A Simplified Method for Calculating the Credit Risk of Lending Portfolios

Akira Ieda, Kohei Marumo, and Toshinao Yoshiba

The common practice for managing the credit risk of lending portfolios is to the calculate the maximum loss within the "value at risk" framework. Most financial institutions use large-scale Monte Carlo simulations to do this. However, such simulations may impose heavy calculation loads. This paper proposes a simplified method that approximates maximum loss with minimal simulation burden.

Our method divides a portfolio into subportfolios at each credit rating level and calculates the maximum loss of each subportfolio. We assume that the subportfolio's structure provokes little fluctuation in the ratio between the maximum loss and the standard deviation. We therefore begin with a subportfolio in which each exposure is of the same amount (a homogeneous subportfolio). Simple calculations provide the standard deviation for both the heterogeneous subportfolio whose risk is to be measured and the homogeneous subportfolio. The maximum loss for the homogeneous subportfolio can be obtained by using analytical techniques rather than simulations. The maximum loss for a heterogeneous subportfolio is then approximated by multiplying the ratio of the maximum loss and standard deviation of the homogeneous subportfolio by the standard deviation of the heterogeneous subportfolio. Simulation examples indicate that this approximation is effective in all portfolios except those including extremely large exposures. This paper also describes a technique for using the total maximum loss of all subportfolios to find the maximum loss for the entire portfolio.

Key words: Credit risk; Lending portfolio; Monte Carlo simulation; Credit concentration/diversification; Correlation between default events

Research Division 1, Institute for Monetary and Economic Studies, Bank of Japan
(E-mail: akira.ieda@boj.or.jp, kouhei.marumo@boj.or.jp, toshinao.yoshiba@boj.or.jp)

This paper was first submitted to a research conference on "New Approaches to Financial Risk Management" sponsored by the Bank of Japan in January and February 2000. We have added to and amended the original paper in the light of comments received at the conference. Our discussions with Mr. Masao Yoneyama (Financial and Payment System Office, Bank of Japan) were particularly beneficial at the beginning of this research. We also wish to acknowledge the many helpful comments received from Professors Masato Kobayashi (Yokohama National University), and Soichiro Moridaira (Keio University), and the participants at the conference.

I. Introduction

Financial institutions in Japan and many other countries are developing and enhancing methods to measure and manage the main risk inherent in their business operations: the credit risk of their lending portfolios. The specific direction that these efforts have taken is to draw on advances in financial engineering and statistics to create computer simulations and analytical methods. These techniques provide a more accurate measurement of risk, which can then be used in bank management, for example, to determine more accurately the pricing of financial instruments and effective credit limits, or even appropriate allocations of capital.

The measurement of the credit risk of lending portfolios usually entails the same basic procedure as the measurement of market risk, i.e. the Value at Risk (VaR) framework is used in a model that calculates the maximum potential loss or expected loss of the portfolio. However, there are several impediments to these measurements: (1) models are harder to handle than those for market risk. In other words, credit risk models deal with a default event for which one cannot assume simple (logarithmic) normality, and particular attention must be paid to data constraints that will impinge on many aspects of parameter estimation and setting, including default rate and recovery rate parameters; and (2) simulations are time-consuming. When a financial institution has tens or hundreds of thousands of credit exposures, simulations for credit risk management require enormous calculation loads. Even powerful computers require a long calculation time before risk results become available.

This paper focuses on simulation problems, especially in credit risk models. We propose a method that roughly captures portfolio credit risks while minimizing the need for simulations, and we consider the impact of this technique on credit risk management.

The structure of this paper is as follows. In Chapter II, we outline the framework for portfolio credit risk management. In Chapter III, we describe the concepts for simplified credit risk measurement. In Chapter IV, we apply these techniques to a sample portfolio and demonstrate their applicability. In Chapter V, we draw some conclusions about these techniques.

II. Framework for the Management and Measurement of Portfolio Credit Risk

A. Framework for the Management of Portfolio Credit Risk

1. Credit ratings

Most financial institutions in Japan and other countries have internal credit rating systems ("internal ratings"), and these systems form an important part of their infrastructure for managing credit risk.[1] There are two main forms that these ratings take: 1) borrower-based ratings that use a borrower default rate (for example, a default rate for the following year) as a basis for assessing creditworthiness (see Table 1 for an example), and 2) facility-based ratings that consider the recovery rate and expected loss for each loan. Facility-based ratings will consider the creditworthiness of the borrower and other factors during the rating process, but at Japanese banks, most ratings are borrower-based,[2] so the remainder of this paper will assume borrower-based ratings.

Table 1 Example of Internal Rating Systems[3]

Rating		Degree of risk	Definition	Borrower category by self-assessment
1		No essential risk	Extremely high degree of certainty of repayment	Normal
2		Negligible risk	High degree of certainty of repayment	
3		Some risk	Sufficient certainty of repayment	
4	A	Better than average	There is certainty of repayment, but substantial changes in the environment in the future may have some impact on this certainty.	
	B			
	C			
5	A	Average	There are no foreseeable problems in the future, but there is a strong likelihood of impact from changes in the environment.	
	B			
	C			
6	A	Tolerable	There are no foreseeable problems in the future, but the future cannot be considered entirely safe.	
	B			
	C			
7		Lower than average	There are no problems at the current time, but the financial position of the borrower is relatively weak.	
8	A	Needs preventive management	There are problems with lending terms or fulfillment, or the borrower's business conditions are poor or unstable, or there are other factors requiring careful management.	Needs attention
	B			
9		Needs serious management	There is a high likelihood of bankruptcy in the future.	In danger of bankruptcy
10	I		The borrower is in serious financial straits and "effectively bankrupt."	Effectively bankrupt
	II		The borrower is bankrupt.	Bankrupt

Ratings (i.e., default rates) assigned to borrowers form the basis for credit risk management. They are the starting point for determining the level of interest rates and credit limits on an individual exposure basis; on a portfolio basis, they are used in simulations to quantify credit risk and calculate the capital required for internal management purposes.

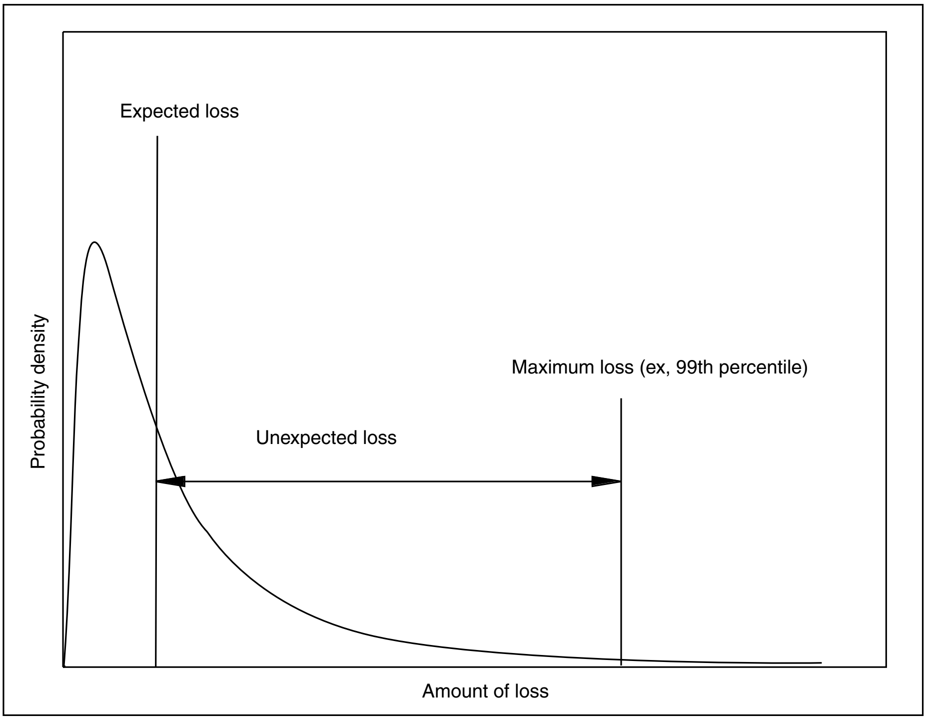
1. See Financial Supervisory Agency/FISC (1999) for an overview of the internal rating systems currently used by financial institutions in Japan; see Treacy and Cary (1998) for a similar study of U.S. institutions.
2. From Financial Supervisory Agency/FISC (1999).
3. Also taken directly from Financial Supervisory Agency/FISC (1999).

2. Outline of credit risk measurement techniques

A variety of input data is required in the measurement of credit risk: the default rate for each exposure, the amount of the exposure, the recovery rate, and the correlations with other exposures.

Simulation techniques use this input data to develop a loss distribution, whereby it is possible to calculate the expected value of loss (expected loss), variance, and maximum loss at the 99th percentile (see Figure 1). The amount remaining when expected loss is deducted from maximum loss is defined as "unexpected loss." Ordinary business practice says that this "unexpected loss" should be covered by economic capital.

Figure 1 Conceptual Diagram of the Density Function of Loss Distribution



B. Credit Risk Measurement Techniques and Their Problems

1. Definition of loss (default mode versus Mark to Market (MTM))

The concept of loss must be defined in order to measure credit risk. The Basel Committee on Banking Supervision provides two definitions of loss.[4] The "default mode" concept defines loss in terms of loss that would be generated only if the borrower defaulted during the risk evaluation period. The MTM concept takes account of declines in the borrower's creditworthiness (rating) in addition to default. The differences are illustrated below by calculating portfolio losses under the two definitions.

4. Basel Committee on Banking Supervision (1999).

a. Default mode

A portfolio has n exposures. The default rate for exposure i up to some point in the future is p_i, the amount of the exposure v_i, and the recovery rate at default r_i ($0 \leq r_i \leq 1$) (all values are fixed).[5] The portfolio loss L can be expressed using a random variable with either 1 or 0 as its value:

$$D_i = \begin{cases} 1 & (\text{Probability} \quad p_i) \\ 0 & (\text{Probability } 1 - p_i) \end{cases},$$

so,

$$L = \sum_{i=1}^{n} D_i v_i (1 - r_i). \tag{1}$$

In Equation (1), the loss is a discrete value, but when n is sufficiently large and the interval between values is sufficiently small, it can be treated as continuously distributed. The expected value for L can be found as:

$$E[L] = \sum_{i=1}^{n} p_i v_i (1 - r_i).$$

b. MTM

The probability that exposure i will migrate to rating k (k=1,$\cdots$, m) is $p_{k,i}$ $(\sum_{k=1}^{m} p_{k,i} = 1)$, and the difference between the present value of exposure i and its value when it migrates to rating k is $dv_{k,i}$ (The market value implicitly incorporates the recovery rate). Calculation of the loss L on the portfolio uses a random variable with values 1,$\cdots$, m (m represents default):

$$D_i = \begin{cases} 1 & (\text{Probability } p_{1,i}) \\ \vdots & \\ m & (\text{Probability } p_{m,i}) \end{cases},$$

such that it can be calculated the same way as in the default mode:

$$L = \sum_{i=1}^{n} dv_{D_i,i}. \tag{2}$$

The expected value is therefore:

5. It is common to set up models so that these parameters are deterministic values, but ordinarily they will have some degree of uncertainty.

$$E[L] = \sum_{i=1}^{n} \sum_{k=1}^{m} p_{k,i}\, dv_{k,i}\,.$$

2. Use of simulation to calculate maximum loss

Quantification of portfolio credit risk begins with a definition of the concept of loss, as shown above. The input parameters for calculation are default rates, exposure amounts, correlations, etc.[6], and the calculation gives the maximum loss of the portfolio for a given confidence level. Generally, Monte Carlo simulation is used to calculate the maximum loss and/or the unexpected loss. Rather than assume a specific loss distribution, this approach uses Monte Carlo simulation to generate a loss distribution, and estimates the maximum loss, etc (See CreditMetrics (J.P. Morgan & Co. [1997]) for an example).[7]

3. Problems with simulation

One problem with simulation is that it takes time. The computing load becomes enormous for financial institutions with tens or hundreds of thousands of exposures. Even with the most powerful computers, a very long calculation time (in some cases, several days) is required before risks and other calculation results are obtained. This becomes a bottleneck for financial institutions when they attempt to use simulation results for such practices as capturing day-to-day changes in the amount of credit risk.

In the light of this problem with simulation, the next chapter discusses the basic concepts needed for a simplified technique for measuring portfolio credit risk.

III. A Framework for Simplified Measurement of Credit Risk

A The Standard Deviation Approach to Risk Measurement

1. Outline of the framework

We noted above that the maximum loss and unexpected loss of a lending portfolio are ordinarily found by simulating loss distribution and assuming a certain confidence level (for example, the 99th percentile). In the discussion that follows, we express this as the Unexpected Loss (UL), but for purposes of simplicity, we do not distinguish between maximum loss and unexpected loss.

In our approach, we do not use simulations to arrive at the UL. Instead, we use the standard deviation of the loss distribution (which we will call the "Volatility of Loss (VL)").[8]

6. Models generally assume these parameters to be mutually independent because of the simulation loads.
7. There is also an analytical approach other than the simulation approach. This approach makes certain assumptions about the loss distribution for individual exposures, and then uses analytical techniques to obtain the loss distribution of the portfolio as a whole. For example, see CreditRisk+ (Credit Suisse Financial Products [1997]).
8. For market risk VaR (variance/covariance method, Monte Carlo simulation method), it is common to assume a (logarithmic) normal distribution for the risk factor fluctuation. Therefore, for example, there is a relationship whereby the maximum loss at the 99th percentile will be approximately 2.33 times the standard deviation of the loss value, but we would caution that, in credit risk VaR, the maximum loss cannot be given *a priori* as a multiple of the standard deviation.

The definition of loss used in this paper is "assessed loss from book value only in the case of default" (i.e., the "default mode"). We do not take account of changes in market values due to changes in ratings. We also use fixed values for the exposure amount and the recovery rate. The exposure amount is the amount remaining when the amount recoverable from collateral etc. (conservatively estimated) is subtracted from the amount of the loan. The recovery rate for this remainder is assumed to be 0 percent. The default rate is the one-year cumulative default rate, assuming a one-year risk evaluation period.

In the discussion that follows, we assume that the default rates that underlie the internal ratings are constant within each rating. From this standpoint, we then consider the subportfolio risks for each rating.

2. The impact of diversification and concentration on credit risk

a. Zero correlation between default events

We begin by assuming a default rate of p_k for all exposures rated k, and an exposure within rating k of $v_{k,i}$ (i = 1,2,$\cdots$). If the VL for each individual exposure is $VL_{k,i}$, then the default is a Bernoulli event expressed by the following equation:

$$VL_{k,i} = \sqrt{p_k(1-p_k)\,v_{k,i}^2} = \sqrt{p_k(1-p_k)}\,v_{k,i}\,. \tag{3}$$

Next, we use VL_k to express the VL for a subportfolio consisting entirely of exposures rated k. If we assume the correlation between default events of individual exposures to be 0, then:

$$VL_k = \sqrt{\sum_i p_k(1-p_k)\,v_{k,i}^2} = \sqrt{p_k(1-p_k)}\,\sqrt{\sum_i v_{k,i}^2}$$

$$= \sqrt{p_k(1-p_k)}\,\sum_i v_{k,i}\,\frac{\sqrt{\sum_i v_{k,i}^2}}{\sum_i v_{k,i}}\,. \tag{4}$$

If the number of exposures included in rating k is fixed, then $\sqrt{\sum_i v_{k,i}^2} / \sum_i v_{k,i}$ (a number between 0 and 1) will be lowest in a homogeneous portfolio in which the amount of individual exposures is equal. The greater the concentration of lending, the closer the figure is to 1. Therefore, $\sqrt{\sum_i v_{k,i}^2} / \sum_i v_{k,i}$ can be considered a factor expressing the degree of concentration or diversification in the portfolio. For the purposes of this paper, we will refer to it as the "Concentration Factor (CF)".

b. Non-zero correlation between default events (extension of A)

We assume the default rate for rating k to be p_k, and individual exposures within rating k to be $v_{k,i}$ (i=1,2,$\cdots$). If VL_k for a subportfolio consisting entirely of the same rating has a correlation ρ_{ij} between default events of exposures i and j ($i \neq j$), then:

$$VL_k = \sqrt{\sum_i VL_{k,i}^{\;2} + 2\sum_{i<j} \rho_{ij} VL_{k,i} VL_{k,j}}\,,$$

$$\text{where } VL_{k,i} = \sqrt{p_k(1-p_k)\,v_{k,i}^2} = \sqrt{p_k(1-p_k)}\;v_{k,i}. \tag{5}$$

This can be transformed as follows:

$$\begin{aligned}
VL_k &= \sqrt{\sum_i p_k(1-p_k)\,v_{k.i}^2 + 2\sum_{i<j} p_k(1-p_k)\,\rho_{ij}\,v_{k,i}\,v_{k,j}} \\
&= \sqrt{p_k(1-p_k)}\;\sqrt{\sum_i v_{k.i}^2 + 2\sum_{i<j}\rho_{ij}\,v_{k,i}\,v_{k,j}} \\
&= \sqrt{p_k(1-p_k)}\;\sum_i v_{k,i}\,\sqrt{\sum_i v_{k.i}^2 + 2\sum_{i<j}\rho_{ij}\,v_{k,i}\,v_{k,j}} \Big/ \sum_i v_{k,i} \\
&= \sqrt{p_k(1-p_k)} = \sum_i v_{k,i}\,\frac{\sqrt{\sum_i v_{k.i}^2}}{\sum_i v_{k,i}}\;\sqrt{1 + \frac{2\sum_{i<j}\rho_{ij}\,v_{k,i}\,v_{k,j}}{\sum_i v_{k.i}^2}}\,.
\end{aligned} \tag{6}$$

We will refer to the final $\mathrm{CF} \times \sqrt{}$ portion on the right hand side of Equation (6) as the "Extended Concentration Factor (ExCF)".

$$\mathrm{ExCF} = \sqrt{\frac{\sum_i v_{k,i}^2 + 2\sum_{i<j}\rho_{ij}\,v_{k,i}\,v_{k,j}}{(\sum_i v_{k,i})^2}}\;\;. \tag{7}$$

Note in relation to the $\mathrm{CF} \times \sqrt{}$ portion (i.e., ExCF) of Equation (6) that the CF and the $\sqrt{}$ cannot be considered separately. For example, the more the portfolio is diversified, the closer the CF will be to zero. But if, at that time, the default correlation between all exposures is considered to be 1, then the ExCF will always be 1. Mere diversification by itself may not be successful in decreasing risk. Therefore, with an ordinary portfolio in which the default correlation between exposures is not zero, it is necessary to evaluate the portfolio's degree of concentration or diversification using the ExCF.

3. The correlation between default events

In order to estimate the correlation coefficient between default events of exposures, it is possible to use correlation coefficients between stock prices, for example.[9] But the more exposures there are, the harder it becomes, in practical terms, to arrive at correlation coefficients between each pair. If the average level of the correlation coefficient can be assumed to be given as ρ, then the ExCF becomes:

$$\mathrm{ExCF} = \sqrt{\frac{\sum_i v_{k,i}^2 + 2\rho\sum_{i<j} v_{k,i}\,v_{k,j}}{(\sum_i v_{k,i})^2}}\;\;. \tag{8}$$

9. This method is used, for example, in CreditMetrics. See Appendix 2 for an explanation of the methods used to calculate correlation coefficients between default events when this is included.

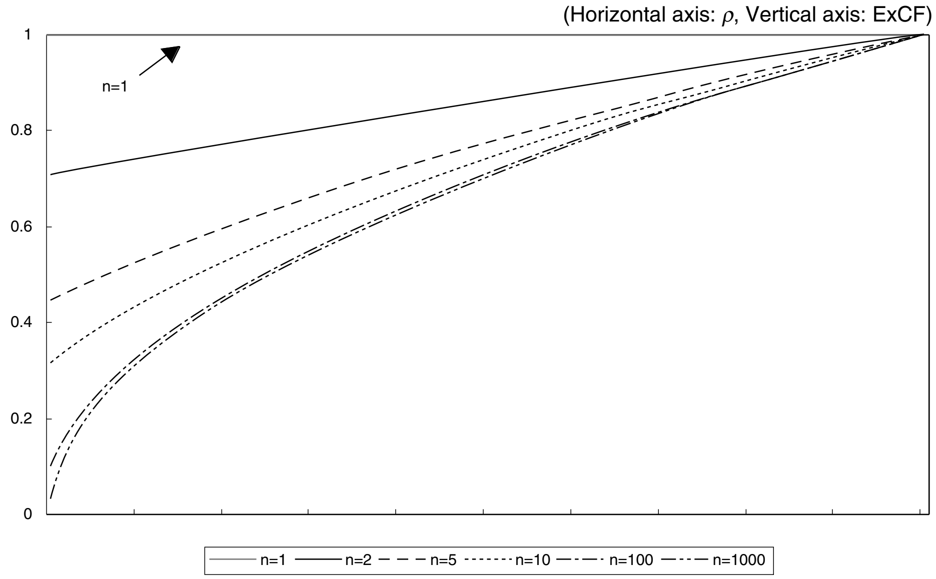
a. Homogeneous portfolio

This section assumes a homogeneous portfolio that contains n exposures each of the same amount. The ExCF from Equation (8) can be expressed by a simple calculation[10] to arrive at the following:

$$\mathrm{ExCF} = \sqrt{\rho + \frac{1-\rho}{n}} \ . \tag{9}$$

Figure 2 was created to illustrate the dependence of the ExCF on the levels of n and ρ.

Figure 2 Relationship between the ExCF and n and ρ (in homogeneous portfolio)



Notes: 1. The ExCF is an increasing function of ρ (however, the ExCF stays at 1 when $n = 1$).
2. The ExCF is a decreasing function of n. When n exceeds about 100, the ExCF remains at almost the same value, except for cases in which ρ is close to 0. When $n \to \infty$, the ExCF $\to \sqrt{\rho}$ (in other words, the diversification effect has a floor of $\sqrt{\rho}$).
3. When ρ is large, even though n is large, the ExCF is close to 1. In other words, the greater the correlation between exposures, the less prevalent the effect of diversification is, as measured by the ExCF.

10. Set $v_{k,i}=v_k$ for all i in Equation (8) (i.e., a homogeneous portfolio), and both the numerator and the denominator in $\sqrt{\ }$ have a factor, so that is canceled out to give Equation (9).

b. Heterogeneous portfolio

Now let us consider a heterogeneous portfolio.[11] We use the CF to express the concentration factor when the correlations between the default events of exposures are not considered. The ExCF is therefore expressed as a simple calculation:[12]

$$\text{ExCF} = \sqrt{\rho + CF^2(1-\rho)}\ . \qquad (10)$$

Note in Equation (10) that the ExCF can be calculated as long as one has information for two parameters: 1) CF, and 2) ρ. As already discussed, if the total amount and number of exposures are fixed for the portfolio, the ExCF (and CF) will be lowest when the portfolio is homogeneous. When the portfolio is heterogeneous, the ExCF is a value between $\sqrt{\rho + (1-\rho)/n}$ and 1 (the CF, between $1/\sqrt{n}$ and 1).

In Table 2, we calculated the ExCF assuming a portfolio with 100 exposures (the lowest possible CF will be 0.1 in a homogeneous portfolio). We assumed ρ to be 0.15, as an example, and set the CF between 0.1 and 0.7 in increments of 0.1. When the CF is relatively low (0.1-0.3) and the portfolio is ostensibly diversified, the ExCF, which takes account of ρ, is 4.0-1.6 times the CF. Therefore, when ρ is not taken into account, the effect of diversification on portfolio risk will be understated by a fraction.

Table 2 Relationship between CF and ExCF (ρ = 0.15)

CF (a)	0.1	0.2	0.3	0.4	0.5	0.6	0.7
ExCF (b)	0.39812	0.42895	0.47592	0.53479	0.60208	0.67528	0.75266
(b)/(a)	3.98	2.14	1.59	1.34	1.20	1.13	1.08

B. Relationship with Maximum Loss

1. A simplified method for calculating maximum loss (approximation with standard deviation)

This section builds on the concepts described above to present a simplified method for calculating the UL for each rating. As noted above, the UL is usually deduced from simulations based on loss distribution models, with an assumed confidence level (for example, the 99th percentile). However, simulations for portfolios that contain large numbers of exposures require a very long time to run, and this becomes a bottleneck in risk management.

Instead of running a model-based simulation every time the portfolio UL was required, it would be possible to obtain an approximate subportfolio UL using an analytic method. The approximation is based on the assumption that the homogeneous portfolio's ratio of UL and ExCF could be nearly equal to the ratio of the heterogeneous portfolio.[13] In short, the basic concept that this paper proposes is that of

11. The portfolios held by financial institutions, i,e, the portfolios whose risk is to be measured, are generally heterogeneous.

12. This calculation uses the relationship $(\sum_i v_{k,i})^2 = \sum_i v_{k,i}^2 + 2\sum_{i<j} v_{k,i} v_{k,j}$.

13. In this calculation, the UL of a homogeneous portfolio can be obtained analytically without simulation, but here we use figures obtained by simulation. See Appendix 1 for a detailed discussion of the analytical techniques used to derive the UL of a homogeneous portfolio.

approximating the UL, rather than running a time-consuming simulation for the portfolio to be measured.

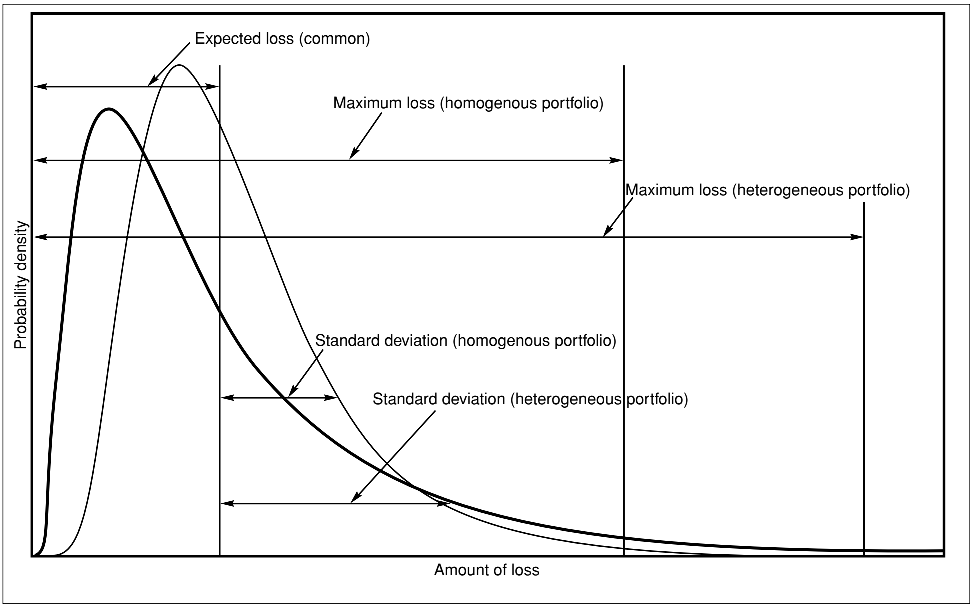
The process is described in more detail below.

1) For a homogeneous portfolio comprising n exposures of the same rating (constant default rate) and amount,[14] the correlation coefficients between default events are assumed to be ρ, and the UL is calculated.
2) It is assumed that the ratio between the VL of the homogeneous portfolio and the VL of the heterogeneous portfolio is the same as the ratio between their ULs. The UL of the heterogeneous portfolio is therefore approximated as shown in Equation (11), using the ExCF of the homogeneous portfolio (with the same rating) and the ExCF of the heterogeneous portfolio (Equation (10)).

UL of heterogeneous portfolio

$$\cong \frac{\sqrt{\rho + CF^2(1-\rho)}}{\sqrt{\rho + \frac{1-\rho}{n}}} \times \text{ UL of homogeneous portfolio.} \qquad (11)$$

Figure 3 Maximum Loss and Standard Deviation in a Homogeneous Portfolio and a Heterogeneous Portfolio



If this approximation can be performed with sufficient precision, then it would be possible simply to approximate the UL of the heterogeneous portfolio using Equation (11). This is the case even if the heterogeneity of the portfolio changes, as long as the number of exposures n is unchanged, or the change in n can be ignored (for example, when $1/n$ <<1). All that is required is a one-time calculation of the UL of a homogeneous portfolio of n exposures. This calculation can be implemented either analytically, as described in Appendix 1, or by simulation.

14. We assume n is sufficiently large.

In the next chapter, we examine the degree of precision that can be achieved when actually using this approximation in practical settings.

2. Limits of the approximation and evaluation of the credit risk of portfolios

a. Limits of the approximation

Before considering the feasibility of the approximation described above, we should first note the limits of our method.

Our method assumes that there is no large difference between the UL/ExCF ratio of the homogeneous portfolio and that of the heterogeneous portfolio. We then calculate the ExCF and the UL of the heterogeneous portfolio, based on the ratio between the UL and ExCF of the homogeneous portfolio. If the composition of the heterogeneous portfolio is vastly different from the homogeneous portfolio that serves as the base, then its UL/ExCF ratio may be vastly different from that of the homogeneous portfolio, so the approximation could not be used. One example might be a portfolio that includes large-lot exposures. As will be discussed more fully when we consider simulations below, the UL of a portfolio in which credits concentrate in certain borrowers will be vastly different when calculated using our approximation than it will be when calculated by simulation.

Also, the approximation may not perform well when n is not very large, or when the loss distribution cannot be regarded as continuous.

What both of these cases have in common is the existence of large-lot exposures in the portfolio. Our method for using the ExCF to approximate the UL may not fully incorporate the impact of large-lot exposures on the UL.

b. Evaluation of the credit risk of portfolios

Generally speaking, at Japanese financial institutions: 1) most borrowers are medium and small-sized companies, so it is unlikely that there will be an extreme lack of borrowers at any level in the internal rating system;[15] and 2) the credit limits set for internal management purposes limit the existence of extremely large exposures. Therefore, the composition of portfolios at financial institutions will, in most cases, probably not be subject to the limitations discussed in A above.

15. During economic slumps such as Japan is currently experiencing, there would be fewer companies with high ratings. In addition, most of the high-rated borrowers would be large companies, and the exposure to them would therefore be relatively large. Certainly, this could be an impediment to calculating the UL for these ratings.

IV. Simulations and Discussions

We perform simulations to derive the maximum loss of a portfolio and compare these results with those obtained by our approximation in order to ascertain the validity of our method. As already described, simulations require complex calculations in order to obtain the maximum loss. We describe the calculations and procedures in Section IV.A. In Section IV.B., we provide our results, together with some discussion.

A. Simulation Methodology

1. Generation of Bernoulli random numbers

It is easy to generate multivariate normal random numbers using the Cholesky decomposition of the variance/covariance matrix, as long as the random variables exhibit a (logarithmic) normal distribution. However, the default mode approach used in this paper assumes a Bernoulli distribution of "default" and "non-default," so it is not possible simply to apply the Cholesky decomposition. We therefore use the following method to generate multivariate Bernoulli random numbers.

a. When the default rates and default correlations are equal

We will first consider a lending portfolio in which the amounts and default rates are equal for individual exposures and the default correlation between individual exposures is constant.

First, we consider the random variable D_i ($i = 1,2,\cdots,n$), which has a Bernoulli distribution.

$$D_i = \begin{cases} 1 \ (\text{Probability} \quad p) \\ 0 \ (\text{Probability} \ 1-p). \end{cases} \tag{12}$$

In other words, D_i ($i = 1,2,\cdots,n$) for exposure i in the portfolio (comprising n exposures) takes the value 1 (default) with probability p and 0 (non-default) with probability $1-p$. Also, the correlation coefficient of each D_i is ρ (constant). As noted above, the process of generating multivariate Bernoulli random numbers that take account of the correlation is not a simple application of the Cholesky decomposition. However, the Cholesky decomposition can be used for normal distributions, so one method is to use the normal distribution as a medium for generating Bernoulli random numbers.

We first consider a random variable X_i ($i = 1,2,\cdots,n$) that follows the standard normal distribution with 0 for its mean and 1 for its variance. (However, individual variables are correlated rather than independent). At this time, D_i is expressed as:

$$D_i = \begin{cases} 1 \ (-\infty < X_i \leq \Phi^{-1}(p)) \\ 0 \ (\Phi^{-1}(p) < X_i < \infty) \quad , \ , \end{cases} \tag{13}$$

where $\Phi^{-1}(\cdot)$ is the inverse function of the cumulative density function of the standard normal distribution.

For the correlation coefficient of D_i ($i = 1,2,\cdots,n$) to be ρ, one need properly set a correlation coefficient $\tilde{\rho}$ for X_i ($i = 1,2,\cdots,n$). ρ can be expressed as:

$$\rho = \frac{E[D_i D_j] - p^2}{\sqrt{p(1-p)}\sqrt{p(1-p)}}\ , \tag{14}$$

where

$$D_i D_j = \begin{cases} 1\ (-\infty < X_i \le \Phi^{-1}(p), -\infty < X_j \le \Phi^{-1}(p)) \\ 0 \qquad\qquad (otherwise) \end{cases} . \tag{15}$$

Therefore, $E[D_i D_j]$ is the cumulative density function of a two-dimensional normal distribution with a correlation coefficient of $\tilde{\rho}$.

$$E[D_i D_j] = \int_{-\infty}^{\Phi^{-1}(p)} \int_{-\infty}^{\Phi^{-1}(p)} \frac{1}{2\pi\sqrt{1-\tilde{\rho}^2}} \exp\left(-\frac{1}{2(1-\tilde{\rho}^2)}\left\{x_i^2 + x_j^2 - 2\tilde{\rho} x_i x_j\right\}\right) dx_i\, dx_j\ . \tag{16}$$

This makes it possible to use Equation (16) and Equation (14) to obtain a $\tilde{\rho}$ that will satisfy Equation (14) (however, numerical calculations will be required to obtain the definite integral above).

It is therefore possible to obtain multivariate Bernoulli random numbers D_i by using Equation (13) after generating multivariate normal random numbers at the n-th dimension with a mean of 0, a variance of 1, and a constant correlation coefficient of $\tilde{\rho}$.

b. When the default rates and default correlations are different

We express default/non-default for exposure i within the portfolio using the Bernoulli random number D_i, as was shown in Section IV.A.1.*a.* above. But when $i \neq j$, one cannot necessarily assume that the individual default rates p_i, p_j will be equal, nor does it necessarily follow that the correlation coefficient between the default events of these exposures ρ_{ij} will be constant either. In this case, random numbers are generated as follows.

First, D_i, D_j ($i \neq j$) is expressed as shown below using random variables X_i, X_j, which follow the standard normal distribution.

$$D_i = \begin{cases} 1\ (-\infty < X_i \le \Phi^{-1}(p_i)) \\ 0\ \ (\Phi^{-1}(p_i) < X_i < \infty) \end{cases} \tag{17}$$

and

$$D_j = \begin{cases} 1\ (-\infty < X_j \le \Phi^{-1}(p_j)) \\ 0\ \ (\Phi^{-1}(p_j) < X_j < \infty) \end{cases} . \tag{18}$$

If the correlation coefficient of D_i, D_j is ρ_{ij}, then the following relationship holds:

$$\rho_{ij}=\frac{E[D_i D_j]-p_i p_j}{\sqrt{p_i(1-p_i)}\sqrt{p_j(1-p_j)}}\ . \tag{19}$$

Likewise, if the correlation coefficient of X_i, X_j is $\tilde{\rho}_{ij}$, then through E[D_iD_j], the relationship between ρ_{ij} and $\tilde{\rho}_{ij}$ is:

$$E[D_i D_j]=\int_{-\infty}^{\Phi^{-1}(p_i)}\int_{-\infty}^{\Phi^{-1}(p_j)}\frac{1}{2\pi\sqrt{1-\tilde{\rho}_{ij}^2}}\exp\left(-\frac{1}{2(1-\tilde{\rho}_{ij}^2)}\left\{x_i^2+x_j^2-2\tilde{\rho}_{ij}x_i x_j\right\}\right)dx_i\,dx_j\,. \tag{20}$$

We can therefore find $\tilde{\rho}_{ij}$ for all $i \neq j$ to arrive at a correlation matrix of standard normal distribution variables $X_1, \cdots, X_n$ such that we can get $D_1, \cdots, D_n$.

2. Correlation between default events

Generally, there are two methods for calculating correlation coefficients between default events. One method uses a corporate asset value model, the other utilizes bond default data. We explain the details of these methods in Appendix 2. Suffice it to say that, for our purposes, we have chosen to use actual default data to confirm the level of correlation coefficients between default events.

Our data comes from historical default data of Moody's ratings (Keenan, Shtogrin and Sobehart [1999]).

We begin with using default rate data for the 1970-1998 period to calculate an average p and variance σ^2 for the default rate at each rating level. Then, assuming that an adequately large sample can be obtained, the average default correlation $\bar{\rho}$ for each rating can be approximated as shown in Table 3.

Table 3 Annual Default Rates for Different Ratings and Correlations within Ratings

Year	Aaa	Aa	A	Baa	Ba	B
1970	0.00%	0.00%	0.00%	0.27%	4.12%	23.38%
1971	0.00%	0.00%	0.00%	0.00%	0.42%	4.00%
1972	0.00%	0.00%	0.00%	0.00%	0.00%	7.41%
1973	0.00%	0.00%	0.00%	0.45%	0.00%	3.92%
1974	0.00%	0.00%	0.00%	0.00%	0.00%	10.34%
1975	0.00%	0.00%	0.00%	0.00%	1.02%	6.15%
1976	0.00%	0.00%	0.00%	0.00%	1.01%	0.00%
1977	0.00%	0.00%	0.00%	0.27%	0.52%	3.39%
1978	0.00%	0.00%	0.00%	0.00%	1.08%	5.56%
1979	0.00%	0.00%	0.00%	0.00%	0.49%	0.00%
1980	0.00%	0.00%	0.00%	0.00%	0.00%	5.06%
1981	0.00%	0.00%	0.00%	0.00%	0.00%	4.60%
1982	0.00%	0.00%	0.26%	0.30%	2.73%	2.41%
1983	0.00%	0.00%	0.00%	0.00%	0.91%	6.36%
1984	0.00%	0.00%	0.00%	0.36%	0.83%	6.78%
1985	0.00%	0.00%	0.00%	0.00%	1.75%	8.28%
1986	0.00%	0.00%	0.00%	1.33%	2.05%	11.80%
1987	0.00%	0.00%	0.00%	0.00%	2.72%	5.86%
1988	0.00%	0.00%	0.00%	0.00%	1.24%	6.02%
1989	0.00%	0.61%	0.00%	0.60%	2.98%	9.17%
1990	0.00%	0.00%	0.00%	0.00%	3.32%	16.11%
1991	0.00%	0.00%	0.00%	0.28%	5.25%	14.66%
1992	0.00%	0.00%	0.00%	0.00%	0.30%	9.00%
1993	0.00%	0.00%	0.00%	0.00%	0.55%	5.76%
1994	0.00%	0.00%	0.00%	0.00%	0.23%	3.81%
1995	0.00%	0.00%	0.00%	0.00%	0.67%	4.84%
1996	0.00%	0.00%	0.00%	0.00%	0.00%	1.45%
1997	0.00%	0.00%	0.00%	0.00%	0.19%	2.10%
1998	0.00%	0.00%	0.00%	0.12%	0.61%	4.08%
p	0.00%	0.02%	0.01%	0.14%	1.21%	6.63%
σ^2	0	1.28E-06	2.33E-07	8.01E-06	1.88E-04	2.49E-03
$\bar{\rho}$	—	0.0061	0.0026	0.0058	0.0158	0.0402

It should be apparent from Table 3 that the default rate for A or better rated bonds is 0.00 percent in most years, so default correlation $\bar{\rho}$ calculated from this is not very reliable. However, for ratings of Baa or below, one finds that the lower the rating, i.e. the higher the default rate, the higher the correlation coefficient between default events. For example, at Baa (average default rate of 0.14 percent), the default correlation is 0.006; at Ba (average default rate of 1.21 percent), the default correlation is 0.016; and at B (average default rate of 6.63 percent), the default correlation is 0.040.

Below, we calculate the average default correlation $\bar{\rho}_{kl}$ for two different rating levels, k and l, both of which are in the Baa range or lower. The small number of samples makes it difficult to arrive at a firm conclusion, but the trend is for the default correlation to be larger, the lower the rating.

$\bar{\rho}_{Baa\,Ba} = 0.0031$,
$\bar{\rho}_{Baa\,B} = 0.0043$, and
$\bar{\rho}_{Ba\,B} = 0.0166$.

3. Profile of sample portfolio[16]

a. Internal ratings

This paper assumes seven rating levels, depending on the level of creditworthiness. Ratings are categorized by default rates (one-year default rates are assumed). We also assume that all exposures within a subportfolio have the same default rate, in other words, that default rates are discrete (Table 4).

Table 4 Default Rates for Ratings Level

Rating	1	2	3	4	5	6	7
Default rate	0.1%	0.5%	1.0%	2.0%	5.0%	10.0%	20.0%

b. Correlation coefficient between default events

The table below shows correlation coefficients assumed for default events within the subportfolio (Table 5). This reflects the conclusion from Section IV.A.2. above that the default correlation will be higher, the higher the default rate.

Table 5 Correlation Coefficient Between Default Events at Different Rating Levels

Ratings	1	2	3	4	5	6	7
Default correlation	0.001	0.005	0.010	0.010	0.015	0.017	0.020

c. Subportfolios for individual rating levels

We set the total exposure included in the subportfolio for any rating at ¥100 billion. We also set three different exposures: a. 100, b. 500, and c. 1,000. Likewise, we set six types of distribution for exposure: (1) homogeneous distribution, (2) concentration on a single borrower, (3) concentration on 10 percent of borrowers, (4) exponential distribution, (5) three levels, and (6) five levels (see Table 6, for details; this provides a total of eighteen combinations, which we numbered 1a-6c).

16. In this paper, we term a portfolio comprising exposures of the same rating a "subportfolio" and an assembly of subportfolios of different ratings a "sample portfolio."

Table 6 Exposure Distributions Etc. for Individual Subportfolios

	a. 100 borrowers	b. 500 borrowers	c. 1,000 borrowers
1) Homogeneous distribution	1a: ¥1 billion each (CF=0.100)	1b: ¥200 million each (CF=0.045)	1c: ¥100 million each (CF=0.032)
2) Concentration in one borrower (100-times others)	2a: ¥50,251 million for one, ¥502 million for the other 99 (CF=0.505)	2b: ¥16,694 million for one, ¥166 million for the other 499 (CF=0.171)	2c: ¥9,099 million for one, ¥90 million for the other 99 (CF=0.095)
3) Concentration in 10% of borrowers (100-times others)	3a: ¥9,174 million for 10, ¥92 million for the other 90 (CF=0.290)	3b: ¥1,835 million for 50, ¥18 million for the other 450 (CF=0.130)	3c: ¥917 million for 100, ¥9 million for the other 900 (CF=0.092)
4) Exponential distribution	4a: exponentially distributed with mean of ¥1 billion (CF=0.142)	4b: exponentially distributed with mean of ¥200 million (CF=0.065)	4c: exponentially distributed with mean of ¥100 million (CF=0.046)
5) 3-level distribution (double at 2nd level, 10-times at 3rd level)	5a: ¥285 million for 50, ¥1,429 million for 40, ¥2,857 million for 10 (CF=0.129)	5b: ¥57 million for 250, ¥286 million for 200, ¥571 million for next 50 (CF=0.058)	5c: ¥29 million for 500, ¥143 million for 400, ¥286 million for 100 (CF=0.041)
6) 5-level distribution (multiplied by 1.5 at each level)	6a: ¥379 million, ¥569 million, ¥853 million, ¥1,280 million, ¥1,919 million, to 20 borrowers at each level (CF=0.114)	6b: ¥76 million, ¥114 million, ¥171 million, ¥256 million, ¥384 million to 100 borrowers at each level (CF=0.051)	6c: ¥38 million, ¥57 million, ¥85 million, ¥128 million, ¥192 million to 200 borrowers at each level (CF=0.036)

Note that Distributions 2 and 3 posit a portfolio with a relatively high degree of concentration. As discussed in the previous chapter, we do not anticipate a very high degree of precision for approximations of maximum loss for such portfolios. Indeed, they were set in order to verify that point.

We noted the CF for these eighteen subportfolios. These figures show the most diversified portfolio to be 1c and the most concentrated to be 2a.

d. Sample portfolios

We created twelve sample portfolios by combining the subportfolios for individual ratings described in Section IV.A.3.c. above. Each sample portfolio is made up of (rating-based) subportfolios with the same distribution and number of exposures. We have numbered the sample portfolios 1A-6A (total of 700 exposures) and 1B-6B (total of 3,500 exposures) (see Table 7, for details).

Table 7 Sample Portfolios

Portfolio \ Rating	1	2	3	4	5	6	7	Total exposures	Total amount (¥100 million)
1A	1a	1a	1a	1a	1a	1a	1a	700	7,000
2A	2a	2a	2a	2a	2a	2a	2a	700	7,000
:	:	:	:	:	:	:	:	:	:
:	:	:	:	:	:	:	:	:	:
5B	5b	5b	5b	5b	5b	5b	5b	3,500	7,000
6B	6b	6b	6b	6b	6b	6b	6b	3,500	7,000

We assume that the correlation between default events of exposures i, j ($i \neq j$) depends only on the combination of rating k, to which i belongs, and rating l, to which j belongs. For the correlation coefficient within individual ratings, we use the numbers from Table 5. Correlation coefficients between exposures with different ratings were set with reference to these levels as shown in Table 8.

Non-diagonal components ρ_{kl} ($k > l$) are determined as follows. First, we set the covariance of default events as $\sigma_{kk} = \rho_{kk}\, p_k\, (1 - p_k)$ using diagonal components. Next, we determine the covariance of non-diagonal components as $\sigma_{kl} = \min\,(\sigma_{kk}, \sigma_{ll})$.[17] Finally, we obtain ρ_{kl} using the relationship $\sigma_{kl} = \rho_{kl}\sqrt{p_k(1-p_k)}\sqrt{p_l(1-p_l)}$.

Table 8 Correlation Coefficients for Rating Combinations

Rating	1	2	3	4	5	6	7
1	0.0010						
2	0.0004	0.0050					
3	0.0003	0.0035	0.0100				
4	0.0002	0.0025	0.0071	0.0100			
5	0.0001	0.0016	0.0046	0.0064	0.0150		
6	0.0001	0.0012	0.0033	0.0047	0.0109	0.0170	
7	0.0001	0.0009	0.0025	0.0035	0.0082	0.0127	0.0200

4. Detailed description of simulation method

We set the subportfolios and sample portfolios that combine them as described above, and calculate the UL using a Monte Carlo simulation with N=100,000.

a. Simulation of subportfolios

We assume a subportfolio of rating k with the number of exposures n, default rate p_k, and a correlation coefficient between default events ρ_{kk}. To use a Monte Carlo simulation to calculate the UL of this subportfolio, one must have a set of correlated n-variate Bernoulli random numbers $d^1 = (d_1^1, \cdots, d_n^1)$, $d^2 = (d_1^2, \cdots, d_n^2), \cdots, d^N = (d_1^N, \cdots, d_n^N)$. The procedure for obtaining this set of random numbers is described in Section IV.A.1.a.

17. We have imposed the condition that the covariance of different ratings (k, l) σ_{kl} cannot be higher than the level of covariance of the individual ratings (k, l) σ_{kk}, σ_{ll} so that the variance/covariance matrix obtained is applicable to the Cholesky decomposition. It is possible that there will be cases in which this condition is not valid, but our purpose is to provide an example of UL calculation for the sample portfolio, so we do not delve into this point deeply in this paper. Rather, we opt to move forward with the discussion, under the assumption that the variance/covariance matrix will be applicable to the Cholesky decomposition.

We set the value of n exposures included in this subportfolio at $v_{k,1},\cdots, v_{k,n}$ (× ¥100 million) each, so that the loss l^1 (× ¥100 million) for the first trial d^1 is found as $l^1 = \sum_{i=1}^{n} d_i^1 v_{k,i}$. For the second trial and beyond, we also derive $l^2,\cdots, l^N$ and draw histograms of $l^1,\ldots, l^N$. We then assume these histograms to represent the true loss distribution, calculate the percentage points, and deem this to be the UL.

b. Simulation of sample portfolios

As described in Section IV.A.3.d, our sample portfolios contain seven levels of ratings. From the perspective of calculating losses, the major difference between the sample portfolios and the subportfolios is that, in the sample portfolios, the default rates and correlations between default events are not constant. See Section IV.A.1.b for a description of the method used to generate multivariate Bernoulli random numbers when default rates and correlations between default events are not constant. The other procedures, as far as the calculation of the UL, are the same as for the subportfolios.

B Simulation Results and Discussions

1. Subportfolio simulations

We use the methods described in the preceding section to calculate the UL for each of the subportfolios. It is normal practice when calculating the VaR for market risk to use the 99 percent point of loss distribution as the UL, but here we calculate both the 99 percent point and the 99.9 percent point of loss distribution obtained from the results of 100,000 calculations. For the remainder of this section, we assume that the UL equals maximum loss.

The example in Figure 4 plots the loss distribution when three ρ values (0.01, 0.10 and 0.20) are set for a homogeneous portfolio with a default rate of 0.1 percent (number of exposures 500, total amount ¥100 billion). The intersections with the horizontal lines indicate the loss amount at that level. At the 99 percent point, there is little difference due to differences in ρ levels. Indeed, $\rho = 0.10$, which could be assumed to have a smaller loss than $\rho = 0.20$, actually has a slightly larger loss. In other words, at the 99 percent point, the subportfolio's risk is not accurately captured. However, at the 99.9 percent point, these problems do not occur. When ρ and other parameters are set for actual analysis, there are no cases like that shown in Figure 4, but we calculate both the 99.0 percent point and the 99.9 percent point just to be sure.

In Figure 5, we have an example of loss probability density distribution based on the results of subportfolio simulation.[18] This sample shows ratings 6 and 7 for subportfolio 1b (homogeneous portfolio, 500 borrowers). Note that the distribution is not symmetric.

18. These distributions are also obtained analytically without simulation. See Appendix 1 for details.

Figure 4 Loss Distribution (number of borrowers: 500, default rate: 0.1 percent)

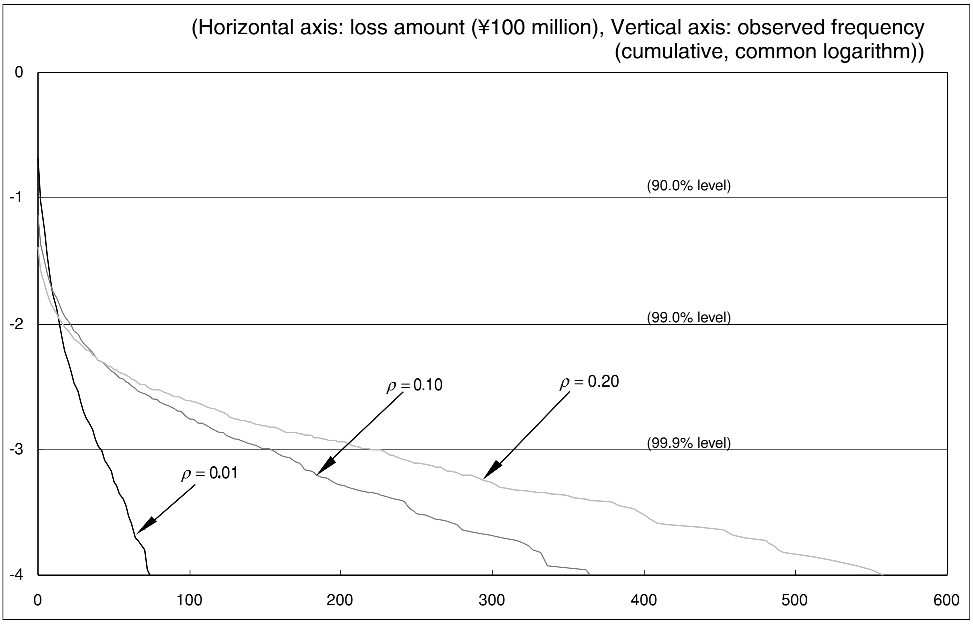
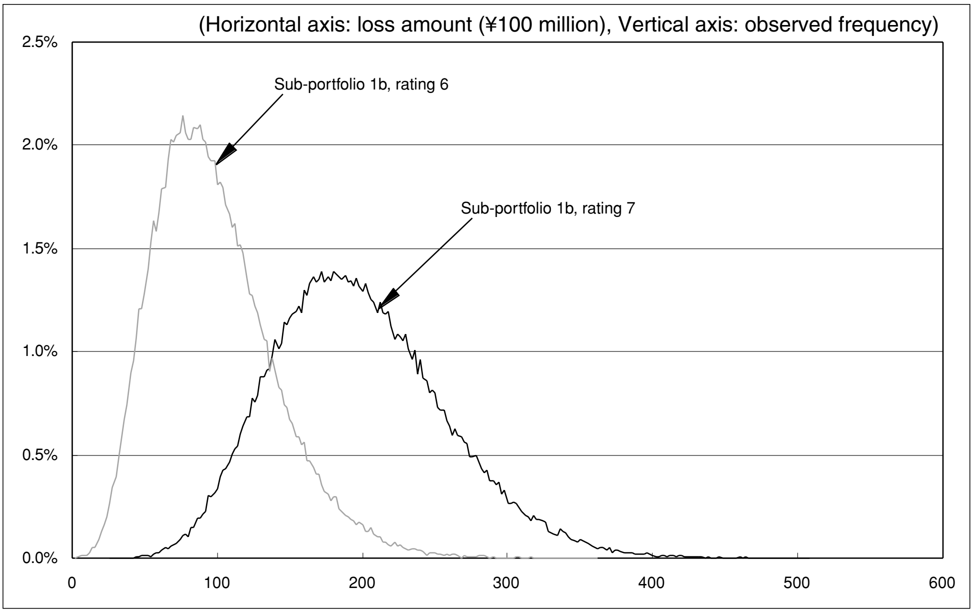


Figure 5 Example of Loss Amount Probability Density Distribution



In Figure 6-10, we show the relative error for the ULs calculated with the approximation in Equation (11) and the ULs calculated with the simulation for subportfolios 2-6.[19,20]

Figure 6 ExCF and Relative Error of Subportfolio 2

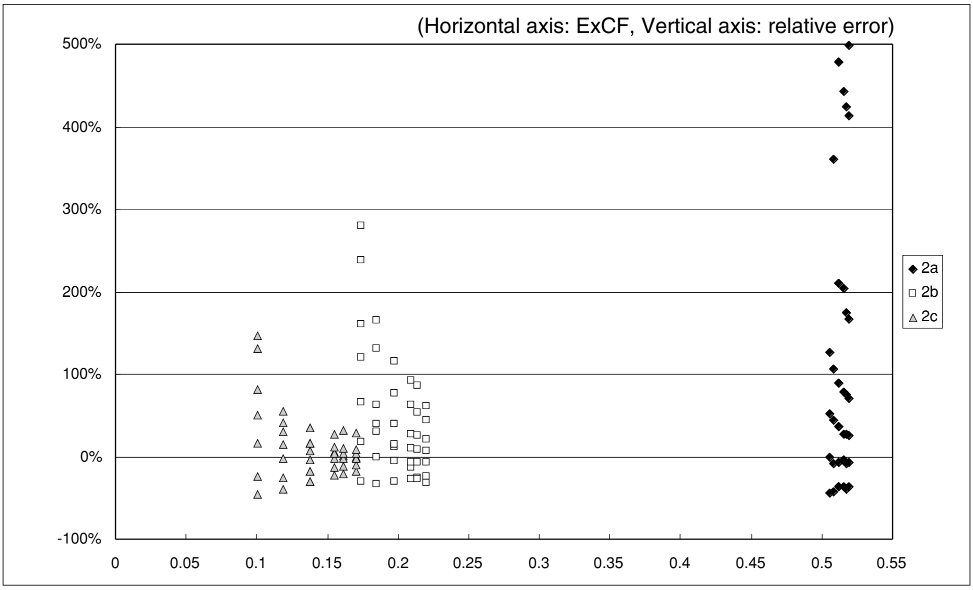
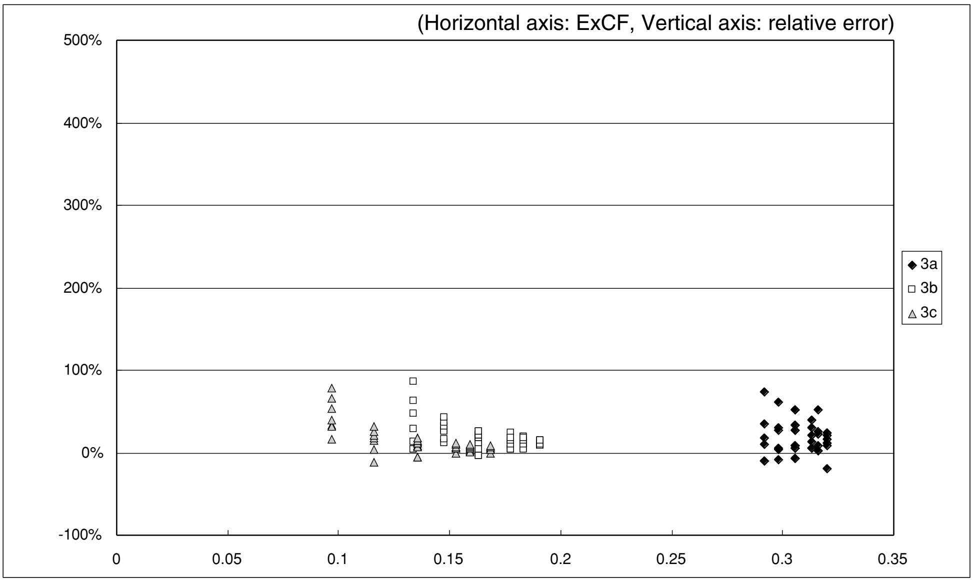


Figure 7 ExCF and Relative Error of Subportfolio 3



19. In this case, relative error is the error in the approximated value for the 99th percentile UL obtained from the simulation.
20. For each subportfolio, we plot data (49 points) showing all combinations of default rates and default correlations. (However, we omit from Figure 6-10 any data with a relative error of 500 percent or more.)

Figure 8 ExCF and Relative Error of Subportfolio 4

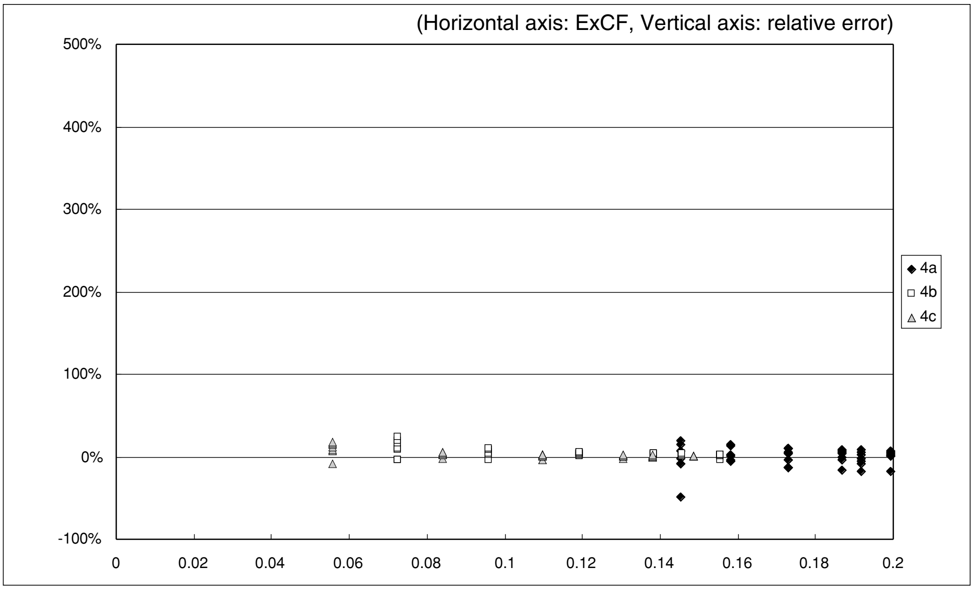


Figure 9 ExCF and Relative Error of Subportfolio 5

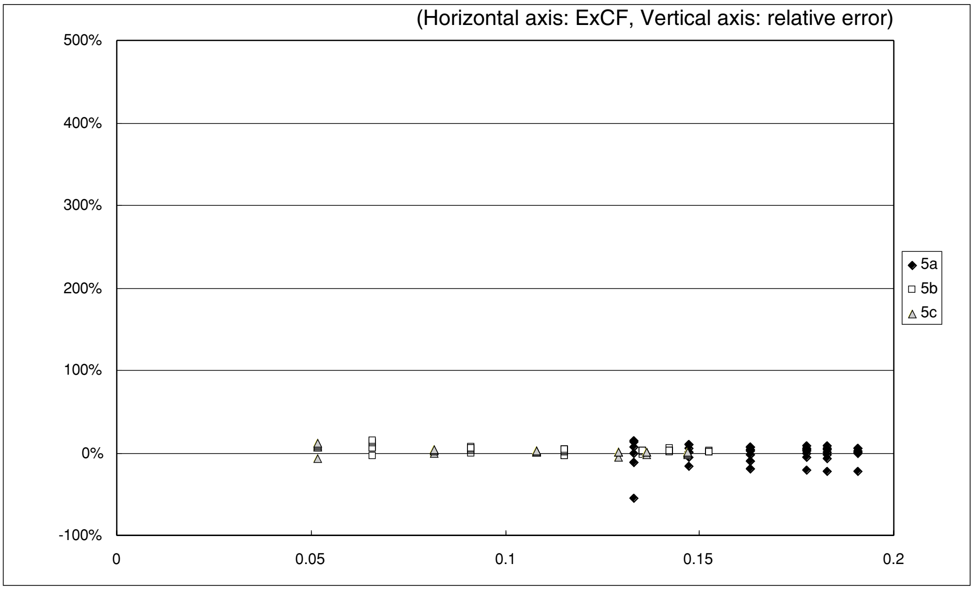


Figure 10 ExCF and Relative Error of Subportfolio 6

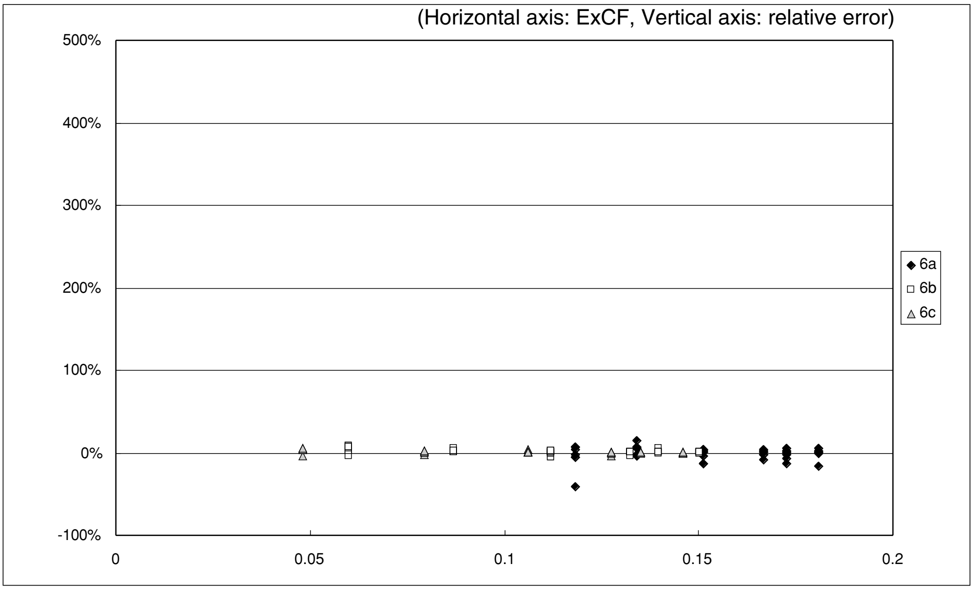


Table 9 Average and Maximum Absolute Values of Relative Error (unit: percent)

	2a	2b	2c	3a	3b	3c	4a	4b	4c
Average absolute value	230.53	53.74	24.42	128.13	18.31	14.73	8.34	4.50	3.34
Maximum value	860.01	280.43	147.00	1,218.43	86.29	78.90	49.41	23.53	18.53

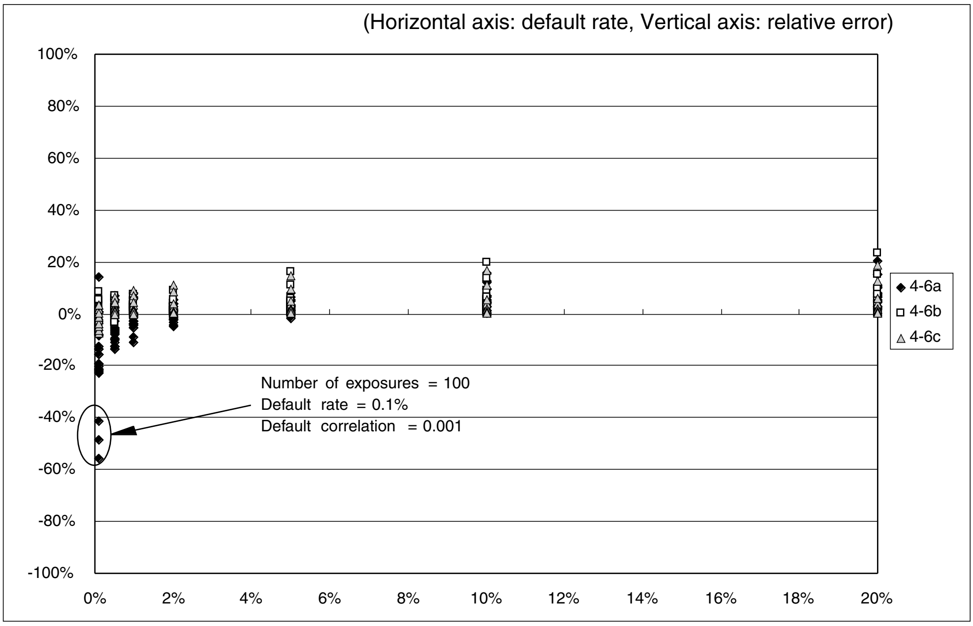
	5a	5b	5c	6a	6b	6c
Average absolute value	8.22	3.46	2.53	5.09	2.10	1.42
Maximum value	55.56	15.15	12.37	41.16	8.23	5.84

In Table 9, we show the average and maximum values of the absolute value of the relative errors for each subportfolio.

These results indicate that the relative error is larger when there is a large exposure to a single borrower (subportfolio 2) and when lending is concentrated on 10 percent of borrowers (subportfolio 3). By contrast, the average absolute value of relative error is only a few percent for the subportfolios in which exposures are fairly diversified (subportfolios 4-6 shown in the shaded cells). There are unlikely to be many cases in which real portfolios of financial institutions are as overly concentrated as in subportfolios 2 and 3, so the UL approximation in Equation (11) would seem to work to some extent in practical applications, within the range of simulations run for these subportfolios.

Note that subportfolios 4a, 5a, and 6a have comparatively large relative errors of -40 percent or more, though for only one example each. In Figure 11, we show the relationship between default rates and relative errors for subportfolios 4a-6a, 4b-6b, and 4c-6c.

Figure 11 Default Rates and Relative Error for Subportfolios 4-6



One can see that a maximum relative error in excess of -40 percent is generated when the number of exposures is 100, the default rate 0.1 percent, and the default correlation 0.001.

As discussed in Footnote 15, during economic slumps such as Japan is currently experiencing, there may not be all that many exposures in lending portfolios of financial institutions that have high internal ratings (i.e., ones that have fairly low default rates). Because of this, it may not be possible to achieve sufficient precision when approximating subportfolios with high internal ratings, as we have done here. For such subportfolios, however, it may be possible to calculate relative error levels from simulation results and use those levels (in the examples above, -40 percent or more) as given to estimate an adjusted UL.

2. Calculation of the ULs for the sample portfolios

We use the methods described in Section IV.A. to calculate the ULs of sample portfolios 1A-6A and 1B-6B. We also calculate the ULs of each of the subportfolios and total them for comparison in Table 10.

Note that the arithmetical total of the ULs calculated by the subportfolio is naturally larger than the ULs of the sample portfolios. In these results, the former is between 1.2 and 2.0 times larger than the latter.[21]

3. Simplified method for calculating portfolio UL

One conceivable simplified method for calculating the UL of a sample portfolio would be to total the ULs of individual subportfolios, and then divide by the multiples shown in Section IV.B.2. above. These multiples can probably be considered constant unless there are substantial changes in the composition of the portfolio.

21. Generally, the more rating categories there are, the more the exposure diversification effect is offset within the subportfolios, so this multiple is larger.

Table 10 ULs of Sample Portfolios

	99.0 percent UL (¥100 million)			99.9 percent UL (¥100 million)		
	(1) Overall total	(2) subportfolio total	(2)/(1)	(1) Overall total	(2) subportfolio total	(2)/(1)
1A	750.00	980.00	1.31	920.00	1,280.00	1.39
2A	1,296.48	2,417.09	1.86	1,788.95	3,537.69	1.98
3A	958.12	1,609.17	1.68	1,164.22	2,273.39	1.95
4A	779.12	1,117.85	1.43	946.62	1,489.89	1.57
5A	771.43	1,074.29	1.39	937.14	1,417.14	1.51
6A	760.66	1,026.07	1.35	918.72	1,352.37	1.47
1B	726.00	876.00	1.21	878.00	1,122.00	1.28
2B	828.05	1,288.82	1.56	1,028.38	1,764.61	1.72
3B	773.95	1,055.60	1.36	953.39	1,394.68	1.46
4B	732.84	905.23	1.24	893.02	1,163.86	1.30
5B	730.85	892.57	1.22	893.14	1,138.85	1.28
6B	729.57	882.94	1.21	887.77	1,126.49	1.27

Given the low probability of substantial changes in the lending portfolios of financial institutions,[22] at least over the short term, it may be sufficient to recalculate the multiples periodically using simulations.

For the subportfolio UL, as described above, even when there are changes in the heterogeneity of the portfolio, as long as the number of exposures n does not change, or the change in n is negligible for practical purposes, the ULs can be approximated by using the UL of a homogeneous portfolio with n exposures as a proxy for the heterogeneous subportfolio.

V. Conclusions

It is common to use computer simulation to calculate the credit risks associated with the lending portfolios of financial institutions. However, these simulations are time-consuming when there is a large number of exposures involved, which makes it difficult to calculate credit risk dynamically.

This paper describes a simplified technique for approximating the credit risk of lending portfolios that attempts to minimize, wherever possible, the simulation burden. There are two main points in this technique: (1) when the number of exposures and the total amount of exposures in the portfolio are constant, the standard deviation of loss is smallest in a homogeneous portfolio in which all exposures are of the same amount; and (2) the heterogeneity of the portfolio can be defined as the CF or the ExCF which adjusts the CF using a default correlation. The loss for a heterogeneous portfolio at an arbitrary confidence level (for example, the 99th percentile) can therefore be approximated by multiplying the loss of a homogeneous portfolio (in which the number of exposures and amounts are the same) by the ratio between the ExCFs of the two portfolios.

22. Currently, the rate of change in the total of lending assets outstanding is extremely low at Japanese banks.

In this paper, we run simulations for several types of hypothetical portfolios to verify whether these assumptions are valid in practical applications. The results of our simulations indicate that it is possible that the simplified technique using the ExCF can approximate the credit risk of a portfolio with some degree of precision (our results indicate a relative error of less than 10 percent on average), except in cases in which there are large exposures and a high degree of concentration in the portfolio. We have verified that our assumptions hold within a range that is generally suited to practical application.

Obviously, the approximations we describe here will contain more error than simulations. However, unlike trading portfolios, the composition of lending portfolios does not change all that rapidly, so it will be effective to use the approximations described in this paper, together with simulations.

APPENDIX 1: ANALYTICAL CALCULATION OF MAXIMUM LOSS FOR A HOMOGENEOUS PORTFOLIO

A. Analytical Calculation of Maximum Loss

In a homogeneous portfolio, the exposure to each borrower is the same, so maximum losses can be calculated in terms of how many borrowers will default at a set confidence level (for example, 99 percent or 99.9 percent). If the probability that n out of N borrowers will default is P_n^N, then the 99 percent maximum loss is mv, where m is the minimum integer which satisfies $\sum_{n=0}^{m} P_n^N \geq 0.99$ and v is the exposure amount per borrower.

One must therefore begin by finding the probability P_n^N that n of N borrowers will default. The default rate of each borrower is p. The default correlation is given as in Section IV.A.1.a.: $\Phi(\cdot)$ is the cumulative density function of the standard normal distribution and $\alpha = \Phi^{-1}(p)$, so that P_n^N can, through the normal distribution, be expressed as shown in Equation (A.1).

$$P_n^N = \Pr\{X_1 \leq \alpha, \ldots, X_n \leq \alpha, \alpha < X_{n+1}, \ldots, \alpha < X_N\}\, {}_N C_n \qquad \text{(A.1)}$$
$$\text{where } X_i \sim N(0,1),\ Cor(X_i, X_j) = \tilde{\rho}\,(i \neq j).$$

In this equation, ${}_N C_n$ is the number of combinations choosing n out of N. The random variable X_i has a correlation as shown in Equation (A.1), so it is not independent, but it can be rewritten using two independent random variables U and V_i:

$$X_i = \sqrt{\tilde{\rho}}\,U + \sqrt{1-\tilde{\rho}}\,V_i,\ U, V_i \sim N(0,1)\ i.i.d. \qquad \text{(A.2)}$$

Using Equations (A.1) and (A.2), P_n^N can be transformed as follows:

$$\begin{aligned} P_n^N &= \Pr\{\sqrt{1-\tilde{\rho}}\,V_1 \leq \alpha - \sqrt{\tilde{\rho}}\,U, \ldots, \sqrt{1-\tilde{\rho}}\,V_n \leq \alpha - \sqrt{\tilde{\rho}}\,U, \\ &\qquad \alpha - \sqrt{\tilde{\rho}}\,U < \sqrt{1-\tilde{\rho}}\,V_{n+1}, \ldots, \alpha - \sqrt{\tilde{\rho}}\,U < \sqrt{1-\tilde{\rho}}\,V_N\}\, {}_N C_n \\ &= \int_{-\infty}^{\infty} \Pr\{\sqrt{1-\tilde{\rho}}\,V_1 \leq \alpha - \sqrt{\tilde{\rho}}\,u, \ldots, \sqrt{1-\tilde{\rho}}\,V_n \leq \alpha - \sqrt{\tilde{\rho}}\,u, \qquad \text{(A.3)} \\ &\qquad \alpha - \sqrt{\tilde{\rho}}\,u < \sqrt{1-\tilde{\rho}}\,V_{n+1}, \ldots, \alpha - \sqrt{\tilde{\rho}}\,u < \sqrt{1-\tilde{\rho}}\,V_N\}\, \phi(u)\, du\, {}_N C_n, \end{aligned}$$

where $\phi(u)$ is the probability density function of the standard normal distribution, that is,

$$\phi(u) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right). \qquad \text{(A.4)}$$

Equation (A.3) can be further transformed:

$$P_n^N = \int_{-\infty}^{\infty} \Pr\{\sqrt{1-\tilde{\rho}}\,V_1 \leq \alpha - \sqrt{\tilde{\rho}}\,u\} \ldots \Pr\{\sqrt{1-\tilde{\rho}}\,V_n \leq \alpha - \sqrt{\tilde{\rho}}\,u\}$$

$$\Pr\{\alpha - \sqrt{\tilde{\rho}}\,U < \sqrt{1-\tilde{\rho}}\,V_{n+1}\} \ldots \Pr\{\alpha - \sqrt{\tilde{\rho}}\,U < \sqrt{1-\tilde{\rho}}\,V_N\}$$

$$\phi(u)\,du\,{}_N C_n$$

$$= \int_{-\infty}^{\infty} \Phi\left(\frac{\alpha - \sqrt{\tilde{\rho}}\,u}{\sqrt{1-\tilde{\rho}}}\right) \cdots \Phi\left(\frac{\alpha - \sqrt{\tilde{\rho}}\,u}{\sqrt{1-\tilde{\rho}}}\right) \left\{1 - \Phi\left(\frac{\alpha - \sqrt{\tilde{\rho}}\,u}{\sqrt{1-\tilde{\rho}}}\right)\right\} \cdots \tag{A.5}$$

$$\left\{1 - \Phi\left(\frac{\alpha - \sqrt{\tilde{\rho}}\,u}{\sqrt{1-\tilde{\rho}}}\right)\right\} \phi(u)\,du\,{}_N C_n.$$

$$= \int_{-\infty}^{\infty} \left\{\Phi\left(\frac{\alpha - \sqrt{\tilde{\rho}}\,u}{\sqrt{1-\tilde{\rho}}}\right)\right\}^n \left\{1 - \Phi\left(\frac{\alpha - \sqrt{\tilde{\rho}}\,u}{\sqrt{1-\tilde{\rho}}}\right)\right\}^{N-n} \phi(u)\,du\,{}_N C_n.$$

This transformation utilizes the fact that V_i s are mutually independent.

Therefore, in order to arrive at the 99 percent maximum loss, one first obtains the smallest m such that:

$$\sum_{n=0}^{m} \int_{-\infty}^{\infty} \left\{\Phi\left(\frac{\alpha - \sqrt{\tilde{\rho}}\,u}{\sqrt{1-\tilde{\rho}}}\right)\right\}^n \left\{1 - \Phi\left(\frac{\alpha - \sqrt{\tilde{\rho}}\,u}{\sqrt{1-\tilde{\rho}}}\right)\right\}^{N-n} \phi(u)\,du\,{}_N C_n \geq 0.99. \tag{A.6}$$

One then multiplies this by the exposure amount of each borrower v to arrive at *mv*, which is the 99 percent maximum loss.

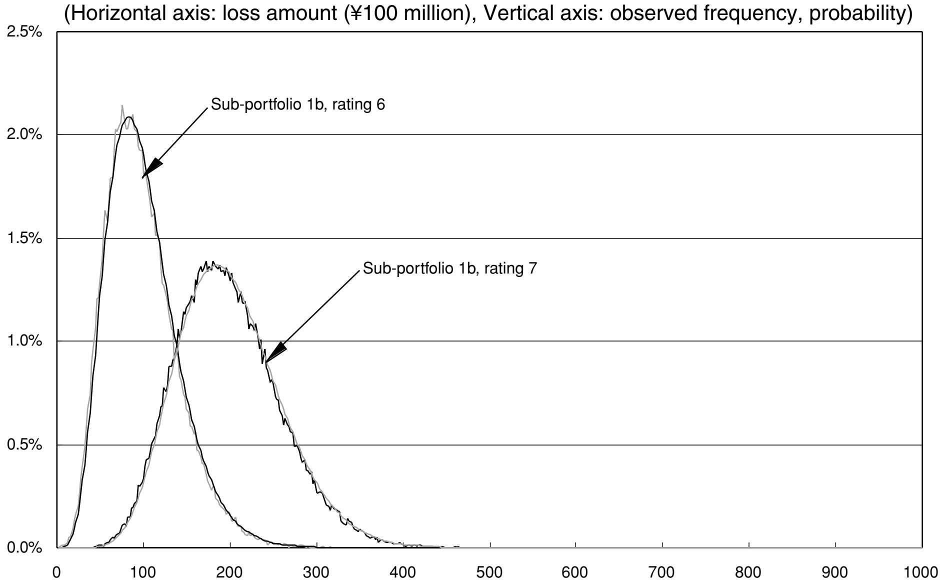
B. Comparison with Results from Loss Distribution Simulation

The concepts described in Section A.A. can be used to analytically derive a loss distribution because P_n^N is the probability that n borrowers will default, i.e. that the loss will be *nv*. Appendix Figure superimposes the results obtained by this analytical technique, together with the loss probability density derived by simulation results from Figure 5 for the same subportfolios. In Appendix Figure, the smooth curves show the results obtained by the analytical method; the jagged curves are the results obtained by simulation.

Appendix Table compares the 99 percent maximum loss obtained by the two methods.

Appendix Figure Probability Density Distribution of Loss Amount (comparison of simulation and analytical methods)

(Horizontal axis: loss amount (¥100 million), Vertical axis: observed frequency, probability)



Note: There is very little difference in the results obtained by simulation and by the analytical method.

Appendix Table 99 Percent UL Obtained by Analytical Method and Simulation

Unit: ¥100 million

Rating of subportfolio 1b	Error (analytical method – simulation)
1	0.00
2	0.00
3	0.00
4	0.00
5	0.00
6	0.00
7	2.00

Note: There is very little difference in the results produced by the two techniques for 99 percent maximum loss.

APPENDIX 2: METHOD OF CALCULATING CORRELATION COEFFICIENTS BETWEEN DEFAULT EVENTS

A. Approach Using the Corporate Asset Value Model

The concepts of Merton (1974) indicate that default will occur if the value of a company's assets falls below a certain level. In other words, corporate asset value contains a threshold value that is the dividing line between default and non-default.

One can therefore create a model that assumes that the rate of return on assets will have the standard normal distribution. In other words, if the default rate of Company i is p_i, then the threshold value for default/non-default is given by $\Phi^{-1}(p_i)$, where $\Phi^{-1}(\cdot)$ is the inverse function of the cumulative density function of the standard normal distribution. This can be used to calculate p_{ij}, which is the simultaneous default rate for Company i and Company j:

$$p_{ij}=\int_{-\infty}^{\Phi^{-1}(p_i)}\int_{-\infty}^{\Phi^{-1}(p_j)}\frac{1}{2\pi\sqrt{1-r^2}}\exp\left(-\frac{1}{2(1-r^2)}\left\{x_i^2+x_j^2-2r\,x_i\,x_j\right\}\right)dx_i\,dx_j, \tag{A.7}$$

where r is the correlation coefficient between the asset profit rates[23] of companies i and j.

Therefore, Equation (A.8)[24] can be used to obtain the correlation coefficient between the default events of Company i and Company j, ρ_{ij}^D.

$$\rho_{ij}^D=\frac{p_{ij}-p_i\,p_j}{\sqrt{p_i(1-p_i)}\sqrt{p_j(1-p_j)}}. \tag{A.8}$$

B. Approach Using Bond Default Data

Let us now turn to a method that uses bond default data. This approach could, for example, be used with the ratings-based bond default data published by ratings agencies to find the average level of correlation between default events within a rating or among different ratings.

1. Correlation within a rating[25]

We first consider N companies with the same default rate (same rating). D_i is a random variable with a value of 1 when Company i defaults and 0 when it does not. The average default rate is p. This produces the following relationship:

$$D_i=\begin{cases}1 & (i: default)\\ 0 & (otherwise)\end{cases} \tag{A.9}$$

$$p=\frac{1}{N}\sum_{i=1}^{N}D_i. \tag{A.10}$$

23. CreditMetrics uses the rate of return on stock price as a proxy for that on assets and provides a framework for calculating its correlation coefficient.
24. In Zhou (1997), this technique is extended into a method that uses a first-passage-time model to calculate the correlation coefficient between default events.
25. We referred to Appendix F of J.P. Morgan & Co. (1997) for this section.

S is the total number of defaults, so $S = \sum_{i=1}^{N} D_i$. The variance of S is therefore:

$$
\begin{aligned}
Var(S) &= \sum_{i=1}^{N}\sum_{j=1}^{N} \rho_{ij}\sigma^2 \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N} \rho_{ij}\, p(1-p) \\
&= p(1-p)\left[N + 2\sum_{i=1}^{N}\sum_{j<i}\rho_{ij}\right].
\end{aligned}
\tag{A.11}
$$

The default correlation between companies is ρ_{ij}, so $\rho_{ii}=1$ and $\rho_{ji}=\rho_{ij}$. When one turns to the average default correlation $\bar{\rho}$ rather than the default correlation between companies ρ_{ij}, one can define $\bar{\rho}$ as follows:

$$
\bar{\rho} = \frac{2\sum_{i=1}^{N}\sum_{j<i}\rho_{ij}}{N(N-1)}. \tag{A.12}
$$

Using this to express the variance of S, we arrive at:

$$
Var(S) = p(1-p)[N + N(N-1)\bar{\rho}]. \tag{A.13}
$$

$\sigma^2 = Var(S/N)$, so

$$
\begin{aligned}
\sigma^2 &= Var\left(\frac{S}{N}\right) = \frac{Var(S)}{N^2} \\
&= p(1-p)\frac{1+(N-1)\bar{\rho}}{N}.
\end{aligned}
\tag{A.14}
$$

This can be transformed to express the average default correlation $\bar{\rho}$ as:

$$
\bar{\rho} = \frac{\dfrac{N\sigma^2}{p(1-p)} - 1}{N-1}. \tag{A.15}
$$

When N is large, Equation (A.14) can be approximated as:

$$
\bar{\rho} \cong \frac{\sigma^2}{p(1-p)}. \tag{A.16}
$$

2. Correlation between different ratings

Similarly, for different ratings k and l, random variables $D_{k,i}$ and $D_{l,j}$ can be defined for Company i and Company j, so their value is 1 in default and 0 otherwise. Assume that N companies have rating k, and M companies have rating l. The total numbers of defaults are defined as S_k and S_l, and the average default rates as p_k and p_l, as shown below.

$$
S_k = \sum_{i=1}^{N} D_{k,i},\ S_l = \sum_{j=1}^{M} D_{l,j} \tag{A.17}
$$

$$p_k = \frac{1}{N}\sum_{i=1}^{N} D_{k,i},\ p_l = \frac{1}{M}\sum_{j=1}^{M} D_{l,j} \quad . \tag{A.18}$$

The average default correlations $\bar{\rho}_{kl}$ for different ratings k and l are defined as:

$$\bar{\rho}_{kl} = \frac{\sum_{i=1}^{N}\sum_{j=1}^{M}\rho_{ij}}{NM}. \tag{A.19}$$

Like Equation (A.13), the covariance of S_k and S_l becomes as follows:

$$\begin{aligned} Cov(S_k, S_l) &= \sum_{i=1}^{N}\sum_{j=1}^{M}\rho_{ij}\sqrt{p_k(1-p_k)}\sqrt{p_l(1-p_l)} \\ &= \sqrt{p_k(1-p_k)}\sqrt{p_l(1-p_l)}\,\bar{\rho}_{kl}\,NM. \end{aligned} \tag{A.20}$$

However, the following also holds true:

$$\frac{Cov(S_k, S_l)}{NM} = Cov\left(\frac{S_k}{N}, \frac{S_l}{M}\right) = Cov(p_k, p_l). \tag{A.21}$$

Therefore, the average default correlation $\bar{\rho}_{kl}$ between ratings can be expressed as follows:

$$\bar{\rho}_{kl} = \frac{Cov(p_k, p_l)}{\sqrt{p_k(1-p_k)}\sqrt{p_l(1-p_l)}}. \tag{A.22}$$

References

Basel Committee on Banking Supervision, *Credit Risk Modelling: Current Practice and Application,* April 1999.

Credit Suisse Financial Products, *CreditRisk+,* 1997.

Financial Supervisory Agency/FISC: "Risk Kanri Model ni Kansuru Kenkyukai Hokokusho Fuzoku Bunsho 2: Shinyo Risk Kanri Model no Rironteki Bunseki (Horon) (Report of the Research Group on Risk Management Models, Annex 2: Theoretical Analysis of Credit Risk Management Models [Appendix])," July 1999 (in Japanese).

J.P. Morgan & Co., *CreditMetrics™ Technical Document,* April 1997.

Keenan, S. C., I. Shtogrin, and J. Sobehart, "Historical Default Rates of Corporate Bond Issues, 1920–98," Moody's Special Comment, March 1999.

Merton, R., "On the Pricing of Corporate Debt: The Risk Structure of Interest Rates," *Journal of Finance,* 28, 1974.

Treacy, W. F., and M. S. Carey, "Credit Risk Rating at Large U.S. Banks," *Federal Reserve Bulletin,* November 1998.

Zhou, C., "Default Correlation: An Analytical Result," Finance and Economics Discussion Series, Federal Reserve Board, May 1997.