Calculating credit risk capital charges with the one-factor model

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Abstract

Even in the simple Vasicek one-factor credit portfolio model, the exact contributions to credit value-at-risk cannot be calculated without Monte-Carlo simulation. As this may require a lot of computational time, there is a need for approximate analytical formulae. In this note, we develop formulae according to two different approaches: the granularity adjustment approach initiated by M. Gordy and T. Wilde[1], and a semi-asymptotic approach. The application of the formulae is illustrated with a numerical example.

KEYWORDS: One-factor model, capital charge, granularity adjustment, quantile derivative.

1 Introduction

In the past two years, one-factor models[2] for credit portfolio risk have become popular. On the one side, they can rather readily be handled from a computational point of view, and, in particular, allow to avoid lengthy Monte Carlo simulations which in general are needed when dealing with more sophisticated credit risk models. On the other side, they

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[1]The authors thank Tom Wilde for stimulating the writing of this paper and for helpful comments on its first version.

[2]See Frey and McNeil (2004) for an overview of credit modeling methodologies.

are capable to incorporate simple dependence structures which are to a certain degree sufficient for a rudimentary form of credit risk management.

Often, one-factor models admit a decomposition of the portfolio loss variable into a monotonic function of the factor and a residual. The former part of the decomposition is called *systematic* risk whereas the latter part is called *specific* or *idiosyncratic* risk. Sometimes, the portfolio loss variable converges in some sense to a monotonic function of the factor. This observation can be used as point of departure for analytic approximations of important statistics of the portfolio like quantiles of the portfolio loss variable *(value-at-risk, VaR)*.

Gordy (2003) was the first to suggest this approach which he called *granularity adjustment* for the so-called *Vasicek one-factor model*. Then it was refined by Wilde (2001) and Pykhtin and Dev (2002). Only recently, Martin and Wilde (2002a) observed that the results by Gouriéroux et al. (2000) make feasible an easier and more systematic way to derive the adjustments. In the following (Section 2), we reexamine the granularity adjustment in the one-factor model and develop the formulae which are necessary to compute adjustments at the transaction level. Additionally, in Section 3 we present an alternative *semi-asymptotic* approach to capital charges at transaction level which relies on a limiting procedure applied to a part of the portfolio only. Section 4 provides a numerical example. We conclude with a short summary in Section 5.

2 The granularity adjustment approach

In order to explain the approach we will follow, we consider two random variables L and X. L denotes the portfolio loss whereas X reflects the value of an economic factor which causes the dependence between the different transactions building up the portfolio. The conditional expectation of L given X, $\mathrm{E}[L\,|\,X]$, is considered the systematic part of the portfolio loss. This can be motivated by the decomposition

$$L \;=\; \mathrm{E}[L\,|\,X] + L - \mathrm{E}[L\,|\,X] \tag{2.1}$$

which leads to

$$\begin{aligned}\mathrm{var}[L] &= \mathrm{E}\big[\mathrm{var}[L\,|\,X]\big] + \mathrm{var}\big[\mathrm{E}[L\,|\,X]\big]\\ &= \mathrm{E}\big[\mathrm{var}[L - \mathrm{E}[L\,|\,X]\,\big|\,X]\big] + \mathrm{var}\big[\mathrm{E}[L\,|\,X]\big],\end{aligned} \tag{2.2}$$

where var denotes variance and $\mathrm{var}[\,\cdot\,|\,X]$ means conditional variance. Under the assumption that conditional on the value of the economic factor X the elementary transactions in the portfolio are independent, the term $\mathrm{E}\big[\mathrm{var}[L\,|\,X]\big]$ in (2.2) is likely to converge to zero with a growing number of transactions in the portfolio. Therefore, the specific risk $L - \mathrm{E}[L\,|\,X]$ of the portfolio may be considered *incremental* and small compared to the systematic risk $\mathrm{E}[L \,|\, X]$. Note that by the factorization lemma $\mathrm{E}[L \,|\, X]$ may be written as $g \circ X$ with some appropriately chosen function g. This becomes interesting when g turns out to be monotonic since in this case values for some statistics of $\mathrm{E}[L \,|\, X]$ like quantiles can be easily computed from the corresponding statistics of X. We fix this as a formal assumption.

Assumption 2.1 *The conditional expectation of L given the factor X can be written as*

$$\mathrm{E}[L \,|\, X] \;=\; g \circ X,$$

where g is continuous and strictly increasing or decreasing. The distribution of X is continuous, i.e. $\mathrm{P}[X = x] = 0$ *for all x.*

For $\alpha \in (0,1)$ and any random variable Y, define the α-quantile of Y by

$$q_\alpha(Y) \;=\; \inf\big\{y \in \mathbb{R} : \mathrm{P}[Y \leq y] \geq \alpha\big\}. \tag{2.3}$$

The granularity adjustment approach to the calculation of the α-quantile $q_\alpha(L)$ of the portfolio loss is essentially a second order Taylor expansion in the following sense:

$$\begin{aligned} q_\alpha(L) &= q_\alpha\big(\mathrm{E}[L \,|\, X] + h\,(L - \mathrm{E}[L \,|\, X])\big)\Big|_{h=1} \\ &\approx q_\alpha\big(\mathrm{E}[L \,|\, X]\big) + \frac{\partial q_\alpha}{\partial h}\big(\mathrm{E}[L \,|\, X] + h\,(L - \mathrm{E}[L \,|\, X])\big)\Big|_{h=0} \\ &\quad + \frac{1}{2}\,\frac{\partial^2 q_\alpha}{\partial h^2}\big(\mathrm{E}[L \,|\, X] + h\,(L - \mathrm{E}[L \,|\, X])\big)\Big|_{h=0}. \end{aligned} \tag{2.4}$$

The derivatives of $q_\alpha\big(\mathrm{E}[L \,|\, X]+h\,(L-\mathrm{E}[L \,|\, X])\big)$ with respect to h can be calculated thanks to results by Gouriéroux et al. (2000) (see Martin and Wilde, 2002b, for the derivatives of arbitrary large orders). This way, we obtain

$$\begin{aligned} \frac{\partial q_\alpha}{\partial h}\big(\mathrm{E}[L \,|\, X] + h\,(L - \mathrm{E}[L \,|\, X])\big)\Big|_{h=0} &= \mathrm{E}\big[L - \mathrm{E}[L \,|\, X] \,\big|\, \mathrm{E}[L \,|\, X] = q_\alpha(\mathrm{E}[L \,|\, X])\big] \\ &= 0, \end{aligned} \tag{2.5}$$

where (2.5) follows from Assumption 2.1, since $q_\alpha(\mathrm{E}[L \,|\, X]) = g\big(q_\alpha(X)\big)$ if g is increasing, and $q_\alpha(\mathrm{E}[L \,|\, X]) = g\big(q_{1-\alpha}(X)\big)$ if g is decreasing. Under Assumption 2.1, $\mathrm{E}[L \,|\, X]$ has a density if X has a density h_X. If this density of $\mathrm{E}[L \,|\, X]$ is denoted by γ_L, it can be determined according to

$$\gamma_L(y) = h_X(g^{-1}(y))\,\left|g'(g^{-1}(y))\right|^{-1}. \tag{2.6}$$

With regard to the second derivative in (2.4), we obtain by means of the corresponding result of Gouriéroux et al. (2000)

$$\begin{aligned} \frac{\partial^2 q_\alpha}{\partial h^2}\big(\mathrm{E}[L \,|\, X] + h\,(L - \mathrm{E}[L \,|\, X])\big)\Big|_{h=0} &= -\frac{\partial \mathrm{var}}{\partial x}\big(L \,|\, X = g^{-1}(x)\big)\Big|_{x=g(q_\beta(X))} \\ &\quad - \frac{\mathrm{var}\big[L \,|\, X = q_\beta(X)\big]}{\gamma_L\big(g(q_\beta(X))\big)}\,\frac{\partial \gamma_L}{\partial x}(x)\Big|_{x=g(q_\beta(X))}, \end{aligned} \tag{2.7}$$

with $\beta = \alpha$, if g is increasing, and $\beta = 1 - \alpha$ if g is decreasing.

In order to be able to carry out numerical calculations by means of (2.4) and (2.7), we have to fix a stochastic model, the one-factor model in our case. We define the portfolio loss L_n by

$$L_n \;=\; L_n(u_1, \ldots, u_n) \;=\; \sum_{i=1}^{n} u_i \, \mathbf{1}_{\{\sqrt{\rho_i}\, X + \sqrt{1-\rho_i}\, \xi_i \le c_i\}}, \tag{2.8}$$

where $u_i \ge 0$, $i = 1, \ldots, n$, denotes the weight or the exposure of asset i in the portfolio, $0 < \rho_i < 1$ and $c_i \ge 0$, $i = 1, \ldots, n$, are constants, and $X, \xi_1, \ldots, \xi_n$ are independent random variables with continuous distributions. As mentioned above, X is interpreted as an economic factor that influences all the assets in the portfolio but to different extents. The so-called *asset correlation* ρ_i measures the degree of the i-th asset's exposure to the systematic risk expressed by X. The random variables ξ_i are assumed to model the idiosyncratic (or specific) risk of the assets.

It is easy to show that in this case Assumption 2.1 holds with decreasing $g = g_n$ where $g_n(x)$ is given by

$$g_n(x) \;=\; \sum_{i=1}^{n} u_i \, \mathrm{P}\big[\xi_i \le \tfrac{c_i - \sqrt{\rho_i}\, x}{\sqrt{1-\rho_i}}\big]. \tag{2.9}$$

(2.8) implies that

$$\mathrm{var}[L_n \,|\, X = x] \;=\; \sum_{i=1}^{n} u_i^2 \, \mathrm{P}\big[\xi_i \le \tfrac{c_i - \sqrt{\rho_i}\, x}{\sqrt{1-\rho_i}}\big] \, \big(1 - \mathrm{P}\big[\xi_i \le \tfrac{c_i - \sqrt{\rho_i}\, x}{\sqrt{1-\rho_i}}\big]\big). \tag{2.10}$$

Under the additional assumption that $\inf_i c_i > -\infty$ and $\sup_i \rho_i < 1$, it follows from (2.10) that in case of i.i.d. $\xi_1, \xi_2, \ldots$ and X we have

$$\lim_{n\to\infty} \mathrm{E}\big[\mathrm{var}[L_n \,|\, X]\big] = 0 \tag{2.11a}$$

if and only if[3]

$$\lim_{n\to\infty} \sum_{i=1}^{n} u_i^2 = 0. \tag{2.11b}$$

By (2.11a) and (2.11b), in case of large portfolios the systematic part $\mathrm{var}\big[\mathrm{E}[L_n \,|\, X]\big]$ of the loss variance $\mathrm{var}[L_n]$ (cf. (2.2)) is the essential part of the variance. Hence, in this case approximating L_n by $\mathrm{E}[L_n \,|\, X]$ seems reasonable. This observation suggests approximating $q_\alpha(L_n)$ with the right-hand side of (2.4), where $\mathrm{E}[L_n \,|\, X]$ is given by $g_n \circ X$ with $g_n(x)$ defined by (2.9).

[3]Of course, here we admit an additional dependence of u_i on n, i.e. $u_i = u_{i,n}$.

Let us now specify the distributions of $X, \xi_1, \xi_2, \ldots, \xi_n$ as all being the standard normal distributions. This implies

$$g_n(x) = \sum_{i=1}^{n} u_i\, \Phi\Big(\frac{c_i - \sqrt{\rho_i}\, x}{\sqrt{1-\rho_i}}\Big) \tag{2.12a}$$

and

$$\mathrm{var}\big[L_n \,|\, X = x\big] = \sum_{i=1}^{n} u_i^2\, \left(\Phi\Big(\frac{c_i - \sqrt{\rho_i}\, x}{\sqrt{1-\rho_i}}\Big) - \Phi\Big(\frac{c_i - \sqrt{\rho_i}\, x}{\sqrt{1-\rho_i}}\Big)^2\right) \tag{2.12b}$$

where Φ denotes the distribution function of the standard normal distribution.

Since g_n is decreasing, we obtain from (2.6) and (2.12a) for the density γ_{L_n} of $\mathrm{E}[L_n \,|\, X]$

$$\gamma_{L_n}(x) = -\frac{\phi\big(g_n^{-1}(x)\big)}{g_n'\big(g_n^{-1}(x)\big)} \tag{2.13a}$$

and

$$\frac{\partial \gamma_{L_n}}{\partial x}(x) = \frac{\phi\big(g_n^{-1}(x)\big)\, g_n''\big(g_n^{-1}(x)\big) - \phi'\big(g_n^{-1}(x)\big)\, g_n'\big(g_n^{-1}(x)\big)}{g_n'\big(g_n^{-1}(x)\big)^3}, \tag{2.13b}$$

with $\phi(x) = (\sqrt{2\,\pi})^{-1} e^{-1/2\, x^2}$ denoting the density of the standard normal distribution.

We are now in a position to specialize (2.4) for the case of normally distributed underlying random variables. Plugging in the expressions from (2.12b), (2.13a), and (2.13b) into (2.7) and then the resulting expression for the second derivative of the quantile into (2.4) yields the following formula.

Proposition 2.2 *Let L_n be the portfolio loss variable as defined in (2.8). Assume that the economic factor X and the idiosyncratic risk factors $\xi_1, \ldots, \xi_n$ are independent and standard normally distributed. Fix a confidence level $\alpha \in (0,1)$. Then the second order Taylor approximation of the quantile $q_\alpha(L_n)$ in the sense of (2.4) can be calculated according to*

$$\begin{aligned} q_\alpha(L_n) \;\approx\;& g_n\big(q_{1-\alpha}(X)\big) \\ &+ \Bigg(\sum_{i=1}^{n} u_i^2 \sqrt{\tfrac{\rho_i}{1-\rho_i}}\, \phi\Big(\tfrac{c_i - \sqrt{\rho_i}\, q_{1-\alpha}(X)}{\sqrt{1-\rho_i}}\Big)\Big(1 - 2\,\Phi\Big(\tfrac{c_i - \sqrt{\rho_i}\, q_{1-\alpha}(X)}{\sqrt{1-\rho_i}}\Big)\Big) \\ &\quad + \Big(q_{1-\alpha}(X) + \tfrac{g_n''\left(q_{1-\alpha}(X)\right)}{g_n'\left(q_{1-\alpha}(X)\right)}\Big) \sum_{i=1}^{n} u_i^2\, \Big(\Phi\Big(\tfrac{c_i - \sqrt{\rho_i}\, q_{1-\alpha}(X)}{\sqrt{1-\rho_i}}\Big) - \Phi\Big(\tfrac{c_i - \sqrt{\rho_i}\, q_{1-\alpha}(X)}{\sqrt{1-\rho_i}}\Big)^2\Big)\Bigg) \\ &\quad \times \big(2\, g_n'(q_{1-\alpha}(X))\big)^{-1}, \end{aligned} \tag{2.14}$$

with ϕ and Φ denoting the density and the distribution function respectively of the standard normal distribution.

Remark 2.3 (Herfindahl index) *Consider the following special case of (2.14):*

$$\rho_i = \rho, \quad \text{and} \quad c_i = c, \quad i = 1, \ldots, n, \quad \sum_{i=1}^n u_i = 1.$$

Here the u_i have to be interpreted as the relative weights of the assets in the portfolio. (2.14) then reads

$$\begin{aligned} q_\alpha(L_n) \;\approx\;& \Phi\left(\tfrac{c-\sqrt{\rho}\,q_{1-\alpha}(X)}{\sqrt{1-\rho}}\right) - \left(2\sqrt{\tfrac{\rho}{1-\rho}}\,\phi\big(\tfrac{c-\sqrt{\rho}\,q_{1-\alpha}(X)}{\sqrt{1-\rho}}\big)\right)^{-1}\sum_{i=1}^n u_i^2 \\ & \times\left(\sqrt{\tfrac{\rho}{1-\rho}}\,\phi\big(\tfrac{c-\sqrt{\rho}\,q_{1-\alpha}(X)}{\sqrt{1-\rho}}\big)\big(1-2\,\Phi\big(\tfrac{c-\sqrt{\rho}\,q_{1-\alpha}(X)}{\sqrt{1-\rho}}\big)\big)\right. \\ & \left. +\,\big(q_{1-\alpha}(X) - \sqrt{\tfrac{\rho}{1-\rho}}\,\tfrac{c-\sqrt{\rho}\,q_{1-\alpha}(X)}{\sqrt{1-\rho}}\big)\Phi\left(\tfrac{c-\sqrt{\rho}\,q_{1-\alpha}(X)}{\sqrt{1-\rho}}\right)\Phi\left(\tfrac{\sqrt{\rho}\,q_{1-\alpha}(X)-c}{\sqrt{1-\rho}}\right)\right). \end{aligned} \tag{2.15}$$

The case $u_i = 1/n$ for all i is particularly important as it describes a completely homogeneous portfolio without concentrations. If the u_i are not all equal, $\big(\sum_{i=1}^n u_i^2\big)^{-1}$ is approximately the equivalent portfolio size that would render the portfolio completely homogeneous. Assume that the i-th asset has absolute exposure v_i. Then its relative weight u_i is given by

$$u_i \;=\; \frac{v_i}{\sum_{j=1}^n v_j}.$$

In this case the above introduced equivalent portfolio size reads $\frac{\left(\sum_{j=1}^n v_j\right)^2}{\sum_{j=1}^n v_j^2}$. This quantity is known as Herfindahl index. *Gordy (2003) suggested the Herfindahl index as a key constituent for constructing equivalent homogeneous portfolios. (2.15) yields strong support for this suggestion.*

Recall that in Proposition 2.2 not only $q_\alpha(L_n) = q_\alpha\big(L_n(u_1, \ldots, u_n)\big)$ is a function of the weights $u_1, \ldots, u_n$ but that also the function g_n depends on $u_1, \ldots, u_n$ via (2.12a). A closer inspection of (2.14) reveals that the right-hand side of the equation (as function of the weight vector $(u_1, \ldots, u_n)$) is positively homogeneous of order 1. By Euler's theorem on the representation of homogeneous functions as a weighted sum of the partial derivatives, we can obtain canonical approximate capital charges on transaction or sub-portfolio level which add up to the approximate value-at-risk (cf. Litterman, 1996; Tasche, 1999). The capital charges will thus be approximations to the terms $u_j\,\frac{\partial q_\alpha(L_n)}{\partial u_j}$, $j = 1, \ldots, n$.

These approximate capital charges do not enjoy the portfolio invariance which is the great advantage of the asymptotic capital charges suggested by Gordy (2003). However, as portfolio invariance has the effect to entail capital charges which are not sensitive to concentrations in the portfolio, the partial derivatives approach has got its own attractiveness.

Corollary 2.4 *The partial derivative with respect to u_j of the approximate quantile $q_\alpha(L_n) = q_\alpha\big(L_n(u_1, \ldots, u_n)\big)$ as provided by the right-hand side of (2.14) is given by*

$$\begin{aligned}
\frac{\partial q_\alpha(L_n)}{\partial u_j} \approx\ & \Phi\left(\frac{c_j - \sqrt{\rho_j}\, q_{1-\alpha}(X)}{\sqrt{1-\rho_j}}\right) \\
& + \big(2\, g_n'(q_{1-\alpha}(X))^2\big)^{-1} \Big(2\, u_j\, A_j\, g_n'(q_{1-\alpha}(X)) - B_j \sum_{i=1}^n u_i^2 A_i \\
& \quad + \big(E_j - \frac{B_j\, g_n''(q_{1-\alpha}(X))}{g_n'(q_{1-\alpha}(X))}\big) \sum_{i=1}^n u_i^2\, D_i \\
& \quad + \big(q_{1-\alpha}(X) + \frac{g_n''(q_{1-\alpha}(X))}{g_n'(q_{1-\alpha}(X))}\big)\Big(2\, u_j\, D_j\, g_n'(q_{1-\alpha}(X)) - B_j \sum_{i=1}^n u_i^2 D_i\Big)\Big),
\end{aligned} \tag{2.16}$$

with

$$\begin{aligned}
A_i &= \sqrt{\frac{\rho_i}{1-\rho_i}}\, \phi\left(\frac{c_i - \sqrt{\rho_i}\, q_{1-\alpha}(X)}{\sqrt{1-\rho_i}}\right)\left(1 - 2\,\Phi\left(\frac{c_i - \sqrt{\rho_i}\, q_{1-\alpha}(X)}{\sqrt{1-\rho_i}}\right)\right), \\
B_i &= -\phi\left(\frac{c_i - \sqrt{\rho_i}\, q_{1-\alpha}(X)}{\sqrt{1-\rho_i}}\right)\sqrt{\frac{\rho_i}{1-\rho_i}}, \\
D_i &= \Phi\left(\frac{c_i - \sqrt{\rho_i}\, q_{1-\alpha}(X)}{\sqrt{1-\rho_i}}\right) - \Phi\left(\frac{c_i - \sqrt{\rho_i}\, q_{1-\alpha}(X)}{\sqrt{1-\rho_i}}\right)^2, \\
E_i &= \phi\left(\frac{c_i - \sqrt{\rho_i}\, q_{1-\alpha}(X)}{\sqrt{1-\rho_i}}\right)\frac{\rho_i}{1-\rho_i}\,\frac{\sqrt{\rho_i}\, q_{1-\alpha}(X) - c_i}{\sqrt{1-\rho_i}}.
\end{aligned}$$

Proposition 2.2 and Corollary 2.4 provide a way for the simultaneous determination of economic capital charges in a top-down approach. First, by means of Proposition 2.2 an approximation for the portfolio value-at-risk is calculated. Then, by means of Corollary 2.4 approximations for the partial derivatives of the portfolio value-at-risk can be found. The derivatives are related to the capital charges via

$$\text{Capital charge of asset } j \;=\; u_j\, \frac{\partial q_\alpha(L_n)}{\partial u_j}. \tag{2.17}$$

However, there is a caveat in using this approach. Its accuracy is subject to the validity of (2.11b) which will be questionable if there are very large exposures in the portfolio. In this case, the semi-asymptotic approach we will explain in Section 3 might be more appropriate.

3 The semi-asymptotic approach

We consider here a special case of (2.8) where $\rho_1 = \tau$, $c_1 = a$ but $\rho_i = \rho$ and $c_i = c$ for $i > 1$, and $\sum_{i=1}^n u_i = 1$. Additionally, we assume that $u_1 = u$ is a constant for all n but that $u_2, u_3, \ldots$ fulfills (2.11b).

In this case, the portfolio loss can be represented by

$$L_n(u, u_2, \ldots, u_n) \;=\; u\,\mathbf{1}_{\{\sqrt{\tau}\,X+\sqrt{1-\tau}\,\xi\le a\}} + (1-u)\sum_{i=2}^{n} u_i\,1_{\{\sqrt{\rho}\,X+\sqrt{1-\rho}\,\xi_i\le c\}} \tag{3.1}$$

with $\sum_{i=2}^{n} u_i = 1$. Transition to the limit for $n \to \infty$ in (3.1) leads to the *semi-asymptotic* percentage loss function

$$L(u) \;=\; u\,\mathbf{1}_D + (1-u)\,Y \tag{3.2}$$

with $D = \{\sqrt{\tau}\,X + \sqrt{1-\tau}\,\xi \le a\}$ and $Y = \mathrm{P}\big[\xi \le \frac{c-\sqrt{\rho}\,x}{\sqrt{1-\rho}}\big]\Big|_{x=X}$. Of course, a natural choice for τ might be $\tau = \rho$, the mean portfolio asset correlation.

Definition 3.1 *The quantity*

$$u\,\mathrm{P}[D\,|\,L(u) = q_\alpha(L(u))]$$

is called semi-asymptotic capital charge *(at level α) of the loan with exposure u (as percentage of total portfolio exposure) and default event D as in (3.1).*

The capital charges we suggest in Definition 3.1 have to be calculated separately, i.e. for each asset an own model of type (3.2) has to be regarded. This corresponds to a bottom-up approach since the total capital requirement for the portfolio must be determined by adding up all the capital charges of the assets. Note that the capital charges of Definition 3.1 are not portfolio invariant in the sense of Gordy (2003). However, in contrast to the portfolio invariant charges, the semi-asymptotic charges take into account concentration effects. In particular, their dependence on the exposure u is not merely linear since also the factor $\mathrm{P}[D\,|\,L(u) = q_\alpha(L(u))]$ depends upon u. Definition 3.1 is in line with the general definition of VaR contributions (cf. Litterman, 1996; Tasche, 1999) since (3.2) can be considered a two-assets portfolio model.

In the following, we will assume that the conditional distribution functions F_0 and F_1 of Y given $\mathbf{1}_D = 0$ and $\mathbf{1}_D = 1$ respectively have densities which are continuous and concentrated on the interval $(0, 1)$. Call these densities f_0 and f_1, i.e.

$$\mathrm{P}[Y \le y\,|\,\mathbf{1}_D = i] \;=\; F_i(y) \;=\; \int_{-\infty}^{y} f_i(t)\,dt, \quad y \in \mathbb{R},\; i = 0, 1. \tag{3.3}$$

In this case, the distribution function and the density of $L(u)$ are given by

$$\mathrm{P}[L(u) \le z] = p\,F_1\big(\frac{z-u}{1-u}\big) + (1-p)\,F_0\big(\frac{z}{1-u}\big) \tag{3.4a}$$

and

$$\frac{\partial \mathrm{P}[L(u) \le z]}{\partial z} = (1-u)^{-1}\left(p\,f_1\big(\frac{z-u}{1-u}\big) + (1-p)\,f_0\big(\frac{z}{1-u}\big)\right) \tag{3.4b}$$

respectively, where $p = \mathrm{P}[D]$ is the *default probability* of the loan under consideration. By means of (3.4a) and (3.4b), the quantile $q_\alpha(L(u))$ can be numerically computed. For the conditional probability which is part of Definition 3.1, we obtain

$$\mathrm{P}[D \,|\, L(u) = z] \;=\; \frac{p\, f_1\big(\frac{z-u}{1-u}\big)}{p\, f_1\big(\frac{z-u}{1-u}\big) + (1-p)\, f_0\big(\frac{z}{1-u}\big)}. \tag{3.5}$$

If, similarly to the situation of (2.12a), we assume that X and ξ are independent and both standard normally distributed, we obtain for the conditional densities in (3.3)

$$f_1(z) = \begin{cases} \frac{f(z)}{p}\,\Phi\Big(\big(\sqrt{1-\tau}\big)^{-1}\big(\Phi^{-1}(p) + \sqrt{\tau}\,\big(\frac{\sqrt{1-\rho}\,\Phi^{-1}(z)-c}{\sqrt{\rho}}\big)\big)\Big), & z \in (0,1)\\ 0, & \text{otherwise} \end{cases} \tag{3.6a}$$

and

$$f_0(z) = \begin{cases} \frac{f(z)}{1-p}\,\Phi\Big(-\big(\sqrt{1-\tau}\big)^{-1}\big(\Phi^{-1}(p) + \sqrt{\tau}\,\big(\frac{\sqrt{1-\rho}\,\Phi^{-1}(z)-c}{\sqrt{\rho}}\big)\big)\Big), & z \in (0,1)\\ 0, & \text{otherwise,} \end{cases} \tag{3.6b}$$

with

$$f(z) = \begin{cases} \sqrt{\rho/(1-\rho)}\,\exp\bigg(1/2\,\bigg(\Phi^{-1}(z)^2 - \Big(\frac{\sqrt{1-\rho}\,\Phi^{-1}(z)-c}{\sqrt{\rho}}\Big)^2\bigg)\bigg), & z \in (0,1)\\ 0, & \text{otherwise.} \end{cases} \tag{3.6c}$$

Note that $p = \Phi(a)$ in (3.6a) and (3.6b). Denote by $\Phi_2(\cdot,\cdot;\theta)$ the distribution function of the bivariate standard normal distribution with correlation θ. Then, for the conditional distribution functions corresponding to (3.6a) and (3.6b), we have

$$F_1(z) = \begin{cases} 1 - p^{-1}\,\Phi_2\big(a, \frac{c-\sqrt{1-\rho}\,\Phi^{-1}(z)}{\sqrt{\rho}}; \sqrt{\tau}\big), & z \in (0,1)\\ 0, & \text{otherwise,} \end{cases} \tag{3.7a}$$

and

$$F_0(z) = \begin{cases} (1-p)^{-1}\,\Phi_2\big(-a, -\frac{c-\sqrt{1-\rho}\,\Phi^{-1}(z)}{\sqrt{\rho}}; \sqrt{\tau}\big), & z \in (0,1)\\ 0, & \text{otherwise.} \end{cases} \tag{3.7b}$$

Unfortunately, the capital charges from Definition 3.1 do not yield reasonable values in all cases. Indeed, considering the relation of the default probability p and the confidence level α, we easily arrive at the following result.

Proposition 3.2 *For $L(u)$ defined by (3.2) with $F_1(0) = 0$, we have*

$$\lim_{u\to 1} q_\alpha(L(u)) \;=\; \begin{cases} 1, & p > 1-\alpha\\ 0, & p \le 1-\alpha. \end{cases}$$

The proof of Proposition 3.2 is presented in Appendix A. Under slightly stronger assumptions than in Proposition 3.2, the limiting behavior of $q_\alpha(L(u))$ is inherited by $\mathrm{P}[D \,|\, L(u) = q_\alpha(L(u))]$.

Proposition 3.3 *Assume that the densities in (3.3) are positive in* $(0,1)$*. Then for* $L(u)$ *defined by (3.2), we have*

$$\lim_{u \to 1} \mathrm{P}[D \,|\, L(u) = q_\alpha(L(u))] = \begin{cases} 1, & p > 1-\alpha \\ 0, & p < 1-\alpha. \end{cases}$$

The proof of Proposition 3.3 is also given in Appendix A. In addition, Remark A.2 explains why there is no statement on the case $\alpha = 1 - p$ in Proposition 3.3.

By Proposition 3.3, we see that in case $p < 1-\alpha$ it would be worthwhile to concentrate all the exposure to one loan. This is another appearance of a well-known deficiency of value-at-risk. Under value-at-risk portfolio risk measurement, putting all the risk into an event with very small probability can quite drastically reduce capital charges. In order to avoid this phenomenon, other risk measures like Expected Shortfall (cf. Acerbi and Tasche, 2002) have to be used. For Expected Shortfall, in Definition 3.1 the conditional probability $\mathrm{P}[D \,|\, L(u) = q_\alpha(L(u))]$ has to be replaced by $\mathrm{P}[D \,|\, L(u) \geq q_\alpha(L(u))]$. See Tasche and Theiler (2003) for details of this approach.

4 Numerical example

For the purpose of illustrating the previous sections, we consider a numerical example in the framework of Section 3. This means that there is a portfolio driven by systematic risk only (the variable Y in (3.2)) which is enlarged with an additional loan (the indicator $\mathbf{1}_D$ in (3.2)).

The portfolio modeled by Y has a quite moderate credit standing which is expressed by its expected loss $\mathrm{E}[Y] = 0.025 = \Phi(c)$. However, we assume that the additional loan enjoys a quite high credit-worthiness as we set $p = \mathrm{P}[D] = 0.002 = \Phi(a)$. The asset correlation ρ of the portfolio with the systematic factor and the asset correlation τ of the additional loan with the systematic factor are chosen according to the current Basel II proposal for corporate loan portfolios (Basel Committee, 2003), namely $\rho = 0.154$ and $\tau = 0.229$.

In order to draw Figure 1, the risk of the portfolio loss variable $L(u)$, as defined by (3.2), is regarded as a function of the relative weight u of the new loan in the portfolio. Risk then is calculated as true Value-at-Risk (VaR) at level 99.9% (i.e. $q_{0.999}(L(u))$), as the granularity adjustment approximation to $q_{0.999}(L(u))$ according to (2.4), and in the manner of the Basel II Accord (cf. Basel Committee, 2003). The values of VaR are computed by solving numerically for z the equation

$$\mathrm{P}[L(u) \le z] \;=\; \alpha, \tag{4.1}$$

with $\mathrm{P}[L(u) \le z]$ given by (3.4a), (3.7a), and (3.7b). The granularity adjustment approximation is computed by applying (2.14) to (3.1) and taking the limit with $n \to \infty$. The Basel II risk weight function can be interpreted as the first order version of (2.4) which yields the straight line

$$u \;\mapsto\; u\,\Phi\left(\frac{a - \sqrt{\tau}\,q_{1-\alpha}(X)}{\sqrt{1-\tau}}\right) + (1-u)\,\Phi\left(\frac{c - \sqrt{\rho}\,q_{1-\alpha}(X)}{\sqrt{1-\rho}}\right). \tag{4.2}$$

Note that for (4.2) we make use of a stylized version of the Basel II risk weight function as we chose arbitrarily the correlation ρ. The Basel I risk weight function is a constant at 8% that should not be compared to the afore-mentioned functions since it was defined without recourse to mathematical methods. For that reason it has been omitted from Figure 1. From Figure 1, we see that, up to a level of 2% relative weight of the new loan, both the

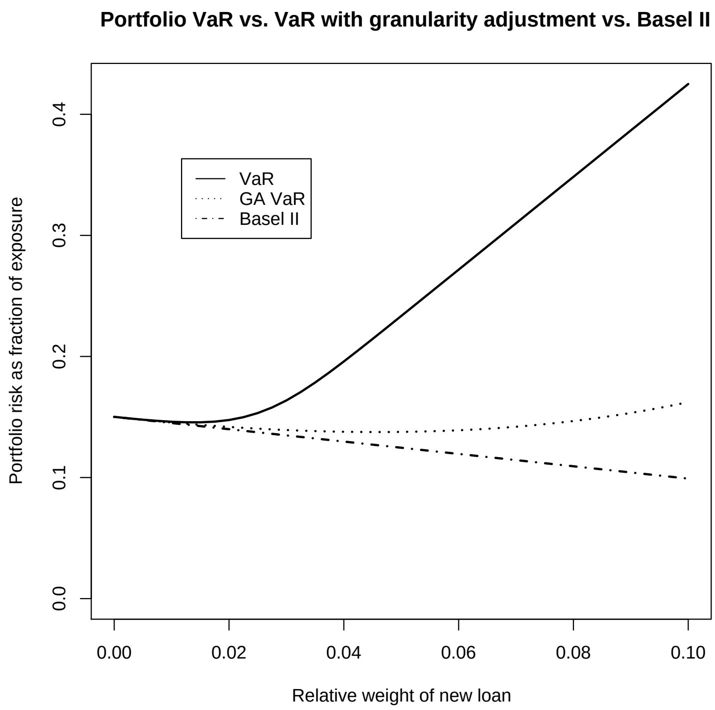

Figure 1: Portfolio risk as function of the relative weight of the additional loan in the setting of Section 3. Comparison of true Value-at-Risk (VaR), granularity adjustment approximation to VaR (GA VaR), and the risk weights according to the Basel II Accord.

granularity adjustment approach and the Basel II approach yield precise approximations to the true portfolio VaR. Beyond that level the approximations lose quality very fast. Of course, this observation does not surprise since both the granularity adjustment approach and the Basel II are based on the assumption that there is no concentration in the portfolio. Intuitively, this condition should be violated when the concentration of the additional loan increases too much.

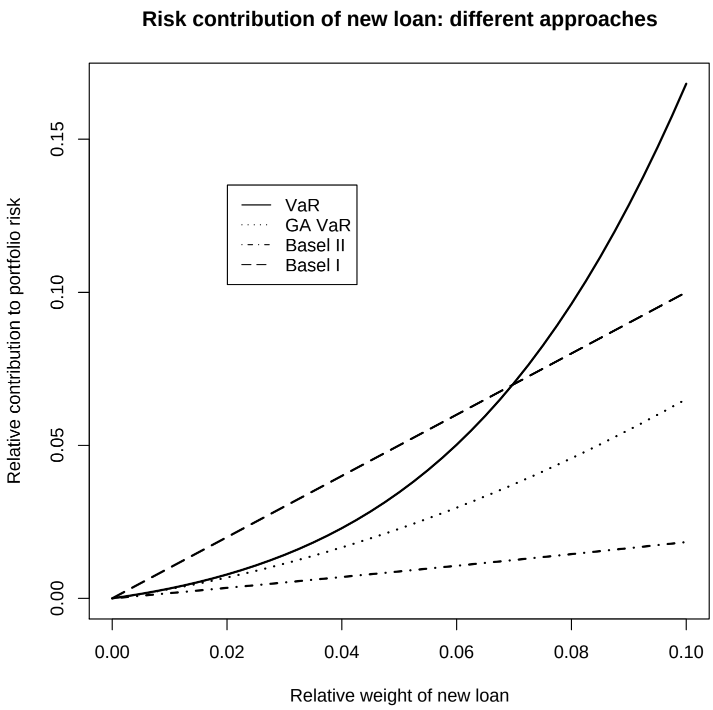

Figure 2: Relative risk contribution of new loan as function of the relative weight of the new loan in the setting of Section 3. Comparison of contribution to true Value-at-Risk (VaR), contribution to granularity adjustment approximation to VaR (GA VaR), and the contributions according to the Basel II and Basel I Accords.

Figure 2 illustrates the relative contribution of the new loan to the risk of the portfolio loss variable $L(u)$. The contribution is considered a function of the relative weight u of the new loan in the portfolio and calculated according to four different methods. The first method relates to the relative contribution to true portfolio VaR, defined as the ratio of the contribution to VaR according to Definition 3.1 and portfolio VaR, i.e. the function

$$u \mapsto \frac{u\,\mathrm{P}[D \mid L(u) = q_\alpha(L(u))]}{q_\alpha(L(u))}, \tag{4.3}$$

where the conditional probability has to be evaluated by means of (3.5), (3.6a), and (3.6b). Secondly, the relative contribution is calculated as the fraction with the granularity adjustment contribution according to Corollary 2.4 applied to (3.1) (with $n \to \infty$) in the numerator and the granularity adjustment approximation to VaR as for Figure 1 in the denominator. Moreover, curves are drawn for the Basel II approach, i.e. the function

$$u \mapsto \frac{u\,\Phi\left(\frac{a-\sqrt{\tau}\,q_{1-\alpha}(X)}{\sqrt{1-\tau}}\right)}{u\,\Phi\left(\frac{a-\sqrt{\tau}\,q_{1-\alpha}(X)}{\sqrt{1-\tau}}\right)+(1-u)\,\Phi\left(\frac{c-\sqrt{\rho}\,q_{1-\alpha}(X)}{\sqrt{1-\rho}}\right)}, \tag{4.4}$$

and the Basel I approach. The latter approach just entails the diagonal as risk contribution curve since it corresponds to purely volume-oriented capital allocation.

Note that in Figure 2 the true VaR curve intersects the diagonal (Basel I curve) just at the relative weight u^* that corresponds to the minimum risk portfolio $L(u^*)$. Whereas the granularity adjustment curve yields, up to a relative risk weight of 2% of the additional loan, a satisfying approximation to the true contribution curve, the Basel II curve loses reliability much earlier. Both these approximation curves are completely situated below the diagonal. This fact could yield the misleading impression that an arbitrarily high exposure to the additional loan still improves the risk of portfolio. However, as the true VaR curve in Figure 2 shows, the diversification effect from pooling with the new loan stops at 7% relative weight. From Figure 1 we see that beyond 7% relative weight of the additional loan the portfolio risk grows dramatically.

5 Conclusions

In the present paper we have extended the granularity approach introduced by Gordy (2003) in a way that takes care of the exact weights of the assets in the portfolio. By calculating the partial derivatives of the resulting approximate portfolio loss quantile and multiplying the results with the asset weights, we can arrive at an approximate capital charge for every asset in the portfolio. In addition, we have introduced the semi-asymptotic approach to one-factor portfolio loss modeling. This approach provides an alternative way to capital charges that accounts for concentrations. By a numerical comparison of these both approaches we have shown that the granularity approach to risk contributions may come up with misleading results if there is a non-negligible concentration in the portfolio. However, also the semi-asymptotic approach yields counter-intuitive results if the probability of default of the loan under consideration is very small. In order to avoid this drawback which is caused by the very nature of value-at-risk, sounder risk measures like Expected Shortfall have to be used.

A Appendix

The proofs of Propositions 3.2 and 3.3 are based on the following simple inequalities for $q_\alpha\big(L(u)\big)$.

Lemma A.1 *Assume $F_1(0) = 0$. Then for $L(u)$ defined by (3.2), we have*

$$q_\alpha\big(L(u)\big) \begin{cases} \le 1-u, & \text{if } \alpha \le 1-p, \\ > u, & \text{if } \alpha > 1-p. \end{cases}$$

Proof. In case $\alpha \le 1-p$ we obtain by means of (3.4a)

$$\mathrm{P}[L(u) \le 1-u] \;=\; (1-p)\, F_0\big(\frac{1-u}{1-u}\big) + p\, F_1\big(\frac{1-2\,u}{1-u}\big) \;\ge\; 1-p \;\ge\; \alpha. \tag{A.1}$$

By the definition of quantiles (see (2.3)), (A.1) implies the first part of the assertion. Assume now $\alpha > 1-p$. Analogously to (A.1), then we calculate

$$\mathrm{P}[L(u) \le u] \;=\; (1-p)\, F_0\big(\frac{u}{1-u}\big) + p\, F_1(0) \;\le\; 1-p \;<\; \alpha. \tag{A.2}$$

Again by the definition of quantiles, (A.2) yields $q_\alpha(L(u)) > u$ as stated above. □

Proof of Proposition 3.2. In case $\alpha \le 1-p$, Lemma A.1 implies

$$\limsup_{u\to 1} q_\alpha\big(L(u)\big) \;\le\; 0, \tag{A.3a}$$

in case $\alpha > 1-p$ we deduce from Lemma A.1

$$\liminf_{u\to 1} q_\alpha\big(L(u)\big) \;\ge\; 1. \tag{A.3b}$$

Since $0 \le q_\alpha\big(L(u)\big) \le 1$, (A.3a) and (A.3b) imply the assertion. □

Proof of Proposition 3.3. By Lemma A.1, in case $\alpha \le 1-p$ we have $q_\alpha\big(L(u)\big) \le 1/2$ for $u > 1/2$ and hence $q_\alpha\big(L(u)\big) - u < 0$. This implies

$$f_1\big(\frac{q_\alpha(L(u)) - u}{1-u}\big) \;=\; 0 \tag{A.4}$$

for $u > 1/2$. A closer inspection of the proof of Lemma A.1 reveals that in case $\alpha < 1-p$ even $0 < q_\alpha(L(u)) < 1-u$ obtains if f_0 is positive. This observation implies $f_0\big(\frac{q_\alpha(L(u))}{1-u}\big) > 0$. Hence, $\lim_{u\to 1} \mathrm{P}[D \,|\, L(u) = q_\alpha(L(u))] = 0$ follows from (3.5) and (A.4). Assume now $\alpha > 1-p$. Then by the positivity assumption on f_1, Lemma A.1 implies $1 > q_\alpha\big(L(u)\big) > u$. Since $u/(1-u) > 1$ for $u > 1/2$ we may deduce that

$$f_0\big(\frac{q_\alpha(L(u))}{1-u}\big) \;=\; 0 \quad \text{and} \quad f_1\big(\frac{q_\alpha(L(u)) - u}{1-u}\big) \;>\; 0. \tag{A.5}$$

This implies $\mathrm{P}[D \,|\, L(u) = q_\alpha(L(u))] = 1$ for u enough close to 1. □

Remark A.2 *By inspection of the proof of Proposition 3.3, one can observe that*

$$q_{1-p}(L(u)) \;=\; 1-u, \qquad u > 1/2. \tag{A.6}$$

As a consequence, $f_1\big(\frac{q_{1-p}(L(u))-u}{1-u}\big)$ and $f_0\big(\frac{q_{1-p}(L(u))}{1-u}\big)$ may equal zero at the same time. Hence, in case $\alpha = 1-p$ according to (3.5) the conditional probability $\mathrm{P}[D \,|\, L(u) = q_{1-p}(L(u))]$ may be undefined for $u > 1/2$. For this reason, Proposition 3.3 does not provide a limit statement for the case $\alpha = 1-p$.

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