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A comparative analysis of current credit risk models ☆

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Abstract

The new BIS 1998 capital requirements for market risks allows banks to use internal models to assess regulatory capital related to both general market risk and credit risk for their trading book. This paper reviews the current proposed industry sponsored Credit Value-at-Risk methodologies. First, the credit migration approach, as proposed by JP Morgan with CreditMetrics, is based on the probability of moving from one credit quality to another, including default, within a given time horizon. Second, the option pricing, or structural approach, as initiated by KMV and which is based on the asset value model originally proposed by Merton (Merton, R., 1974. Journal of Finance 28, 449–470). In this model the default process is endogenous, and relates to the capital structure of the firm. Default occurs when the value of the firm's assets falls below some critical level. Third, the actuarial approach as proposed by Credit Suisse Financial Products (CSFP) with CreditRisk+ and which only focuses on default. Default for individual bonds or loans is assumed to follow an exogenous Poisson process. Finally, McKinsey proposes CreditPortfolioView which is a discrete time multi-period model where default probabilities are conditional on the macro-variables like unemployment, the level of interest rates, the growth rate in the economy, … which to a large extent drive the credit cycle in the economy. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

BIS 1998 is now in place, with internal models for market risk, both general and specific risk, implemented at the major G-10 banks, and used every day to report regulatory capital for the trading book. The next step for these banks is to develop a VaR framework for credit risk. The current BIS requirements for "specific risk" are quite loose, and subject to broad interpretation. To qualify as an internal model for specific risk, the regulator should be convinced that "concentration risk", "spread risk", "downgrade risk" and "default risk" are appropriately captured, the exact meaning of "appropriately" being left to the appreciation of both the bank and the regulator. The capital charge for specific risk is then the product of a multiplier, whose minimum volume has been currently set to 4, times the sum of the VaR at the 99% confidence level for spread risk, downgrade risk and default risk over a 10-day horizon.

There are several issues with this piecemeal approach to credit risk. First, spread risk is related to both market risk and credit risk. Spreads fluctuate either, because equilibrium conditions in capital markets change, which in turn affect credit spreads for all credit ratings, or because the credit quality of the obligor has improved or deteriorated, or because both conditions have occurred simultaneously. Downgrade risk is pure credit spread risk. When the credit quality of an obligor deteriorates then the spread relative to the Treasury curve widens, and vice versa when the credit quality improves. Simply adding spread risk to downgrade risk may lead to double counting. In addition, the current regime assimilates the market risk component of spread risk to credit risk, for which the regulatory capital multiplier is 4 instead of 3.

Second, this issue of disentangling market risk and credit risk driven components in spread changes is further obscured by the fact that often market participants anticipate forthcoming credit events before they actually happen. Therefore, spreads already reflect the new credit status when the rating agencies effectively downgrade an obligor, or put him on "credit watch".

Third, default is just a special case of downgrade, when the credit quality has deteriorated to the point where the obligor cannot service anymore its debt obligations. An adequate credit-VaR model should therefore address both migration risk, i.e. credit spread risk, and default risk in a consistent and integrated framework.

Finally, changes in market and economic conditions, as reflected by changes in interest rates, the stock market indexes, exchange rates, unemployment rates, etc. may affect the overall profitability of firms. As a result, the exposures of the various counterparts to each obligor, as well as the probabilities of default and of migrating from one credit rating to another. In fact, the ultimate framework to analyze credit risk calls for the full integration of market risk and credit risk. So far no existing practical approach has yet reached this stage of sophistication.

During the last two years a number of initiatives have been made public. CreditMetrics from JP Morgan, first published and well publicized in 1997, is reviewed in the next section. CreditMetrics' approach is based on credit migration analysis, i.e. the probability of moving from one credit quality to another, including default, within a given time horizon, which is often taken arbitrarily as 1 year. CreditMetrics models the full forward distribution of the values of any bond or loan portfolio, say 1 year forward, where the changes in values are related to credit migration only, while interest rates are assumed to evolve in a deterministic fashion. Credit-VaR of a portfolio is then derived in a similar fashion as for market risk. It is simply the percentile of the distribution corresponding to the desired confidence level.

KMV Corporation, a firm specialized in credit risk analysis, has developed over the last few years a credit risk methodology, as well as an extensive database, to assess default probabilities and the loss distribution related to both default and migration risks. KMV's methodology differs somewhat from CreditMetrics as it relies upon the "Expected Default Frequency", or EDF, for each issuer, rather than upon the average historical transition frequencies produced by the rating agencies, for each credit class.

Both approaches rely on the asset value model originally proposed by Merton (1974), but they differ quite substantially in the simplifying assumptions they require in order to facilitate its implementation. How damaging are, in practice, these compromises to a satisfactory capture of the actual complexity of credit measurement stays an open issue. It will undoubtedly attract many new academic developments in the years to come. KMV's methodology is reviewed in Section 3.

At the end of 1997, Credit Suisse Financial Products (CSFP) released a new approach, CreditRisk+, which only focuses on default. Section 4 examines briefly this model. CreditRisk+ assumes that default for individual bonds, or loans, follows a Poisson process. Credit migration risk is not explicitly modeled in this analysis. Instead, CreditRisk+ allows for stochastic default rates which partially account, although not rigorously, for migration risk.

Finally, McKinsey, a consulting firm, now proposes its own model, CreditPortfolioView, which, like CreditRisk+, measures only default risk. It is a discrete time multi-period model, where default probabilities are a function of macro-variables such as unemployment, the level of interest rates, the growth rate in the economy, government expenses, foreign exchange rates, which also drive, to a large extent, credit cycles. CreditPortfolioView is examined in Section 5.

From the actual comparison of these models on various benchmark portfolios, it seems that any of them can be considered as a reasonable internal model to assess regulatory capital related to credit risk, for straight bonds and loans without option features. [1] All these models have in common that they assume deterministic interest rates and exposures. While, apparently, it is not too damaging for simple "vanilla" bonds and loans, these models are inappropriate to measure credit risk for swaps and other derivative products. Indeed, for these instruments we need to propose an integrated framework that allows to derive, in a consistent manner, both the credit exposure and the loss distribution. Currently, none of the proposed models offers such an integrated approach. In order to measure credit risk of derivative securities, the next generation of credit models should allow at least for stochastic interest rates, and possibly default and migration probabilities which depend on the state of the economy, e.g. the level of interest rates and the stock market. According to Standard & Poor's, only 17 out of more than 6700 rated corporate bond issuers it has rated defaulted on US $4.3 billion worth of debt in 1997, compared with 65 on more than US $20 billion in 1991. In Fig. 1 we present the record of defaults from 1985 to 1997. It can be seen that in 1990 and 1991, when the world economies were in recession, the frequency of defaults was quite large. In recent years, characterized by a sustained growth economy, the default rate has declined dramatically.

2. CreditMetrics [2] and CreditVaR I [3]

CreditMetrics/CreditVaR I are methodologies based on the estimation of the forward distribution of the changes in value of a portfolio of loan and bond type products [4] at a given time horizon, usually 1 year. The changes in value

[1] IIF (the International Institute of Finance) and ISDA (the International Swap Dealers Association) have conducted an extensive comparison of these models on several benchmark portfolios of bonds and loans. More than 20 international banks participated in this experiment. A detailed account of the results will be published in the fall of 1999.

[2] CreditMetrics is a trademark of JP Morgan. The technical document, CreditMetrics (1997) provides a detailed exposition of the methodology, illustrated with numerical examples.

[3] CreditVaR is CIBC's proprietary credit value at risk model that is based on the same principles as CreditMetrics for the simple version implemented at CIBC, CreditVaR I, to capture specific risk for the trading book. A more elaborate version, CreditVaR II, extends CreditMetrics framework to allow for stochastic interest rates in order to assess credit risk for derivatives, and incorporates credit derivatives. Note that to price credit derivatives we need to use "risk neutral" probabilities which are consistent with the actual probabilities of default in the transition matrix.

[4] CreditMetrics' approach applies primarily to bonds and loans which are both treated in the same manner, and it can be easily extended to any type of financial claims as receivables, loan commitments, financial letters of credit for which we can derive easily the forward value at the risk horizon, for all credit ratings. For derivatives, like swaps or forwards, the model needs to be somewhat tweaked, since there is no satisfactory way to derive the exposure and the loss distribution in the proposed framework, which assumes deterministic interest rates.

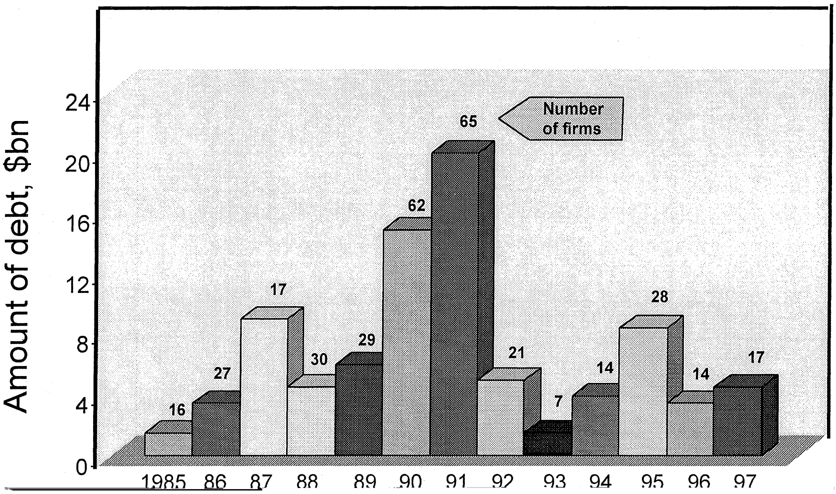


Fig. 1. Corporate defaults, worldwide (*source:* Standard & Poor's).

are related to the eventual migrations in credit quality of the obligor, both up and downgrades, as well as default.

In comparison to market-VaR, credit-VaR poses two new challenging difficulties. First, the portfolio distribution is far from being normal, and second, measuring the portfolio effect due to credit diversification is much more complex than for market risk.

While it was legitimate to assume normality of the portfolio changes due to market risk, it is no longer the case for credit returns which are by nature highly skewed and fat-tailed as shown in Figs. 2 and 6. Indeed, there is limited upside to be expected from any improvement in credit quality, while there is substantial downside consecutive to downgrading and default. The percentile levels of the distribution cannot be any longer estimated from the mean and variance only. The calculation of VaR for credit risk requires simulating the full distribution of the changes in portfolio value.

To measure the effect of portfolio diversification we need to estimate the correlations in credit quality changes for all pairs of obligors. But, these correlations are not directly observable. CreditMetrics/CreditVaR I base their evaluation on the joint probability of asset returns, which itself results from strong simplifying assumptions on the capital structure of the obligor, and on the generating process for equity returns. This is clearly a key feature of CreditMetrics/CreditVaR I on which we will elaborate in the next section.

Finally, CreditMetrics/CreditVaR I, as the other approaches reviewed in this paper, assumes no market risk since forward values and exposures are simply derived from deterministic forward curves. The only uncertainty in CreditMetrics/CreditVaR I relates to credit migration, i.e. the process of

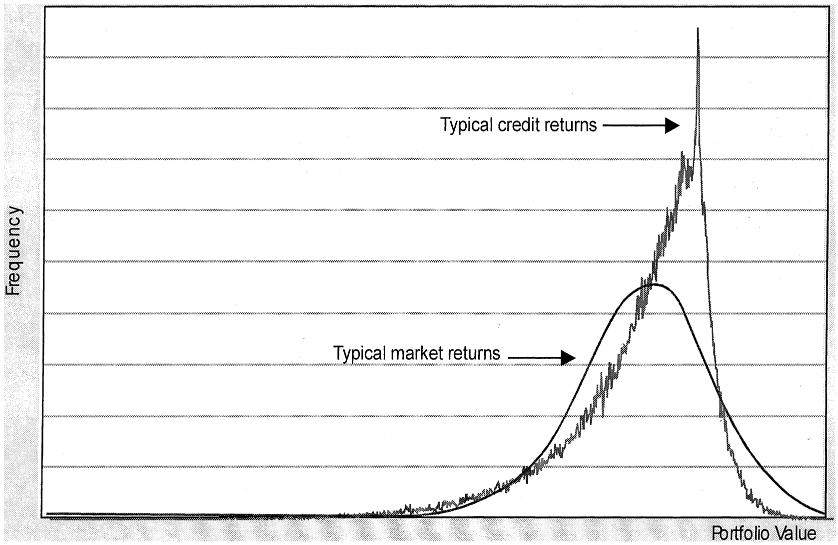


Fig. 2. Comparison of the distributions of credit returns and market returns (*source:* CIBC).

moving up or down the credit spectrum. In other words, credit risk is analyzed independently of market risk, which is another limitation of this approach.

2.1. CreditMetrics/CreditVaR I framework

CreditMetrics/CreditVaR I risk measurement framework is best summarized by Fig. 3 which shows the two main building blocks, i.e. "value-at-risk due to credit" for a single financial instrument, then value-at-risk at the portfolio level which accounts for portfolio diversification effects ("Portfolio Value-at-Risk due to Credit"). There are also two supporting functions, "correlations" which derives the asset return correlations which are used to generate the joint migration probabilities, and "exposures" which produces the future exposures of derivative securities, like swaps.

2.2. Credit-Var for a bond (building block #1)

The first step is to specify a rating system, with rating categories, together with the probabilities of migrating from one credit quality to another over the credit risk horizon. This transition matrix is the key component of the credit-VaR model proposed by JP Morgan. It can be Moody's, or Standard & Poor's, or the proprietary rating system internal to the bank. A strong assumption made by CreditMetrics/CreditVaR I is that all issuers are credit-homogeneous

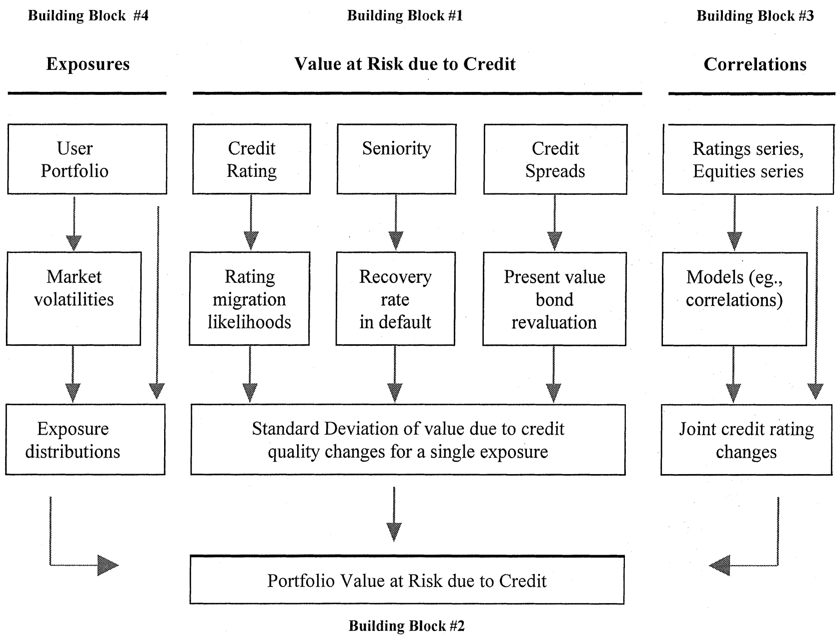


Fig. 3. CreditMetrics/CreditVaR I framework: The 4 building blocks (*source:* JP Morgan).

within the same rating class, with the same transition probabilities and the same default probability. KMV departs from CreditMetrics/CreditVaR I in the sense that in KMV's framework each issuer is specific, and is characterized by his own asset returns distribution, its own capital structure and its own default probability.

Second, the risk horizon should be specified. It is usually 1 year, although multiple horizons could be chosen, like 1–10 years, when one is concerned by the risk profile over a longer period of time as it is needed for long dated illiquid instruments.

The third phase consists of specifying the forward discount curve at the risk horizon(s) for each credit category, and, in the case of default, the value of the instrument which is usually set at a percentage, named the "recovery rate", of face value or "par".

In the final step, this information is translated into the forward distribution of the changes in portfolio value consecutive to credit migration.

The following example taken from the technical document of CreditMetrics illustrates the four steps of the credit-VaR model.

Example 1. Credit-VaR for a senior unsecured BBB rated bond maturing exactly in 5 years, and paying an annual coupon of 6%.

Step 1: Specify the transition matrix.

The rating categories, as well as the transition matrix, are chosen from a rating system (Table 1).

In the case of Standard & Poor's there are 7 rating categories, the highest credit quality being AAA, and the lowest, CCC; the last state is default. Default corresponds to the situation where an obligor cannot make a payment related to a bond or a loan obligation, whether it is a coupon or the redemption of principal. "Pari passu" clauses are such that when an obligor defaults on one payment related to a bond or a loan, he is technically declared in default on all debt obligations.

The bond issuer has currently a BBB rating, and the italicized line corresponding to the BBB initial rating in Table 1 shows the probabilities estimated by Standard & Poor's for a BBB issuer to be, in 1 year from now, in one of the 8 possible states, including default. Obviously, the most probable situation is for the obligor to stay in the same rating category, i.e. BBB, with a probability of 86.93%. The probability of the issuer defaulting within 1 year is only 0.18%, while the probability of being upgraded to AAA is also very small, i.e. 0.02%. Such transition matrix is produced by the rating agencies for all initial ratings. Default is an absorbing state, i.e. an issuer who is in default stays in default.

Moody's also publishes similar information. These probabilities are based on more than 20 years of history of firms, across all industries, which have migrated over a 1 year period from one credit rating to another. Obviously, this data should be interpreted with care since it represents average statistics across a heterogeneous sample of firms, and over several business cycles. For this reason many banks prefer to rely on their own statistics which relate more closely to the composition of their loan and bond portfolios.

Moody's and Standard & Poor's also produce long-term average cumulative default rates, as shown in Table 2 in a tabular form and in Fig. 4 in a graphical form. For example, a BBB issuer has a probability of 0.18% to default within 1 year, 0.44% to default in 2 years, 4.34% to default in 10 years.

Table 1
Transition matrix: Probabilities of credit rating migrating from one rating quality to another, within 1 year[a]

Initial rating	Rating at year-end (%)							
	AAA	AA	A	BBB	BB	B	CCC	Default
AAA	90.81	8.33	0.68	0.06	0.12	0	0	0
AA	0.70	90.65	7.79	0.64	0.06	0.14	0.02	0
A	0.09	2.27	91.05	5.52	0.74	0.26	0.01	0.06
BBB	*0.02*	*0.33*	*5.95*	*86.93*	*5.30*	*1.17*	*1.12*	*0.18*
BB	0.03	0.14	0.67	7.73	80.53	8.84	1.00	1.06
B	0	0.11	0.24	0.43	6.48	83.46	4.07	5.20
CCC	0.22	0	0.22	1.30	2.38	11.24	64.86	19.79

[a] *Source*: Standard & Poor's CreditWeek (April 15, 1996).

Table 2
Average cumulative default rates (%)[a]

Term	1	2	3	4	5...	7...	10...	15
AAA	0.00	0.00	0.07	0.15	0.24...	0.66...	1.40...	1.40
AA	0.00	0.02	0.12	0.25	0.43...	0.89...	1.29...	1.48
A	0.06	0.16	0.27	0.44	0.67...	1.12...	2.17...	3.00
BBB	*0.18*	*0.44*	*0.72*	*1.27*	*1.78...*	*2.99...*	*4.34...*	*4.70*
BB	1.06	3.48	6.12	8.68	10.97...	14.46...	17.73...	19.91
B	5.20	11.00	15.95	19.40	21.88...	25.14...	29.02...	30.65
CCC	19.79	26.92	31.63	35.97	40.15...	42.64...	45.10...	45.10

[a] *Source*: Standard & Poor's CreditWeek (April 15, 1996).

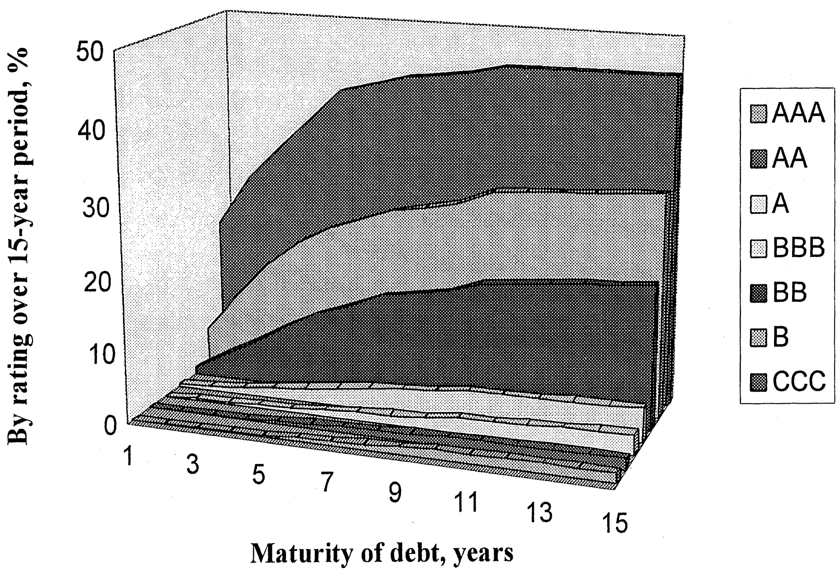


Fig. 4. Average cumulative default rates (%) (*source:* Standard & Poor's CreditWeek April 15, 1996).

Tables 1 and 2 should in fact be consistent with one another. From Table 2 we can back out the transition matrix which best replicates, in the least square sense, the average cumulative default rates. Indeed, assuming that the process for default is Markovian and stationary, then multiplying the 1-year transition matrix n times generates the n-year matrix. The n-year default probabilities are simply the values in the last default column of the transition matrix, and should match the column in year n of Table 2.

Actual transition and default probabilities vary quite substantially over the years, depending whether the economy is in recession, or in expansion. (See

Fig. 1 for default rates.) When implementing a model which relies on transition probabilities, one may have to adjust the average historical values as shown in Table 1, to be consistent with one's assessment of the current economic environment. Moody's study by Carty and Lieberman (1996) provides historical default statistics, both the mean and standard deviation, by rating category for the population of obligors they have rated during the period 1920–1996 (see Table 3).

Step 2: Specify the credit risk horizon.

The risk horizon is usually 1 year, and is consistent with the transition matrix shown in Table 1. But this horizon is purely arbitrary, and is mostly dictated by the availability of the accounting data and financial reports processed by the rating agencies. In KMV's framework, which relies on market data as well as accounting data, any horizon can be chosen from a few days to several years. Indeed, market data can be updated daily while assuming the other firm characteristics stay constant until new information becomes available.

Step 3: Specify the forward pricing model.

The valuation of a bond is derived from the zero-curve corresponding to the rating of the issuer. Since there are 7 possible credit qualities, 7 "spread" curves are required to price the bond in all possible states, all obligors within the same rating class being marked-to-market with the same curve. The spot zero curve is used to determine the current spot value of the bond. The forward price of the bond in 1 year from now is derived from the forward zero-curve, 1 year ahead, which is then applied to the residual cash flows from year one to the maturity of the bond. Table 4 gives the 1-year forward zero-curves for each credit rating.

Empirical evidence shows that for high grade investment bonds the spreads tend to increase with time to maturity, while for low grade, like CCC the spread tends to be wider at the short end of the curve than at the long end, as shown in Fig. 5.

Table 3
One-year default rates by rating, 1970–1995[a]

Credit rating	One-year default rate	
	Average (%)	Standard deviation (%)
Aaa	0.00	0.0
Aa	0.03	0.1
A	0.01	0.0
Baa	0.13	0.3
Ba	1.42	1.3
B	7.62	5.1

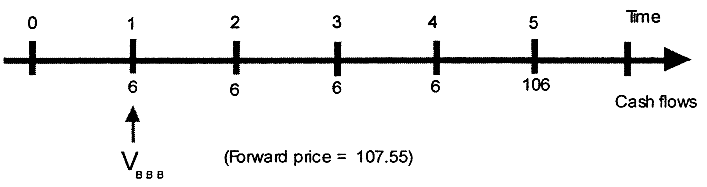
[a] *Source*: Carty and Lieberman (1996).

Table 4
One-year forward zero-curves for each credit rating (%)[a]

Category	Year 1	Year 2	Year 3	Year 4
AAA	3.60	4.17	4.73	5.12
AA	3.65	4.22	4.78	5.17
A	3.72	4.32	4.93	5.32
BBB	4.10	4.67	5.25	5.63
BB	5.55	6.02	6.78	7.27
B	6.05	7.02	8.03	8.52
CCC	15.05	15.02	14.03	13.52

[a] *Source*: CreditMetrics, JP Morgan.

The 1-year forward price of the bond, if the obligor stays BBB, is then:



$$V_{\mathrm{BBB}} = 6 + \frac{6}{1.0410} + \frac{6}{(1.0467)^2} + \frac{6}{(1.0525)^3} + \frac{106}{(1.0563)^4} = 107.55$$

If we replicate the same calculations for each rating category we obtain the values shown in Table 5. [5]

If the issuer defaults at the end of the year, we assume that not everything is lost. Depending on the seniority of the instrument, a recovery rate of par value is recuperated by the investor. These recovery rates are estimated from historical data by the rating agencies. Table 6 shows the recovery rates for bonds by different seniority classes as estimated by Moody's. [6] In our example the recovery rate for senior unsecured debt is estimated to be 51.13%, although the estimation error is quite large and the actual value lies in a fairly large confidence interval.

In the Monte Carlo simulation used to generate the loss distribution, it is assumed that the recovery rates are distributed according to a beta distribution with the same mean and standard deviation as shown in Table 6.

Step 4: Derive the forward distribution of the changes in portfolio value.

[5] CreditMetrics calculates the forward value of the bonds, or loans, cum compounded coupons paid out during the year.

[6] Cf. Carty and Lieberman (1996). See also Altman and Kishore (1996, 1998) for similar statistics.

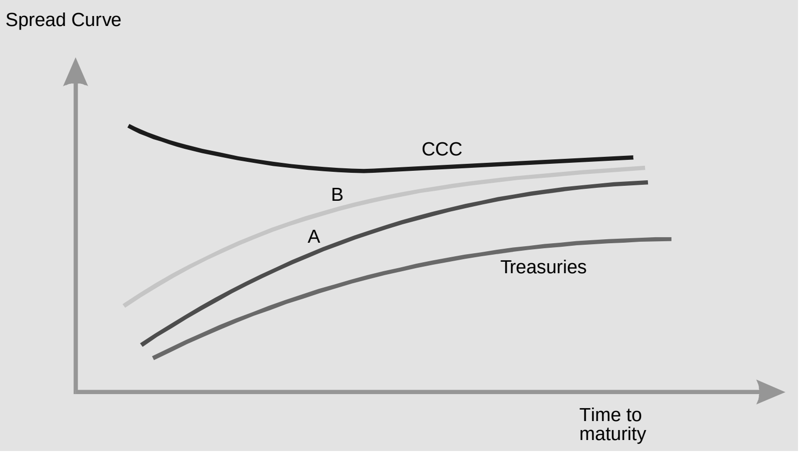


Fig. 5. Spread curves for different credit qualities.

Table 5
One-year forward values for a BBB bond[a]

Year-end rating	Value ($)
AAA	109.37
AA	109.19
A	108.66
BBB	*107.55*
BB	102.02
B	98.10
CCC	83.64
Default	51.13

[a] *Source*: CreditMetrics, JP Morgan.

The distribution of the changes in the bond value, at the 1-year horizon, due to an eventual change in credit quality is shown Table 7 and Fig. 6. This distribution exhibits long downside tails. The first percentile of the distribution of ΔV, which corresponds to credit-VaR at the 99% confidence level is -23.91. It is much larger than if we computed the first percentile assuming a normal distribution for ΔV. In that case credit-VaR at the 99% confidence level would be only -7.43. [7]

[7] The mean, m, and the variance, σ^2, of the distribution for ΔV are: $m = \text{mean}(\Delta V) = \sum_i p_i \Delta V_i = 0.02\% \times 1.82 + 0.33\% \times 1.64 + \cdots + 0.18\% \times (-56.42) = -0.46$; $\sigma^2 = \text{variance}(\Delta V) = \sum_i p_i (\Delta V_i - m)^2 = 0.02\%(1.82 + 0.46)^2 + 0.33\%(1.64 + 0.46)^2 + \cdots + 0.18\%(-56.42 + 0.46)^2 = 8.95$ and $\sigma = 2.99$. The first percentile of a normal distribution $M(m, \sigma^2)$ is $(m - 2.33\sigma)$, i.e. -7.43.

Table 6
Recovery rates by seniority class (% of face value, i.e., "par")[a]

Seniority class	Mean (%)	Standard deviation (%)
Senior secured	53.80	26.86
Senior unsecured	51.13	25.45
Senior subordinated	38.52	23.81
Subordinated	32.74	20.18
Junior subordinated	17.09	10.90

[a] *Source*: Carty and Lieberman (1996).

Table 7
Distribution of the bond values, and changes in value of a BBB bond, in 1 year[a]

Year-end rating	Probability of state: p (%)	Forward price: V (\$)	Change in value: ΔV (\$)
AAA	0.02	109.37	1.82
AA	0.33	109.19	1.64
A	5.95	108.66	1.11
BBB	86.93	107.55	0
BB	5.30	102.02	−5.53
B	1.17	98.10	−9.45
CCC	0.12	83.64	−23.91
Default	0.18	51.13	−56.42

[a] *Source*: CreditMetrics, JP Morgan.

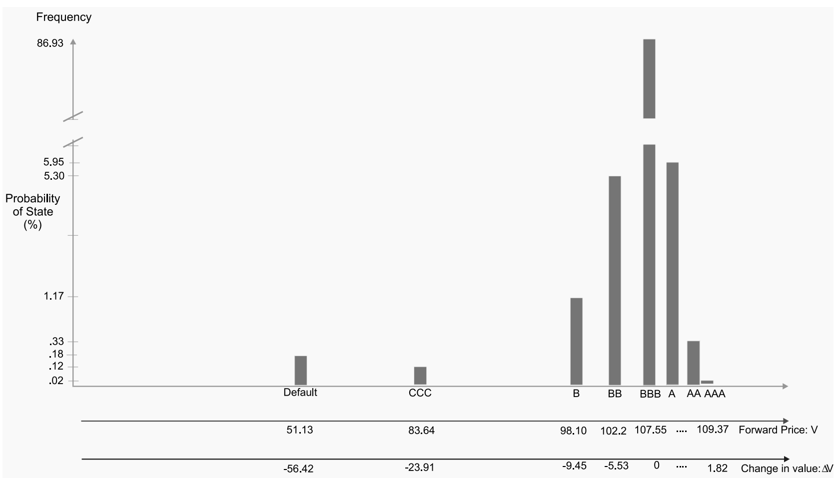


Fig. 6. Histogram of the 1-year forward prices and changes in value of a BBB bond.

2.3. Credit-VaR for a loan or bond portfolio (building block #2)

First, consider a portfolio composed of 2 bonds with an initial rating of BB and A, respectively. Given the transition matrix shown in Table 1, and assuming no correlation between changes in credit quality, we can then derive easily the joint migration probabilities shown in Table 8. Each entry is simply the product of the transition probabilities for each obligor. For example, the joint probability that obligor #1 and obligor #2 stay in the same rating class is

$$73.32\% = 80.53\% \times 91.05\%,$$

where 80.53% is the probability that obligor #1 keeps his current rating BB, and 91.05% is the probability that obligor #2 stays in rating class A.

Unfortunately, this table is not very useful in practice when we need to assess the diversification effect on a large loan or bond portfolio. Indeed, the actual correlations between the changes in credit quality are different from zero. And it will be shown in Section 5 that the overall credit-VaR is in fact quite sensitive to these correlations. Their accurate estimation is therefore determinant in portfolio optimization from a risk–return perspective.

Correlations are expected to be higher for firms within the same industry or in the same region, than for firms in unrelated sectors. In addition, correlations vary with the relative state of the economy in the business cycle. If there is a slowdown in the economy, or a recession, most of the assets of the obligors will decline in value and quality, and the likelihood of multiple defaults increases substantially. The contrary happens when the economy is performing well: default correlations go down. Thus, we cannot expect default and migration probabilities to stay stationary over time. There is clearly a need for a structural model that bridges the changes of default probabilities to fundamental variables whose correlations stay stable over time. Both CreditMetrics and KMV derive the default and migration probabilities from a correlation model of the firm's assets that will be detailed in the next section.

Contrary to KMV, and for the sake of simplicity, CreditMetrics/CreditVaR I have chosen the equity price as a proxy for the asset value of the firm that is not directly observable. This is another strong assumption in CreditMetrics that may affect the accuracy of the method.

Table 8
Joint migration probabilities (%) with zero correlation for 2 issuers rated BB and A

Obligor #1 (BB)		Obligor #2 (single-A) AAA 0.09	AA 2.27	*A* *91.05*	BBB 5.52	BB 0.74	B 0.26	CCC 0.01	Default 0.06
AAA	0.03	0.00	0.00	*0.03*	0.00	0.00	0.00	0.00	0.00
AA	0.14	0.00	0.00	*0.13*	0.01	0.00	0.00	0.00	0.00
A	0.67	0.00	0.02	*0.61*	0.40	0.00	0.00	0.00	0.00
BBB	7.73	0.01	0.18	*7.04*	0.43	0.06	0.02	0.00	0.00
BB	*80.53*	*0.07*	*1.83*	*73.32*	*4.45*	*0.60*	*0.20*	*0.01*	*0.05*
B	8.84	0.01	0.20	*8.05*	0.49	0.07	0.02	0.00	0.00
CCC	1.00	0.00	0.02	*0.91*	0.06	0.01	0.00	0.00	0.00
Default	1.06	0.00	0.02	*0.97*	0.06	0.01	0.00	0.00	0.00

First, CreditMetrics estimates the correlations between the equity returns of various obligors, then the model infers the correlations between changes in credit quality directly from the joint distribution of equity returns.

The proposed framework is the option pricing approach to the valuation of corporate securities initially developed by Merton (1974). The firm's assets value, V_t, is assumed to follow a standard geometric Brownian motion, i.e.:

$$V_t = V_0 \exp\left\{ \left(\mu - \frac{\sigma^2}{2} \right) t + \sigma \sqrt{t} Z_t \right\} \tag{1}$$

with $Z_t \sim \mathrm{N}(0,1)$, μ and σ^2 being respectively the mean and variance of the instantaneous rate of return on the assets of the firm, $\mathrm{d}V_t/V_t$. [8] V_t is lognormally distributed with expected value at time t, $E(V_t) = V_0 \exp\{\mu t\}$.

It is further assumed that the firm has a very simple capital structure, as it is financed only by equity, S_t, and a single zero-coupon debt instrument maturing at time T, with face value F, and current market value B_t. The firm's balance-sheet can be represented as in Table 9.

In this framework, default only occurs at maturity of the debt obligation, when the value of assets is less than the promised payment, F, to the bond holders. Fig. 7 shows the distribution of the assets' value at time T, the maturity of the zero-coupon debt, and the probability of default which is the shaded area below F.

Merton's model is extended by CreditMetrics to include changes in credit quality as illustrated in Fig. 8. This generalization consists of slicing the distribution of asset returns into bands in such a way that, if we draw randomly from this distribution, we reproduce exactly the migration frequencies shown in the transition matrix. Fig. 8 shows the distribution of the normalized assets' rates of return, 1 year ahead, which is normal with mean zero and unit variance. The credit rating thresholds correspond to the transition probabilities in Table 1 for a BB rated obligor. The right tail of the distribution on the right-hand side of Z_{AAA} corresponds to the probability for the obligor of being upgraded from BB to AAA, i.e. 0.03%. Then, the area between Z_{AA} and Z_{AAA} corresponds to the probability of being upgraded from BB to AA, etc. The left tail of the distribution, on the left-hand side of Z_{CCC}, corresponds to the probability of default, i.e. 1.06%.

Table 10 shows the transition probabilities for two obligors rated BB and A, respectively, and the corresponding credit quality thresholds.

This generalization of Merton's model is quite easy to implement. It assumes that the normalized log-returns over any period of time are normally distributed with mean 0 and variance 1, and it is the same for all obligors within the

[8] The dynamics of $V(t)$ is described by $\mathrm{d}V_t/V_t = \mu \mathrm{d}t + \sigma \mathrm{d}W_t$, where W_t is a standard Brownian motion, and $\sqrt{t}Z_t \equiv W_t - W_0$ being normally distributed with zero mean and variance equal to t.

Table 9
Balance sheet of Merton's firm

	Assets		Liabilities/Equity	
	Risky assets:	V_t	Debt:	B_t (F)
			Equity:	S_t
Total		V_t		V_t

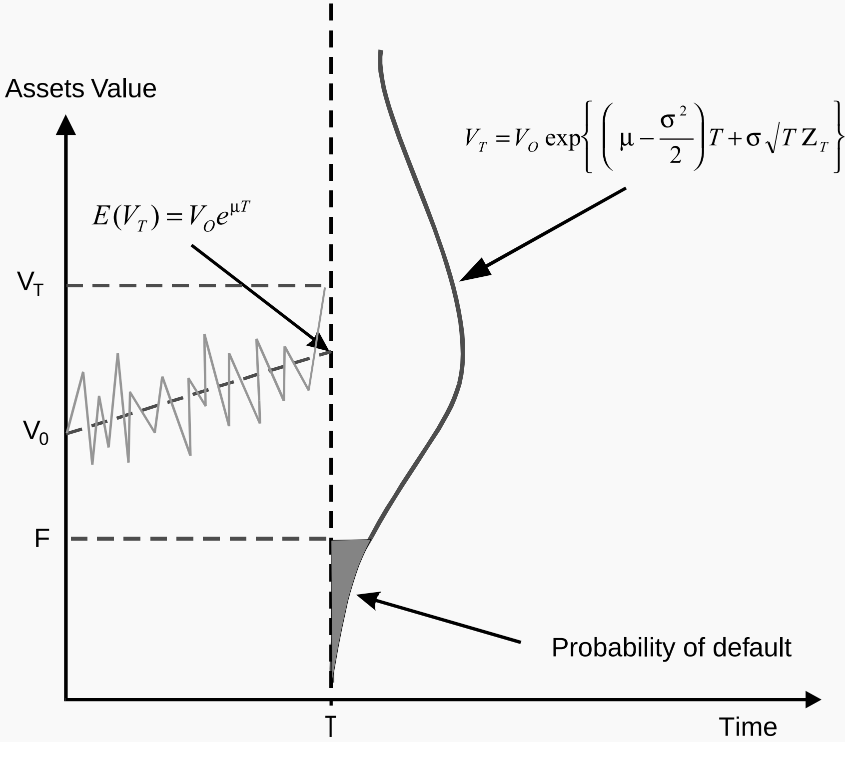


Fig. 7. Distribution of the firm's assets value at maturity of the debt obligation.

same rating category. If p_{Def} denotes the probability for the BB-rated obligor of defaulting, then the critical asset value V_{Def} is such that

$$p_{\text{Def}} = \Pr\left[V_t \leqslant V_{\text{Def}}\right]$$

which can be translated into a normalized threshold Z_{CCC}, such that the area in the left tail below Z_{CCC} is p_{Def}. Indeed, according to (1), default occurs when Z_t satisfies

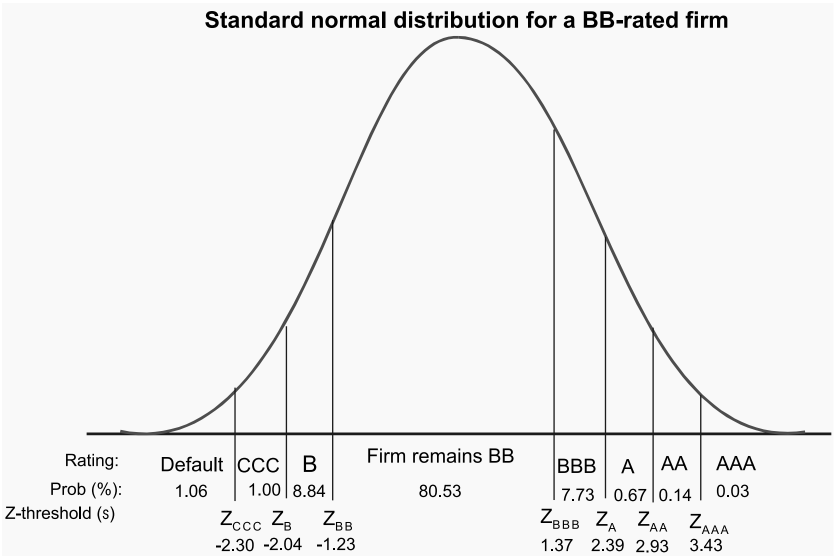


Fig. 8. Generalization of the Merton model to include rating changes.

Table 10
Transition probabilities and credit quality thresholds for BB and A rated obligors

	Rated-A obligor		Rated-BB obligor	
Rating in 1 year	Probabilities (%)	Thresholds: $Z(\sigma)$	Probabilities (%)	Thresholds: $Z(\sigma)$
AAA	0.09	3.12	0.03	3.43
AA	2.27	1.98	0.14	2.93
A	91.05	−1.51	0.67	2.39
BBB	5.52	−2.30	7.73	1.37
BB	0.74	−2.72	80.53	−1.23
B	0.26	−3.19	8.84	−2.04
CCC	0.01	−3.24	1.00	−2.30
Default	0.06		1.06	

$$p_{\mathrm{Def}} = \Pr\left[\frac{\ln(V_{\mathrm{Def}}/V_0) - (\mu - (\sigma^2/2))t}{\sigma\sqrt{t}} \geqslant Z_t\right]$$
$$= \Pr\left[Z_t \leqslant -\frac{\ln[V_0/V_{\mathrm{Def}}] + [\mu - (\sigma^2/2)]t}{\sigma\sqrt{t}}\right] \equiv \mathrm{N}(-d_2), \tag{2}$$

where the normalized return

$$r = \frac{\ln(V_t/V_0) - (\mu - (\sigma^2/2))t}{\sigma\sqrt{t}} \tag{3}$$

is N[0, 1]. Z_{CCC} is simply the threshold point in the standard normal distribution corresponding to a cumulative probability of p_{Def}. Then, the critical asset value V_{Def} which triggers default is such that $Z_{CCC} = -d_2$ where

$$d_2 \equiv \frac{\ln(V_0/V_{Def}) + (\mu - (\sigma^2/2))t}{\sigma\sqrt{t}} \tag{4}$$

and is also called "distance-to-default". [9] Note that only the threshold levels are necessary to derive the joint migration probabilities, and they are calculated without the need to observe the asset value, and to estimate its mean and variance. Only to derive the critical asset value V_{Def} we need to estimate the expected asset return μ and asset volatility σ.

Accordingly Z_B is the threshold point corresponding to a cumulative probability of being either in default or in rating CCC, i.e., $p_{Def} + p_{CCC}$, etc.

Further, since asset returns are not directly observable, CreditMetrics/CreditVaR I chose equity returns as a proxy, which is equivalent to assume that the firm's activities are all equity financed.

Now, for the time being, assume that the correlation between asset rates of return is known, and is denoted by ρ, which is assumed to be equal to 0.20 in our example. The normalized log-returns on both assets follow a joint normal distribution:

$$f(r_{BB}, r_A; \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{ \frac{-1}{2(1-\rho^2)} \left[r_{BB}^2 - 2\rho r_{BB} r_A + r_A^2 \right] \right\}.$$

We can then easily compute the probability for both obligors of being in any combination of ratings, e.g. that they remain in the same rating classes, i.e. BB and A, respectively:

$$\Pr(-1.23 < r_{BB} < 1.37,\ -1.51 < r_A < 1.98) = \int_{-1.23}^{1.37} \int_{-1.51}^{1.98} f(r_{BB}, r_A; \rho)\, dr_{BB}\, dr_A = 0.7365.$$

If we implement the same procedure for the other 63 combinations we obtain Table 11. We can compare Table 11 with Table 8, the later being derived assuming zero correlation, to notice that the joint probabilities are different.

Fig. 9 illustrates the effect of asset return correlation on the joint default probability for the rated BB and A obligors. To be more specific, consider two obligors whose probabilities of default are $P1(P_{Def1})$ and $P2(P_{Def2})$, respectively. Their asset return correlation is ρ. The events of default for obligors 1

[9] Note that d_2 is different from its equivalent in the Black–Scholes formula since, here, we work with the "actual" instead of the "risk neutral" return distributions, so that the drift term in d_2 is the expected return on the firm's assets, instead of the risk-free interest rate as in Black–Scholes.

Table 11
Joint rating probabilities (%) for BB and A rated obligors when correlation between asset returns is 20%[a]

Rating of first company (BB)	Rating of second company (A)								
	AAA	AA	A	BBB	BB	B	CCC	Def	Total
AAA	0.00	0.00	0.03	0.00	0.00	0.00	0.00	0.00	0.03
AA	0.00	0.01	0.13	0.00	0.00	0.00	0.00	0.00	0.14
A	0.00	0.04	0.61	0.01	0.00	0.00	0.00	0.00	0.67
BBB	0.02	0.35	7.10	0.20	0.02	0.01	0.00	0.00	7.69
BB	0.07	1.79	73.65	4.24	0.56	0.18	0.01	0.04	80.53
B	0.00	0.08	7.80	0.79	0.13	0.05	0.00	0.01	8.87
CCC	0.00	0.01	0.85	0.11	0.02	0.01	0.00	0.00	1.00
Def	0.00	0.01	0.90	0.13	0.02	0.01	0.00	0.00	1.07
Total	0.09	2.29	91.06	5.48	0.75	0.26	0.01	0.06	100

[a] *Source*: CreditMetrics, JP Morgan.

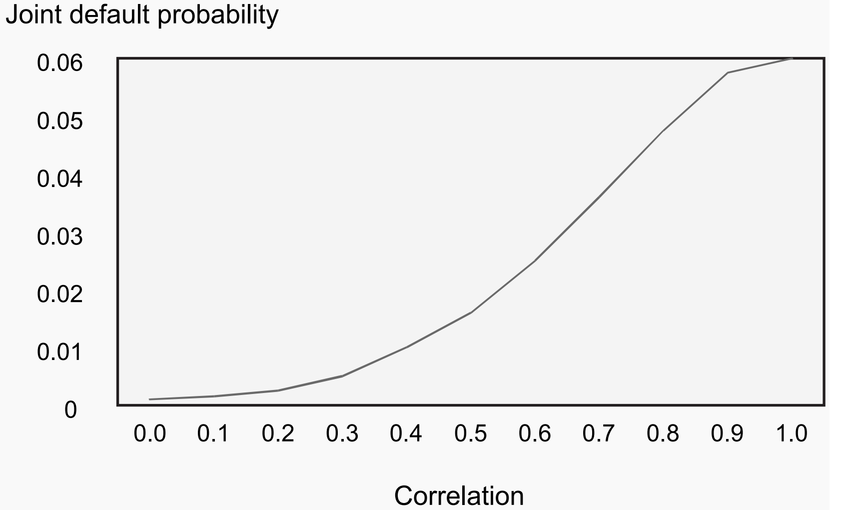


Fig. 9. Probability of joint defaults as a function of asset return correlation (*source:* CreditMetrics, JP Morgan).

and 2 are denoted DEF1 and DEF2 , respectively, and $P(\mathrm{DEF1}, \mathrm{DEF2})$ is the joint probability of default. Then, it can be shown that the default correlation is [10]

[10] See Lucas (1995).

$$\mathrm{corr}(\mathrm{DEF1},\mathrm{DEF2}) = \frac{P(\mathrm{DEF1},\mathrm{DEF2}) - P1 \cdot P2}{\sqrt{P1(1-P1)\cdot P2(1-P2)}}. \tag{5}$$

The joint probability of both obligors defaulting is, according to Merton's model,

$$P(\mathrm{DEF1},\mathrm{DEF2}) = \Pr\left[V_1 \leqslant V_{\mathrm{Def1}}, V_2 \leqslant V_{\mathrm{Def2}}\right], \tag{6}$$

where V_1 and V_2 denote the asset values for both obligors at time t, and V_{Def1} and V_{Def2} are the corresponding critical values which trigger default. Expression (6) is equivalent to

$$P(\mathrm{DEF1},\mathrm{DEF2}) = \Pr\left[r_1 \leqslant -d_2^1, r_2 \leqslant -d_2^2\right] \equiv \mathrm{N}_2\left(-d_2^1, -d_2^2, \rho\right), \tag{7}$$

where r_1 and r_2 denote the normalized asset returns as defined in (3) for obligors 1 and 2, respectively, and d_2^1 and d_2^2 are the corresponding distant to default as in (4). $\mathrm{N}_2(x,y,\rho)$ denotes the cumulative standard bivariate normal distribution where ρ is the correlation coefficient between x and y. Fig. 9 is simply the graphical representation of (7) for the asset return correlation varying from 0 to 1.

Example 1 (*Continuation*).

$$\begin{aligned}
&\rho = 20\%,\\
&P(\mathrm{DEF1},\mathrm{DEF2}) = N_2\left(-d_2^1, -d_2^2, \rho\right) = N_2(-3.24, -2.30, 0.20)\\
&= 0.000054,\\
&P1(\mathrm{A}) = 0.06\%,\\
&P2(\mathrm{BB}) = 1.06\%;\\
&\text{it then follows:}\\
&\mathrm{corr}\ (\mathrm{DEF1},\ \mathrm{DEF2}) = 0.019 = 1.9\%.
\end{aligned}$$

The ratio of asset returns correlations to default correlations is approximately 10–1 for asset correlations in the range of 20–60%. This shows that the joint probability of default is in fact quite sensitive to pairwise asset return correlations, and it illustrates the necessity to estimate correctly these data to assess precisely the diversification effect within a portfolio. In Section 5 we show that, for the benchmark portfolio we selected for the comparison of credit models, the impact of correlations on credit-VaR is quite large. It is larger for low credit quality than for high grade portfolios. Indeed, when the credit quality of the portfolio deteriorates the expected number of defaults increases, and this number is magnified by an increase in default correlations.

The statistical procedure to estimate asset return correlations is discussed in the next section dedicated to KMV. [11]

2.4. Analysis of credit diversification (building block #2, continuation)

The analytic approach that we just sketched out for a portfolio with bonds issued by 2 obligors is not doable for large portfolios. Instead, CreditMetrics/CreditVaR I implement a Monte Carlo simulation to generate the full distribution of the portfolio values at the credit horizon of 1 year. The following steps are necessary.

1. Derivation of the asset return thresholds for each rating category.
2. Estimation of the correlation between each pair of obligors' asset returns.
3. Generation of asset return scenarios according to their joint normal distribution. A standard technique to generate correlated normal variables is the Cholesky decomposition. [12] Each scenario is characterized by n standardized asset returns, one for each of the n obligors in the portfolio.
4. For each scenario, and for each obligor, the standardized asset return is mapped into the corresponding rating, according to the threshold levels derived in step 1.
5. Given the spread curves which apply for each rating, the portfolio is revalued.
6. Repeat the procedure a large number of times, say 100 000 times, and plot the distribution of the portfolio values to obtain a graph which looks like Fig. 2.
7. Then, derive the percentiles of the distribution of the future values of the portfolio.

2.5. Credit-VaR and calculation of the capital charge

Economic capital stands as a cushion to absorb unexpected losses related to credit events, i.e. migration and/or default. Fig. 10 shows how to derive the capital charge related to credit risk.

$V(p)$ = value of the portfolio in the worst case scenario at the p% confidence level.

FV = forward value of the portfolio = $V_0(1 + \mathrm{PR})$.

V_0 = current mark-to-market value of the portfolio.

[11] The correlation models for CreditMetrics and KMV are different but the approaches being similar, we detail only KMV's model which is more elaborated.

[12] A good reference on Monte Carlo simulations and the Cholesky decomposition is Fishman (1997, p. 223).

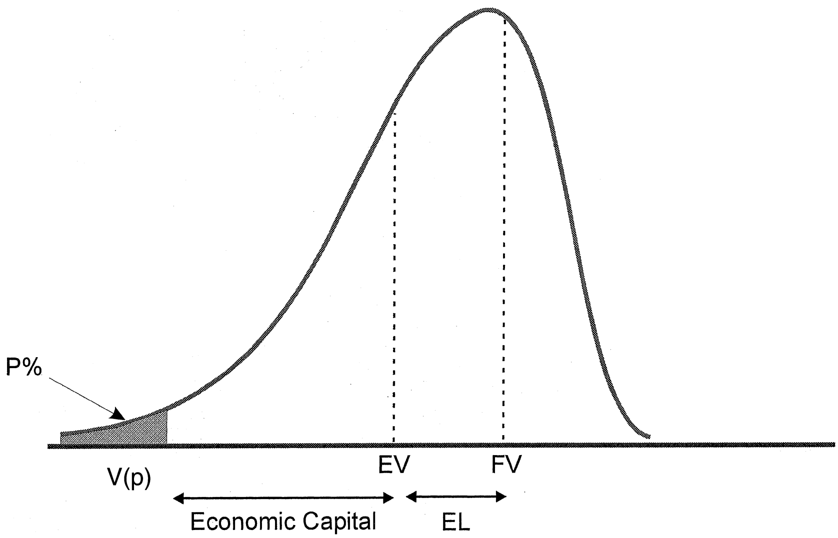


Fig. 10. Credit-VaR and calculation of economic capital.

PR = promised return on the portfolio. [13]
EV = expected value of the portfolio = $V_0(1 + \text{ER})$.
ER = expected return on the portfolio.
EL = expected loss = FV − EV.
The expected loss does not contribute to the capital allocation, but instead goes into reserves and is imputed as a cost into the RAROC calculation. The capital charge comes only as a protection against unexpected losses:

$$\text{Capital} = \text{EV} - V(p).$$

2.6. CreditMetrics/CreditVaR I as a loan/bond portfolio management tool: Marginal risk measures (building block #2, continuation)

In addition to the overall credit-VaR analysis for the portfolio, CreditMetrics/CreditVaR I offer the interesting feature of isolating the individual marginal risk contributions to the portfolio. For example, for each asset, CreditMetrics/CreditVaR I calculate the marginal standard deviation, i.e. the impact of each individual asset on the overall portfolio standard deviation. By comparing the marginal standard deviation to the stand-alone standard devi-

[13] If there were only one bond in the portfolio, PR would simply be the 1-year spot rate on the corporate curve corresponding to the rating of the obligor.

ation for each loan, one can assess the extent of the benefit derived from portfolio diversification when adding the instrument in the portfolio. Fig. 11 shows the marginal standard deviation for each asset, expressed in percentage of the overall standard deviation, plotted against the marked-to-market value of the instrument.

This is an important pro-active risk management tool as it allows one to identify trading opportunities in the loan/bond portfolio where concentration, and as a consequence overall risk, can be reduced without affecting expected profits. Obviously, for this framework to become fully operational it needs to be complemented by a RAROC model which provides information on the adjusted return on capital for each deal.

The same framework can also be used to set up credit risk limits, and monitor credit risk exposures in terms of the joint combination of market value and marginal standard deviation, as shown in Fig. 12.

2.7. Estimation of asset correlations (building block #3)

Since asset values are not directly observable, equity prices for publicly traded firms are used as a proxy to calculate asset correlations. For a large portfolio of bonds and loans, with thousand of obligors, it would still require the computation of a huge correlation matrix for each pair of obligors. To reduce the dimensionality of the this estimation problem, CreditMetrics/ CreditVaR I use a multi-factor analysis. This approach maps each obligor to the countries and industries that most likely determine its performance. Equity returns are correlated to the extent that they are exposed to the same industries

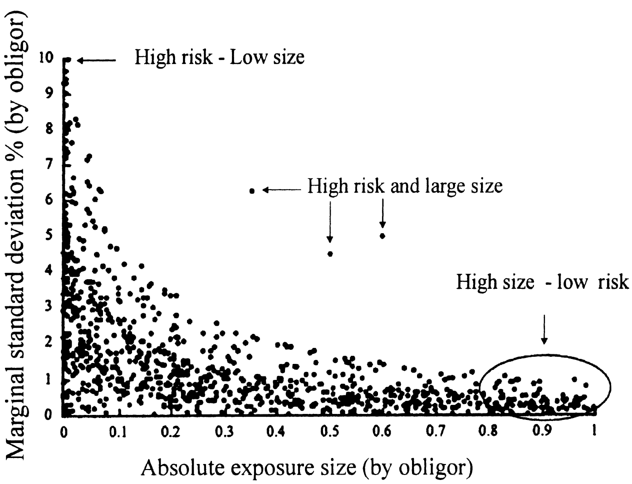


Fig. 11. Risk versus size of exposures within a typical credit portfolio.

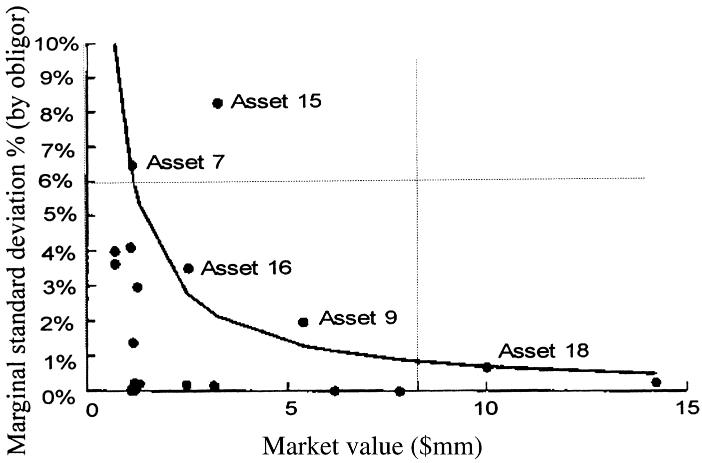


Fig. 12. Example of risk limits for a portfolio (*source:* CreditMetrics, JP Morgan).

and countries. In CreditMetrics/CreditVaR I the user specifies the industry and country weights for each obligor, as well as the "firm-specific risk", which is idiosyncratic to each obligor and neither correlated to any other obligor nor any index. [14]

2.8. Exposures (building block #4)

What is meant by "exposures" in CreditMetrics/CreditVaR I is somewhat misleading since market risk factors are assumed constant. This building block is simply the forward pricing model that applies for each credit rating. For bond-type products like bonds, loans, receivables, commitments to lend, letters of credit, exposure simply relates to the future cash flows at risk, beyond the 1-year horizon. Forward pricing is derived from the present value model using the forward yield curve for the corresponding credit quality. The example presented in Section 2.2 illustrates how the exposure distribution is calculated for a bond.

For derivatives, like swaps and forwards, the exposure is conditional on future interest rates. Contrary to a bond, there is no simple way to derive the future cash flows at risk without making some assumptions on the dynamics of interest rates. The complication arises since the risk exposure for a swap can be either positive if the swap is in-the-money for the bank, or negative if it is out-of-the-money. In the later case it is a liability and it is the counterparty who is

[14] See also KMV's correlation model presented in the next section.

at risk. Fig. 13 shows the exposure profiles of an interest rate swap for different interest rate scenarios, assuming no change in the credit ratings of the counterparty, and of the bank. The bank is at risk only when the exposure is positive.

At this stage we assume the average exposure of a swap given and it is supposed to have been derived from an external model. In CreditMetrics/CreditVaR I interest rates being deterministic, the calculation of the forward price distribution relies on an ad hoc procedure:

$$\begin{aligned}&\text{Value of swap in 1 year, in rating } R\\&\quad = \text{Forward risk-free value in 1 year}\\&\qquad - \text{Expected loss in years 1 to maturity for the given rating } R,\end{aligned} \tag{8}$$

where

$$\begin{aligned}&\text{Expected loss in years 1 to maturity for the given rating } R\\&\quad = \text{Average exposure from year 1 to maturity}\\&\qquad \times \text{Probability of default in years 1 through maturity}\\&\qquad \text{for the given rating } R \times (1 - \text{recovery rate}).\end{aligned} \tag{9}$$

The forward risk-free value of the swap is calculated by discounting the future net cash flows of the swap, based on the forward curve, and discounting them using the forward Government yield curve. This value is the same for all credit ratings.

The probability of default in year 1 through maturity either comes directly from Moody's or Standard & Poor's, or can be derived from the transition

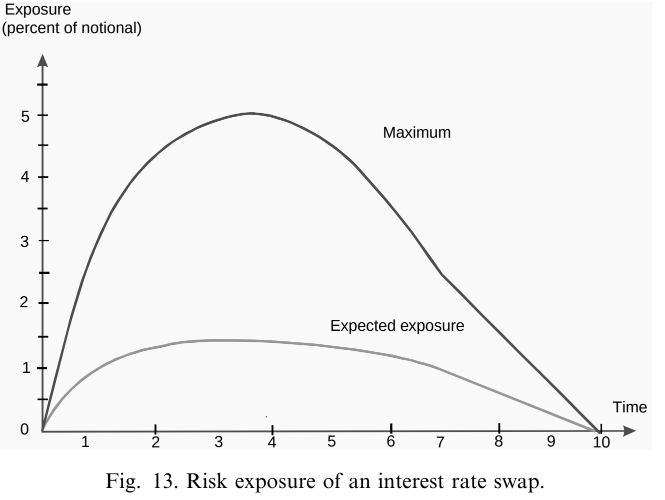


Fig. 13. Risk exposure of an interest rate swap.

Table 12
Distribution of the 1-year forward values of a 3-year interest rate swap[a]

Year-end rating	Two-year default likelihood (%)	Forward value ($)
AAA	0.00	1
AA	0.02	6
A	0.15	46
BBB	0.48	148
BB	2.59	797
B	10.41	3209
CCC	33.24	10,304
Default	–	50,860

[a] *Source*: CreditMetrics, JP Morgan.

matrix as previously discussed in Section 1. The recovery rate comes from the statistical analyses provided by the rating agencies.

Example 2. Consider a 3-year interest rate swap with a $10 million notional value. The average expected exposure between year 1 and 3 is supposed to be $61 627. Given the 2-year probability of default, the distribution of 1-year forward values for the swap can be calculated according to the above formulas (4) and (5). The results are shown in Table 12, where FV denotes the forward risk-free value of the swap.

Obviously, this ad hoc calculation of the exposure of an interest rate swap is not satisfactory. Only a model with stochastic interest rates will allow a proper treatment of exposure calculations for swaps as well as other derivative securities.

3. KMV [15] model

The major weakness of CreditMetrics/CreditVaR I is not the methodology, which is rather appealing, but the reliance on transition probabilities based on average historical frequencies of defaults and credit migration. The accuracy of CreditMetrics/CreditVaR I calculations relies upon two critical assumptions: first, all firms within the same rating class have the same default rate, and second, the actual default rate is equal to the historical average default rate. The same assumptions also apply to the other transition probabilities. In other words, credit rating changes and credit quality changes are identical, and credit

[15] KMV is a trademark of KMV Corporation. Stephen Kealhofer, John McQuown and Oldrich Vasicek founded KMV Corporation in 1989.

rating and default rates are synonymous, i.e. the rating changes when the default rate is adjusted, and vice versa.

This view has been strongly challenged by KMV. Indeed, this cannot be true since default rates are continuous, while ratings are adjusted in a discrete fashion, simply because rating agencies take time to upgrade or downgrade companies whose default risk have changed. KMV has shown through a simulation exercise that the historical average default rate and transition probabilities can deviate significantly from the actual rates. In addition, KMV has demonstrated that substantial differences in default rates may exist within the same bond rating class, and the overlap in default probability ranges may be quite large with, for instance, some BBB and AA rated bonds having the same probability of default. KMV has replicated 50 000 times, through a Monte Carlo simulation, Moody's study of default over a 25-year period. For each rating they have assumed a fixed number of obligors which is approximately the same as in Moody's study. For each rating they have assumed that the true probability of default is equal to the reported Moody's average default rate over the 25-year period. KMV has also run the simulation for several levels of correlation among the asset returns, ranging from 15% to 45%. A typical result is illustrated in Fig. 14 for a BBB obligor. Given an exact default probability of 13 bp, the 25-year average historical default rate ranges between 4 and 27 bp at the 95% confidence level, for an asset correlation of 15%.

The distribution is quite skewed so that the mean default rate usually exceeds the typical (median) default rate for each credit class. Thus the average historical default probability overstates the default rate for a typical obligor. [16]

Unlike CreditMetrics/CreditVaR I, KMV does not use Moody's or Standard & Poor's statistical data to assign a probability of default which only depends on the rating of the obligor. Instead, KMV derives the actual probability of default, the Expected Default Frequency (EDF), for each obligor based on a Merton (1974)'s type model of the firm. The probability of default is thus a function of the firm's capital structure, the volatility of the asset returns and the current asset value. The EDF is firm-specific, and can be mapped into any rating system to derive the equivalent rating of the obligor. [17] EDFs can be viewed as a "cardinal ranking" of obligors relative to default risk, instead of the more conventional "ordinal ranking" proposed by rating agencies and which

[16] This can lead to adverse selection of corporate customers in banks. Indeed, if the pricing of loans is based on this average historical default rate, then a typical customer will be overcharged and may have an incentive to leave, while the worst obligors in the class will benefit from an advantageous pricing with regard to their actual credit risk.

[17] See Section 2.1.4.

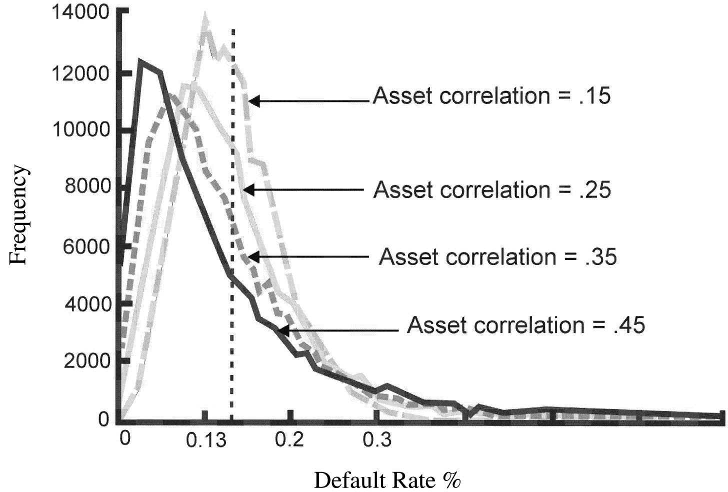


Fig. 14. Monte Carlo simulated distribution of average default rate for a BBB bond with a true default rate of 0.13%.

relies on letters like AAA, AA, etc. Contrary to CreditMetrics/CreditVaR I, KMV's model does not make any explicit reference to the transition probabilities which, in KMV's methodology, are already imbedded in the EDFs. Indeed, each value of the EDF is associated with a spread curve and an implied credit rating.

As for CreditMetrics/CreditVaR I, KMV's model is also based on the option pricing approach to credit risk as originated by Merton (1974). [18] Thus, credit risk is essentially driven by the dynamics of the asset value of the issuer. Given the current capital structure of the firm, i.e. the composition of its liabilities: equity, short-term and long-term debt, convertible bonds, etc., once the stochastic process for the asset value has been specified, then *the actual probability of default* for any time horizon, 1 year, 2 years, etc. can be derived. Fig. 7 in the previous section depicts how the probability of default relates to the distribution of asset returns and the capital structure of the firm, in the simple case where the firm is financed by equity and a zero-coupon bond.

KMV best applies to publicly traded companies for which the value of equity is market determined. The information contained in the firm's stock price and balance sheet can then be translated into an implied risk of default as shown in the next section.

[18] See Vasicek (1997) and Kealhofer (1995, 1998). See also the previous section.

3.1. Actual probabilities of default: EDFs (expected default frequencies)

The derivation of the probabilities of default proceeds in 3 stages which are discussed below: estimation of the market value and volatility of the firm's assets; calculation of the distance-to-default, which is an index measure of default risk; and scaling of the distance-to-default to actual probabilities of default using a default database.

3.1.1. Estimation of the asset value, V_A, and the volatility of asset return, σ_A

In the contingent claim approach to the pricing of corporate securities, the market value of the firm's assets is assumed to be lognormally distributed, i.e. the log-asset return follows a normal distribution. [19] This assumption is quite robust and, according to KMV's own empirical studies, actual data conform quite well to this hypothesis. [20] In addition the distribution of asset return is stable over time, i.e. the volatility of asset return stays relatively constant.

If all the liabilities of the firm were traded, and marked-to-market every day, then the task of assessing the market value of the firm's assets and their volatility would be straightforward. The firm's assets value would be simply the sum of the market values of the firm's liabilities, and the volatility of the asset return could be simply derived from the historical time series of the reconstituted assets value.

In practice, however, only the price of equity for most public firms is directly observable, and in some cases part of the debt is actively traded. The alternative approach to assets valuation consists in applying the option pricing model to the valuation of corporate liabilities as suggested in Merton (1974). [21] In order to make the model tractable, KMV assumes that the capital structure is only composed of equity, short-term debt which is considered equivalent to

[19] Financial models consider essentially market values of assets, and not accounting values, or book values, which only represent the historical cost of the physical assets, net of their depreciation. Only the market value is a good measure of the value of the firm's ongoing business and it changes as market participants revise the firm's future prospects. KMV models the market value of liabilities based on the assumed distribution of assets value, and the estimation of the current value of the firm's assets. In fact, there might be huge differences between both the market and the book values of total assets. For example, as of February 1998, KMV has estimated the market value of Microsoft assets to US $228.6 billion versus US $16.8 billion for their book value, while for Trump Hotel and Casino the book value which amounts to US $2.5 billion is higher than the market value of US $1.8 billion.

[20] The exception is when the firm's portfolio of businesses has changed substantially through mergers and acquisitions, or restructuring.

[21] See also Crouhy and Galai (1994), Bensoussan et al. (1994, 1995), and Vasicek (1997) for the valuation of equity for more complex capital structures which, for example, include equity warrants and convertible bonds.

cash, long-term debt which is assumed to be a perpetuity, and convertible preferred shares. [22] With these simplifying assumptions it is then possible to derive analytical solutions for the value of equity, V_E, and its volatility, σ_E:

$$V_E = f(V_A, \sigma_A, K, c, r), \tag{10}$$

$$\sigma_E = g(V_A, \sigma_A, K, c, r), \tag{11}$$

where K denotes the leverage ratio in the capital structure, c is the average coupon paid on the long-term debt and r the risk-free interest rate.

If σ_E were directly observable, like the stock price, we could resolve, simultaneously (10) and (11) for V_A and σ_A. But the instantaneous equity volatility, σ_E, is relatively unstable, and is in fact quite sensitive to the change in assets value, and there is no simple way to measure precisely σ_E from market data. [23] Since only the value of equity, V_E, is directly observable, we can back out V_A from (10) which becomes a function of the observed equity value, or stock price, and the volatility of asset returns:

$$V_A = h(V_E, \sigma_A, K, c, r). \tag{12}$$

To calibrate the model for σ_A, KMV uses an iterative technique.

3.1.2. Calculation of the DD

In the option pricing framework default, or equivalently bankruptcy, occurs when assets value falls below the value of the firm's liabilities. In practice, default is distinct from bankruptcy which corresponds to the situation where the firm is liquidated, and the proceeds from the assets sale is distributed to the various claim holders according to pre-specified priority rules. Default is the event when a firm misses a payment on a coupon and/or the reimbursement of principal at debt maturity. Cross-default clauses on debt contracts are such that when the firm misses a single payment on a debt, it is declared in default on all its obligations. Fig. 15 shows the number of bankruptcies and defaults for the period 1973–1994.

KMV has observed from a sample of several hundred companies that firms default when the asset value reaches a level somewhere between the value of total liabilities and the value of short-term debt. Therefore, the tail of the

[22] In the general case the resolution of this model may require the implementation of complex numerical techniques, with no analytical solution, due to the complexity of the boundary conditions attached to the various liabilities. See, for example, Vasicek (1997).

[23] It can be shown that $\sigma_E = \eta_{E,A}\sigma_A$ where $\eta_{E,A}$ denotes the elasticity of equity to asset value, i.e., $\eta_{E,A} = (V_A/V_E)(\partial V_E/\partial V_A)$ (cf. Bensoussan et al., 1994). In the simple Merton's framework, where the firm is financed only by equity and a zero coupon debt, equity is a call option on the assets of the firm with striking price the face value of the debt and maturity the redemption date of the bond. Then, the partial derivative $\partial V_E/\partial V_A$ is simply the delta of the call with respect to the underlying asset of the firm.

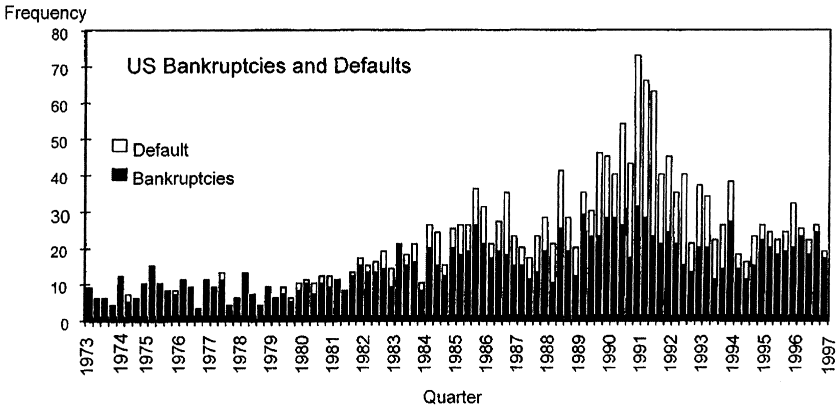


Fig. 15. Bankruptcies and defaults, quarterly from 1973 to 1997 (*source:* KMV Corporation).

distribution of asset value below total debt value may not be an accurate measure of the actual probability of default. Loss of accuracy may also result from other factors such as the non-normality of asset return distribution, the simplifying assumptions about the capital structure of the firm. This can be further aggravated by the fact that there are unknown undrawn commitments (lines of credit) which, in case of distress, will be used and as a consequence may unexpectedly increase liabilities while providing the necessary cash to honor promised payments.

For all these reasons, KMV implements an intermediate phase before computing the probabilities of default. As shown in Fig 16, which is similar to Fig. 7, KMV computes an index called "distance-to-default" (DD). DD is the number of standard deviations between the mean of the distribution of the asset value, and a critical threshold, the "default point", set at the par value of current liabilities including short term debt to be serviced over the time horizon, plus half the long-term debt. Formally DD is defined as follows:

STD	short-term debt,
LTD	long-term debt,
DPT	default point = STD + 1/2 LTD,
DD	distance-to-default which is the distance between the expected asset value in 1 year, $E(V_1)$, and the default point, DPT expressed in standard deviation of future asset returns: $$\mathrm{DD} = \frac{E(V_1) - \mathrm{DPT}}{\sigma_{\mathrm{A}}}.$$

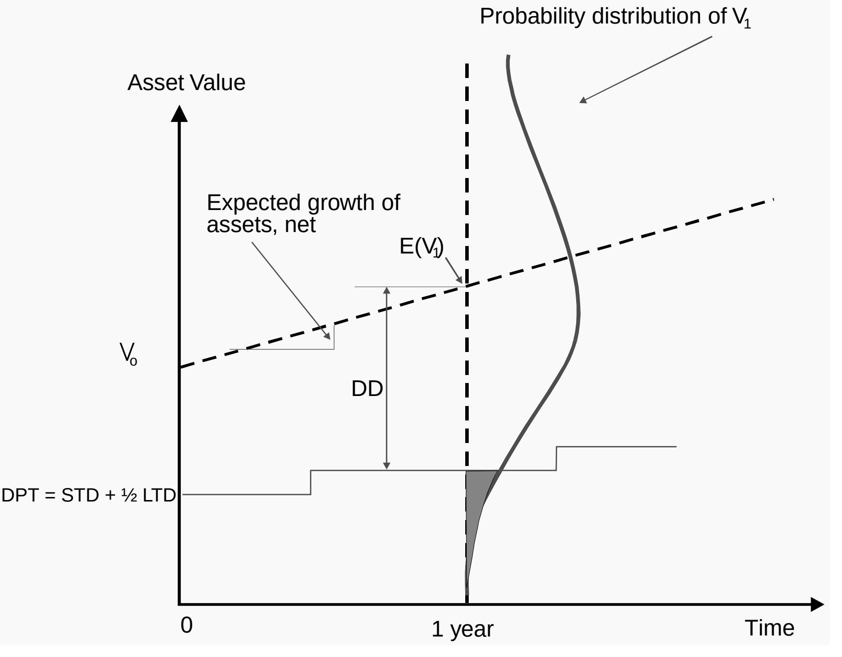


Fig. 16. Distance-to-Default (DD).

Given the lognormality assumption of asset values as specified in (1) then, according to (4), the DD expressed in unit of asset return standard deviation at time horizon T, is

$$\mathrm{DD} = \frac{\ln(V_0/\mathrm{DPT}_T) + (\mu - (1/2)\sigma^2)T}{\sigma\sqrt{T}}, \tag{19}$$

where V_0 is current market value of assets, DPT_T the default point at time horizon T, μ the expected net return on assets, σ the annualized asset volatility.

It follows that the shaded area below the default point is equal to $N(-\mathrm{DD})$.

3.1.3. Derivation of the probabilities of default from the DD

This last phase consists of mapping the DD to the actual probabilities of default, for a given time horizon. These probabilities are called by KMV, EDFs, for Expected Default Frequencies.

Based on historical information on a large sample of firms, which includes firms which defaulted one can estimate, for each time horizon, the proportion of firms of a given ranking, say $\mathrm{DD} = 4$, which actually defaulted after 1 year. This proportion, say 40 bp, or 0.4%, is the EDF as shown in Fig. 17.

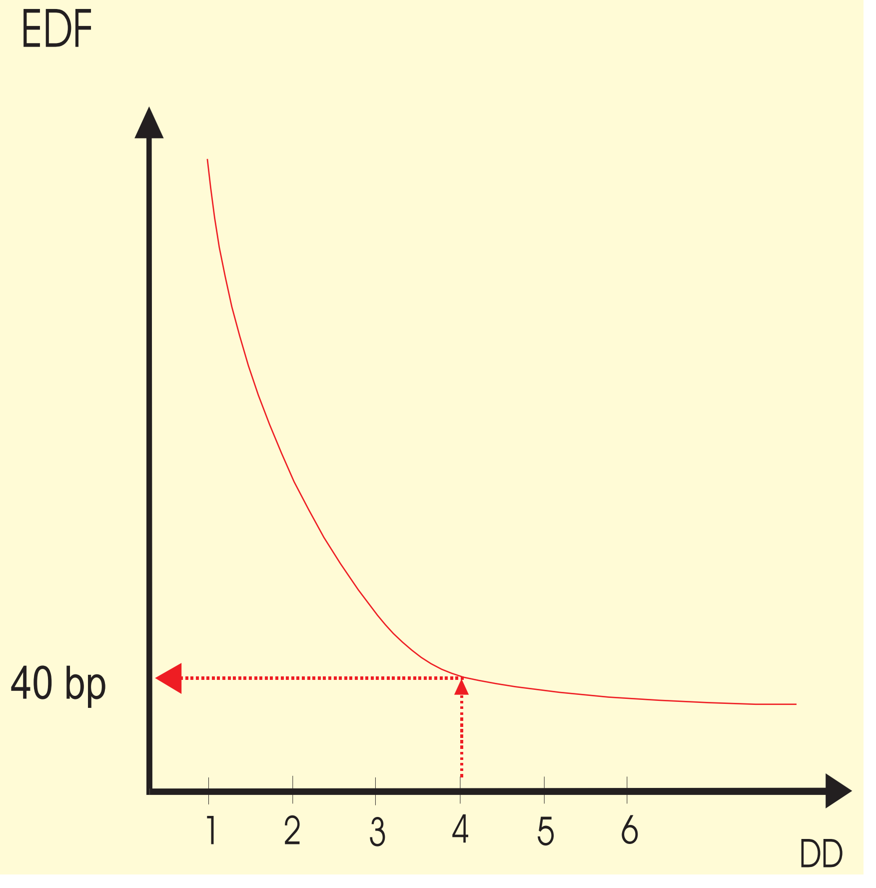


Fig. 17. Mapping of the "distance-to-default" into the "expected default frequencies", for a given time horizon.

Example 3.

Current market value of assets	$V_0 = 1000$
Net expected growth of assets per annum	20%
Expected asset value in 1 year	$V_0\ (1.20) = 1200$
Annualized asset volatility, σ	100
Default point	800

then

$$\mathrm{DD} = \frac{1200 - 800}{100} = 4.$$

Assume that among the population of all the firms with a DD of 4 at one point in time, say 5000 firms, 20 defaulted 1 year later, then:

$$\mathrm{EDF}_{1\,\mathrm{yr}} = \frac{20}{5000} = 0.004 = 0.4\% \text{ or } 40 \text{ bp}.$$

The implied rating for this probability of default is BB^+.

The next example is provided by KMV and relates to Federal Express on two different dates: November 1997 and February 1998.

Example 4. Federal Express ($ figures are in billions of US$).

	November 1997	February 1998
Market capitalization (price × shares outstanding)	$7.7	$7.3
Book liabilities	$4.7	$4.9
Market value of assets	$12.6	$12.2
Asset volatility	15%	17%
Default point	$3.4	3.5
DD	$\frac{12.6 - 3.4}{0.15 \cdot 12.6} = 4.9$	$\frac{12.2 - 3.5}{0.17 \cdot 12.2} = 4.2$
EDF	0.06% (6 bp) $\equiv AA^-$	0.11% (11 bp) $\equiv A^-$

This last example illustrates the main causes of changes for an EDF, i.e. the variations in the stock price, the debt level (leverage ratio), and the asset volatility which is the expression of the perceived degree of uncertainty on the business value.

3.1.4. EDF as a predictor of default

KMV has provided the service "Credit Monitor" of estimated EDFs since 1993. EDFs have proved to be a useful leading indicator of default, or at least of the degradation of the creditworthiness of issuers. When the financial situation of a company starts to deteriorate, EDFs tend to shoot up quickly until default occurs as shown in Fig. 18. Fig. 19 shows the evolution of equity value, asset value, as well as the default point during the same period. On the vertical axis of both graphs the EDF in percent, and the corresponding Standard & Poor's rating are shown.

KMV has analyzed more than 2000 US companies that have defaulted or entered into bankruptcy over the last 20 years, these firms belonging to a large sample of more than 100 000 company-years with data provided by Compustat. In all cases KMV has shown a sharp increase in the slope of the EDF between 1 and 2 years prior to default.

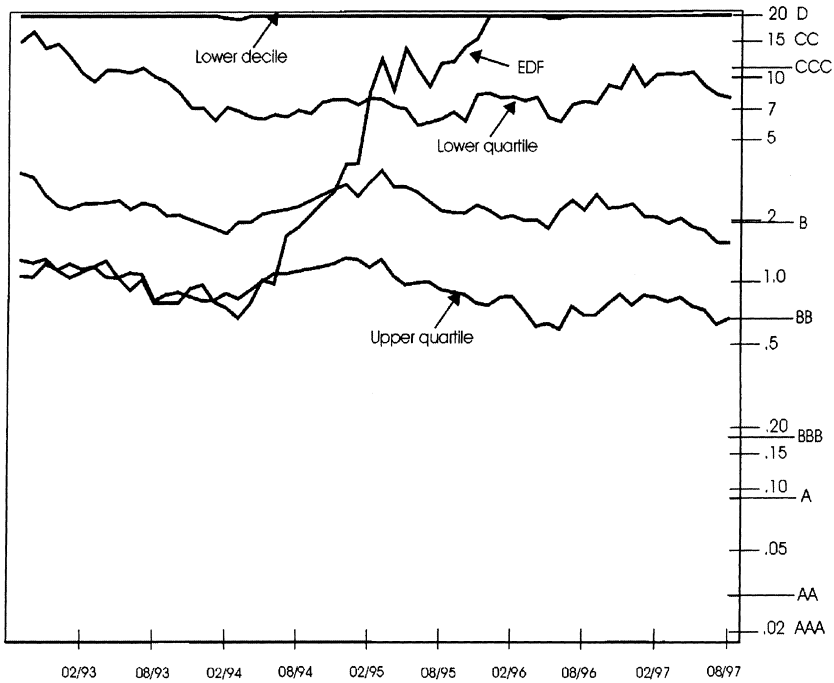


Fig. 18. EDF of a firm which actually defaulted versus EDFs of firms in various quartiles and the lower decile. (The quartiles and decile represent a range of EDFs for a specific credit class.)

Changes in EDFs tend also to anticipate at least by 1 year the downgrading of the issuer by rating agencies like Moody's and Standard & Poor's, as shown in Fig. 20. Contrary to Moody's and Standard & Poor's historical default statistics, EDFs are not biased by periods of high or low defaults. Distant-to-default can be observed to decrease during recession periods where default rates are high, and increase during periods of prosperity characterized by low default rates.

3.1.5. EDFs and ratings

Standard & Poor's risk ratings represent default probabilities only, while Moody's factors also include a measure of the probability of loss, i.e. EDF $\times$ LGD. Table 13 shows the correspondence between EDFs and the ratings of Standard & Poor's, Moody's, as well as the internal ratings of CIBC, Nationbank and Swiss Bank Corp. The ratings of Nationbank and Swiss Bank were published in their recent CLO transactions.

Within any rating class the default probabilities of issuers are clustered around the median. However, as we discussed earlier, the average default rate for each class is considerably higher than the default rate of the typical firm.

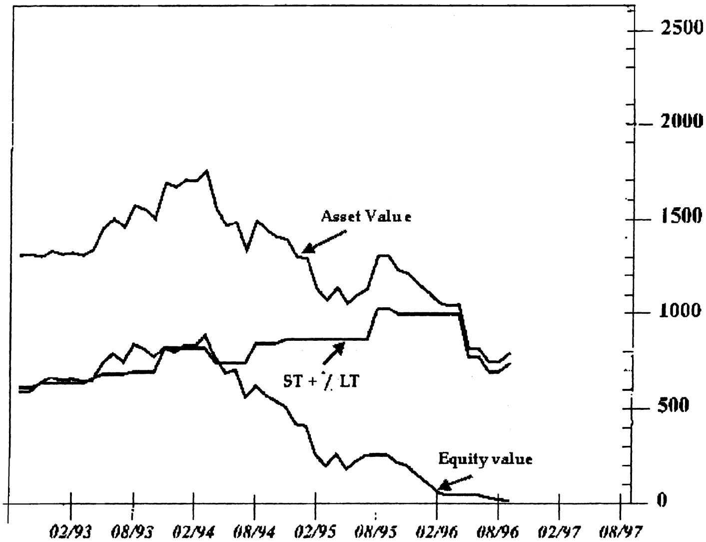


Fig. 19. Assets value, equity value, short-term debt and long-term debt of a firm which actually defaulted.

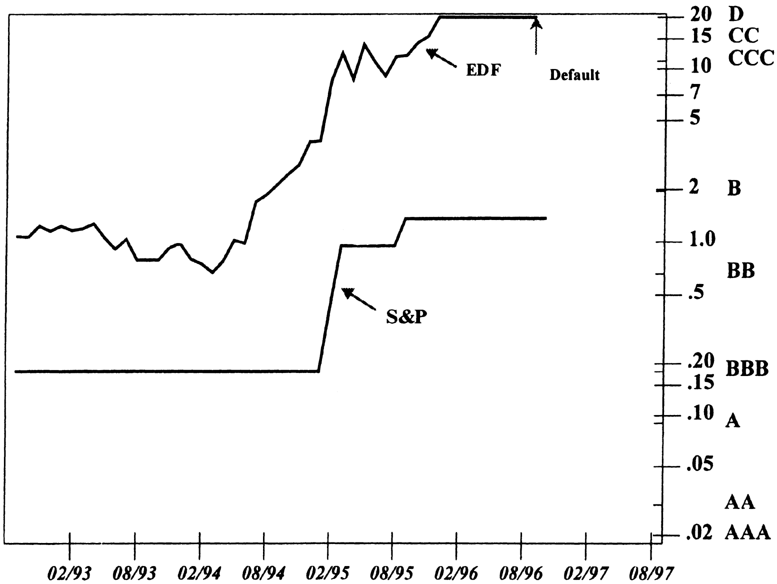


Fig. 20. EDF of a firm which actually defaulted versus Standard & Poor's rating.

Table 13
EDFs and risk rating comparisons

EDF (bp)	S&P	Moody's	CIBC	Nationbank	SBC
2–4	$\geqslant$ AA	$\geqslant$ Aa2	1	AAA	C1
4–10	AA/A	A1	2	AA	C2
10–19	A/BBB+	Baa1	3	A	C3
19–40	BBB+/BBB−	Baa3	4	A/BB	C4
40–72	BBB−/BB	Ba1	4.5	BBB/BB	C5
72–101	BB/BB−	Ba3	5	BB	C6
101–143	BB−/B+	B1	5.5	BB	C7
143–202	B+/B	B2	6	BB/B	C8
202–345	B/B−	B2	6.5	B	C9

This is because each rating class contains a group of firms which have much higher probabilities of default, due to the approximate exponential change in default rates as default risk increases. These are firms which should have been downgraded, but as of yet no downgrade has occurred. There are also firms that should have been upgraded. Table 14 shows the variation of the EDFs within each rating class.

Three conclusions follow from the previous analysis. First, since the rating agencies are slow to change their ratings, the historical frequency of staying in a rating class should overstate the true probability of keeping the same credit quality. Second, the average historical probability of default overstates the true probability of default for typical firms within each rating class, due to the difference between the mean and the median default rates. Third, if both the probability of staying in a given rating class, and the probability of default are too large, then the transition probabilities must be too small.

KMV has constructed a transition matrix based upon default rates rather than rating classes. They start by ranking firms into groups based on non-overlapping ranges of default probabilities that are typical for a rating class. For instance all firms with an EDF less than 2 bp are ranked AAA, then those with an EDF comprised between 3 and 6 bp are in the AA group, firms with an EDF of 7–15 bp belong to A rating class, and so on. Then using the history of

Table 14
Variation of EDFs within rating classes[a]

Quantiles	10	25	50	75	90	Mean
AAA	0.02	0.02	0.02	0.02	0.10	0.04
AA	0.02	0.02	0.02	0.04	0.10	0.06
A	0.02	0.03	0.08	0.13	0.28	0.14
BBB	0.05	0.09	0.15	0.33	0.71	0.30
BB	0.12	0.22	0.62	1.30	2.53	1.09
B	0.44	0.87	2.15	3.80	7.11	3.30
CCC	1.43	2.09	4.07	12.24	18.82	7.21

[a] *Source*: KMV Corporation.

changes in EDFs we can produce a transition matrix shown in Table 15 which is similar in structure to the one produced as Table 1 and reproduced as Table 16.

However, the difference in the various probabilities between the two tables is striking, but as expected. According to KMV, except for AAA, the probability of staying in the same rating class is between half and one-third of historical rates produced by the rating agencies. KMV's probabilities of default are also lower, especially for the low grade quality. Migration probabilities are also much higher for KMV, especially for the grade above and below the current rating class.

These differences may have a considerable impact on the VaR calculations such as those derived in the previous section related to CreditMetrics.

3.2. Valuation model for cash flows subject to default risk

In CreditMetrics/CreditVaR I the valuation model is quite simplistic and has already been described in Section 1. If 1 year is the time horizon, then the

Table 15
KMV 1-year transition matrix based on non-overlapping EDF ranges[a]

Initial rating	Rating at year-end (%)							
	AAA	AA	A	BBB	BB	B	CCC	Default
AAA	66.26	22.22	7.37	2.45	0.86	0.67	0.14	0.02
AA	21.66	43.04	25.83	6.56	1.99	0.68	0.20	0.04
A	2.76	20.34	44.19	22.94	7.42	1.97	0.28	0.10
BBB	0.30	2.80	22.63	42.54	23.52	6.95	1.00	0.26
BB	0.08	0.24	3.69	22.93	44.41	24.53	3.41	0.71
B	0.01	0.05	0.39	3.48	20.47	53.00	20.58	2.01
CCC	0.00	0.01	0.09	0.26	1.79	17.77	69.94	10.13

[a] *Source*: KMV Corporation.

Table 16
Transition matrix based on actual rating changes[a]

Initial rating	Rating at year-end (%)							
	AAA	AA	A	BBB	BB	B	CCC	Default
AAA	90.81	8.33	0.68	0.06	0.12	0	0	0
AA	0.70	90.65	7.79	0.64	0.06	0.14	0.02	0
A	0.09	2.27	91.05	5.52	0.74	0.26	0.01	0.06
BBB	0.02	0.33	5.95	86.93	5.30	1.17	1.12	0.18
BB	0.03	0.14	0.67	7.73	80.53	8.84	1.00	1.06
B	0	0.11	0.24	0.43	6.48	83.46	4.07	5.20
CCC	0.22	0	0.22	1.30	2.38	11.24	64.86	19.79

[a] *Source*: Standard & Poor's CreditWeek (April 15, 1996).

forward value of a bond is the discounted value of the future cash flows beyond 1 year, where the discount factors are derived from the forward yield curve. To each credit rating is associated a specific spread curve, and the distribution of future values follows from the transition probabilities.

In KMV the approach is quite different, and is consistent with the option pricing methodology to the valuation of contingent cash flows. Given the term structure of EDFs for a given obligor, we can derive the net present value of any stream of contingent cash flows. The final step, discussed in the next section, consists of deriving the loss distribution for the entire portfolio.

More specifically, KMV's pricing model is based upon the "risk neutral" valuation model, also named the Martingale approach to the pricing of securities, which derives prices as the discounted expected value of future cash flows. The expectation is calculated using the so-called risk neutral probabilities and not the actual probabilities as they can be observed in the market place from historical data or the EDFs. [24] Assuming, for the time being, that we know how to derive the "risk neutral probabilities" from the EDFs, then the valuation of risky cash flows proceeds in two steps, first the valuation of the default-free component, and second, the valuation of the component exposed to credit risk.

(i) *Case of a single cash flow*

Example 5. Valuation of a zero coupon bond with a promised payment in 1 year of \$100, with a recovery of (1 − LGD) if the issuer default, i.e. LGD is the loss given default, assumed to be 40% in this example illustrated in Fig. 21.

The risk-free component, \$100(1 − LGD) is valued using the default-free discount curve, i.e.

$$\mathrm{PV}_1 = \mathrm{PV}(\text{risk-free cash flow}) = 100(1 - \mathrm{LGD})/(1 + r) = \$54.5,$$

where r denotes the 1-year risk-free rate assumed to be 10%.

The risky cash flow is valued using the Martingale approach, i.e.

$$\mathrm{PV}_2 \text{ (risky cash flow)} = E_Q \text{ (discounted risky cash flow)},$$

where the expectation is calculated using the risk neutral probability. Denote by Q, the risk neutral probability that the issuer defaults in 1 year from now, and it is assumed to be 20%, then:

[24] See, for example, Jarrow and Turnbull (1997, Ch. 5 and 6).

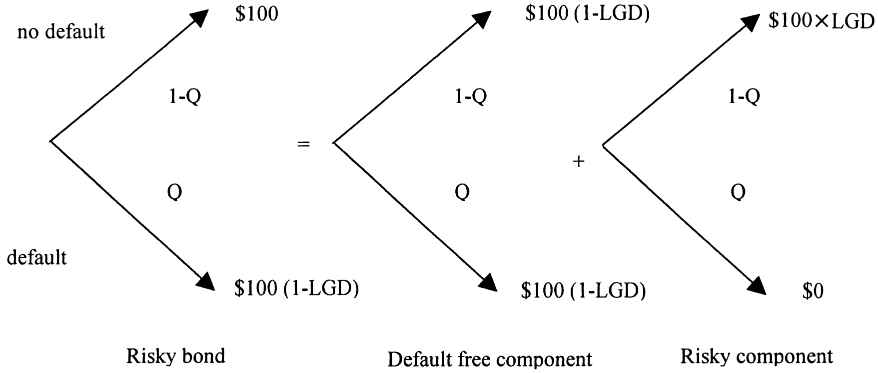


Fig. 21. Valuation of a single cash flow subject to default risk.

$$\mathrm{PV}_2 = \mathrm{PV}\ (\text{risky cash flow}) = \frac{100\mathrm{LGD}(1-Q) + 0 \cdot Q}{1+r}$$
$$= \frac{100\mathrm{LGD}(1-Q)}{1+r} = \$29.1.$$

The present value of this zero coupon bond subject to default risk is the sum of the default-free component and the risky component, i.e.

$$\mathrm{PV} = \mathrm{PV}_1 + \mathrm{PV}_2 = \$54.5 + \$29.1 = \$83.6.$$

If the zero coupon bond were default free, its present value would simply be its discounted value using the default-free interest rate, i.e.

$$\$100/(1+r) = \$90.9.$$

We can then compute the implicit discount rate, R, which accounts for default risk, i.e.

$$R = r + \mathrm{CS},$$

where CS denotes the credit spread. It is solution of

$$\frac{100(1-\mathrm{LGD})}{1+r} + \frac{100\mathrm{LGD}(1-Q)}{1+r} = \frac{100}{1+r+\mathrm{CS}}. \tag{10}$$

Solving (10) for CS gives:

$$\mathrm{CS} = \frac{\mathrm{LGD} \cdot Q \cdot (1+r)}{1 - \mathrm{LGD} \cdot Q}. \tag{11}$$

For this example, $R = 19.6\%$, so that the 1-year credit spread for this issuer is 9.6%.

(ii) *Generalized pricing model for a bond or a loan subject to default risk*:

The previous approach can be easily generalized to the valuation of a stream of cash flows $[C_1, \ldots, C_i, \ldots, C_n]$:

$$\mathrm{PV} = (1 - \mathrm{LGD})\sum_{i=1}^{n} \frac{C_i}{(1 + r_i)^{t_i}} + \mathrm{LGD}\sum_{i=1}^{n} \frac{(1 - Q_i)C_i}{(1 + r_i)^{t_i}}, \tag{12}$$

or in continuous time notation,

$$\mathrm{PV} = (1 - \mathrm{LGD})\sum_{i=1}^{n} C_i\,\mathrm{e}^{-\tilde{r}_i t_i} + \mathrm{LGD}\sum_{i=1}^{n}(1 - Q_i)C_i\,\mathrm{e}^{-\tilde{r}_i t_i}, \tag{13}$$

where Q_i denotes the cumulative "risk neutral" EDF at the horizon t_i and $\tilde{r}_i = 1n(1 + r_i)$.

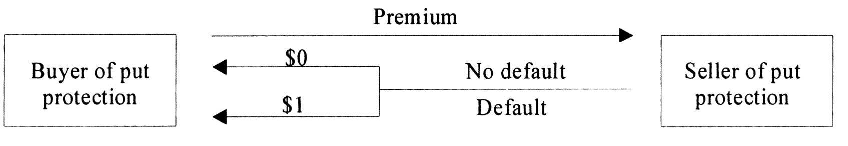
Example 6. What is the value of a 5-year bond with a face value of \$100, which pays an annual coupon of 6.25%? Let us assume that the risk-free interest rate is 5%, the LGD is 50% and the cumulative risk neutral probabilities are given in the table below.

Time	Q_i (%)	Discount factor $1/(1 + r_i)t_i$	Cash flow	PV_1 (risk-free cash flows)	PV_2 (risky cash flows)
(1)	(2)	(3)	(4)	$(5) = \frac{1}{2}(4) \times (3)$	$(6) = (5)[1 - (2)]$
1	1.89	0.9512	6.25	2.97	2.92
2	4.32	0.9048	6.25	2.83	2.71
3	6.96	0.8607	6.25	2.69	2.50
4	9.69	0.8187	6.25	2.56	2.31
5	12.47	0.7788	106.25	41.37	36.21
Total				52.42	46.65

$$\mathrm{PV} = \mathrm{PV}_1 + \mathrm{PV}_2 = 99.07.$$

This methodology also applies to simple credit derivatives like a default put:

Example 7. What is the premium of a 1-year default put which pays \$1 in case the underlying bond defaults?



Assume a risk neutral probability $Q = 0.39\%$ and an interest rate $r = 5.8\%$, then

$$\text{Premium} = Q/(1+r) = 0.0039/1.058 = 0.37\%.$$

3.3. Derivation of the risk neutral EDFs

Under the risk neutral probability measure the expected return on all securities is the default free interest rate, r, for any horizon, say T. Therefore, the risk neutral EDF, or Q, is defined as the probability of default, i.e. the probability that the value of the assets at time T falls below the default point DPT_T, under the modified risk neutral process for the asset value, V_t^*:

$$\begin{aligned} Q &= \Pr\left[V_T^* \leqslant \mathrm{DPT}_T\right] \\ &= \Pr\left[\ln V_0 + \left(r - (1/2)\sigma^2\right)T + \sigma\sqrt{T}Z_T \leqslant \ln \mathrm{DPT}_T\right] \\ &= \Pr\left[Z_T \leqslant -\frac{\ln[V_0/\mathrm{DPT}_T] + [r - (1/2)\sigma^2]T}{\sigma\sqrt{T}}\right] = \mathrm{N}(-d_2^*), \end{aligned} \tag{14}$$

where $\mathrm{N}(\cdot)$ is the cumulative standard normal distribution and

$$d_2^* = \frac{\ln[V_0/\mathrm{DPT}_T] + [r - \frac{1}{2}\sigma^2]T}{\sigma\sqrt{T}},$$

with $(\mathrm{d}V_t^*/V_t^*) = r\,\mathrm{d}t + \sigma\,\mathrm{d}W_t$ where W_t is a standard Brownian motion, and $\sqrt{T}Z_T = W_T - W_0$ is normally distributed with zero mean and variance equal to T.

If the EDF was precisely the shaded area under the default point in Fig. 8, then we would have exactly:

$$\mathrm{EDF}_T = N(-d_2)$$

where d_2 has already been defined in (3), i.e.

$$d_2 = \frac{\ln[V_0/\mathrm{DPT}_T] + (\mu - (1/2)\sigma^2)T}{\sigma\sqrt{T}}.$$

Since $-d_2 + ((\mu - r)\sqrt{T})/\sigma = -d_2^*$, it thus follows that the cumulative risk neutral EDF, Q_T, at horizon T can be expressed as

$$Q_T = N\left[N^{-1}(\mathrm{EDF}) + \frac{(\mu - r)}{\sigma}\sqrt{T}\right]. \tag{15}$$

Since $\mu \geqslant r$ it follows that $Q_T \geqslant \mathrm{EDF}_T$ i.e. the risk neutral probability of default, after adjusting for the price of risk, is higher than the actual probability of default.

According to the CAPM

$$\mu - r = \beta\pi \tag{16}$$

with

$$\beta = \text{beta of the asset with the market} = \frac{\operatorname{cov}(R, R_{\mathrm{M}})}{\operatorname{var}(R_{\mathrm{M}})} = \mathrm{Rho}\,\frac{\sigma}{\sigma_{\mathrm{M}}}, \tag{17}$$

where R and R_{M} denote the return of the firm's asset and the market portfolio, respectively; σ and σ_{M} are the volatility of the asset return and the market return, respectively; Rho is the correlation between the asset's return and the market's return.

$$\pi = \text{market risk premium for a unit of beta risk} = \mu_{\mathrm{M}} - r, \tag{18}$$

where μ_{M} and μ denote the expected return on the market portfolio and the firm's assets, respectively, and r is the risk-free rate.

It follows that

$$\frac{\mu - r}{\sigma} = \frac{\beta\pi}{\sigma} = \mathrm{Rho}\,\frac{\pi}{\sigma_{\mathrm{M}}} = \mathrm{Rho}\,U, \tag{19}$$

where $U = \pi/\sigma_M$ denotes the market Sharpe ratio, i.e. the excess return per unit of market volatility for the market portfolio.

Substituting (19) into (15) we obtain:

$$Q_T = N\left[N^{-1}(\mathrm{EDF}_T) + \mathrm{Rho}\,\frac{\pi}{\sigma_{\mathrm{M}}}\sqrt{T}\right]. \tag{20}$$

Rho is estimated by the linear regression of asset returns against market returns:

$$R = \alpha + \beta R_{\mathrm{M}} + \varepsilon, \tag{21}$$

where α is the intercept of the regression and ε the error term. Rho is simply the square root of the R-squared of this regression.

In practice, π, the market risk premium is difficult to estimate statistically, and it varies over time. In addition, the EDF is not precisely the shaded area under the default point in Fig. 7, and the asset return distribution is not exactly normal. For all these reasons, KMV estimates the risk neutral EDF, Q_T, by calibrating the market Sharpe ratio, U, and θ in the following relation, using bond data:

$$Q_T = N[N^{-1}(\mathrm{EDF}_T) + \mathrm{Rho}\,U\ T^{\theta}], \tag{22}$$

where θ is a time parameter which should be, in theory, equal to 1/2.

Assuming we have derived the zero-coupon curve for an obligor, then according to the pricing model (13) presented earlier:

$$\mathrm{e}^{-\tilde{R}_i t_i} = [(1 - \mathrm{LGD}) + (1 - Q_i)\mathrm{LGD}]\,\mathrm{e}^{-\tilde{r}_i t_i} \tag{23}$$

for $i = 1, \ldots, n$, where $\tilde{R}_i$ is the continuously compounded zero-coupon interest rate for the obligor, i.e. $\tilde{R}_i = \ln(1 + R_i)$, for maturity t_i, $\tilde{r}_i$ the continuously compounded zero-coupon risk-free rate i.e. $\tilde{r}_i = \ln(1 + r_i)$, for maturity t_i, so that

$$\tilde{R}_i - \tilde{r}_i = -\frac{1}{t_i}\ln[1 - Q_i\mathrm{LGD}]. \tag{24}$$

Combining (22) and (24) we obtain:

$$\tilde{R}_i - \tilde{r}_i = -\frac{1}{t_i}\ln\left[1 - N(N^{-1}(\mathrm{EDF}_{t_i}) + \mathrm{Rho}\,U\ T^{\theta})\mathrm{LGD}\right], \tag{25}$$

where $\tilde{R}_i - \tilde{r}_i$ is the obligor's corporate spread for maturity t_i, which is directly extracted from corporate bond data. [25] U and θ are calibrated to produce the best fit of (25) in the least square sense.

3.4. Credit-VaR and calculation of the capital charge for a portfolio

KMV does not simulate the full forward distribution of the portfolio values at the credit horizon, H. Instead, KMV derives analytically the loss distribution of the portfolio at this horizon. For the sake of simplicity assume that all bonds mature at time T, greater than the credit horizon, H.

Denote by $V_{H/ND}$ the discounted value of the portfolio at time H, assuming no default, and V_H, the equilibrium value of the portfolio at time H, derived from the valuation model presented in Section 2.2. The portfolio loss at time H is defined as the difference between the riskless value of the portfolio and its market value at that time:

$$L = V_{H/ND} - V_H.$$

Note that V_H is unknown at time 0, only its probability distribution can be derived so that the loss, L, is a random variable.

Under some simplifying assumption it can be shown that the limiting distribution of the portfolio loss, when the portfolio is widely diversified across issuers, is a normal inverse for which it is relatively easy to compute the percentiles. The normal inverse distribution is highly skewed and leptokurtic. Table 17 shows some values of the α-percentile, L_α, expressed as the number of standard deviations from the mean, for several values of parameters. The α-percentiles of the standard normal distribution are shown for comparison.

p is the probability of default of one bond in the portfolio (all bonds are assumed to have the same probability of default).

ρ is the pairwise asset correlation, constant across all issuers.

α is the confidence level.

The expected loss for the portfolio is $\mathrm{EL} = p$ and its standard deviation is denoted by s.

[25] An empirical issue is whether the corporate spread should be calculated over the Treasury curve, or instead over the LIBOR (swap) curve. It seems that the best fits are obtained when spreads are calculated over LIBOR.

Table 17
Values of $(L_\alpha - p)/s$ for the normal inverse distribution

p	ρ	$\alpha = 0.1$	$\alpha = 0.01$	$\alpha = 0.001$	$\alpha = 0.0001$
0.01	0.1	1.19	3.8	7.0	10.7
0.01	0.4	0.55	4.5	11.0	18.2
0.001	0.1	0.98	4.1	8.8	15.4
0.001	0.4	0.12	3.2	13.2	31.7
Normal		1.28	2.3	3.1	3.7

These values manifest the extreme non-normality of the distribution. Suppose a lender holds a large portfolio of bonds issued by obligors where pairwise asset correlation is $\rho = 0.4$ and where probability of default is $p = 0.01$. If the desired confidence level is $\alpha = 0.001$ (10 bp), then the required capital should be enough to cover 11 times the portfolio loss standard deviation.

To be more specific the capital charge is

α-percentile − expected spread revenue,

where

α- percentile is expressed in absolute value,

expected spread revenue = total spread revenue − expected loss

and

total spread revenue = annualized expected revenue over funding cost.

3.5. Asset return correlation model

CreditMetrics/CreditVaR I and KMV derive asset return correlations from a structural model which links correlations to fundamental factors. By imposing a structure on the return correlations, sampling errors inherent in simple historical correlations are avoided, and a better accuracy in forecasting correlations is achieved. In addition, there is a practical need to reduce dramatically the number of correlations to be calculated. Assume that a bank is dealing with $N = 1000$ different counterparties. Then, we have $N(N-1)/2$ different correlations to estimate, i.e., 499 500. This number is staggering. Multi-factor models of asset returns reduce correlations to be calculated to those between the limited number of common factors affecting asset returns.

It is assumed that the firm's asset returns are generated by a set of common, or systematic risk factors, and idiosyncratic factors. The idiosyncratic factors are either firm-, or country- or industry-specific, and do not contribute to asset return correlations, since they are not correlated with each other and not correlated with the common factors. Asset return correlations between two firms are only explained by the common factors to all firms. Only the risks associated with the idiosyncratic risk factors can be diversified away through portfolio diversification, while the risk contribution of the common factors is, on the contrary, non diversifiable.

For the sake of illustration, assume the asset return generating process for all firms is

$$r_k = \alpha_k + \beta_{1k} I_1 + \beta_{2k} I_2 + \varepsilon_k \quad \text{for } k = 1, \ldots, N, \tag{26}$$

where N is the number of obligors (firms), r_k the asset return for firm k, α_k the component of asset return independent of common factors, I_1, I_2 are the common factors, β_{1k}, β_{2k} are the expected changes in r_k, given a change in common factors 1 and 2, respectively, ε_k the idiosyncratic risk factor with zero mean, and assumed to be uncorrelated with all the common factors, as well as with the idiosyncratic risk factors of the other firms.

Then, from elementary statistics we can derive the well-known results in portfolio theory: [26]

$$\begin{aligned} \mathrm{var}(r_k) &\equiv \sigma_k^2 \\ &= \beta_{1k}^2 \mathrm{var}(I_1) + \beta_{2k}^2 \mathrm{var}(I_2) + \mathrm{var}(\varepsilon_k^2) + 2\beta_{1k}\ \beta_{2k} \mathrm{cov}(I_1, I_2), \end{aligned} \tag{27}$$

$$\begin{aligned} \mathrm{cov}(r_i, r_j) &\equiv \sigma_{ij} \\ &= \beta_{1i}\beta_{2j} \mathrm{var}(I_1) + \beta_{2i}\beta_{2j} \mathrm{var}(I_2) + (\beta_{1i}\beta_{2j} + \beta_{2i}\beta_{1j}) \mathrm{cov}(I_1, I_2). \end{aligned} \tag{28}$$

If we denote by ρ_{ij} the asset return correlation between firm i and firm j, then

$$\rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \sigma_j}. \tag{29}$$

To derive the asset return correlation between any number of firms we only need, according to (27)–(29), to estimate the β_{ik}, i.e., $2N$ parameters, and the covariance matrix for the common factors, i.e., 3 parameters. In the previous example where we considered $N = 1000$ firms, the implementation of this 2-factor model would only require an estimate of 2003 parameters instead of 499 500 different historical asset return correlations. For K common factors the number of parameters to be estimated is $KN + K(K-1)/2$. If $K = 10$ then this number becomes 10 045. This result can be easily generalized to any number of common factors and idiosyncratic risk components.

[26] See, for example, Elton and Gruber (1995, Ch. 8). While a multi-factor model can be implemented directly, the model gains very convenient mathematical properties if the factors are uncorrelated, i.e., orthogonal. There are simple techniques to convert any set of correlated factors into a set of orthogonal factors. In that case we would have $\mathrm{cov}(I_1, I_2) = 0$.

The issue now is to specify the factor structure. CreditMetrics and KMV are proposing relatively similar models, and in the following we will only present KMV's model which is more comprehensive and elaborated. [27]

KMV constructs a three-layer factor structure model as shown in Fig. 22:

- *First level*: a composite company-specific factor, which is constructed individually for each firm based on the firm's exposure to each country and industry,
- *second level*: country and industry factors,
- *third level*: global, regional and industrial sector factors.

The first level of the structure divides between firm specific, or idiosyncratic risk, and common, or systematic risk. The first, systematic risk is captured by a single composite index, which is firm specific, and which is constructed as a weighted sum of the firm's exposure to country and industry factors defined at the second level of the structure:

$$r_k = \beta_k \mathrm{CF}_k + \varepsilon_k \quad \text{for all firms } k = 1, \ldots, N,$$

where r_k is asset return for firm k, CF_k the composite factor for firm k, β_k the firm k's response to composite factor, i.e., expected change in r_k given a change in composite factor and ε_k firm k's specific risk factor.

The composite factor is constructed as the sum of the weighted country and industry factors specified at the second level of the structure:

$$\mathrm{CF}_k = \sum_m \alpha_{km} C_m + \sum_n \alpha_{kn} I_n,$$

where C_m is rate of change on country risk factor m, I_n the rate of change on industry risk factor n, α_{km} the weight of firm k in country m, with the constraint that $\sum_m \alpha_{km} = 1$ and α_{kn} the weight of firm k in industry n, with the constraint that $\sum_n \alpha_{kn} = 1$.

[27] For a review of multi-factor models, see Elton and Gruber (1995) and Rudd and Clasing (1988). The most widely used technique in portfolio management is the single index model, or market model, which assumes that the co-movement between stock returns is due to a single common index, the market portfolio. The return generating process for this model is described by $r_k = \alpha_k + \beta_k r_{\mathrm{M}} + \varepsilon_k$ where r_{M} denotes the rate of return on the market portfolio. This model can then be extended to capture industry effects beyond the general market effects. Rosenberg (1979) has developed a model for predicting extra market covariance which relates not only on industry factors, but also company specific descriptors like market variability which captures the risk of the firm as perceived by the market, earnings variability, index of low valuation and unsuccess, immaturity and smallness, growth orientation, and financial risk. (See Rudd and Clasing, 1988.) Finally, a number of multi-factor models have been proposed which relate security returns to macroeconomic variables. Excess returns are explained by the unexpected changes, or innovations, in variables like inflation, economic growth as measured by unexpected change in industrial production, business cycles as proxied by the corporate spread over Treasuries, long-term interest rates, short-term interest rates, and currency fluctuations. (See Chen et al., 1986; Berry et al., 1988.)

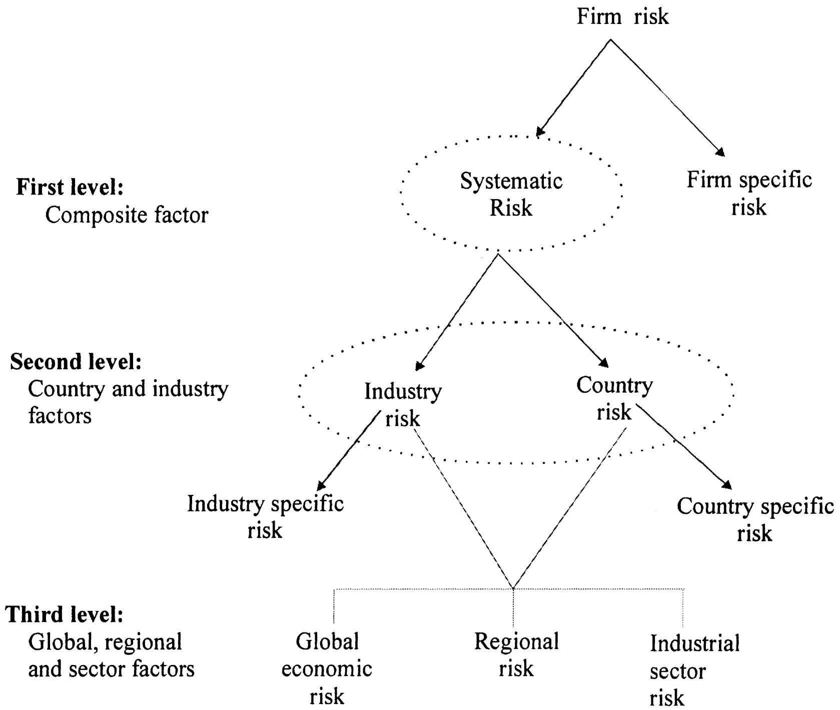


Fig. 22. Factor model for asset return correlations (*source:* KMV Corporation).

For example, consider a Canadian firm which has two lines of business, and assume that the data is extracted from Compustat: [28]

Business line	SIC	Assets (%)	Sales (%)
Lumber and forestry	2431	35	45
Paper production	2611	65	55
Total		100	100

In the above table SIC denotes the Standard Industrial Classification which is a US based business classification system.

To determine the weight by industry we average the asset and sales breakdowns. Thus for the above example the weight for Lumber and Forestry is

$$40\% = (35\% + 45\%)/2,$$

and for Paper it is

[28] Compustat is a database of financial and economic information on firms.

$$60\% = (65\% + 55\%)/2.$$

Note that by construction the weights add up to 100%. The country exposures are calculated in a similar manner and should also sum up to 100%. In this example, we assume a 100% exposure to Canada. Then the composite factor can be written as

$$\mathrm{CF} = 1.0C_{\mathrm{Canada}} + 0.6I_{\mathrm{paper}} + 0.4I_{\mathrm{lumber}}.$$

At the third level of the factor structure the risk of countries and industries is further decomposed into systematic and idiosyncratic components. The systematic component is captured by basic factors like: global economic effect, regional factor effect and sector factor effect. While the common factor is firm-specific, the third level factors are the same for all countries and all industries:

$$\begin{bmatrix}\text{Country}\\\text{returns}\end{bmatrix} = \begin{bmatrix}\text{Global}\\\text{economic}\\\text{effect}\end{bmatrix} + \begin{bmatrix}\text{Regional}\\\text{factor}\\\text{effect}\end{bmatrix} + \begin{bmatrix}\text{Sector}\\\text{factor}\\\text{effect}\end{bmatrix} + \begin{bmatrix}\text{Country}\\\text{specific}\\\text{risk}\end{bmatrix},$$

$$\begin{bmatrix}\text{Industry}\\\text{return}\end{bmatrix} = \begin{bmatrix}\text{Global}\\\text{economic}\\\text{effect}\end{bmatrix} + \begin{bmatrix}\text{Regional}\\\text{factor}\\\text{effect}\end{bmatrix} + \begin{bmatrix}\text{Sector}\\\text{factor}\\\text{effect}\end{bmatrix} + \begin{bmatrix}\text{Industry}\\\text{specific}\\\text{risk}\end{bmatrix}.$$

We can now express this factor structure into a form similar to (26) from which it is easy to derive the asset return correlations (29).

4. CreditRisk+ model

CreditRisk+ [29] applies an actuarial science framework to the derivation of the loss distribution of a bond/loan portfolio. Only default risk is modeled, not downgrade risk. Contrary to KMV, default risk is not related to the capital structure of the firm. In CreditRisk+ no assumption is made about the causes of default: an obligor A is either in default with probability P_A, or it is not in default with probability $1 - P_A$. It is assumed that:

- for a loan, the probability of default in a given period, say 1 month, is the same for any other month;
- for a large number of obligors, the probability of default by any particular obligor is small, and the number of defaults that occur in any given period is independent of the number of defaults that occur in any other period.

[29] CreditRisk+ is a trademark of Credit Suisse Financial Products (CSFP). CreditRisk+ is described in CSFP's publication (Credit Suisse, 1997).

Under those circumstances, the probability distribution for the number of defaults, during a given period of time (say 1 year) is well represented by a Poisson distribution: [30]

$$P(n \text{ defaults}) = \frac{\mu^n \mathrm{e}^{-\mu}}{n!} \quad \text{for } n = 0, 1, 2, \ldots, \tag{30}$$

where

$$\mu = \text{average number of defaults per year,}$$
$$\mu = \sum_A P_A, \text{ where } P_A \text{ denotes the probability of default for obligor } A.$$

The annual number of defaults, n, is a stochastic variable with mean μ, and standard deviation $\sqrt{\mu}$. The Poisson distribution presents the nice property to be fully specified by only one parameter μ. [31] For example, if we assume $\mu = 3$ then the probability of no default in the next year is

$$P(0 \text{ default}) = \frac{3^0 \mathrm{e}^{-3}}{0!} = 0.05 = 5\%$$

and the probability of exactly 3 defaults is

$$P(3 \text{ defaults}) = \frac{3^3 \mathrm{e}^{-3}}{3!} = 0.224 = 22.4\%.$$

4.1. CreditRisk+ framework

The distribution of default losses for a portfolio is derived in two stages, as shown in Fig. 23.

4.1.1. Frequency of default events (building block #1)

So far we have assumed that a standard Poisson distribution approximates the distribution of the number of default events. Then we should expect the standard deviation of the default rate to be approximately equal to the square root of the mean, i.e., $\sqrt{\mu}$, where μ is the average default rate. According to Table 3, for obligors in rating category B, we expect a standard deviation of the default rate to be close to $\sqrt{7.27}$, i.e., 2.69, while Table 3 reports an actual standard deviation of 5.1. We derive similar observations for Baa and Ba obligors. Under

[30] In a portfolio there is, obviously, a finite number of obligors, say m, therefore, the Poisson distribution which specifies the probability of n defaults, for $n = 1, \ldots, \infty$ is only an approximation. However, if the number of obligors, m, is large enough, then the sum of the probabilites of $n+1,\ n+2, \ldots$ defaults become negligible.

[31] Expression (30) can be derived from the probability generating function for a portfolio of independent obligors (see Credit Suisse, 1997, pp. 34, 35).

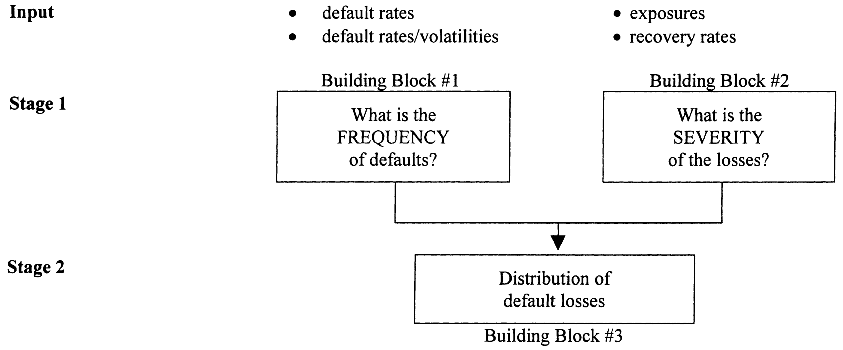


Fig. 23. CreditRisk+ risk measurement framework (*source:* CreditRisk+).

those circumstances the Poisson distribution will underestimate the actual probability of default. This is not a surprising result when we observe the variability of default rates over time (see Fig. 1). As a matter of fact, we expect the mean default rate to change over time depending on the business cycle.

Still the Poisson distribution can be used to represent the default process, but with the additional assumption that the mean default rate is itself stochastic with mean μ and standard deviation σ_μ. [32]

Assuming a stochastic default rate makes the distribution of defaults more skewed with a fat right tail (see Fig. 24).

4.1.2. Severity of the losses (building block #2)

In the event of default of an obligor, the counterparty incurs a loss equal to the amount owned by the obligor (the exposure, i.e., the marked-to-market value if positive, and zero otherwise, at the time of default) less a recovery amount (see Table 6).

In CreditRisk+ the exposure for each obligor is adjusted by the anticipated recovery rate, in order to calculate the loss given default. These adjusted exposures are exogenous to the model, and are independent of market risk and downgrade risk.

4.1.3. Distribution of default losses for a portfolio (building block #3)

In order to derive the loss distribution for a well-diversified portfolio, the losses (exposures, net of the recovery adjustments) are divided into bands, with the level of exposure in each band being approximated by a single number.

[32] CreditRisk+ assumes that the mean default rate is Gamma distributed. Mean default rate volatility may also reflect the influence of default correlation and background factors, such as a change in the rate of growth in the economy which may in turn affect the correlation of default events.

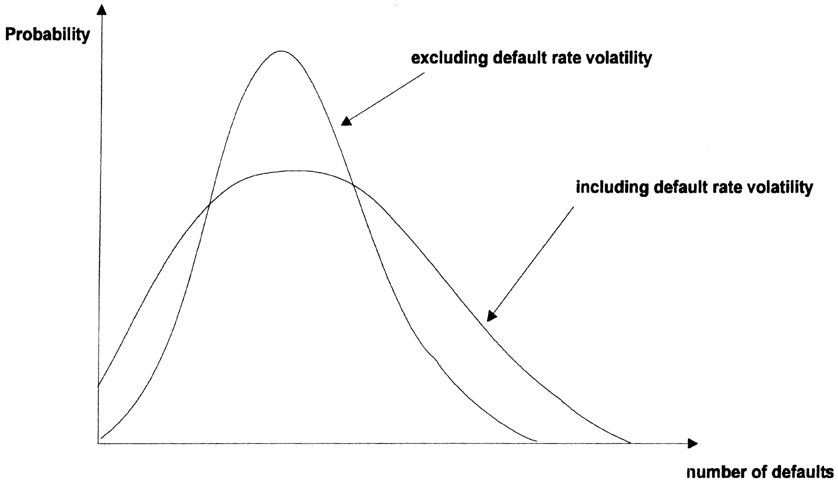


Fig. 24. Distribution of default events (source: CreditRisk+).

Example. Suppose the bank holds a portfolio of loans and bonds from 500 different obligors, with exposures between \$50 000 and \$1 million.

	Notation
Obligor	A
Exposure	L_A
Probability of default	P_A
Expected loss	$\lambda_A = L_A \times P_A$

Note: In CreditRisk+ the exposure is the forward value of the facility times the loss given default rate.
In the following table we only show the exposures for the first 6 obligors.

Obligor A	Exposure (\$) (loss given default) L_A	Exposure (in \$100 000) $\overline{v}_j$	Round-off exposure (in \$100 000) v_j	Band j
1	150 000	1.5	2	2
2	460 000	4.6	5	5
3	435 000	4.35	5	5
4	370 000	3.7	4	4
5	190 000	1.9	2	2
6	480 000	4.8	5	5

The unit of exposure is assumed to be $L = \$100\,000$. Each band j, $j = 1, \ldots, m$, with $m = 10$, has an average common exposure: $v_j = \$100\,000 \times j$.

In CreditRisk+ each band is viewed as an independent portfolio of loans/bonds, for which we introduce the following notation:

	Notation
Common exposure in band j in units of L	v_j
Expected loss in band j in units of L	ε_j
Expected number of defaults in band j	μ_j

Then, by definition we have

$$\varepsilon_j = v_j \times \mu_j.$$

Hence,

$$\mu_j = \frac{\varepsilon_j}{v_j}. \tag{31}$$

Denote by ε_A the expected loss for obligor A in units of L, i.e.,

$$\varepsilon_A = \frac{\lambda_A}{L}$$

then, the expected loss over a 1-year period in band j, ε_j, expressed in units of L, is just the sum of the expected losses ε_A of all the obligors belonging to band j, i.e.,

$$\varepsilon_j = \sum_{A:v_A=v_j} \varepsilon_A.$$

From (31) it follows that the expected number of defaults per annum in band j is

$$\mu_j = \frac{\varepsilon_j}{v_j} = \sum_{A:v_A=v_j} \frac{\varepsilon_A}{v_j} = \sum_{A:v_A=v_j} \frac{\varepsilon_A}{v_A}.$$

The table below provides an illustration of the results of those calculations:

Band j	Number of obligors	ε_j	μ_j
1	30	1.5	1.5
2	40	8	4
3	50	6	2
4	70	25.2	6.3
5	100	35	7
6	60	14.4	2.4
7	50	38.5	5.5
8	40	19.2	2.4
9	40	25.2	2.8
10	20	4	0.4

To derive the distribution of losses for the entire portfolio we proceed as follows:

Step 1: Probability generating function for each band.

Each band is viewed as a portfolio of exposures by itself. The probability generating function for any band, say band j, is by definition:

$$G_j(z) = \sum_{n=0}^{\infty} P(\text{loss} = nL) z^n = \sum_{n=0}^{\infty} P(n \text{ defaults}) z^{n v_j},$$

where the losses are expressed in the unit L of exposure.

Since we have assumed that the number of defaults follows a Poisson distribution (see expression (30)) then:

$$G_j(z) = \sum_{n=0}^{\infty} \frac{e^{-\mu_j} \mu_j^n}{n!} z^{n v_j} = \exp\{-\mu_j + \mu_j z^{v_j}\}. \tag{32}$$

To derive the distribution of losses for the entire portfolio we proceed as follows:

Step 2: Probability generating function for the entire portfolio.

Since we have assumed that each band is a portfolio of exposures, independent from the other bands, the probability generating function for the entire portfolio is just the product of the probability generating function for each band:

$$G(z) = \prod_{j=1}^{m} \exp\{-\mu_j + \mu_j z^{v_j}\} = \exp\left\{ -\sum_{j=1}^{m} \mu_j + \sum_{j=1}^{m} \mu_j z^{v_j} \right\}, \tag{33}$$

where $\mu = \sum_{j=1}^{m} \mu_j$ denotes the expected number of defaults for the entire portfolio.

Step 3: Loss distribution for the entire portfolio.

Given the probability generating function (33) it is straightforward to derive the loss distribution, since

$$P(\text{loss of } nL) = \frac{1}{n!} \frac{d^n G(z)}{dz^n} \Big|_{z=0} \quad \text{for } n = 1, 2, \ldots,$$

these probabilities can be expressed in closed form and depend only on 2 sets of parameters: ε_j and v_j (see Credit Suisse, 1997, p. 26).

4.2. Extensions of the basic modelg

CreditRisk+ proposes several extensions of the basic one period, one factor model. First, the model can be easily extended to a multi-period framework, and second, the variability of default rates can be assumed to result from a number of "background" factors, each representing a sector of activity. Each factor, k, is represented by a random variable, X_k, which is the number of defaults in sector k, and which is assumed to be Gamma distributed. The mean default rate for each obligor is then supposed to be a linear function of the background factors, X_k. These factors are further assumed to be independent.

In all cases CreditRisk+ derives a closed form solution for the loss distribution of a bond/loan portfolio which makes this approach very attractive from a computational standpoint.

4.3. Advantages and limits of CreditRisk+

CreditRisk+ presents the advantage of being relatively easy to implement. First, closed form expressions are derived for the probability of portfolio bond/loan losses, which make CreditRisk+ computationally attractive. In addition, marginal risk contributions by obligor can be easily computed. Second, CreditRisk+ focuses only on default, requiring relatively few inputs to estimates. For each instrument only the probability of default and the exposure are required.

The same limitations as for CreditMetrics and KMV apply, i.e., the methodology assumes no market risk. In addition, CreditRisk+ ignores migration risk so that the exposure for each obligor is fixed and does not depend on eventual changes in the credit quality of the issuer, as well as the variability of future interest rates. Even in its most general form where the probability of default depends upon several stochastic background factors, exposures are still constant and not related to changes in these factors.

Finally, like CreditMetrics and KMV, CreditRisk+ does not deal with nonlinear products such as, e.g., options and foreign currency swaps.

5. CreditPortfolioView [33]

CreditPortfolioView is a multi-factor model which is used to simulate the joint conditional distribution of default and migration probabilities for various rating groups in different industries, for each country, conditional on the value of macroeconomic factors like the unemployment rate, the rate of growth in GDP, the level of long-term interest rates, foreign exchange rates, government expenditures and the aggregate savings rate. [34]

CreditPortfolioView is based on the casual observation that default probabilities, as well as migration probabilities, are linked to the economy (see Fig. 1). When the economy worsens both downgrades as well as defaults increase. It is the contrary when the economy becomes stronger. In other words,

[33] CreditPortfolioView is a risk measurement model developed by Wilson (1987, 1997) and proposed by McKinsey.

[34] This model applies best to speculative grade obligors for which default probabilities are more sensitive to the credit cycle than investment grade obligors.

credit cycles follow business cycles closely. Since the state of the economy is, to a large extent, driven by macroeconomic factors, CreditPortfolioView proposes a methodology to link those macroeconomic factors to the default and migration probabilities.

Provided that data is available this methodology can be applied in each country, to different sectors and various classes of obligors which react differently over the business cycle like construction, financial institutions, agriculture, services, etc.

5.1. Default prediction model

Default probabilities are modeled as a logit function where the independent variable is a country speculative grade specific index which depends upon current and lagged macroeconomic variables:

$$P_{j,t} = \frac{1}{1 + \mathrm{e}^{-Y_{j,t}}}, \tag{34}$$

where $P_{j,t}$ is the conditional probability of default in period t, for speculative grade obligors in country/industry j, $Y_{j,t}$ is the index value derived from a multi-factor model described below.

Note that the logit transformation ensures that the probability (34) takes a value between 0 and 1.

The macroeconomics index, which captures the state of the economy in each country, is determined by the following multi-factor model:

$$Y_{j,t} = \beta_{j,0} + \beta_{j,1}X_{j,1,t} + \beta_{j,2}X_{j,2,t} + \cdots + \beta_{j,m}X_{j,m,t} + v_{j,t}, \tag{35}$$

where $Y_{j,t}$ is the index value in period t for the jth country/industry/speculative grade, $\beta_j = (\beta_{j,0}, \beta_{j,1}, \beta_{j,2}, \ldots, \beta_{j,m})$ are coefficients to be estimated for the jth country/industry/speculative grade, $X_{j,t} = (X_{j,1,t}, X_{j,2,t}, \ldots, X_{j,m,t})$ are period t values of the macroeconomics variables for the jth country/industry, $v_{j,t}$ is the error term assumed to be independent of $X_{j,t}$ and identically normally distributed, i.e.

$$v_{j,t} \sim \mathrm{N}(0, \sigma_j), \text{ and } v_t \sim \mathrm{N}(0, \Sigma_v),$$

where v_t denotes the vector of stacked index innovations $v_{j,t}$, and Σ_v is the $j \times j$ covariance matrix of the index innovations.

The macroeconomics variables are specified for each country. When sufficient data is available the model can be calibrated at the country/industry level. Both the probability of default $P_{j,t}$, and the index, $Y_{j,t}$, are then defined at the country/industry level, and the coefficient β_j are calibrated accordingly.

In the proposed implementation, each macroeconomics variable is assumed to follow a univariate, auto-regressive model of order 2 (AR2):

$$X_{j,i,t} = \gamma_{j,i,0} + \gamma_{j,i,1}X_{j,i,t-1} + \gamma_{j,i,2}X_{j,i,t-2} + e_{j,i,t}, \tag{36}$$

where $X_{j,i,t-1}, X_{j,i,t-2}$ denote the lagged values of the macroeconomic variable $X_{j,i,t}$, $\gamma_j = (\gamma_{j,i,0}\gamma_{j,i,1}\gamma_{j,i,2})$ are coefficients to be estimated, $e_{j,i,t}$ is the error term assumed to be independent and identically distributed, i.e.

$$e_{j,i,t} \sim \mathrm{N}(0, \sigma_{e_{j,i,t}}) \text{ and } e_t \sim \mathrm{N}(0, \Sigma_e),$$

where e_t denotes the vector of stacked error terms $e_{j,i,t}$ of the $j \times i$ AR(2) equations Σ_e is the $(j \times i)$ $(j \times i)$ covariance matrix of the error terms e_t.

To calibrate the default probability model defined by (34)–(36), one has to solve the system

$$\begin{aligned} P_{j,t} &= \frac{1}{1 + \mathrm{e}^{-Yj,t}}, \\ Y_{j,t} &= \beta_{j,0} + \beta_{j,1}X_{j,1,t} + \cdots + \beta_{j,m}X_{j,m,t} + v_{j,t}, \\ X_{j,i,t} &= \gamma_{j,i,0} + \gamma_{j,i,1}X_{j,i,t-1} + \gamma_{j,i,2}X_{j,i,t-2} + e_{j,i,t}, \end{aligned} \tag{37}$$

where the vector of innovations E_t is

$$E_t = \begin{bmatrix} v_t \\ e_t \end{bmatrix} \sim \mathrm{N}(0, \Sigma)$$

with

$$\Sigma = \begin{bmatrix} \Sigma_v & \Sigma_{v,e} \\ \Sigma_{e,v} & \Sigma_e \end{bmatrix},$$

where $\Sigma_{v,e}$ and $\Sigma_{e,v}$ denote the cross correlation matrices.

Once the system (37) has been calibrated, then one can use the Cholesky decomposition of Σ, i.e., [35]

$$\Sigma = AA' \tag{38}$$

to simulate the distribution of speculative default probabilities. First, draw a vector of random variables $Z_t \sim \mathrm{N}(0, I)$ where each component is normally distributed $\mathrm{N}(0, 1)$.

Then, calculate

$$E_t = A'Z_t$$

which is the stacked vector of error terms $v_{j,t}$ and $e_{j,i,t}$. Using these realizations of the error terms one can derive the corresponding values for $Y_{j,t}$ and $P_{j,t}$.

5.2. Conditional transition matrix

The starting point is the unconditional Markov transition matrix based on Moody's or Standard & Poor's historical data, which we denote by ϕM.

[35] See footnote 12.

Transition probabilities are unconditional in the sense that they are historical averages based on more than 20 years of data covering several business cycles, across many different industries.

As we discussed earlier, default probabilities for non-investment grade obligors is higher than average during a period of recession. Also downgrade migrations increase, while upward migrations decrease. It is the opposite during a period of economic expansion:

$$\begin{aligned} &\frac{\mathrm{SDP}_t}{\phi\mathrm{SDP}} > 1 \quad \text{in economic recession,} \\ &\frac{\mathrm{SDP}_t}{\phi\mathrm{SDP}} < 1 \quad \text{in economic expansion,} \end{aligned} \tag{39}$$

where SDP_t is the simulated default probability for a speculative grade obligor, $\phi\mathrm{SDP}_t$ the unconditional (historical average) probability of default for a speculative grade obligor.

CreditPortfolioView proposes to use these ratios (39) to adjust the migration probabilities in ϕM in order to produce a transition matrix, M, conditional on the state of the economy:

$$M_t = M(P_{j,t}/\phi\mathrm{SDP}),$$

where the adjustment consists of shifting the probability mass into downgraded and defaulted states when the ratio $P_{j,t}/\phi\mathrm{SDP}$ is greater than 1, and vice versa if the ratio is less than 1. Since one can simulate $P_{j,t}$ over any time horizon $t = 1, \ldots, T$, this approach can generate multi-period transition matrices:

$$M_T = \prod_{t=1,\ldots,T} M\big(P_{j,t}/\phi\,\mathrm{SDP}\big). \tag{40}$$

One can simulate many times the transition matrix (40) to generate the distribution of the cumulative conditional default probability for any rating, over any time period.

The same Monte Carlo methodology can be used to produce the conditional cumulative distributions of migration probabilities over any time horizon.

5.3. Conclusion

KMV and CreditPortfolioView base their approach on the same empirical observation that default and migration probabilities vary over time. KMV adopts a microeconomic approach which relates the probability of default of any obligor, to the market value of its assets. CreditPortfolioView proposes a methodology which links macroeconomics factors to default and migration probabilities. The calibration of this model necessitates reliable default data for each country, and possibly for each industry sector within each country. Another limitation of the model is the ad-hoc procedure to adjust the migration matrix. It is not clear that the proposed methodology performs better than a simple Bayesian model where the revision of the transition probabilities would be based on the internal expertise accumulated by the credit department of the bank, and the internal appreciation of where we are in the credit cycle given the quality of the bank's credit portfolio.

These two approaches are somewhat related since the market value of the firms' assets depends on the shape of the economy. It would then be interesting to compare the transition matrices produced by both models.

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