AN EXTENSION OF THE JARROW-LANDO-TURNBULL MODEL TO RANDOM RECOVERY RATE

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Abstract. We extend the Markovian rating model of Jarrow, Lando and Turnbull for pricing defaultable zero-coupon bonds and other credit sensitive instruments such as credit spread options, allowing for a stochastic recovery rate. The extension is performed by expanding the default state into multiple states to which correspond possibly different recovery rates. We analyze the extended model and generalize the calibration procedures introduced by Jarrow, Lando and Turnbull [17] and Kijima and Komoribayashi [19].

1 - INTRODUCTION

Recent years have seen a dramatic growth in the total number of corporate defaults, as witnessed by empirical studies (S&P 2002 special report [7]). Moreover, since their introduction in the early 90s, the market of credit derivatives has rapidly risen to reach the size of $1189 billion and, according to the British Bankers Association, the predicted size, by the end of 2004, is about $5 trillion.

Hence, the introduction of models for evaluating risky securities, such as risky bonds, vulnerable claims, and for pricing credit derivatives, such as credit spread options, credit linked notes, credit default swaps, is widely recognized as one of the major issues in modern finance.

The first contribution on the pricing of risky debt dates back to the pioneering work of Black and Scholes [6] and Merton [23], as an application of

Option Pricing Theory. This approach was further explored and extended in several directions (see for instance Black and Cox [5], Shimko et al. [26], Longstaff and Schwarz [22], and, more recently, Buffet [8]). This class of models is often referred to as *Structural* or *Firm-Based Models*, in that they consider the firm value as the fundamental state variable and all defaultable securities issued by the firm are viewed as claims contingent on this variable.

Although of high economic content, the Structural approach encounters many drawbacks; for instance, since many of the firm's assets are typically not traded, the firm's value process is hardly observable; this complicates the implementation of the model. Moreover, the term structure of yield spreads implied by Firm Based Models seems not to be comparable with actual spreads. This happens because in the Structural approach the default time has a sort of predictability property: when the maturity of the debt is short and the firm value is large compared to the debt amount or to the default boundary, then the probability of defaulting is small, so that the implied yield spreads underestimate the observed ones.

Another approach, which has become popular in the last decade, avoids to make appeal to the value of the firm, and is therefore known as *Reduced Form Approach*. In this approach the default time is either an exogenous variable as in Jarrow and Turnbull [18], Duffie and Singleton [14], or it is linked to some other fundamental variables as in Jarrow, Lando and Turnbull [17] (henceforth JLT). In every case the default time has the property of being, in some sense, an unexpected event. In their paper, which is a refinement of Jarrow and Turnbull [18], JLT consider the rating as the fundamental variable driving the default process. The rating is a measure of the credit quality of an institution; it can be firm-specific or issue-specific. Commercial agencies such as Moody's, Standard & Poor's and Fitch assign ratings, on a periodic basis, to a wide range of corporations, sovereign entities and insurance companies. Moreover some major financial institutions use their own rankings based on internally developed methodologies ("internal ratings")[1]. In this paper the rating is meant to be any measure of the credit quality of an institution. A rating assessment is traditionally an element of a finite set (the set of *rating classes* or *grades*). For instance, the S&P Long Term Issuer Credit Ratings have the following classes: AAA, AA, A, BBB, BB, B, CCC, CC, C, D, where classes are ordered from the "best" (AAA) to the "worst" (C), and D stands for default. Of course the rating of a given corporate may change over time according

[1]We refer to Crouhy et al. [11] and Krahnen and Weber [20] for further information on rating agencies and internal ratings

to the improvement or deterioration of its credit quality. This phenomenon is sometimes referred to as *credit migration*.

In their model (which is developed in both discrete and continuous time) JLT assume that the rating of the firm issuing the debt (or the rating specific to that debt issue) follows, under the historical probability, a homogeneous Markov chain, with the default state being absorbing. Hence, under further assumptions, they supply a methodology for pricing and hedging risky bonds and credit derivatives.

The theoretical literature on credit risk has focused mainly on the specification of the default time and probability of default, while the issue of recovery rate has been somewhat disregarded[2]. For instance, JLT and Longstaff and Schwarz [22] assume in their models that the recovery rate is a known constant. A notable exception is the paper of Das and Tufano [14], who extend the JLT approach by assuming that the recovery rate follows a stochastic process correlated with the default-free forward rates (modelled as in Heath et al. [16]). Our approach differs from the one in [12] in that we assume the recovery rate to be linked to the rating and be embedded in the Markov chain process; more precisely, the default state is composed by several default classes, each one corresponding to a (possibly different) recovery rate. When the firm bankrupts, it is assigned to a default class with its recovery rate. Hence, in our model, the recovery upon default is a random variable correlated to the next-to-last rating previous to default. This feature is supported by some empirical facts: first, it is generally accepted that ratings are judgements about the expected loss in case of default, therefore depending both on probability of default and on severity of loss. Typically, the higher is the rating, the lower is the loss in case of default (see Berthault et al. [2]). Second, Fitch International Long-Term Credit Ratings adopt the same non-default classes as Standard & Poor's, while they identify three default grades, DDD, DD and D, corresponding to different types of bankruptcy proceedings; each of these classes is associated with a recovery rate in the range, respectively, of $90 \div 100\%$, $50 \div 90\%$ and $0 \div 50\%$ (see Fitch Ratings Definitions).

The approach in this paper provides a generalization of that of JLT. In particular, we get similar formulae for the price of a defaultable bond and for the term structure of defaultable rates. Here, the constant recovery rate is replaced by the expected recovery rate conditional on default and on the rating at the valuation time. Moreover, we extend the calibration procedures

[2] This is not the case for Structural models of the Merton-type, because the recovery rate is determined by the firm value and is therefore stochastic.

introduced by JLT and Kijima and Komoribayashi [19].

The paper is organized as follows : in the next section we introduce the market structure and recall the JLT model. In section 3 we provide our extension and in section 4 we address the issue of calibration for this model. Finally in section 5 we suggest a further extension.

2 - THE JARROW-LANDO-TURNBULL MODEL

In this section we review the (discrete time version of the) JLT model. Consider an economy with trading dates $0, 1, \dots, T$. The uncertainty is modeled by a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t \le T}, \mathbb{P})$, where the probability $\mathbb{P}$ is the so called historical or empirical probability. The σ-field $\mathcal{F}_t$ contains the information accumulated up to time t, available to all agents in the economy. Traded in this economy are default riskless zero-coupon bonds (e.g. treasury bonds) of all maturities, a default free money market instrument and defaultable zero-coupon bonds of all maturities issued by a corporation. Moreover, it is assumed that markets are arbitrage free. For $0 \le s \le t \le T$ let $B(s,t)$ be the time s price of the default-free zero coupon bond with unit face value at maturity t (i.e. $B(t,t) = 1$). The default-free (compounded) forward rates are defined for $0 \le s \le t \le T-1$ by

$$f(s,t) = -\log(B(s,t+1)/B(s,t)). \tag{1}$$

The default-free spot rate relative to the period $(t, t+1)$, denoted by $r(t)$, is defined as $r(t) = f(t,t)$ for $0 \le t \le T-1$. Finally, the money market account is given by $B(t) = \exp \sum_{0 \le s < t} r(s)$ for $1 \le t \le T$ and $B(0) = 1$.

Let $D(s,t)$ denote the price at time s of the corporate zero-coupon bond with unit face value at maturity t, for $0 \le s \le t \le T$. Since the firm is subject to default risk, at maturity the bondholders will receive the face value only if the firm has not defaulted. *If the firm has defaulted prior to maturity, then bondholders will receive a fraction* $0 \le \delta < 1$ *of the face value of the bond at maturity*[3]. The fraction δ is known as *recovery rate* and it is assumed to be deterministic and exogenously given. This assumption will be relaxed in the next section.

All price processes introduced so far are assumed to be adapted to the reference filtration $(\mathcal{F}_t)$. Note that this implies that $r(s)$ and $f(s,t)$ are $\mathcal{F}_s$-measurable.

[3] This recovery rule is sometimes referred to as *Fractional Recovery of Treasury Value*. Other recovery schemes could be considered: see Bielecki and Rutkowski [4, p. 7] for a review.

It is assumed that *there exists a unique probability measure* $\mathbb{Q}$ *(referred to as risk neutral probability or martingale measure) equivalent to* $\mathbb{P}$ *such that the discounted (i.e. normalized by the money market instrument) price of all traded securities are* $\mathbb{Q}$*-martingales.* It is well known that the existence of such probability derives from the absence of arbitrage opportunities, while its uniqueness corresponds to the market completeness hypothesis (see for instance Duffie [13]). In the following we will denote by $\mathbb{E}^{\mathbb{Q}}$ the expectation taken under $\mathbb{Q}$ and with $\mathbb{E}_s^{\mathbb{Q}}$ the expectation under $\mathbb{Q}$ conditional upon $\mathcal{F}_s$, for $0 \leq s \leq T$; moreover we let $\mathbb{Q}_s(A) = \mathbb{E}_s^{\mathbb{Q}}(\mathbb{I}_A)$ for all $A \in \mathcal{F}$, where, as usual, $\mathbb{I}_A$ denotes the indicator function of the set A (similar conventions apply with respect to the probability $\mathbb{P}$). As a consequence, the price of a traded security equals the expectation under the risk neutral probability $\mathbb{Q}$ of its discounted final payoff. In particular for a default-free bond with maturity t we have

$$B(s,t) = \mathbb{E}_s^{\mathbb{Q}} \left[\frac{B(s)}{B(t)} \right] \tag{2}$$

Denoting by τ the default time of the firm[4], assume that τ is an exogenously given random variable taking values in the set $\{0, 1, \ldots, T, +\infty\}$, where $\tau = +\infty$ means that the firm has not defaulted before time T. Therefore, since $D(t,t) = \mathbb{I}_{\{\tau > t\}} + \delta \mathbb{I}_{\{\tau \leq t\}}$,

$$\begin{aligned} D(s,t) &= \mathbb{E}_s^{\mathbb{Q}} \left[\frac{B(s)}{B(t)} (\mathbb{I}_{\{\tau > t\}} + \delta \mathbb{I}_{\{\tau \leq t\}}) \right] = \\ &= \mathbb{E}_s^{\mathbb{Q}} \left[\frac{B(s)}{B(t)} (\delta + (1-\delta) \mathbb{I}_{\{\tau > t\}}) \right]. \end{aligned} \tag{3}$$

In the JLT model, moreover, it is assumed that, in a sense, interest rate risk and default risk are independent: more precisely, *under the risk neutral probability* $\mathbb{Q}$ *the default time* τ *and the process* $(r(t))_{0 \leq t \leq T}$ *are independent conditionally on* $\mathcal{F}_s$*, for all* $0 \leq s \leq T$. Discussions on the implications and significance of this hypothesis can be found in Jarrow and Turnbull [18] and JLT. It follows that equation (3) simplifies to

$$\begin{aligned} D(s,t) &= \mathbb{E}_s^{\mathbb{Q}} \left[\frac{B(s)}{B(t)} \right] (\delta + (1-\delta) \mathbb{E}_s^{\mathbb{Q}} [\mathbb{I}_{\{\tau > t\}})] = \\ &= B(s,t)(\delta + (1-\delta) \mathbb{Q}_s(\tau > t)), \end{aligned} \tag{4}$$

[4]Default should be meant as any event resulting in the bondholders receiving less than the promised payment.

where we have used (2). As a result, the price of the bond is determined once we know the (risk neutral) survival probability $\mathbb{Q}_s(\tau > t)$ and the term structure of interest rates. The assumptions we have made so far are essentially those of Jarrow and Turnbull [18]. The JLT model refines the model of Jarrow and Turnbull by considering the rating of the firm as the factor driving the default. Let $\mathcal{C} = \{1, 2, \dots, K, K+1\}$ be the set of rating classes ordered from highest to lowest, with $K+1$ denoting the unique default class. For instance in S&P Long-Term Issuer Credit Ratings grade 1 is AAA and K corresponds to CCC. For $0 \le t \le T$ denote by $\eta(t)$ the rating at time t of (the firm issuing) the defaultable bond. The process $(\eta(t))$ is assumed to be adapted to the reference filtration $(\mathcal{F}_t)$. The default time is then defined as the first time the process $\eta(t)$ enters the default state

$$\tau = \inf\{t \ge 0 | \eta(t) = K+1\}.$$

The main feature of the JLT model is that it assumes the rating to be a Markov process under the historical probability. More specifically, *the process* $(\eta(t))_{0\le t\le T}$ *is a homogeneous* $(\mathcal{F}_t)$*-Markov chain under* $\mathbb{P}$ *with state space* $\mathcal{C}$*. The default state* $K+1$ *is the unique absorbing state.* Since the default state is absorbing, the one period transition matrix $\mathbf{P} = (p_{ij})_{i,j\in\mathcal{C}}$ can be written as

$$\mathbf{P} = \begin{pmatrix} \mathbf{A} & \mathbf{R} \\ \mathbf{0}' & 1 \end{pmatrix}$$

where

$$\mathbf{A} = \begin{pmatrix} p_{1,1} & \cdots & p_{1,K} \\ \dots\dots & \dots\dots & \dots\dots \\ p_{1,K} & \cdots & p_{K,K} \end{pmatrix}, \; \mathbf{R} = \begin{pmatrix} p_{1,K+1} \\ \vdots \\ p_{K,K+1} \end{pmatrix}$$

and $\mathbf{0}$ is the vector of $\mathbb{R}^K$ having all zero components[5]. Under the risk neutral probability $\mathbb{Q}$ the process $(\eta(t))$ need not be Markovian; Bielecki and Rutkowski [4, chapter 11] provide a condition guaranteeing that the process under the risk neutral measure is Markovian and investigate the relationship between the historical and risk neutral transition probabilities. For what follows, the next assumption is then necessary: *the process* $(\eta(t))_{0\le t\le T}$ *is a* $(\mathcal{F}_t)$*-Markov chain under* $\mathbb{Q}$. Note that the Markov chain is not supposed to be homogeneous. Therefore the transition matrix from time t to $t+1$,

[5] Here and in the sequel all vectors will be column. Moreover the transposed of a vector $\mathbf{v}$ is denoted by $\mathbf{v}'$.

denoted by $\mathbf{Q}(t) = (q_{ij}(t))_{i,j\in\mathcal{C}}$, for $0 \le t \le T-1$, depends on the time t. Since $\mathbb{P}$ and $\mathbb{Q}$ are equivalent, we must have $p_{ij} = 0$ if and only if $q_{ij}(t) = 0$ for all $0 \le t \le T-1$, for all $1 \le i, j \in \mathcal{C}$. In particular this implies that the default state is absorbing also under $\mathbb{Q}$ so that

$$\mathbf{Q}(t) = \begin{pmatrix} \mathbf{A}(t) & \mathbf{R}(t) \\ \mathbf{0}' & 1 \end{pmatrix}$$

where

$$\mathbf{A}(t) = \begin{pmatrix} q_{1,1}(t) & \dots & q_{1,K}(t) \\ \dots & \dots & \dots \\ q_{1,K}(t) & \dots & q_{K,K}(t) \end{pmatrix}, \ \mathbf{R}(t) = \begin{pmatrix} q_{1,K+1}(t) \\ \vdots \\ q_{K,K+1}(t) \end{pmatrix}.$$

Therefore the transition matrix $\mathbf{Q}(s,t) = (q_{i,j}(s,t))_{i,j\in\mathcal{C}}$ from time s to time t, for $0 \le s \le t \le T$, is given by

$$\mathbf{Q}(s,t) = \mathbf{Q}(s)\mathbf{Q}(s+1)\cdots\mathbf{Q}(t-1).$$

Hence, keeping in mind that the default state is absorbing and the Markov property, the survival probability can be computed in terms of this matrix:

$$\begin{aligned}\mathbb{Q}_s(\tau > t) &= \mathbb{Q}(\eta(t) \ne K+1|\mathcal{F}_s) = \\ &= 1 - \mathbb{Q}(\eta(t) = K+1|\mathcal{F}_s) = \\ &= 1 - \mathbb{Q}(\eta(t) = K+1|\eta(s)) = \\ &= 1 - q_{\eta(s),K+1}(s,t).\end{aligned}$$

Substituting this expression in (4) gives the price of the risky bond:

$$\begin{aligned} D(s,t) &= B(s,t)\Big(\delta + (1-\delta)(1 - q_{\eta(s),K+1}(s,t))\Big) = \\ &= B(s,t)\Big(1 - (1-\delta)q_{\eta(s),K+1}(s,t)\Big). \end{aligned} \tag{5}$$

Therefore, writing $D_i(s,t)$ for $D(s,t)$ when the rating at time s is i (that is on the event $\{\eta(s) = i\}$), we have

$$D_i(s,t) = B(s,t)(1 - (1-\delta)q_{i,K+1}(s,t)). \tag{6}$$

In particular, the post-default value of the bond is $D_{K+1}(s,t) = \delta B(s,t)$.

3 - RANDOM RECOVERY RATE IN DEFAULT

In this section we extend the JLT model introducing randomness for the recovery rate in case of default. We preserve the market structure, terminology and assumptions of the previous section, unless they are not explicitly modified.

The claimed extension is realized by enlarging the set of rating classes to include multiple default classes, one for each possible recovery rate. More precisely, we assume that the set of rating classes is

$$\mathcal{C} = \{1, 2, \dots, K, K+1, \dots, K+L\}.$$

Letting $\mathcal{B} = \mathcal{C} - \mathcal{S}$, with $\mathcal{S} = \{1, 2, \dots, K\}$ denoting the set of non-default grades, then each element in $\mathcal{B}$ represents a default class corresponding to a possibly different recovery rate. Consider a nonnegative function δ defined on $\mathcal{B}$ where, for $i \in \mathcal{B}$, δ_i is the recovery rate when the default state is i. We can extend δ to the whole state space $\mathcal{C}$ by setting $\delta_i = 1$ on $\mathcal{S}$, so that $\delta_{\eta(t)}$ is a well defined random variable for $0 \leq t \leq T$. Without loss of generality, assume that $\delta_{K+1} \geq \delta_{K+2} \geq \dots \geq \delta_{K+L} \geq 0$ so that the class $K+1$ corresponds to the highest recovery rate in default and $K+L$ to the lowest. For instance Fitch International Long-Term Credit Ratings have three default classes DDD, DD, D corresponding to a potential recovery in the range respectively $90 \div 100\%$, $50 \div 90\%$, $0 \div 50\%$. Note that the case $\delta_{K+L} = 0$ (zero recovery) is not excluded and we do not assume $\delta_{K+1} < 1$ (situations where the recovery rate is larger than 100%, i.e. the bondholders benefit from default, can actually occur; see Carty and Hamilton [9] and the the discussion in Longstaff and Schwarz [22]). The default time is now defined by

$$\tau = \inf\{0 \leq t \leq T | \eta(t) \in \mathcal{B}\}$$

Note that τ is a stopping time (with respect to $(\mathcal{F}_t)$). That being stated, we then assume

(A1) *If the firm defaults in state $i \in \mathcal{B}$, the bondholders receive a fraction δ_i of the face value. In other words the final payoff of the defaultable bond is $D(t,t) = \mathbb{I}_{\{\tau > t\}} + \delta_{\eta(\tau)} \mathbb{I}_{\{\tau \leq t\}}$.*

As in the previous section, we require that under $\mathbb{P}$ the process $(\eta(t))$ is Markov:

(A2) *The process $(\eta(t))_{0 \leq t \leq T}$ is a homogeneous $(\mathcal{F}_t)$-Markov chain under $\mathbb{P}$ with state space $\mathcal{C}$. Each class in $\mathcal{B}$ is absorbing.*

Therefore, the one-period transition matrix $\mathbf{P} = (p_{i,j})_{i,j\in\mathcal{C}}$ has the form

$$\mathbf{P} = \begin{pmatrix} \mathbf{A} & \mathbf{R} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}$$

where

$$\mathbf{A} = \begin{pmatrix} p_{1,1} & \cdots & p_{1,K} \\ \cdots & \cdots & \cdots \\ p_{K,1} & \cdots & p_{K,K} \end{pmatrix}, \ \mathbf{R} = \begin{pmatrix} p_{1,K+1} & \cdots & p_{1,K+L} \\ \cdots & \cdots & \cdots \\ p_{K,K+1} & \cdots & p_{K,K+L} \end{pmatrix},$$

$\mathbf{0}$ is the $L \times K$ matrix with all zero elements and $\mathbf{I}$ is the identity matrix of rank L. Note that the matrix $\mathbf{R}$ collects the default probabilities, i.e. $p_{i,j}, i \in \mathcal{S}, j \in \mathcal{B}$ denotes the probability of defaulting at a generic date with recovery rate δ_j given that the rating is i at the previous time. To exemplify, we consider the two following simple cases:

Example 1. Assume that for all $i \in \mathcal{S}$ there is at most one $j \in \mathcal{B}$ such that $p_{i,j} > 0$, i.e. the matrix $\mathbf{R}$ has at most one nonzero element in each row. In other words, if the firm has defaulted then the recovery rate is just a function of the last rating previous to default. A similar specification of the recovery rate can be found in Arvanitis et al. [1] and in Bielecki and Rutkowski [3].

In the next example and in the sequel, let $p_{i,A} = \sum_{j\in A} p_{i,j}$ for all set $A \subset \mathcal{C}$.

Example 2. Suppose that for all $0 \le t \le T-1$, $i \in \mathcal{S}$, $j \in \mathcal{B}$ we have

$$\mathbb{P}(\eta(t+1) = j | \eta(t+1) \in \mathcal{B}, \eta(t) = i) =$$

$$= \mathbb{P}(\eta(t+1) = j | \eta(t+1) \in \mathcal{B}) := p_{j|\mathcal{B}}.$$

This hypothesis is equivalent to the independence (under $\mathbb{P}$) of $\eta(t)$ and $\eta(t+1)$ conditionally on $\eta(t+1) \in \mathcal{B}$ (for all $0 \le t \le T-1$)[6], which in turn

[6]In fact we have, for all real functions f, g defined on $\mathcal{C}$

$$\begin{aligned} &\mathbb{E}^{\mathbb{P}}[f(\eta(t+1))g(\eta(t))|\eta(t+1) \in \mathcal{B}] = \\ &= \sum_{j\in\mathcal{B}}\sum_{i\in\mathcal{C}} f(j)g(i)\mathbb{P}(\eta(t+1) = j, \eta(t) = i|\eta(t+1) \in \mathcal{B}) = \\ &= \sum_{j\in\mathcal{B}} f(j)p_{j|\mathcal{B}} \sum_{i\in\mathcal{C}} g(i)\mathbb{P}(\eta(t) = i|\eta(t+1) \in \mathcal{B}) = \\ &= \mathbb{E}^{\mathbb{P}}[f(\eta(t+1))|\eta(t+1) \in \mathcal{B}]\mathbb{E}^{\mathbb{P}}[g(\eta(t))|\eta(t+1) \in \mathcal{B}]. \end{aligned}$$

implies the conditional independence of $\delta_{\eta(t+1)}$ and $\eta(t)$. Moreover we have $p_{i,j} = p_{i,\mathcal{B}}\, p_{j|\mathcal{B}}$; hence any two rows (and any two columns) of the matrix $\mathbf{R}$ are proportional. The probability of defaulting with recovery δ_j conditional on being in state i can be computed in two steps: first compute the probability of default conditional on being in state i, and then the probability that the recovery be δ_j given bankruptcy.

As in the previous section, we need to require the Markovianity of $(\eta(t))$ under the (unique) risk neutral probability $\mathbb{Q}$:

(A3) *The process $(\eta(t))_{0\leq t\leq T}$ is a $(\mathcal{F}_t)$-Markov chain under $\mathbb{Q}$.*

Since we do not assume the homogeneity of the process, we denote by $\mathbf{Q}(t) = (q_{i,j}(t))_{i,j\in\mathcal{C}}$ the risk neutral one-period transition matrix at time t (for $0 \leq t \leq T-1$). The equivalence between $\mathbb{P}$ and $\mathbb{Q}$ implies that we must have $p_{i,j} = 0$ if and only if $q_{i,j}(t) = 0$ for $0 \leq t \leq T-1$, for all $i, j \in \mathcal{C}$. Therefore each state in $\mathcal{B}$ is absorbing, so that we have

$$\mathbf{Q}(t) = \begin{pmatrix} \mathbf{A}(t) & \mathbf{R}(t) \\ \mathbf{0} & \mathbf{I} \end{pmatrix}$$

where

$$\mathbf{A}(t) = \begin{pmatrix} q_{1,1}(t) & \dots & q_{1,K}(t) \\ \dots & \dots & \dots \\ q_{K,1}(t) & \dots & q_{K,K}(t) \end{pmatrix}, \ \mathbf{R} = \begin{pmatrix} q_{1,K+1}(t) & \dots & q_{1,K+L}(t) \\ \dots & \dots & \dots \\ q_{K,K+1}(t) & \dots & q_{K,K+L}(t) \end{pmatrix}.$$

Given the one-period transition matrices, we can compute the transition matrix $\mathbf{Q}(s,t) = (q_{i,j}(s,t))_{i,j\in\mathcal{C}}$ from time s to time t (for all $0 \leq s \leq t \leq T$), where $q_{i,j}(s,t) = \mathbb{Q}(\eta(t) = j|\eta(s) = i)$, as

$$\mathbf{Q}(s,t) = \mathbf{Q}(s)\mathbf{Q}(s+1)\cdots\mathbf{Q}(t-1).$$

In the sequel we let $q_{i,A}(s,t) = \sum_{j\in A} q_{i,j}(s,t)$ for all set $A \subset \mathcal{C}$. Assume further that

(A4) *Under the risk neutral probability $\mathbb{Q}$ the processes $(r(t))_{0\leq t\leq T}$ and $(\eta(t))_{0\leq t\leq T}$ are independent conditionally on $\mathcal{F}_s$ (for all $0 \leq s \leq T$)*[7]

[7]That is, for all measurable $f : \mathbb{R}^{T+1} \to \mathbb{R}$, $g : \mathcal{C}^{T+1} \to \mathbb{R}$, we have

$$\mathbb{E}_s^{\mathbb{Q}}\left[f(r(.))g(\eta(.))\right] = \mathbb{E}_s^{\mathbb{Q}}\left[f(r(.))\right]\mathbb{E}_s^{\mathbb{Q}}\left[g(\eta(.))\right]$$

Note that this is a bit stronger than assuming the independence of τ and $(r(t))_{0\le t\le T}$. Therefore, noting that on the event $\{\tau \le t\}$ we have $\delta_{\eta(\tau)} = \delta_{\eta(t)}$ (recall the states in $\mathcal{B}$ are absorbing), the price of the risky bond can be computed as

$$\begin{aligned} D(s,t) &= \mathbb{E}_s^{\mathbb{Q}}\left[\frac{B(s)}{B(t)}(\mathbb{I}_{\{\tau>t\}} + \delta_{\eta(t)}\mathbb{I}_{\{\tau\le t\}})\right] = \\ &= B(s,t)(1 - \mathbb{E}_s^{\mathbb{Q}}\left[\mathbb{I}_{\{\tau\le t\}}(1-\delta_{\eta(t)})\right]) \end{aligned} \tag{7}$$

Since $\{\tau \le t\} = \{\eta(t) \in \mathcal{B}\}$, by the Markov property we get

$$\begin{aligned} \mathbb{E}_s^{\mathbb{Q}}[\mathbb{I}_{\tau\le t}(1-\delta_{\eta(t)})] &= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{I}_{\{\eta(t)\in\mathcal{B}\}}(1-\delta_{\eta(t)})|\mathcal{F}_s\right] = \\ &= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{I}_{\{\eta(t)\in\mathcal{B}\}}(1-\delta_{\eta(t)})|\eta(s)\right] = \\ &= \sum_{j\in\mathcal{B}} q_{\eta(s),j}(s,t)(1-\delta_j), \end{aligned}$$

and hence

$$\begin{aligned} D(s,t) &= B(s,t)\Big(1 - \sum_{j\in\mathcal{B}} q_{\eta(s),j}(s,t)(1-\delta_j)\Big) = \\ &= B(s,t)\Big(q_{\eta(s),\mathcal{S}}(s,t) + \sum_{j\in\mathcal{B}} q_{\eta(s),j}(s,t)\delta_j\Big). \end{aligned} \tag{8}$$

Writing $D_i(s,t)$ for $D(s,t)$ on the event $\{\eta(s) = i\}$, $i \in \mathcal{C}$, we have

$$\begin{aligned} D_i(s,t) &= B(s,t)\Big(1 - \sum_{j\in\mathcal{B}} q_{i,j}(s,t)(1-\delta_j)\Big) = \\ &= B(s,t)\Big(q_{i,\mathcal{S}}(s,t) + \sum_{j\in\mathcal{B}} q_{i,j}(s,t)\delta_j\Big). \end{aligned} \tag{9}$$

In particular, in default state $j \in \mathcal{B}$ we have $D_j(s,t) = \delta_j B(s,t)$. Note that if $q_{i,\mathcal{B}}(s,t) > 0$, then formula (9) can be written as

$$\begin{aligned} D_i(s,t) &= B(s,t)\Big(1 - q_{i,\mathcal{B}}(s,t)\sum_{j\in\mathcal{B}}(1-\delta_j)\frac{q_{i,j}(s,t)}{q_{i,\mathcal{B}}(s,t)}\Big) = \\ &= B(s,t)\Big(1 - q_{i,\mathcal{B}}(s,t)\sum_{j\in\mathcal{B}}(1-\delta_j)\mathbb{Q}(\eta(t)=j|\eta(t)\in\mathcal{B},\eta(s)=i)\Big) = \\ &= B(s,t)\Big(1 - q_{i,\mathcal{B}}(s,t)\big(1 - \mathbb{E}^{\mathbb{Q}}[\delta_{\eta(t)}|\eta(t)\in\mathcal{B},\eta(s)=i]\big)\Big). \end{aligned} \tag{10}$$

Therefore in each rating class the price of a defaultable bond is equal to the value of a default-free zero-coupon bond less the present value of the loss given default (i.e. the probability of default times the expected loss in case of default). Comparing (6) with (10), the difference is that the deterministic recovery rate in (6) is substituted in (10) with the (risk neutral) expected recovery rate at time t conditional on being in class i at time s and in default at time t. By looking at (10) we deduce that $D_i(s,t) < B(s,t)$ if and only if $\mathbb{E}^{\mathbb{Q}}[\delta_{\eta(t)} | \eta(t) \in \mathcal{B}, \eta(s) = i] < 1$, which usually happens even if $\delta_{K+1} > 1$ (indeed the probability that the bondholders benefit from default is very small).

We can now compute other quantities of interest implied by the term structure of defaultable bonds. By (9) and (1), the implied defaultable forward rate in state $i \in \mathcal{C}$ at time s for time t ($0 \leq s \leq t \leq T-1$) can be expressed (for positive recovery rates when $i \in \mathcal{B}$) as

$$\begin{aligned} h_i(s,t) &:= -\log(D_i(s,t+1)/D_i(s,t)) = \\ &= f(s,t) + \log \frac{q_{i,\mathcal{S}}(s,t) + \sum_{j\in\mathcal{B}} q_{i,j}(s,t)\delta_j}{q_{i,\mathcal{S}}(s,t+1) + \sum_{j\in\mathcal{B}} q_{i,j}(s,t+1)\delta_j}. \end{aligned} \tag{11}$$

Taking $s = t$, the defaultable short rate when $i \in \mathcal{S}$ is

$$s_i(t) := h_i(t,t) = r(t) - \log\Big(1 - \sum_{j\in\mathcal{B}} (1-\delta_j) q_{i,j}(t)\Big).$$

while in default ($i \in \mathcal{B}$) the term structure of defaultable interest rates coincides with the default-free term structure (i is absorbing).

Recalling the standard definition of the default-free yield to maturity, $Y(s,t) = -\frac{1}{t-s}\log B(s,t)$ with $0 \leq s \leq t \leq T$, analogously one can define the defaultable yield to maturity as

$$\begin{aligned} H_i(s,t) &:= -\frac{1}{t-s}\log D_i(s,t) = \\ &= Y(s,t) - \frac{1}{t-s}\log\Big(q_{i,\mathcal{S}}(s,t) + \sum_{j\in\mathcal{B}} q_{i,j}(s,t)\delta_j\Big), \end{aligned}$$

when $i \in \mathcal{S}$, while in default ($i \in \mathcal{B}$) the yield coincide with the default-free yield (provided $\delta_i > 0$). The difference between the two yields is called *credit yield spread* and is

$$\Delta_i(s,t) := H_i(s,t) - Y(s,t) = -\frac{1}{t-s}\log\Big(q_{i,\mathcal{S}}(s,t) + \sum_{j\in\mathcal{B}} q_{i,j}(s,t)\delta_j\Big)$$

when $i \in \mathcal{S}$ and zero otherwise. Note that the credit yield spread can be written also as

$$\Delta_i(s,t) = -\frac{1}{t-s}\log\Big(1 - q_{i,\mathcal{B}}(s,t)\big(1 - \mathbb{E}^{\mathbb{Q}}[\delta_{\eta(t)}|\eta(t) \in \mathcal{B}, \eta(s) = i]\big)\Big),$$

showing explicitly the dependence on the average recovery rate and the probability of default. Therefore, as expected, the credit spread is positive if an only if the average recovery rate is less than unity.

The knowledge of the term structure of defaultable interest rates allows one to price contingent claims (in particular credit derivatives) written on this structure. For instance one can evaluate an option written on the credit yield spread with a procedure analogous to the one adopted in section III in Kijima and Komoribayashi [19].

4 - CALIBRATING THE MODEL

In this section we extend the fitting procedures introduced by JLT and Kijima and Komoribayashi [19] (hereafter KK). Suppose to have the following data

1. The observed term structure of default-free zero-coupon bond prices $\{\widetilde{B}(0,t),\ 1 \le t \le T\}$ at initial time.

2. The observed term structure of defaultable zero-coupon bond prices $\{\widetilde{D}_i(0,t),\ 1 \le t \le T\}$ at initial time, for any (non default) rating class $i \in \mathcal{S}$.

3. The recovery rate in each class of default, i.e. δ_i, $i \in \mathcal{B}$.

   See Carty and Hamilton [9] for information on recovery rates of defaulted debt.

4. The historical one-period transition matrix $\widetilde{\mathbf{P}} = (\widetilde{p}_{i,j})_{i,j\in\mathcal{C}} = \left(\begin{smallmatrix}\widetilde{\mathbf{A}} & \widetilde{\mathbf{R}} \\ \mathbf{0} & \mathbf{I}\end{smallmatrix}\right)$.

   We recall that some commercial rating agencies periodically supply estimates of rating transition matrices specific to some economic sectors (See Nickel et al. [24] for the properties of estimated rating transition matrices and Lando and Skødeberg [21] on estimation methodologies) for the case of a unique default class ($L = 1$). While with the unique default class it is enough to evaluate the matrix $\widetilde{\mathbf{A}}$, in our extended model, to get $\widetilde{\mathbf{P}}$, we need to evaluate also the matrix $\widetilde{\mathbf{R}}$. More precisely, we have to estimate the following probabilities

$$\mathbb{P}(\eta(t+1) = j|\eta(t+1) \in \mathcal{B}, \eta(t) = i),$$

for $i \in \mathcal{S}, j \in \mathcal{B}$, in order to compute $\widetilde{p}_{i,j}$ by

$$\widetilde{p}_{i,j} = \widetilde{p}_{i,\mathcal{B}}\widetilde{\mathbb{P}}(\eta(t+1) = j | \eta(t+1) \in \mathcal{B}, \eta(t) = i).$$

Given these data, the calibrating procedure seeks to identify the risk neutral transition matrices $\mathbf{Q}(t)$ (for $0 \leq t \leq T-1$) in such a way that the observed market prices $\{\widetilde{D}_i(0,t),\, 1 \leq t \leq T\}$ coincide with the values implied by (9) (where we have plugged the observed prices of default-free zero-coupon bonds). More precisely, we require that

$$\widetilde{D}_i(0,t) = \widetilde{B}(0,t)\Big(1 - \sum_{j \in \mathcal{B}} q_{i,j}(0,t)(1-\delta_j)\Big) \tag{12}$$

for all $i \in \mathcal{S}$ and $1 \leq s \leq T$. To ensure the implementation of the model, JLT suggest to add some restrictions linking the risk neutral transition probabilities (that we want to estimate) and the historical ones. Since $\mathbb{P}$ and $\mathbb{Q}$ are equivalent, we can always write

$$q_{i,j}(t) = \pi_{i,j}(t)\widetilde{p}_{i,j} \text{ for all } i,j \in \mathcal{C} \text{ and } 0 \leq t \leq T-1,$$

where coefficients $(\pi_{i,j}(t))$ are referred to as *risk premium adjustments*. Note that these adjustments are arbitrary when $i \in \mathcal{B}$ and $i \neq j$. Consider the framework of Section 2, that is $L = 1$; in the literature it is assumed that the adjustments $\pi_{i,j}(t)$ do not depend on j, that is

$$\pi_{i,j}(t) = \pi_i(t) \text{ for all } i \in \mathcal{S}, j \neq k(i) \text{ and } 0 \leq t \leq T-1, \tag{13}$$

where k is a suitable mapping of $\mathcal{C}$ into itself. In the JLT paper $k(i) = i$, while KK suggest to use $k(i) = K+1$ in order to avoid some numerical problem that could arise in the JLT procedure. See both papers for comments on these issues. The fitting procedures act recursively estimating at each step the premium adjustment and then computing the transition matrix using (13). We are going now to extend both procedures to our model. First note that the transition matrix $\mathbf{Q}(0,t+1)$ from time 0 to time $t+1$ can be written as

$$\mathbf{Q}(0,t+1) = \begin{pmatrix} \mathbf{A}(0,t+1) & \mathbf{R}(0,t+1) \\ \mathbf{0} & \mathbf{I} \end{pmatrix}$$

where the matrices $\mathbf{A}(0,t+1)$, $\mathbf{R}(0,t+1)$ are expressed in terms of $\mathbf{A}(s)$, $\mathbf{R}(s)$ for $0 \leq s \leq t$. indeed, from $\mathbf{Q}(0,t+1) = \mathbf{Q}(0,t)\mathbf{Q}(t)$ it easily follows

$\mathbf{A}(0,t+1) = \mathbf{A}(0,t)\mathbf{A}(t)$ and $\mathbf{R}(0,t+1) = \mathbf{A}(0,t)\mathbf{R}(t)+\mathbf{R}(0,t)$. Moreover, formula (12) can be written, for $i \in \mathcal{S}$, as

$$\frac{\widetilde{B}(0,t) - \widetilde{D}_i(0,t)}{\widetilde{B}(0,t)} = \sum_{j\in\mathcal{B}} q_{i,j}(0,t)(1-\delta_j)$$

and in matrix form as

$$\widetilde{\mathbf{b}}(t) = \mathbf{R}(0,t)\mathbf{c} \tag{14}$$

where

$$\widetilde{\mathbf{b}}(t) = \begin{pmatrix} \frac{\widetilde{B}(0,t)-\widetilde{D}_1(0,t)}{\widetilde{B}(0,t)} \\ \vdots \\ \frac{\widetilde{B}(0,t)-\widetilde{D}_K(0,t)}{\widetilde{B}(0,t)} \end{pmatrix}, \ \mathbf{c} = \begin{pmatrix} 1-\delta_{K+1} \\ \vdots \\ 1-\delta_{K+L} \end{pmatrix}.$$

Finally, remark that

$$\mathbf{R}(0,t+1) = \mathbf{A}(0,t)\mathbf{R}(t) + \mathbf{R}(0,t),$$

so that, multiplying both sides by $\mathbf{c}$, and using (14) we get

$$\mathbf{A}(0,t)\mathbf{R}(t)\mathbf{c} = \widetilde{\mathbf{b}}(t+1) - \widetilde{\mathbf{b}}(t). \tag{15}$$

Extending the JLT fitting procedure. Assume that for any $i \in \mathcal{S}$ and $0 \le t \le T-1$,

$$q_{i,j}(t) = \pi_i(t)\widetilde{p}_{i,j} \text{ for all } j \ne i. \tag{16}$$

Since $\sum_{j\in\mathcal{C}} q_{i,j}(t) = 1$ we get $q_{i,i}(t) = 1-\pi_i(t)(1-\widetilde{p}_{i,i})$. In matrix form this is written as

$$\mathbf{A}(t) = \mathbf{I}_K + \Pi(t)(\widetilde{\mathbf{A}} - \mathbf{I}_K), \ \mathbf{R}(t) = \Pi(t)\widetilde{\mathbf{R}}, \tag{17}$$

where $\Pi(t) = \mathrm{diag}(\pi_1(t), \dots, \pi_K(t))$[8] and $\mathbf{I}_K$ is the identity matrix of rank K. Since $\mathbb{P}$ and $\mathbb{Q}$ are equivalent, by (16) one easily gets

$$\begin{cases} 0 < \pi_i(t) < \frac{1}{1-\widetilde{p}_{i,i}} \text{ for all } t & \text{if } 0 < \widetilde{p}_{i,i} < 1 \\ \pi_i(t) \text{ arbitrary for all } t & \text{if } \widetilde{p}_{i,i} = 1 \\ \pi_i(t) = 1 \text{ for all } t & \text{if } \widetilde{p}_{i,i} = 0. \end{cases} \tag{18}$$

[8] $\mathrm{diag}(\alpha_1, \dots, \alpha_n)$ denotes the diagonal matrix with diagonal $(\alpha_1, \dots, \alpha_n)$.

Note that it is always the case that $0 < \widetilde{p}_{i,i} < 1$ for all $i \in \mathcal{S}$: indeed, both the event of remaining in the same class and of migrating have positive probabilities. Therefore, at each step, after estimating the risk premium adjustments, one must check that (18) is satisfied; if this does not happen, then the procedure stops. The procedure is initialized as follows: consider (12) with $t = 1$ and substitute $q_{i,j}(0) = \pi_i(0)\widetilde{p}_{i,j}$; solving for $\pi_i(0)$ gives

$$\pi_i(0) = \frac{\widetilde{B}(0,1) - \widetilde{D}_i(0,1)}{\widetilde{B}(0,1)\sum_{j\in\mathcal{B}}(1-\delta_j)\widetilde{p}_{i,j}},$$

(provided the denominator is positive) so that, by (17), we can compute the time 0 risk neutral transition matrix. Suppose now to have computed $\mathbf{Q}(0), \dots, \mathbf{Q}(u-1)$ in such a way that (14) (or equivalently (12)) holds for $1 \le t \le u$; we want to compute $\mathbf{Q}(u)$ so that (14) holds for $t = u + 1$. To this end, we now compute $\Pi(u)$. From (15) and (17), we can write

$$\widetilde{\mathbf{b}}(u+1) - \widetilde{\mathbf{b}}(u) = \mathbf{A}(0,u)\Pi(u)\widetilde{\mathbf{R}}\mathbf{c}.$$

Hence, if the matrix $\mathbf{A}(0,u)$ is invertible, we have

$$\mathbf{A}^{-1}(0,u)(\widetilde{\mathbf{b}}(u+1) - \widetilde{\mathbf{b}}(u)) = \Pi(u)\widetilde{\mathbf{R}}\mathbf{c}.$$

Note that the left hand side is known, while on the right hand side only $\Pi(u)$ is unknown. Denoting by $w_{i,j}(0,u)$ the generic entry of $\mathbf{A}^{-1}(0,u)$, the last equation can be written in extensive form as

$$\begin{pmatrix} \pi_1(u)\sum_{j\in\mathcal{B}}\widetilde{p}_{1,j}(1-\delta_j) \\ \vdots \\ \pi_K(u)\sum_{j\in\mathcal{B}}\widetilde{p}_{K,j}(1-\delta_j) \end{pmatrix} = \begin{pmatrix} w_{1,1}(0,u) & \dots & w_{1,K}(0,u) \\ \dots & \dots & \dots \\ w_{K,1}(0,u) & \dots & w_{K,K}(0,u) \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \vdots \\ \zeta_K \end{pmatrix},$$

where

$$\zeta_h = \frac{\widetilde{B}(0,u+1)\big(q_{h,\mathcal{S}}(0,u) + \sum_{j\in\mathcal{B}} q_{h,j}(0,u)\delta_j\big) - \widetilde{D}_h(0,u+1)}{\widetilde{B}(0,u+1)}.$$

When $L = 1$ this coincides with formula (12.13) of Bielecki and Rutkowski [4]. If $\sum_{j\in\mathcal{B}}\widetilde{p}_{i,j}(1-\delta_j) > 0$ we can solve for $\pi_i(u)$ and get

$$\pi_i(u) = \sum_{h\in\mathcal{S}} \frac{w_{i,h}(0,u)}{\widetilde{B}(0,u+1)\sum_{j\in\mathcal{B}}\widetilde{p}_{i,j}(1-\delta_j)} \times \Big(\widetilde{B}(0,u+1)\big(q_{h,\mathcal{S}}(0,u) + \sum_{j\in\mathcal{B}} q_{h,j}(0,u)\delta_j\big) - \widetilde{D}_h(0,u+1)\Big).$$

Finally we can compute $\mathbf{Q}(u)$ by (16).

Extending the KK fitting procedure. Assume that for any $i \in \mathcal{S}$, $j \in \mathcal{C}$ and $0 \leq t \leq T-1$

$$q_{i,j}(t) = \begin{cases} \pi_i(t) p_{i,j} & \text{if } j \in \mathcal{S} \\ \gamma_i(t) p_{i,j} & \text{if } j \in \mathcal{B}. \end{cases} \tag{19}$$

Note that, for a given row, there are two (possibly) distinct risk premium adjustments depending on the arrival state being default or not. Moreover, this relationship is equivalent, when $L = 1$, to the KK condition for risk premium adjustments (see formula (15) in [19]). In matrix form, (19) can be written as

$$\mathbf{A}(t) = \Pi(t)\widetilde{\mathbf{A}},\ \mathbf{R}(t) = \Gamma(t)\widetilde{\mathbf{R}}. \tag{20}$$

where $\Pi(t) = \operatorname{diag}(\pi_1(t), \ldots, \pi_K(t))$ and $\Gamma(t) = \operatorname{diag}(\gamma_1(t), \ldots, \gamma_K(t))$. Since $\mathbf{Q}(t)$ is a stochastic matrix, we have $\pi_i(t)\widetilde{p}_{i,\mathcal{S}} + \gamma_i(t)\widetilde{p}_{i,\mathcal{B}} = 1$ for any $i \in \mathcal{S}$ and hence

$$\gamma_i(t) = \frac{1 - \pi_i(t)(1 - \widetilde{p}_{i,\mathcal{B}})}{\widetilde{p}_{i,\mathcal{B}}},\ \pi_i(t) = \frac{1 - \gamma_i(t)(1 - \widetilde{p}_{i,\mathcal{S}})}{\widetilde{p}_{i,\mathcal{S}}}, \tag{21}$$

provided $0 < \widetilde{p}_{i,\mathcal{B}} < 1$, or, in matrix form,

$$\begin{aligned} \Pi(t)\widetilde{\mathbf{A}}\mathbf{e}_K &= \mathbf{e}_K - \Gamma(t)(\mathbf{e}_K - \widetilde{\mathbf{A}}\mathbf{e}_K) \\ \Gamma(t)\widetilde{\mathbf{R}}\mathbf{e}_L &= \mathbf{e}_K - \Pi(t)(\mathbf{e}_K - \widetilde{\mathbf{R}}\mathbf{e}_L), \end{aligned} \tag{22}$$

where $\mathbf{e}_K$, $\mathbf{e}_L$ are the vectors of dimensions K and L respectively, with all unitary components. The equivalence of $\mathbb{P}$ and $\mathbb{Q}$ implies that

$$\begin{cases} 0 < \pi_i(t) < \frac{1}{1 - \widetilde{p}_{i,\mathcal{B}}} \text{ for all } t & \text{if } 0 < \widetilde{p}_{i,\mathcal{B}} < 1 \\ \pi_i(t) \text{ arbitrary, } \gamma_i(t) = 1 \text{ for all } t & \text{if } \widetilde{p}_{i,\mathcal{B}} = 1 \\ \pi_i(t) = 1,\ \gamma_i(t) \text{ arbitrary for all } t & \text{if } \widetilde{p}_{i,\mathcal{B}} = 0. \end{cases} \tag{23}$$

Note that the only actual case is $0 \leq \widetilde{p}_{i,\mathcal{B}} < 1$ for all $i \in \mathcal{S}$ (indeed $\widetilde{p}_{i,\mathcal{B}} = 1$ means that all rated i corporate bonds will default in the next period). Hence, at each step, after estimating the risk premium adjustments, one must check that (23) is satisfied; if this does not happen, then the procedure stops.

The procedure starts with the following step: from (12) and $q_{i,j}(0) = \gamma_i(0)\widetilde{p}_{i,j}$, we get

$$\gamma_i(0) = \frac{\widetilde{B}(0,1) - \widetilde{D}_i(0,1)}{\widetilde{B}(0,1)\sum_{j\in\mathcal{B}}\widetilde{p}_{i,j}(1-\delta_j)},$$

provided the denominator is positive. Hence, from (21), we get

$$\pi_i(0) = \frac{1}{\widetilde{p}_{i,\mathcal{S}}}\frac{\sum_{j\in\mathcal{B}}\big(\widetilde{D}_i(0,1) - \widetilde{B}(0,1)\delta_j\big)\widetilde{p}_{i,j}}{\widetilde{B}(0,1)\sum_{j\in\mathcal{B}}\widetilde{p}_{i,j}(1-\delta_j)}.$$

Finally, by (19) one can compute the matrix $\mathbf{Q}(0)$. Suppose now that we have computed $\mathbf{Q}(0),\ldots,\mathbf{Q}(u-1)$ in such a way that (14) (or equivalently (12)) holds for $1 \le t \le u$, and we want to compute $\mathbf{Q}(u)$ so that (14) holds for $t = u+1$. First, we compute $\Gamma(u)$. From (15), and (20), we have

$$\widetilde{\mathbf{b}}(u+1) - \widetilde{\mathbf{b}}(u) = \mathbf{A}(0,u)\Gamma(u)\widetilde{\mathbf{R}}\mathbf{c}.$$

Hence, if the matrix $\mathbf{A}(0,u)$ is invertible, we have

$$\mathbf{A}^{-1}(0,u)(\widetilde{\mathbf{b}}(u+1) - \widetilde{\mathbf{b}}(u)) = \Gamma(u)\widetilde{\mathbf{R}}\mathbf{c}.$$

Denoting by $w_{i,j}(0,u)$ the generic entry of $\mathbf{A}^{-1}(0,u)$, the last equation can be written in extensive form as

$$\begin{pmatrix} \gamma_1(u)\sum_{j\in\mathcal{B}}\widetilde{p}_{1,j}(1-\delta_j) \\ \vdots \\ \gamma_K(u)\sum_{j\in\mathcal{B}}\widetilde{p}_{K,j}(1-\delta_j) \end{pmatrix} = \begin{pmatrix} w_{1,1}(0,u) & \ldots & w_{1,K}(0,u) \\ \ldots & \ldots & \ldots \\ w_{K,1}(0,u) & \ldots & w_{K,K}(0,u) \end{pmatrix}\begin{pmatrix} \zeta_1 \\ \vdots \\ \zeta_K \end{pmatrix},$$

where

$$\zeta_h = \frac{\widetilde{B}(0,u+1)\big(q_{h,\mathcal{S}}(0,u) + \sum_{j\in\mathcal{B}} q_{h,j}(0,u)\delta_j\big) - \widetilde{D}_h(0,u+1)}{\widetilde{B}(0,u+1)}.$$

Therefore, if $\sum_{j\in\mathcal{B}}\widetilde{p}_{i,j}(1-\delta_j) > 0$, we get

$$\gamma_i(u) = \sum_{h\in\mathcal{S}} \frac{w_{i,h}(0,u)}{\widetilde{B}(0,u+1)\sum_{j\in\mathcal{B}}\widetilde{p}_{i,j}(1-\delta_j)} \times \Big(\widetilde{B}(0,u+1)\big(q_{i,\mathcal{S}}(0,u) + \sum_{j\in\mathcal{B}} q_{i,j}(0,u)\delta_j\big) - \widetilde{D}_h(0,u+1)\Big).$$

Using (21), after some algebra, we have

$$\pi_i(u) = \frac{1}{\widetilde{p}_{i\mathcal{S}}} \sum_{h\in\mathcal{S}} \frac{w_{i,h}(0,u)}{\widetilde{B}(0,u+1)\sum_{j\in\mathcal{B}}\widetilde{p}_{i,j}(1-\delta_j)} \sum_{j\in\mathcal{B}} \widetilde{p}_{i,j}\Big(\widetilde{D}_h(0,u+1) - \widetilde{B}(0,u+1)\big(\sum_{l\in\mathcal{B}} q_{h,l}(0,u)\delta_l + \delta_j q_{h,\mathcal{S}}(0,u)\big)\Big),$$

which coincides, when $L = 1$, with formula (19) in [19]. We can finally compute $\mathbf{Q}(u)$ by (19).

Some remarks about these procedures are required. At each step, one must check if the computed risk premium adjustments assure that the corresponding matrices, given respectively by (17) and (20), are stochastic matrices for a probability $\mathbb{Q}$ equivalent to $\mathbb{P}$; in other words one must check at each step that (18) and (23), respectively, hold. Moreover, note that it's not unusual for a high rated (AAA, say) bond to have a 0 estimated (one period) probability of default, that is $\widetilde{p}_{i,\mathcal{B}} = 0$. In this case both fitting procedures are ill-posed since the model implies that the risky bond's price must be equal to the default-free bond's price (see (12)) while the actual spread is always positive.

5 - A FURTHER EXTENSION

The model introduced in Section 3 can be extended requiring in (A2) that the set $\mathcal{B}$ is a closed class for the homogeneous $\mathbb{P}$-Markov chain $(\eta(t))$ (i.e. $p_{i,\mathcal{B}} = 1$), without assuming that each of these states be absorbing. In other words, once the chain visits a state in $\mathcal{B}$ it can migrate only to a (possibly different) state of $\mathcal{B}$. The transition matrix of the chain $(\eta(t))$ has therefore the following form:

$$\mathbf{P} = \begin{pmatrix} \mathbf{A} & \mathbf{R} \\ \mathbf{0} & \mathbf{C} \end{pmatrix}$$

where $\mathbf{A}$ and $\mathbf{R}$ are as in Section 3, and $\mathbf{C}$ is a $L \times L$ stochastic matrix containing the transition probabilities between the states of $\mathcal{B}$. Such a situation can occur, for instance, when a firm default can be ranked in two classes, one corresponding to reorganization and the other to liquidation. In the former case, the firm agrees with its creditors on a reorganization plan (which must be approved by the court) for payment of their claims; in the latter the firm ceases to operate and its assets are sold to pay the creditors according to their priorities. Of course not all reorganization plans are successful and it may happen that a reorganization eventually becomes a liquidation or that the firm files for another reorganization plan. For instance $\mathcal{B}$ could have two classes ($L = 2$), reorganization and liquidation, with the second state being absorbing, i.e the matrix $\mathbf{C}$ could have the form $\mathbf{C} = \begin{pmatrix} p & 1-p \\ 0 & 1 \end{pmatrix}$. We modify also the recovery rule: it seems more reasonable now that the recovery rate is not known at the time of bankruptcy but only successively, for instance after the firm has emerged from reorganization or after the liquidation sales have taken place (Franks and Torous [15] report that the average number spent in reorganization by thirty firms in their sample is 3.67 years, while Carty and Hamilton report an average of 1.3 years in their sample of 155 firms). We assume here that the recovery rate for a bond maturing at t is given by $\delta_{\eta(t)}$ (hence we have modified also assumption (A1)), while we keep unchanged the other assumptions. Remark that now it is no longer necessarily true that $\eta(t) = \eta(\tau)$ on the set $\{\tau \leq t\}$. Inspection of the derivation of the defaultable bond price in Section 3 shows that the same reasoning remains true in the present setting, and the value of the bond is again given by (9) for a firm rated i at time s. However, note that if the bond has already defaulted at time s, that is $i \in \mathcal{B}$, then the bond's value is

$$D_i(s,t) = B(s,t) \sum_{j \in \mathcal{B}} \delta_j q_{i,j}(s,t),$$

i.e. the value of a default free bond times the expected recovery rate in default $E(\delta_{\eta(t)} | \eta(s) = i)$.

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