A Continuous Time Compound Credit Rating Migration Model for Bond and Loan Valuations

Abstract

In financial industry, including banking, investment and insurance, a need for solid methods to valuate contingent future payments is becoming stronger and stronger. In many of the methods developed, stochastic process is employed to build credit rating migration models in order to fully considerate possible situations of migration among different credit quality states. To do valuation at any time, this paper extends the discrete time transition matrix to a continuous time transition intensity model, and adds in a payment ratio variable, turning the valuation model into a compound model. The goal of this paper is to provide a convenient approach (and software) for the frequent updating of the valuation of future contingent cash flows (for example, bonds and loan payments) using easily accessible data. Thus any change of risk can be reflected directly and quickly in the pricing of financial assets so that risk management can be more flexible and timely.

1. Introduction

1.1 Background

Credit risk has become perhaps the key risk management challenge of the late 1990s. Globally, institutions are taking on an increasing amount of credit risk. As credit exposures have multiplied, the need for more sophisticated risk management techniques on credit risk has also increased. More specifically, a need for solid methods to valuate contingent future payments with credit risks is becoming stronger and stronger. This is most obvious for the valuation of bonds and bank facilities such as loans: risk comes not only from default but also from deterioration of value due to the decrease of credit rating.

Investment banks have been making evaluations of credit qualities for a long time. Systematic credit rating criteria have been developed based on the assumption that financial instruments that have the same risk should have the same credit rating. To valuate credit risks, a natural approach would be to consider all the possible credit qualities that a financial instrument might have at each specific time in the future. Morgan developed transition matrices for this purpose as early as 1987. Since then, a broad literature of work has been built upon which applies migration analysis to credit risk evaluation. The first publication of transition matrices was in 1991 by both Professor Edward Altman of New York University and separately by Lucas & Lonski of Moody's Investors Service. They have since been published regularly (see Moody's Carty & Lieberman and Standard & Poor's *Creditweek*) and are also calculated by firms such as KMV.

Many practical events (e.g., calls, enforced collateral provisions, spread resets) can be triggered by a credit rating change. These events can directly affect the realized value within each credit quality state. For instance, a pricing grid **-** which predetermines a credit spread schedule given changes in credit quality state **-** can reduce the volatility of asset value across up (down) grades. Thus, we find it very convenient to explicitly incorporate awareness of rating migrations into our risk models.

1.2 A Comparison of Discrete Time and Continuous Time Stochastic Models

There are many studies about stochastic models of defaults either in single instrument or portfolio context, such as CreditMetrics, CreditRisk+, and CreditPortfolioView, etc. We call them as the credit rating migration models (CRMM). One common thing of many of these models is that they build up their models in a single period and on a discrete time basis. And their differences lie in subtle choices of the driving distributions and in the data sources one would naturally use to feed the models. CreditMetrics is a typical discrete time stochastic valuation model.

Unlike market risks where daily liquid price observations allow a direct calculation of value-at-risk (VaR), CreditMetrics seeks to construct what it cannot directly observe: the volatility of value due to credit quality changes. It seeks to balance the best of all sources of information in a model which examines broad historical data rather than only recent market moves and across the full range of credit quality migration - upgrades and downgrades - rather than just default. As we will emphasize later, the main purpose of applying credit rating migration model to bond and loan valuation is to comprehensively valuate credit risks, so that timely adjustment can be made to the pricing of these assets to reflect their risk status, and thus take these risks under better control. The market prices of financial instruments already reflect investors' expectations of credit risks. If the market is efficient, then the market prices should be able to reflect the premiums for credit risks. Using stochastic model is just a quantitative way of valuating credit risks referring to historical data. With enough information, CreditMetrics can provide valuation of bonds and loans at each anniversary of issue. In this paper, we seek a way to do valuation at any time between issue date and maturity date. To do that, a continuous time stochastic model is necessary.

A fundamental technique that CreditMetrics uses is migration analysis, that is, the study of changes in the credit quality of names through time. Since CreditMetrics uses a discrete time model, its transition matrix gives the probability of migrations among states each period. However, if we want to valuate the cash flows at any time, we would need to know the probability of credit quality of a certain state at any time t. So, we employ a different transition matrix, called the force of migration matrix, to calculate these probabilities.

Besides CreditMetrics, Christopher C. Finger surveyed a number of extensions of the standard single-period models that allow for a treatment of default timing over longer horizons. Five models were compared in his paper, these including Discrete CreditMetrics extension, Diffusion CreditMetrics extension, the copula functions, the stochastic intensity model-----slow mean reversion, and the stochastic intensity model-----fast mean reversion. We list some important results below. For details of these comparisons, see Finger [2000].

The first three approaches can all be thought of as extensions of the single period CreditMetrics framework. Each approach relies on standard normal random variables to drive defaults and calibrates thresholds for these variables. Moreover, the first three approaches are identical over the first period. They only differ in their behavior over multiple periods. The fourth and the fifth model take a different approach to the construction of correlated defaults over time and can be thought of as an extension of the single period CreditRisk+ framework.

The differences stated above can be attributed to the model structures, and to some extent, to the choice of distributions that drive the models. In a single period case, a number of studies have concluded that when calibrate to the same first and second order information, the various models do not produce vastly different conclusions. Among these models, the copula method requires significantly fewer calculations and produces results in the middle of the range observed, it is the most practical choice of the models tested for simulating defaults over multiple horizons.

The conclusions of Finger [2000] helped us to understand better how a continuous time CRMM can be built. We can use data of a single period model and extend our model to multiple periods on a continuous time basis. The time-homogeneous continuous time Markov chain is the model we use in this paper.

2 Valuation Models

2.1 Bond and Loan Models With Default

We start building our valuation models by first looking at the simplest situation with “default” the only type of credit risk. That is, the only situation that complete payment can not made is when the obligor defaults. (However, this is not true in real life. That’s why we introduce the payment ratio distribution later.) Consider the following example: Suppose the initial credit rating of the obligor is i. The amount of loan is $1,000, maturity 3 years, annual interest rate 5%, with annually interest payments. (Same setting as an annual coupon bond.)

If we don't consider the possibility of credit rating migration, the only risk factor left would be default. In order to calculate the actuarial present value of the future cash flows, we need to know the survival model for the time until default variable, T. So, to do the valuation, we need some assumptions:

Assumption 1:

Assume the time until default is exponentially distributed, with constant force of default μ.

Assumption 2:

If default happens, there will be no principal repayment, and no further interest payments would be made in the future. (That is, the recovery rate is always 0 at default and full amount of payment (interest or principal) will be made if not default at time of payment.)

View loan cash flows as a "life" annuity and employ actuarial notations:

$$ {}_{n}P_{0} = \Pr(T > n) = \Pr\ (\textit{default occurs after n years}) \tag{2.1.1}$$

Using the current payment method, we can calculate APV of the random present value of the loan cash flows, Y as:

$$ Price = APV(Y) = \sum_{n=1}^{3} 50 \cdot (1+i)^{-n} \cdot {}_{n}P_{0} + 1000 \cdot (1+i)^{-3} \cdot {}_{3}P_{0} \tag{2.1.2}$$

To avoid confusion about the interest rate i used here and state i used in stochastic models, we use force of interest in this paper. Assume constant force of default μ and force of interest δ, (2.1) can be written as:

$$ Price = APV(Y) = \sum_{n=1}^{3} 50 \cdot e^{-n(\mu+\delta)} + 1000 \cdot e^{-3(\mu+\delta)} \tag{2.1.3}$$

If there is no rating migration, (2.2) can be generalized to evaluate cash flows of any time length. However, since credit rating can change, and a lower credit rate usually means worse financial situation and greater likeliness of default, the probability of default for a certain obligor will not be exponentially distributed with constant force of default. Still, based on systematic credit rating methods used by credit rating companies, it is reasonable to assume similar financial situations among obligors with same credit rating, thus similar distributions of default. Then comes the necessity to evaluate future cash flows using CRMM (Credit Rating Migration Model). CreditMetrics is a typical discrete time CRMM. Next we introduce its valuation method.

2.2 CreditMetrics' Valuation Method

Suppose we have a mortgage. The mortgage's value is decided by the actuarial present value of related future cash flows. Thus, credit risk arises because the future cash flows can vary depending on the credit quality of the obligor. For example: an obligor may ask for prepayment, partial payment, deferral of payment or even default. The possibilities of such activities are related to the obligor's credit rate. In general, we know that the value of this mortgage will decline with a downgrade or default of the obligor - and appreciate if the credit quality of the obligor improves. Value changes will be relatively small with minor up (down) grades, but could be substantial — 50 % to 90% are common — if there is a default.

To illustrate, consider a five-year AAA bond. Say the face value of this bond is $1000 and the coupon rate is 10%, maturity 5 years. A key point for CreditMetrics' method is: CreditMetrics use the yield rates derived from the forward zero curve for each specific rating category to discount cash flows. This forward curve is calculated as of the end of the risk horizon.

For example, we want to find the value of the bond at year-end if it upgrades to single-A. There will be a \$100 coupon at the end of the year, still 4 more coupons (also \$100) in the following years and \$1000 principal at maturity. To obtain the value of the bond assuming a downgrade to double-A, simply discount these five cash flows (four coupons and one principal) with interest rates derived from the forward zero double-A curve. Similar calculations can be done for each possible credit rating at the end of the year. In case of default, a recovery rate distribution is taken into consideration. After these calculations, the final actuarial present value of the bond would be the average of all these present values weighted by the probabilities of correspondent credit rating migrations. These probabilities are derived from the one-year transition matrix.

In CreditMetrics, different discount rates are used according to different yield curves of those credit quality states. However, if we take into consideration the probabilities of default, partial payments arrangements and recovery rates, a risk-free yield curve should be used for discount rates. The reason different credit states have different shapes of the yield curves is that different levels of risks (including default, partial payments, etc) have been taken into consideration. If the market is efficient, these risk factors should have already been sufficiently reflected in the yield curve. So, if we use these discount rates with risk factors in our CRMM model, there will "double discount" effect on credit risk. For the same reason, it is worth asking whether it is necessary for CreditMetrics to take into consideration of recovery rates in valuation.

2.3 Continuous Time Markov Chains

Using discrete time model, we can define:

P_{ij} = Probability that a process presently in state *i* will next be in state *j*.

Thus, we can find out the probability a process will be in state *j* after n steps (or at the end of n periods) by:

$$P_{ij}(n) = (\mathbf{P}^n)_{ij} \tag{2.3.1}$$

However, this model does not tell us information about the process in those time periods. We don't know the exact time that default occurs, same for time of other credit rating migration events. If we want to evaluate the future cash flow of a loan at any time during the loaning period, a continuous time Markov model will be indispensable.

Suppose F is a finite set which can be written as F = {1, 2, …, n}. Think of a random walk on the set of F. Let $X(t)$, t = 0, 1, 2 … be the position of the walk at time t. If we are at the point *i* at time *t*,

$i \in F$, the probability of being at state *j* at time *u* is:

$$P_{ij}\left(t,u;F_t\right)=\Pr\left[X\left(u\right)=j\middle|X\left(t\right)=i,F_t\right],\ 0\le t\le u \text{ , and } i,\ j\in F. \qquad (2.3.2)$$

Same as in Jones [1997], "F_t represents the history of the process X up to time t and any other relevant information that is known at time t." Under the time-homogeneous Markov chain, the future of X is independent of all information contained in F_t except the state at time t. Then, can define the transition probability function as follows:

$$P_{ij}(t)=P\left\{X(t+s)=j\middle|X(s)=i\right\} \qquad (2.3.3)$$

Which is the probability that a process presently in state *i* will be in state *j* a time t later.

Also, we can define the state transition rate v_i and instantaneous transition rate q_{ij} (See Ross [1997]). For any pair of states *i* and *j*, let

$$r_{ij}=q_{ij} \text{ for } i\neq j \text{ and } r_{ii}=v_i$$

Then the r_{ij} form a new matrix **R**, which we call the force of transition matrix. Using this notation, we can rewrite the Kolmogorov backward equations

$$P_{ij}^{'}(t)=\sum_{k\neq i}q_{ik}P_{kj}\left(t\right)-v_iP_{ij}(t)$$

as:

$$P_{ij}^{'}(t)=\sum_{k\neq i}r_{ik}P_{kj}\left(t\right)$$

$$\text{with boundary condition } P_{ij}(0)=0,\ \ i\neq j \text{ and } P_{ii}(0)=1 \qquad (2.3.4)$$

In matrix form,

$$\mathbf{P}^{'}(t)=\mathbf{RP}(t) \qquad (2.3.5)$$

Solving the matrix differential equation to get:

$$\mathbf{P}(t)=\mathbf{P}(0)e^{\mathbf{R}t} \qquad (2.3.6)$$

There are several ways to calculate $\mathbf{P}(t)$ as discussed in Jones [1997] and Ross [1997], for example, diagonolization method introduced by Cox and Miller [1965] and a numerical approximation method introduced in Ross [1997], using the following approximation formula:

$$e^{\mathbf{R}t} = \lim_{n\to\infty}(\mathbf{I} + \mathbf{R}\frac{t}{n})^n$$

Also, with mathematical software $\mathbf{P}(t)$ can be easily found. But the final matrix would be quite "messy" because of the symbolic computations involved. In this paper, we first diagonolize **R** using Matlab and then built up a valuation spreadsheet with MS Excel that has a much more user-friendly interface.

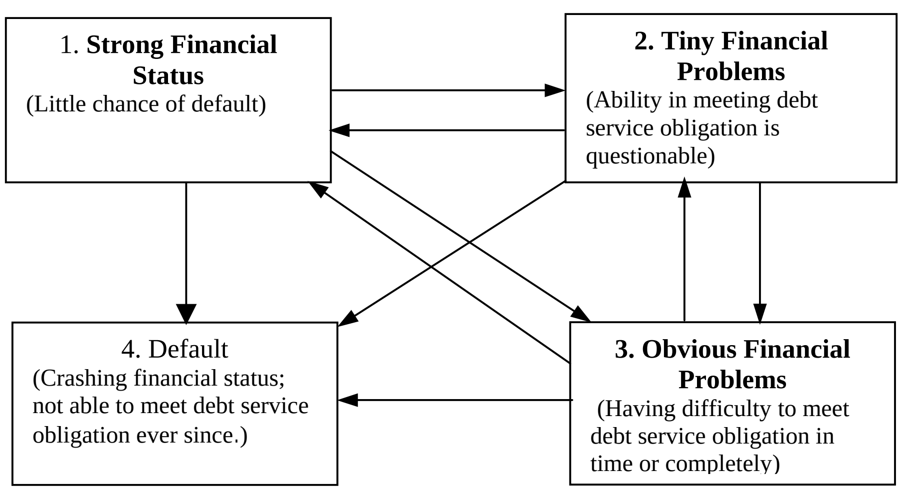
3. Homogeneous Compound Continuous Time CRMM

3.1 The Basic CRMM

We begin with a simple stochastic process for the CRMM. For the purpose of simplicity, first there are only four different states, corresponding to the four credit levels. Possible migrations among these four states are described in Figure 1. In practice, there will be more credit quality states. For example, we use S&P's credit rating method, in which there are 8 states, including AAA, AA, A, BBB, BB, B, CCC and Default. For notational convenience we mark these ratings under the order of number 1 to 8. Then, a more complicated graph will be needed to show the possible migrations among these states.

Figure 2.1

Rating Migration Diagram For Individual Obligor



To evaluate the APV of Y using the current payment method shown in **2.1**, we need to find out ${}_nP_0$ = Pr (loan "survives" at the end of n years). In the continuous time CRMM, we need to update the assumptions in **2.1**:

Assumption 1':

For a certain credit state *i*, assume the time until default is exponentially distributed, with constant force of default μ_i.

Assumption 2':

Credit rating migrations are only possible among the four credit states.

Before introducing the compound model, we still keep Assumption 3:

Assumption 3:

If default happens, there will be no principal repayment, and no further interest payments would be made in the future. (That is, the recovery rate is always 0 at default and full amount of payment (interest or principal) will be made if not default at time of payment.)

Later we will change this assumption into a more realistic one in 3.2, where a compound distribution is considered.

${}_nP_0$ can be easily calculated as $1-P_{i4}(n)$, since state 4 is default. Suppose we want to calculate the APV of the future cash flow of a 3-year loan with initial credit rating 1. Based on Assumption 4, (2.2) can be rewritten as:

$$APV(Y_0)=\sum_{n=1}^{3}50\cdot e^{-n\delta}\cdot[1-P_{14}(n)]+1000\cdot e^{-3\delta}\cdot[1-P_{14}(3)] \qquad (3.1.1)$$

Here we can see more clearly the need for continuous-time Markov model. If we want to know the APV of future cash flows of the loan at some time t during the payment period:

$$APV(Y_t)=\sum_{n=\lfloor t\rfloor+1}^{3}50\cdot e^{-(n-t)\delta}\cdot[1-P_{14}(n-t)]+1000\cdot e^{-(3-t)\delta}\cdot[1-P_{14}(3-t)] \qquad (3.1.2)$$

The notation $\lfloor t\rfloor$ means the largest integer that is smaller than *t*.

3.2 Compound Distribution

As mentioned earlier, a more realistic approach to deal with payments of obligors with different credit rating levels would be to use probability distributions. This is because that, in practice, creditors and obligors would usually negotiate to make special arrangements if the obligor has difficulty in making payments on time. For example, there might be special clauses in the loan contract, permitting the obligor to pay part of the amount due and defer the payment of the rest amount when financial problems appear.

We choose the ratio of the actual payment to the whole amount due as the random variable. The ratio is denoted by $\theta(\mathrm{i})$, $0 \le \theta(\mathrm{i}) \le 1$, where *i* refers to the credit quality state of the obligor when payment is made. Apparently, different distributions are needed for different credit rating levels. (In 3.3, we choose to use Beta distribution for $\theta(\mathrm{i})$ because Beta distributions are flexible as to their shape and can be fully specified by stating the desired mean and standard deviation.) Then the whole cash flow model becomes a compound distribution:

a. A stochastic process $X(t)$ with state space *F* is used to describe rating migrations.
b. Once the credit rating is determined, random variable $\theta(\mathrm{i})$ with pdf $f_i(\theta)$ will be the ratio of payment made at time *t*.

Then, the ratio of payment made at any time t is determined by $\theta(X(t))$. Let $C(t)$ be payment due at time *t*, then the actual payment at time *t* will be $C(t) \cdot \theta(X(t))$.

The present value Y of future cash flow will be:

$$Y_0 = \int_0^{\infty} \theta(X(t)) \cdot C(t) \cdot e^{-\delta t} dt \qquad (3.2.1)$$

The actuarial present value of Y can be calculated as:

$$APV(Y_0) = \int_0^{\infty} e^{-\delta t} dt \sum_{j \in F} [P_{ij}(t) \cdot \int_0^1 f_j(\theta) \cdot C(t) d\theta] \qquad (3.2.2)$$

Because Markov chain starts again at any time of the process, (3.2.2) can also be used to calculate APV at any time.

3.3 Numerical Illustrations

To get data on credit quality state migrations, the easiest way is to change the one-year transition matrix published by major investment banks regularly into the force of transition matrix. In this paper, we choose to use the One-year transition matrix **P** by Standard & Poor's (See Table 3.1). Since this is the discrete time transition matrix, the first thing we need to do is to convert it into force of transition matrix **R**. Below are the three steps for the conversion:

Table 3.1

	One-year transition matrix (%)							
	AAA	**AA**	**A**	**BBB**	**BB**	**B**	**CCC**	**D**
AAA	90.81	8.33	0.68	0.06	0.12	0	0	0
AA	0.7	90.65	7.79	0.64	0.06	0.14	0.02	0
A	0.09	2.27	91.05	5.52	0.74	0.26	0.01	0.06
BBB	0.02	0.33	5.95	86.93	5.3	1.17	0.12	0.18
BB	0.03	0.14	0.67	7.73	80.53	8.84	1	1.06
B	0	0.11	0.24	0.43	6.48	83.46	4.07	5.2
CCC	0.22	0	0.22	1.3	2.38	11.24	64.86	19.79
D	0	0	0	0	0	0	0	100

Source: Standard & Poor's Credit Week (15 April 96)

First, find the probability that the next state will be j given that current state is *i*. Let:

$$p_{ij} = \Pr(X(t+1) = j \mid X(t) = i)$$

= Pr (the process will go to state j at t+1given current state is *i* at time t)

$m_{ij} =$ Pr (the process will next go to state j given current state is *i*)

In discrete time Markov chain,

$$m_{ij} = p_{ij} + p_{ii} \cdot p_{ij} + p_{ii}^2 \cdot p_{ij} + \cdots = \frac{p_{ij}}{1 - p_{ii}} \qquad (3.3.1)$$

Thus we can calculate the matrix **M** (see Table 3.2) with the diagonal elements equal to 0 (because $m_{ii} = 0$).

Table 3.2

	M --- Probability Process Will Next Go to State j							
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**
1	0.0000	0.9064	0.0740	0.0065	0.0131	0.0000	0.0000	0.0000
2	0.0749	0.0000	0.8332	0.0684	0.0064	0.0150	0.0021	0.0000
3	0.0101	0.2536	0.0000	0.6168	0.0827	0.0291	0.0011	0.0067
4	0.0015	0.0252	0.4552	0.0000	0.4055	0.0895	0.0092	0.0138
5	0.0015	0.0072	0.0344	0.3970	0.0000	0.4540	0.0514	0.0544
6	0.0000	0.0067	0.0145	0.0260	0.3918	0.0000	0.2461	0.3144
7	0.0063	0.0000	0.0063	0.0370	0.0677	0.3199	0.0000	0.5632
8	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000

In continuous time Markov chain,

$$m_{ij} = \frac{q_{ij}}{v_i}, \text{ where } v_i = \sum_{j \neq i} q_{ij} \tag{3.3.2}$$

Second, in continuous time Markov chain, $e^{-v_i(t-s)}$ is the probability that the process will remain in state i during time $[t, s]$ given the state is i at time t. So,

$$e^{-v_i} = p_{ii}$$

$$v_i = -\log(p_{ii}) \tag{3.3.3}$$

Third, through (3.3.2),

$$q_{ij} = m_{ij} \cdot v_i \tag{3.3.4}$$

Then we can find R by $r_{ij} = q_{ij}$ for $i \neq j$ and $r_{ii} = v_i$ (see Table 3.3).

Table 3.3

	R --- Force of Transition Matrix							
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**
1	-0.0964	0.0874	0.0071	0.0006	0.0013	0.0000	0.0000	0.0000
2	0.0073	-0.0982	0.0818	0.0067	0.0006	0.0015	0.0002	0.0000
3	0.0009	0.0238	-0.0938	0.0578	0.0078	0.0027	0.0001	0.0006
4	0.0002	0.0035	0.0638	-0.1401	0.0568	0.0125	0.0013	0.0019
5	0.0003	0.0016	0.0075	0.0860	-0.2165	0.0983	0.0111	0.0118
6	0.0000	0.0012	0.0026	0.0047	0.0708	-0.1808	0.0445	0.0568
7	0.0027	0.0000	0.0027	0.0160	0.0293	0.1385	-0.4329	0.2438
8	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Although it is quite convenient to calculate $e^{\mathbf{R}t}$ in mathematical software like Matlab, the final expression is quite messy. Considering the need for easy access, we choose to use an Excel spreadsheet to do the calculations automatically.

Next, consider the distributions of $\theta(i)$.

Payment ratios are hard to predict, especially for obligors of lower credit qualities. "When a banker makes a loan or an investor buys a bond, it is in the belief that the obligor will go bankrupt but that the instrument will outperform."---CreditMetrics (P77). So, it is not easy to predict what arrangements will be in case the obligor has trouble making full payments. For high credit qualities, the most likely situation is theta =1, meaning full payment of interest or/and principal. It is also possible for minor and temporary financial problems to occur, in which case the banker may allow the obligor to make partial payments, but theta should be close to 1. However, as credit qualities goes down, more serious problems may occur, thus lower expected value and larger volatility for $\theta(i)$. Finally, if default occurs, "will it be a catastrophe of entirely default which leaves no value to recover, or will it be a regrettable but well behaved wrapping up which affects only shareholders but leave debt holders whole?"

It is in the remote chance of an outright default that a credit instrument will realize its greatest potential loss. According to CreditMetrics, recovery rates are affected by the following factors: (i) seniority ranking of debt; (ii) instrument type or use; (iii) credit rating X-years before default, and (iv) size and/or industry of the obligor. Many studies refine their estimates of the distributions of recovery rate according to seniority type among bonds. Asarnow & Edwards [95] shows the recovery rates have a mean of 65.21% and median 78.79%. CreditMetrics estimated a standard deviation 32.70% for recovery rates.

We adopt distributions of $\theta(i)$ similar to the recovery rate distribution in CreditMetrics. Wide uncertainties and the general shape of the payment ratio distribution can be shown -- while staying within the bounds of 0% to 100% -- by utilizing a beta distribution. Beta distributions are flexible as to their shape and can be fully specified by staying the desired mean and standard deviation. With the help of simulation software **@Risk**, different shapes of beta distributions can be easily captured. We used a distribution generating function in **@Risk**, specifying the minimum, maximum, mode and mean of the beta distributions. The following charts are the beta distributions with parameters used in simulation shown in Table 3.4:

Figure 3.1 shows that $\theta(1)$ is between 0.91161 and 0.9984 90% of the time. The standard deviation is quite small. Figure 3.2 shows the beta distribution at default. Apparently it has a much larger standard deviation, which is just what we have assumed.

Table 3.4

Mode and Mean Used in Simulation for Beta Distributions

	Mode	Mean
1	0.9999	0.97
2	0.98	0.9521
3	0.975	0.921
4	0.97	0.9
5	0.96	0.8521
6	0.95	0.8
7	0.94	0.7521
8	0.92	0.6521

Figure 3.1

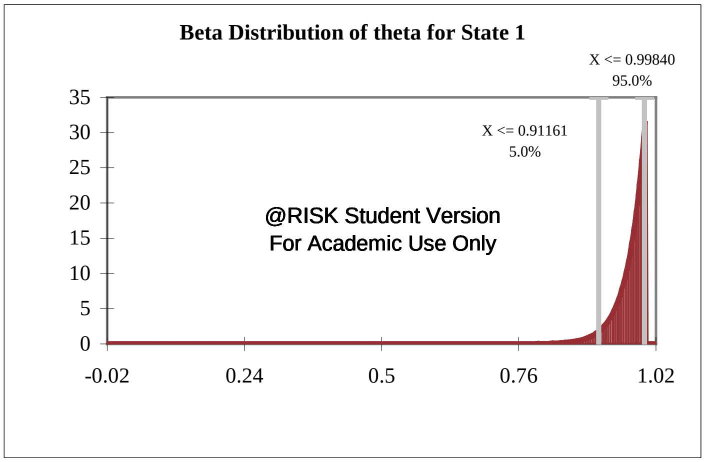
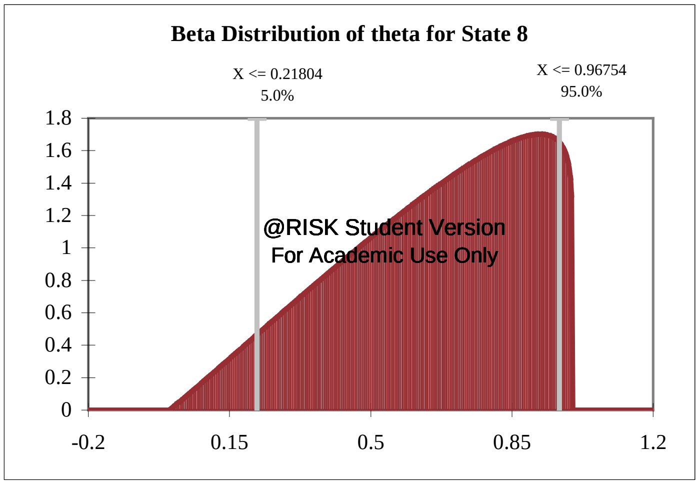


Figure 3.2



3.4 Simulation

We made a simulation of 5-year AAA annual coupon bond same as in 2.1 (face value $1000, coupon rate 10%) with spreadsheet and **@Risk**. The simulation reports generated by **@Risk** are in the appendix. To find the actuarial present value of the bond at issue, we have run five simulations of the present value of $100 for five different payment times. Then, the APV of the bond is simply the sum of the APV of the five payments. Table 3.6 shows the main results of the simulations:

Table 3.6

Simulation Results of the Present Value of $100 at Different Payment Time

Payment Time	Minimum	Maximum	Mean	Std Dev	5%	95%
1	70.4860	94.8768	**90.3675**	3.1356	84.1452	94.0194
2	68.0425	89.8103	**85.4593**	2.7912	79.9327	88.8764
3	65.9199	85.5425	**81.2596**	2.5392	76.2707	84.4790
4	63.5237	81.2824	**77.0646**	2.3513	72.6926	80.1315
5	61.0435	77.1616	**73.0323**	2.2069	68.8685	75.9733

Table 3.7 shows a comparison of valuations of the bond with and without credit risks. Although the results shown here are calculated on integer times (where a discrete time CRMM will also work), our spreadsheet can calculate the APV of future contingent cash flows at any time.

Table 3.7

Time	C(t)	APV of C(t) (with credit risk)	PV of C(t) (no credit risk)
1	100	90.3675	95.1229
2	100	85.4593	90.4837
3	100	81.2596	86.0708
4	100	77.0646	81.8731
5	1100	803.3552	856.6809
Total	**1500**	**1137.506**	**1210.231**

4. Conclusions and Areas for Further Research

In this paper, we built a compound credit rating migration model on the continuous time basis to calculate the actuarial present value of future contingent cash flows. This model can be applied to evaluate credit risks of bonds, bank loans and many other bank facilities. We employed the continuous time Markov chain in order to extend valuation to a continuous time basis. Introducing the payment ratio distributions into the valuation model is also an important aspect of the differences of our model from CreditMetrics. While the yield curves in the market reflect investors' expectation of the credit risks, we are actually trying to quantify these estimations using available information. Finally, we build a spreadsheet[1] with very friendly interface using Excel, and calculated the APV of simple coupon bond through simulation with **@Risk**.

However, there are many things worthy of serious consideration and further research. First, (3.2.2) gives the formula of the APV of future contingent cash flows. We tried to develop formula to calculate the variance of future contingent cash flows, but the compound distribution really brings trouble. A possible approach might be to use the generalized version of Hattendorf's theorem presented by Ramlau-Hansen [1988] as is used in Jones [1997]. The next thing we will do is to find the variance through simulations and add this function into our spreadsheet.

Second, our model is still on the stage of single financial instrument. So, another important thing to do is to develop Continuous Time Compound CRMM for portfolios. In simulation the parameters of the Beta Distributions for payment ratios are chosen quite arbitrarily, not based on empirical study. With more information, much more accurate payment ratio distributions can be found. What's more, employing a non-homogeneous Markov model can make the model more realistic (probably also more complicated).

[1] A copy of the spreadsheet can be downloaded at: http://www.math.lsa.umich.edu/~chenhz/CRMM.html

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