desire. When two bonds have perfect dependency, the default of the more risky one should completely rely on the less risky one; but not the other way around. This desired one-sided dependency is captured by the perfect correlation of two normal variables. Note that if the two probabilities are equal, then the dependency becomes symmetric. In this case, the two bonds should behave identically, just like one bond.

In the other extreme where $\rho = -1$, we obtain another desired result that the default of one bond implies the complete survival of the other. To see that, we first note that the joint default probability generated by the joint normal distribution under perfectly negative correlation is 0. Consequently, the conditional probability of one surviving given the other has defaulted is:

A 4 $$p(X_j^C \mid X_k) = \frac{p(X_j^C \cap X_k)}{p(X_k)} = \frac{p(X_k) - p(X_j \cap X_k)}{p(X_k)} = \frac{p(X_k) - 0}{p(X_k)} = 1$$

where *j* and *k* can be either bond.[18]

B. CALIBRATION

As delineated in Lando (1998), the hazard rate, λ, can itself be stochastic. As we have shown earlier, this λ is not only the forward default probability, but it is also the forward spread. In the static model, this λ is solved for from the periodic default probability. Now we shall simulate this λ with a mean equal to the periodic probability. In particular, we shall simulate λ from a log normal distribution with

A 5 $$\lambda_{t+\Delta t} = \lambda_t \alpha_t e^{\sigma \sqrt{\Delta t} z_t}$$

[18] The default dependency is clearly 0 on either direction.

where z is normal(0,1) and $\alpha(t)$ is a time dependent parameter used for calibrating the following:

A 6 $$Q(t) = E_0\left[e^{-\int_0^t \lambda_s ds}\right]$$

In the simulation, we set a piece-wise flat α according to

A 7 $$Q(t+\Delta t) = E_0\left[e^{-\int_0^t \lambda_s ds} e^{-\lambda_{t+\Delta t}\Delta t}\right]$$

Every path in the simulation is a draw from an *n*-dimensional joint log normal distribution, each of which follows the process described as such. For every period, α is calculated for that period so that the forward probability curve is properly calibrated. Then, an *independent* random draw from an *n*-dimensional joint uniform distribution is used to determine if any of the assets default.

Finally, the first-to-default protection value of a basket is the present value of the risk-neutral expectation of the payoff (notional value minus recovery value):

A 8 $$E\left[e^{-\int_0^t r_s ds} 1_{\{t<T\}}\right]$$

where t is the default time, T is the maturity time of the basket, and 1 is the indicator function which returns a value of zero if its subscript is untrue and one if it is true. The valuation is achieved by calculating the average of the simulations.

We first simulate 10,000 independent normal random numbers and then use the Chelosky matrix to correlate them.[19] Then we transform these normal variates to uniform ones by using the cumulative normal function. This allows us to determine whether a bond has defaulted or not. Finally we count the number of events that have at least one bond defaults to acquire the basket default probability.

In the second period, the Poisson intensity parameters are log-normally distributed. Hence, we simulate 10,000 correlated log-normal numbers calibrate the model using the default probabilities of the second period. We then simulate 10,000 default events according to the newly obtained (and calibrated) intensity parameter values

[19] Each time we generate 10,000 random numbers, 5,000 of them are random paths and the other 5,000 are antithetically generated. The same paths are used throughout the paper.

using the procedure described above. Finally we count the number of events where at least one bond defaults, given that it did not default in the first period.

The base case is such that there are two bonds in the basket. The respective default probabilities are 10% for the first bond issuer and 20% for the second bond issuer. Their respective spread volatilities are both 100% and the default correlation and spread correlation between the two bonds are zero.

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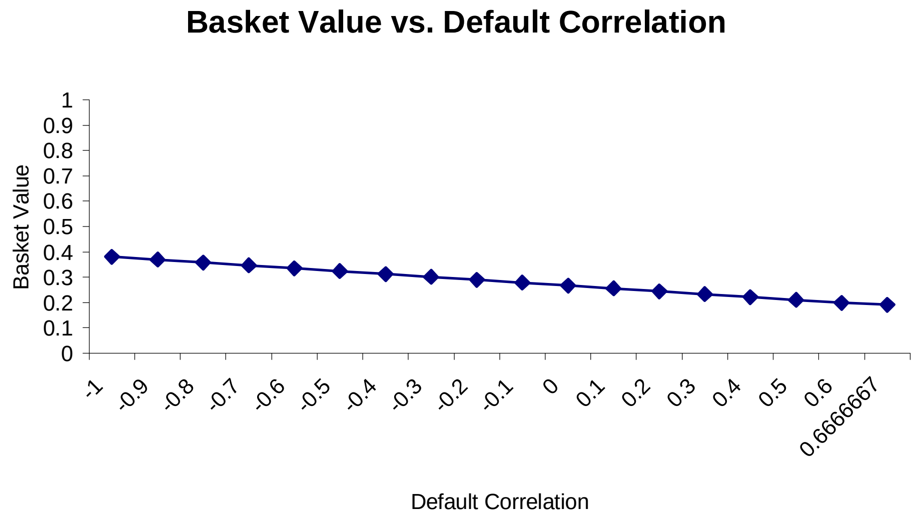
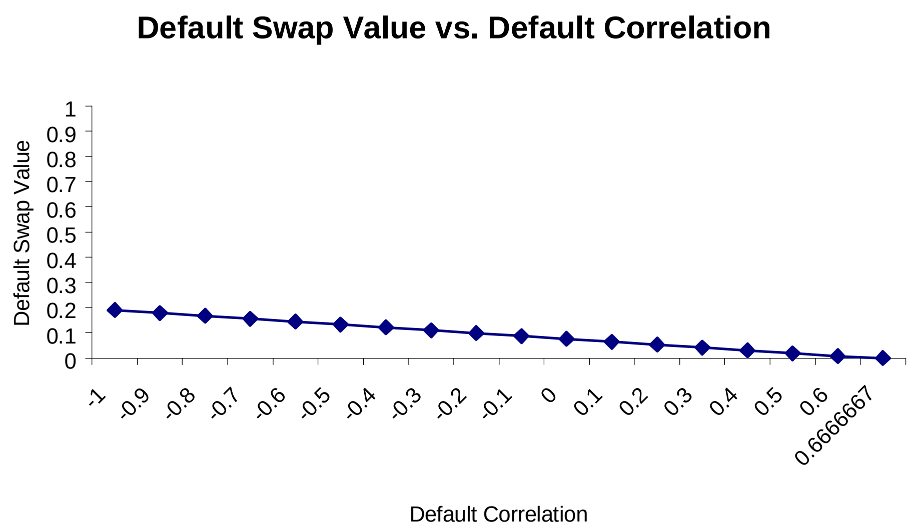
Figure 1



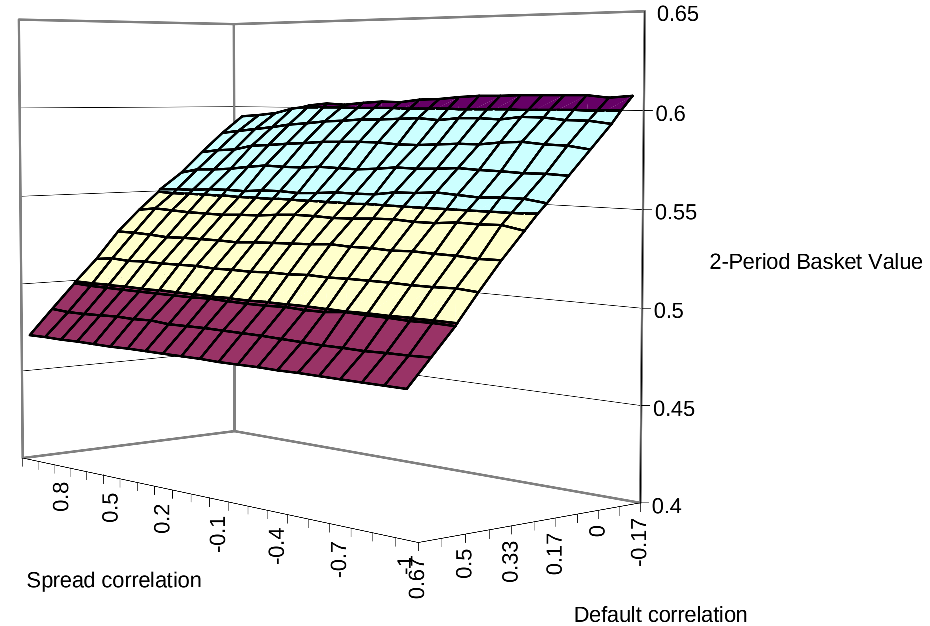
The risk-free rate is 5% and the default probability is 10% for the first bond and 20% for the second bond.

Figure 2



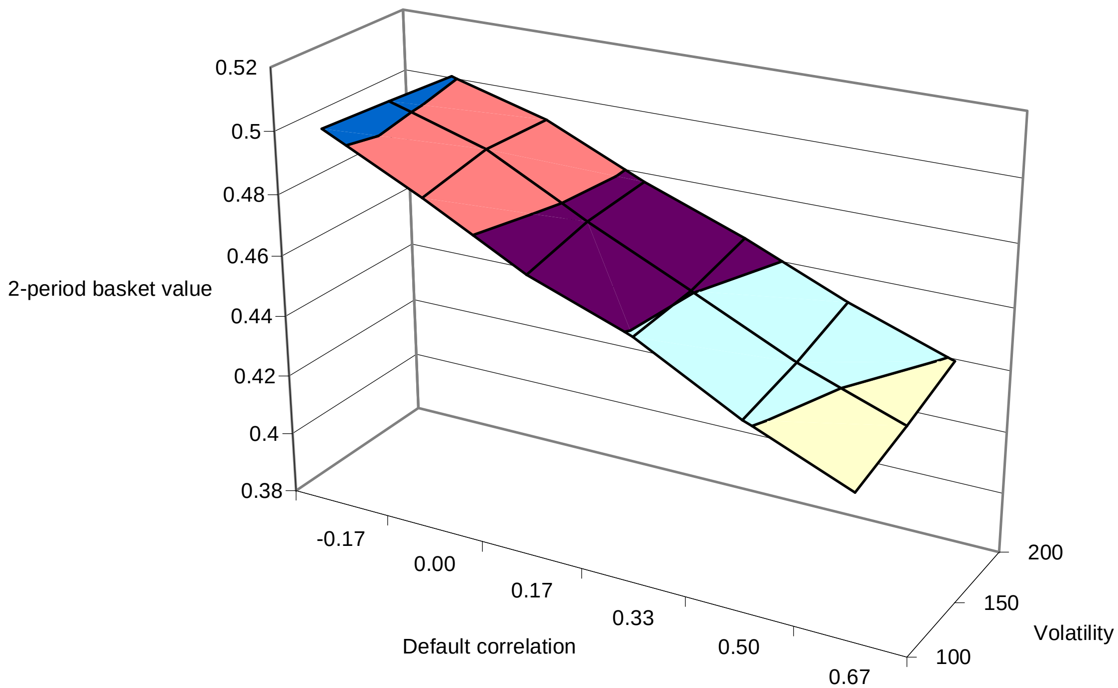
The risk-free rate is 5% and the default probability is 10% for the first bond and 20% for the counterparty.

Figure 3. Interaction Between Default Correlation and Spread Correlation



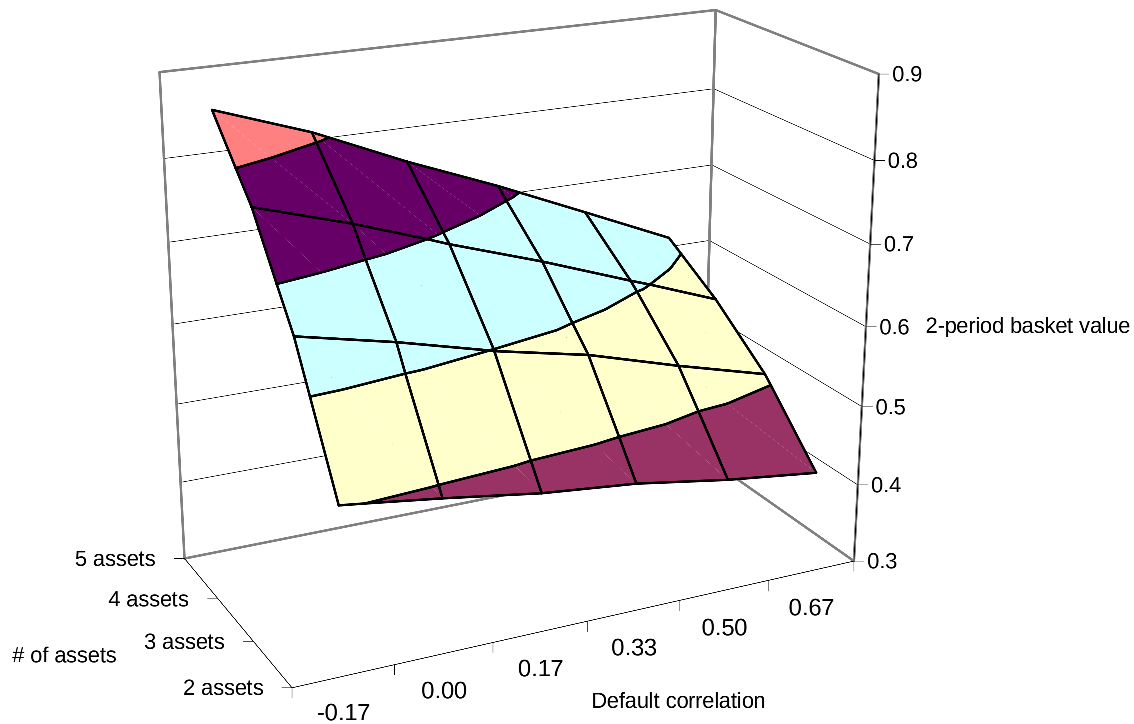
In generating this figure, the risk-free rate is assumed to be 5%. The default probability is 10% for the first bond and 20% for the second bond. The same default probabilities apply in both periods. The volatility is 100% for each bond.

Figure 4. Interaction between Vol and Default Correlation



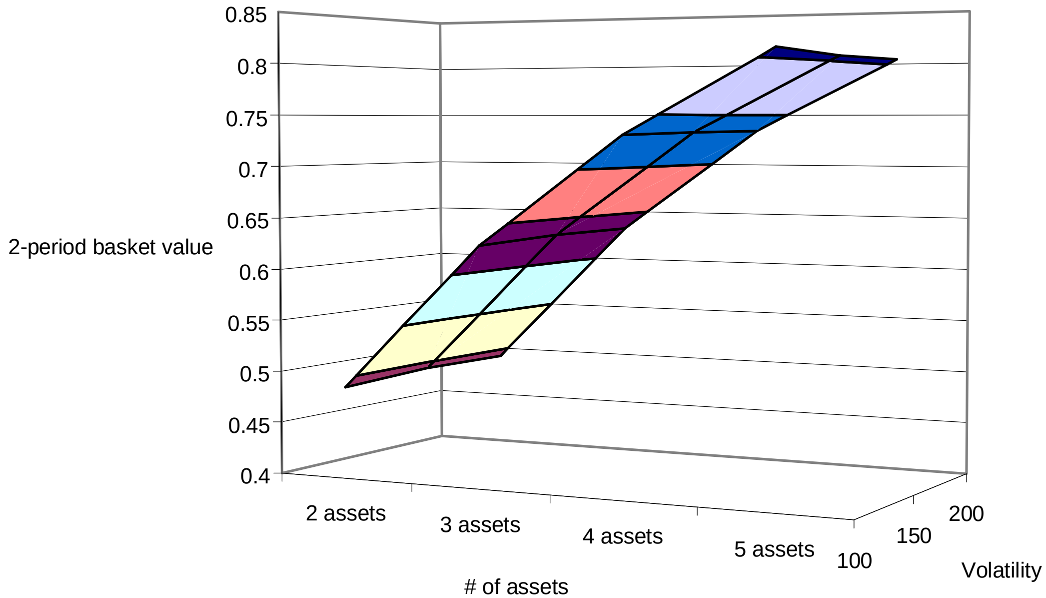
In generating this figure, the risk-free rate is assumed to be 5%. The default probability is 10% for the first bond and 20% for the second bond. The same default probabilities apply in both periods. The spread correlation is zero.

Figure 5. Interaction between # of Assets and Default Correlation



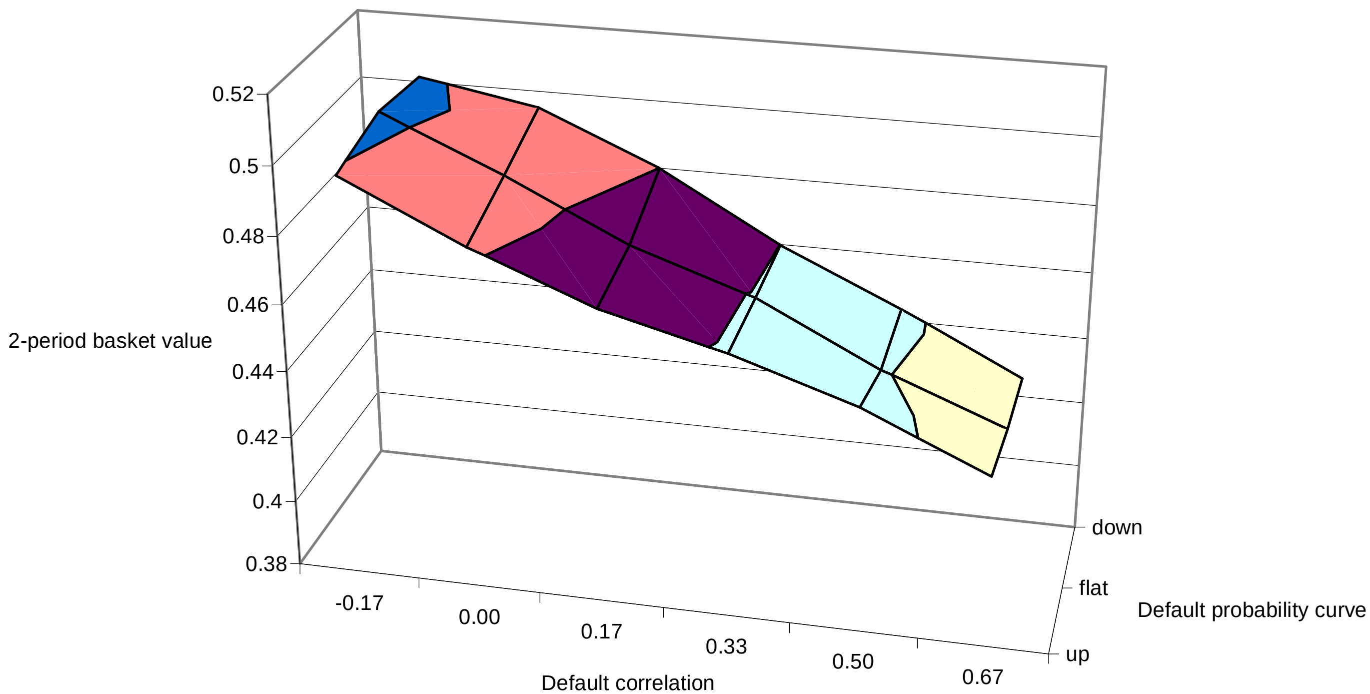
In generating this figure, the risk-free rate is assumed to be 5%. The default probability is 10% for the first bond, 20% for the second bond, and 15% for the remaining three bonds, respectively. The same default probabilities apply in both periods. The volatility is 100% for each bond. The spread correlation is zero.

Figure 6. Interaction between Vol and # of Assets



In generating this figure, the risk-free rate is assumed to be 5%. The default probability is 10% for the first bond, 20% for the second bond, and 15% for the remaining three bonds, respectively. The same default probabilities apply in both periods. All of the default correlations and the spread correlations are zero.

Figure 7. Interaction between Probability Curve and Default Correlation



In generating this figure, the risk-free rate is assumed to be 5%. The spread correlation is zero. The volatility is 100% for each bond. The upward sloping forward default probability curve for the first bond is 5% in period one and 15% in period two and for the second bond is 15% and 25%, respectively. The downward sloping curve is 15% and 5% for the first bond and 25% and 15% for the second bond. The flat curve is the base case of 10% and 10% for the first bond and 20% and 20% for the second bond.