A New Approach for Credit Risk

Hyungsok Ahn, **Varqa Khadem**, and **Paul Wilmott**
MFG, Oxford University

Abstract. Credit risk is one of the key issues facing the financial industry, spectacular losses have made newspaper headlines. The emphasis of this paper is to derive a sensible credit risk model for risky bonds that incorporates a realistic assessment of inherent risks and difficulties with hedging. As it is in the real world, the market we consider is incomplete. Every non-trivial portfolio is exposed to risk and its valuation should allow for a risk preference; we will show how this introduces non-linearity. This forms a basis for further issues of risk management, such as optimal portfolio selection and static hedging. This is a brief paper sets out as aompactly as possible this new paradigm. We discuss the modelling principles for this new paradigm, methods of calibration, and make comparisons with traditional models.

1 Introduction

A firm issues a corporate bond as a means of borrowing money and the issue is rated according to the estimated ability of the issuer to live up to its future contractual obligations. A bond of a low credit rating yields more than a bond of high credit rating, *ceteris paribus*, because the bond is perceived as being more likely to default. In fact, studies of the historical risk and return of the corporate bond market indicate that high-yield bonds have outperformed default-free bonds such as U.S. Treasuries. In other words the investment on high-yield bonds is lucrative *on average*. However, default risk is not wholly diversifiable as it propagates rapidly through the global economy. Besides, the holder will lose much or even all of his investment when default occurs. Thus studies of sensible investment policies based upon a correct recognition of the default risk have been an important issue to professional money managers who trade corporate bonds.

Ideas of the risk-neutral pricing have been adopted for credit risk, but have not been adapted successfully. Here, we propose a non-linear model for risky bonds that we believe to be more sensible than the existing models, incorporating a realistic assessment of inherent risks and difficulties with hedging.

2 Existing models

Most of the existing credit risk models are in one of the two categories: (i) the structural model; (ii) the reduced-form model. For models not in these categories, see Skora (1998). The structural model attempts to describe the default time through the issuer's assets and liabilities, while the reduced-form model simplifies the default mechanism by modeling the default time directly as an exogenous shock. Up-to-date models attempt to accommodate both principles, and yet these models are still under the umbrella of risk-neutral pricing.

2.1 The structural model

The origin of the structural model goes back to Black and Scholes (1973), which is followed by the works of Merton (1974), Geske (1977), Leland (1994), Longstaff and Schwartz (1994), and Anderson and Sundaresan (1996). In this approach, default is triggered whenever the firm value reaches a critical, prescribed, level. The firm value is a state variable that indicates the current financial status of the firm, for example the ratio of assets to liabilities, and is typically described by a diffusion process. Thus one can anticipate a firm's bankruptcy as its value drifts to the point of no return. A risk-free portfolio is essentially the one that is insensitive to the fluctuation of the firm value and is achieved by successive rebalancings of the firm's stock as in the conventional theory of option pricing.

Although it is encouraging that the model attempts to relate the likelihood of default to the financial status of the issuing firm, the model is vulnerable as the firm value itself fails to provide an appropriate measure for the risk that arises from the financial dependencies among firms in a dynamic modern economy. A meltdown of a large financial entity could easily cause a chain reaction, and the unexpected loss from the abrupt change is devastating. Unfortunately the model provides no clue to this at all. The Asian crisis is a case in point. Implementation of the model is another disappointment. It requires a thorough audit of the firm's account each time they measure the firm value while it is often the case that only a limited information is available to public market investors. A filtering process can be employed (see Duffie and Lando (1997) and Epstein, Mayor, Schönbucher, Whalley, and Wilmott (1997), for example), but the performance is poor unless the coefficients of the underlying diffusion are *a priori* known.

2.2 The reduced-form model

The reduced-form model requires less thought to implement. In fact, implementation boils down to estimating the hazard rate (i.e., the instantaneous rate of default) over the time period in question. Some of the key references are Artzner and Delbaen (1992), Jarrow, Lando, and Turnbull (1993), Duffie and Singleton (1994), Schönbucher (1996), and Blauer and Wilmott (1998). The model possesses another advantage as it efficiently accommodates prevailing term structure models of interest rates, adopting the complete market assumption generously; this is essentially a multi-factor model under market completeness. A trader can maintain a risk-free portfolio involving the trading of the bonds of the same issuer. A no-arbitrage price must exist for each bond issue and the model exploits the feedback from this postulated price to resolve the necessary hedge. To illustrate this, we give the following simple example with a constant interest rate r and a constant hazard rate p. Consider two zero coupon bonds of the same issuer but with different maturities τ_1 and $\tau_2 > \tau_1$. Par values are assumed one dollar. No-arbitrage price must be a function of the maturity only in this particular case, and hence we may write $v(\tau)$ as the price of the bond matures in τ. Suppose that an investor shorts a share of the first bond and holds $v(\tau_2 - \tau_1)^{-1}$ shares of the second. If default occurs before τ_1 his portfolio is null. If not, he can close his position on the second bond with unit price $v(\tau_2 - \tau_1)$ and pay back his liability. As a result the value of this portfolio is zero regardless of the default time and v must satisfy $v(\tau_1)v(\tau_2 - \tau_1) = v(\tau_2)$ for each $\tau_1 < \tau_2$, or equivalently[1]

$$\frac{\partial v}{\partial t} = (r + \tilde{p})v \tag{1}$$

for some $\tilde{p}$, the risk-neutral default rate. But what $\tilde{p}$? To price complex credit derivatives will require knowledge of $\tilde{p}$. Fashionable at the moment is parameter fitting or calibration. This amounts to choosing model parameters in such a way that model prices match with the prices of contracts quoted in the market. To facilitate this fitting it obviously helps to have enough parameters. A stochastic default rate is an attractive choice for this reason.

It is often neglected that the complete market assumption for a multi-factor model is entirely hypothetical unless model prices agree with market prices not only at the present

[1] The equivalence is first shown by Cauchy. Essentially this says a multiplication semigroup in the real line must be an exponential function.

time but also consistently in the future. In the previous example, the portfolio will not be risk-free if the market price of the second bond at τ_1 is realized differently from the postulated price $v(\tau_2 - \tau_1)$. A good calibration is challenging even for the Treasury yield curve and certainly the task will not be any easier for corporate bonds. In practice such models perform poorly. (See Dumas, Fleming and Whaley (1998) for results on the calibration of deterministic volatility models.)

A fatal error is that, according to the model, managing a risk-free portfolio requires continuous trading of assets, but this is controversial because of the transaction costs that arise naturally from the bid-ask spreads. This is indeed a serious matter as some high-yield bonds are not as liquid as investment-grade bonds. The illiquid nature of credit markets have two other negative consequences on attempts to adopt risk-neutral pricing methods: short selling of contracts is often very difficult because of the regulatory features of issuing credit, and in particular instances it can be very cumbersome to identify a suitable product with which credit risk can be hedged.

3 Modelling principles and their consequences

We wish to develop a credit risk model that is sensible and that delivers safety and practicality. To this end we adopt the following principles:

- A sensible model should avoid the naive premise of market completeness as this would only conceal the existing risk in a portfolio.
- The model should adopt the more practical reduced-form methodology, but adapted to capture economic dependencies.
- Every non-trivial portfolio possesses risk and hence its valuation must abide by an individual risk preference.
- Transaction costs should be taken into account.
- The model does not necessarily attempt to explain market prices, but complies with them by treating them as exogenous variables.

A risk preference is quantified by a measurement or a combination of measurements. For example, the expected utility, the recursive utility (see Kreps and Porteus (1981), for example), or a combination of the mean and the variance. A reasonable risk preference should be non-linear and capture the typical investor's risk aversion. Thus, for example, an investor will evaluate a \$1 bet on a coin toss differently from a \$1000 bet although the mean return is null for both bets. There are two fundamental consequences of a non-linear risk preference:

- There will generally be a range of prices for each contract that makes it appealing to either buy or sell.

- For a given market price of a contract there will be an optimal quantity that one would like to hold.

In fact, these properties are very useful in risk management. First of all, the range of prices for a contract depends on what other contracts are currently in the portfolio. Thus, if other instruments with the same default exposure are available in the market, a trader can discharge tradings for a safer (or more lucrative) position. This is essentially a static hedge. A pursuit of an optimal static hedging is also conceivable, where optimality depends upon the choice of a risk preference. There is no more fear of transaction costs, as long as one discriminates bid prices and ask prices in the optimisation procedure. In short, one obtains

- optimal portfolio selection
- optimal static hedge and pricing

given the market prices of instruments and the trader's budget constraints.

In the future, as the market unveils the prices of contracts, the investor may revise his portfolio to achieve a better position. This is not incorporated in the current estimate of risk yet because future market prices are not available at present.

4 Model Specification, Calibration, and Equations

It is not too difficult to describe the physical dynamics of the short rate r as a diffusion process. See Wilmott (1998), for example. If necessary, we can enhance the goodness-of-fit using additional factors. Even if there is doubt about the accuracy of such models, this is not the issue of our paper. The choice of the hazard rate p is another story. The reason is that the credit ratings from service institutions are more qualitative than quantitative and that the lack of default history precludes statistical inference. We suggest the following two methodologies to improve the quantification of the hazard rate:

- To pose reasonable limits for the hazard rate and to investigate the best and worst scenario. Similar work has been done for uncertain volatility model for derivative hedging and pricing; see Avellaneda, Lévy and Parás (1995).
- To accomodate the fact that a corporate's credit quality is also affected by the status of the global economy. For example, p is a function of SP500 or its volatility (under a stochastic volatility model).

In the second approach, the construction of such p is purely statistical. For example one can pose a regression problem: the response surface is the number of firms defaulted in each category of Moody's rating, while SP500 is the independent (i.e. control) variable. This will be the subject of future work.

In the remainder of this paper, we will assume that the physical dynamics of the short rate r follows a diffusion:

$$dr = \alpha(r)dt + \beta(r)dW \,,$$

and that the hazard rate p is a function of SP500 for which dynamics are given by another diffusion:

$$dS = \mu(S)dt + \sigma(S)dB$$

where W and B are two standard Brownian motions with correlation ρ. Nevertheless, the scope of our discussion in the paper will not be restricted to this particular, simple diffusion. Also we will assume that the recovery rate is zero for simplicity. We designate $\mathcal{L}$ to be the differential operator associated with (r, S):

$$\mathcal{L} = \mu\frac{\partial}{\partial S} + \alpha\frac{\partial}{\partial r} + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial S^2} + \rho\sigma\beta\frac{\partial^2}{\partial S\partial r} + \frac{1}{2}\beta^2\frac{\partial^2}{\partial r^2}\,. \tag{2}$$

In addition, we define a non-linear operator:

$$\mathcal{Q}(f) = \sigma^2\Big(\frac{\partial f}{\partial S}\Big)^2 + 2\rho\sigma\beta\Big(\frac{\partial f}{\partial S}\Big)\Big(\frac{\partial f}{\partial r}\Big) + \beta^2\Big(\frac{\partial f}{\partial r}\Big)^2. \tag{3}$$

Now we are ready to present some sensible models:

4.1 Mean-Variance

First we consider the certainty equivalent given by a combination of mean and variance

$$\text{mean} - \kappa(\text{variance})^c. \tag{4}$$

The certainty equivalent (4) does not satisfy the usual axioms for the individual risk preference: for example, the monotonicity with respect to the first order stochastic dominance fails. However this is a popular choice, being tractable.

Let Π_t be the sum of discounted future cash flows starting at time t given that default has not occurred yet, and let τ be the default time. In addition, we let Ω_t be the discounted non-defaultable cash flow included in Π_t. For example Ω may include cash flows of US Treasury bills or SP500 index options. Provided that no contract in the portfolio matures in $[t, t+dt]$, we obtain

$$\begin{aligned}(1+rdt)\Pi_t &= \Pi_{t+dt}\,\mathbb{I}\,(\tau > t+dt) + \Omega_{t+dt}\,\mathbb{I}\,(\tau \le t+dt) &(5)\\ (1+rdt)\Omega_t &= \Omega_{t+dt} &(6)\end{aligned}$$

Here we simply assumed that the risky bond has a zero recovery rate. If a contract matures in $[t, t+dt]$, both Π and Ω must satisfy a jump condition. Equations (5) and (6) are the starting point when one tries to find the differential equation for mean and variance. It can be shown that the mean and the variance of Ω (m_Ω and v_Ω, respectively) satisfy

$$\begin{aligned}\frac{\partial}{\partial t}m_\Omega + \mathcal{L}m_\Omega &= rm_\Omega\\ \frac{\partial}{\partial t}v_\Omega + \mathcal{L}v_\Omega + \mathcal{Q}(m_\Omega) &= 2rv_\Omega\end{aligned}$$

and that

$$\begin{aligned}
\frac{\partial}{\partial t} m_\Pi + \mathcal{L} m_\Pi &= r m_\Pi + p(m_\Pi - m_\Omega) \\
\frac{\partial}{\partial t} v_\Pi + \mathcal{L} v_\Pi + \mathcal{Q}(m_\Pi) &= 2 r v_\Pi + p(v_\Pi - v_\Omega) - p(m_\Pi - m_\Omega)^2
\end{aligned}$$

for the mean (m_Π) and the variance (v_Π) of Π. Here $\mathcal{L}$ and $\mathcal{Q}$ are operators defined in (2) and (3). This completes the formation of the mean-variance model.

4.2 Recursive-Utility

Another approach would be to use recursive utility. The recursive-utility model was initiated by Selden (1978) and Kreps and Porteus (1978). The premise underlying the model is that the utility at the moment is the result of both the immediate payoff and the rational expectation of the utility of the next time period. This resolves the certainty equivalent of temporal uncertainty in the form of a dynamic programming, which facilitates finding the optimal strategy over the time period in question. In continuous-time setting, its development is relatively new, due to Duffie and Epstein (1992). An example is the additive utility in the classical Merton's problem. See Duffie (1996) for details.

Let ϕ and ω be the certainty equivalents of Π and Ω, respectively. We will consider that a trader's risk preference is given by the followings:

$$U((1+r dt)\omega) = E_t\Big[U(\omega + d\omega)\Big] \tag{7}$$

$$U((1+r dt)\phi) = E_t\Big[U(\phi + d\phi)\Big](1 - p dt) + E_t\Big[U(\omega + d\omega)\Big] p dt \tag{8}$$

where the temporal utility function U is twice continuously differentiable and strictly increasing, and where E_t is the (conditional) expectation given the information up to time t (i.e. the values of r_t and S_t). The equations (7) and (8) are valid when there is no contracts mature in $[t, t+dt]$. Otherwise, we must impose the corresponding jump conditions. The differential equations for ω and ϕ are:

$$\begin{aligned}
\frac{\partial}{\partial t}\omega + \mathcal{L}\omega + \frac{1}{2}\frac{U''}{U'}(\omega)\mathcal{Q}(\omega) &= r\omega \\
\frac{\partial}{\partial t}\phi + \mathcal{L}\phi + \frac{1}{2}\frac{U''}{U'}(\phi)\mathcal{Q}(\phi) &= r\phi + p\Big[\frac{U(\phi) - U(\omega)}{U'(\phi)}\Big]
\end{aligned}$$

Note that the coefficient of absolute risk aversion (U''/U') is multiplied by the measure of temporal uncertainty (the $\mathcal{Q}$ term). Thus the more risk averse a trader is, the more he should discount the portfolio due to uncertainty. This completes the recursive utility credit pricing model.

5 Competition and a Moving Boundary

As we stated in the modelling principles, our approach does not necessarily attempt to explain market prices, but complies with them by treating them as exogenous variables.

The reason for adopting this is because a practical mathematical model would almost certainly not accomodate the peculiarities of the credit market, and consequently there would remain a gap between the model price and the market price. In this section, we study the friction between a trader equipped with our recursive utility model (call this AKW) and another trader with a market price model, for example, a model with the complete market assumption (call this CMA).

Suppose that the market price at time t is contained in an interval with center being the CMA's model price: that is, $x(t) \pm \varepsilon$ where $x(t)$ is the predicted price of the portfolio calculated from CMA's model. x would depend upon state variables such as short rate, but we will use this abbreviated notations for this general discussion. The market price is the price that AKW can enter or close a position and he will discharge the trade if it increases his utility. Therefore the dynamics of AKW's certainty equivalent ϕ has another feature, a moving boundary:

$$\phi \geq \max(x + \varepsilon, x - \varepsilon)\,.$$

6 Concluding Remarks

The current state of the art of credit-risk modelling has been briefly discussed. At present these models fail to capture the distinguishing illiquidity and incompleteness of credit markets. Consequently all of these methods are always subject to modelling error.

Utility based pricing, which intrinsically respects the incompleteness of credit markets in its provisions, constitutes an enormous advance in the practice of correctly valuing defaultable instruments. Although this new, utility-based approach for credit risk is still in its infancy, there are many possibilities and disections for its development and modification. Some interesting areas of future research include: in-depth study of recovery structures including the uncertainty in its outcome, more elaborate default structures, and more statistical studies of default probabilities. We hope that this note will stimulate discussion and lead to more sophisticated and realistic credit risk pricing.

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