

# CREDIT BARRIER MODELS

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**ABSTRACT.** The model introduced in this article is designed to provide a consistent representation for both the real-world and pricing measures for the credit process. We find that good agreement with historical and market data can be achieved across all credit ratings simultaneously. The model is characterized by an underlying stochastic process that represents credit quality and default events are associated to barrier crossings. The stochastic process has state dependent volatility and jumps which are estimated by using empirical migration and default rates. A risk-neutralizing drift and implied recovery rates are estimated to consistently match the average spread curves corresponding to all the various ratings.

## 1. INTRODUCTION

Rating agencies rank the credit-worthiness of sovereigns and corporations into a number of credit classes and provide migration and default rates based on historical data. A broad array of applications requires accurate pricing models which are consistent with empirical credit migration and default rates and which capture the relevant components of the price of risk. We introduce new credit barrier models extending Hull and White [14] and Avellaneda and Zhu [5]. These are structural models based on a notion of *distance to default* related to credit ratings. One can think of the distance to default as a measure of an obligor's leverage relative to the volatility of its asset values. The model in this article attempts to capture aggregate information including migration rates and average spread curves for all ratings within a unified framework.

Pricing models for credit sensitive assets can be divided into three main categories: (i) structural-form models, (ii) reduced-form models and (iii) credit barrier models. The first category of credit risk models is based on work by Merton [24]. In its many variants proposed by Kim et al. [18], Nielsen et al. [25], Longstaff and Schwartz [21] and others, this approach attempts to directly model the borrower's balance sheet. The firm's fixed liabilities constitute a barrier point for the value of its assets. If assets drop below that barrier, the firm is unable to support its debt and default occurs. Estimating these models is a subtle art as extracting a default barrier from accounting statements requires significant exercise of judgment to account for complex liability structures.

Reduced-form models were considered by Litterman and Iben [20], Madan and Unal [23], Jarrow and Turnbull [17], Jarrow, Lando and Turnbull [16], Lando [19], Duffie and Singleton [10], and Duffie [9]. Reduced-form models differ fundamentally from structural-form models in the degree of predictability of the default. A typical reduced-form model assumes that an exogenous random variable drives default and that the probability of default over any time interval is nonzero. Empirical evidence concerning reduced-form models is rather limited. Using the Duffie and Singleton [10] framework, Duffie [8] finds that these models have difficulty in explaining the observed term structure of credit spreads across firms of different qualities. In particular, such models have difficulty generating both relatively flat yield spreads when firms have low credit risk and steeper yield spreads when firms have higher credit risk.

The literature on credit barrier models is more recent and includes working papers by Hyer et al. [15], Gordy and Heitfield [12], Douady and Jeanblanc [7] and articles by Hull and White [14] and Avellaneda and Zhu [5]. In [14] and [5], the underlying variable is a firm specific measurement of distance to default  $x_t$  which undergoes a Wiener process and triggers default events by hitting a time dependent

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*Date:* January 6, 2003.

The authors were supported in part by the National Science and Engineering Council of Canada. We thank the participants to the 2002 Annual Derivatives Conference, Cornell University, the Field's Conference on Computational Finance, Risk 2002 USA, the 2002 Annual Options Conference at Warwick University and the 2002 MSRI Meeting on Event Risk. In particular we thank Marco Avellaneda, Peter Carr, Raphael Douady, Stewart Hodges, Jing-zhi Huang, Ming Huang and Petter Wiberg for useful discussions. All remaining errors are our own.

barrier. These models are related to structural models as the Hull and White model for interest rates are related to the Vasicek model: the explicitly time dependent barrier is mathematically equivalent to a risk neutralizing drift and its term-structure is adapted to achieve a good fit with the observed spread curves. The distance to default in these models is not directly observable, but rather it is a mathematical abstraction introduced for the sake of building a risk-neutral measure which is consistent with a particular firm's specific yield spread curve. The ease of calibration makes it possible to build meaningful models for basket credit default swaps by correlating processes corresponding to different names.

In the credit barrier models in this paper, the driving process is identified with the observable credit rating and is estimated consistently to both the migration rates and pricing information on yield spreads. Credit quality is modelled as a continuous variable  $f_t$  varying between 0 and 1 which undergoes a pure jump process with state dependent volatility in continuous time. This variable captures the firm's fundamental information and is directly related to credit ratings, as credit migrations and default events correspond to barrier crossings. Rich statistical information on the process followed by the credit quality under the real-world measure is given by the historical credit migration and default rates. To explain this data, we find that it is necessary to allow our explanatory variable to have a volatility dependent on the distance to default (as higher quality ratings are less volatile than lower quality ones). In the context of a simple diffusion model, as already noticed in [12], it is not possible to achieve accurate fits for migration rates of retaining the same rating and of changing two or three ratings. To overcome this difficulty we have the option to either include jumps as in [26] or a stochastic volatility component. We opt for the former solution since a stationary process under the real-world measure can be estimated in terms of observable quantities. On the other hand, a stochastic volatility model would require marking to market an hidden volatility variable.

To calibrate under the risk-neutral measure, we allow for a time-dependent risk-neutralizing drift. Since our model attempts to capture average spread curves for all credit ratings at once, the estimation requires to disentangle spreads due uniquely to the market price of credit risk from the *liquidity* and the *differential taxation* premia. This difficult problem is studied in a recent article by Elton, Gruber, Agrawal and Mann [11] and in a working paper by Huang and Huang [13]. The latter authors, make use of a variety of different structural models and estimates on the equity risk premium to disentangle the liquidity from the credit premium. Rather than delving into a statistical analysis along similar lines, in this article we simply attempt to achieve qualitative consistency with the estimates in [13] by using our credit barrier models and make use of the estimates of the tax premia in [11].

We observe that our model is capable of reproducing the qualitative features of observed term structures of credit spreads across the full range of credit ratings, with curves for high quality ratings being fairly flat and curves for lower quality ratings being inverted and steep. To achieve a quantitatively accurate match with market spread data, a necessary step for pricing purposes, one can adjust the term structure of *implied recovery rates* depending on the initial ratings. Similarly to implied volatility, implied recovery rates capture the non-modelled market imperfections related to liquidity premia and have characteristic features different from the historical counterparts. For instance, high quality borrowers have low implied recovery rates since yields on their debt is significantly affected by the liquidity premium. The lowest grade borrowers also have relatively low implied recovery to reflect the positive correlation between historical recovery rates and credit quality, see Figure 8.

For technical reasons, we ask that our models be integrable in terms of special functions. This condition turns out to be of crucial technical importance for the estimation and calibration aspects of the problem, as accurate expressions for transition probabilities are required for the numerical fitting procedure. Analytical tractability eases the calculation of other important quantities such as first passage times across the credit barriers. It also simplifies tasks such that the generation of Monte Carlo scenarios for possibly correlated credit histories under both the real-world and risk-neutral measures. These tasks are the elementary building blocks required to assemble pricing algorithms for applications ranging from pricing of basket credit derivative to risk assessment for credit exposures.

## 2. MODEL DESCRIPTION AND ESTIMATION

The main features of our credit barrier models are a state dependent volatility, jumps and a risk-neutralizing drift. This next subsection introduces a framework for the first only, while the next two subsections complete the model description.

**2.1. The case of a diffusion process.** According to both Moody's and Standard&Poor's schemes, credit classes are categorized into either a finer partition of 17 or into a coarser partition of 7 ratings. The examples in this paper refer to the finer 17-ratings scheme, while the methods are more general.

The main underlying driver of our pricing processes is a credit quality process  $f_t$  taking values in the interval  $[0, 1]$  and satisfying a diffusion equation of the form

$$(1) \quad df_t = \mu(f_t)dt + \sigma(f_t)dW_t$$

The functional forms of the drift and volatility are restricted so that we can use results in [3] to find the transition probability density function  $U_f(f, f_0, t)$  by solving the forward Kolmogorov equation

$$(2) \quad \frac{\partial U_f}{\partial t} = \frac{1}{2} \frac{\partial^2}{\partial f^2} (\sigma^2 U_f) - \frac{\partial}{\partial f} (\mu U_f).$$

explicitly in closed form. The process is defined in such a way that the variable  $f_t$  takes values only in the interval  $[0, 1]$ . The upper boundary at 1 is unattainable and the lower boundary at 0 has an absorbing boundary condition. We refer the reader to Appendix I for a detailed process specification.

We subdivide the interval  $[0, 1]$  with equally spaced credit barriers located at  $b_i = \frac{i}{17}$  for  $i = 0, \dots, 17$ . The 17 subintervals  $(b_i, b_{i+1}]$ ,  $i = 0, \dots, 17$  correspond to the various credit classes. Let  $f_1, \dots, f_{17}$  denote the midpoints of these intervals. That is,  $f_i = \frac{1}{34} + \frac{i-1}{17}$ . Let  $p_{ij}(t)$  denote the conditional (transition) probability that an index with given initial rating  $f_i$  will have a rating in the interval  $[b_{j-1}, b_j]$  at a later time  $t > 0$ . These transition probabilities are expressed in terms of integrals of the conditional probability density over a *region* about the final rating level  $f_j$  while assuming a fixed initial credit level  $f_i$ :

$$(3) \quad p_{ij}(t) = \int_{b_{j-1}}^{b_j} U_f(f, f_i, t) df$$

where the probability kernel is given in Appendix I. The probability that starting from the initial rating  $f_i$  and reaching a state of default by time  $t$  is given by the following difference:

$$(4) \quad p_i^D(t) = 1 - \int_0^1 U_f(f, f_i, t) df.$$

We attempted unsuccessfully to fit such a diffusion model to the historical migration rates. The problem we encountered was that although transition probabilities  $p_{ij}(t)$  for nearest rating migrations can be well reproduced, it is not possible to simultaneously fit migrations involving a rating change of two or three levels and to accurately reproduce default probabilities across credit ratings.

**2.2. Jumps in the Real-world Measure.** The difficulties above can be overcome by allowing for either jumps or a stochastic volatility component. We choose to include jumps rather than stochastic volatility for two reasons. First, to remain within the class of stationary processes and reflect the effort by credit rating agencies to revise ratings in a timely fashion so that migration probabilities remain consistent over time and through economic cycles. Second, for the sake of having a parsimonious specification which can be estimated without having to infer the value of hidden variables such as the initial condition for a volatility parameter.

To include jumps, we use stochastic time changes similarly to what is done in the Variance-Gamma (VG) model by Madan et al. [22]. These authors focus on jump versions of the geometric Brownian motion in the Black-Scholes pricing framework. The generalization we introduce in this papers regards a VG process  $V_t(\nu)$  obtained by evaluating a diffusion process  $f_t$  satisfying equation (1) at a random time given by a gamma process  $\gamma(t, 1, \nu)$ , i.e.

$$V_t(\nu) = f_{\gamma(t, 1, \nu)},$$

where  $\nu$  is called the variance rate and has the dimension of time.

The process  $\gamma(t, 1, \nu)$  can be interpreted as a mapping from calendar time  $t$  to a financial time  $\tau$ , defined as a measure of total financial activity up to a certain point in calendar time, see [23]. The variable  $\gamma(t, 1, \nu)$  does not take all values, but instead proceeds with a series of upward jumps of various size. Jumps arise in the model because financial time may proceed rapidly with respect to physical time without information about these excursions being observable. Let's remark though that the notion of financial activity is just a useful support for the intuition behind stochastic time changes: the actual estimation of the parameter  $\nu$  does not involve any inter-temporal analysis but only the consideration of transition probabilities.

Our model specification is that a default event occurs whenever the process  $f_\tau$  crosses the barrier  $f_\tau = 0$ , possibly during an unobservable excursion. With this interpretation, probability kernel  $\tilde{U}_f(f, f_0, t)$  can be evaluated as time changed versions of probability kernels for the corresponding diffusion process, namely

$$(5) \quad \tilde{U}_f(f, f_0, t) = \int_0^\infty U_f(f, f_0, s) \tilde{\Gamma}(s, t) ds,$$

with

$$(6) \quad \tilde{\Gamma}(s, t) = \frac{s^{t/\nu-1} e^{-s/\nu}}{\Gamma(t/\nu) \nu^{t/\nu}}$$

the gamma density function and  $\Gamma(x)$  the Gamma function.

**2.3. Risk-neutralizing Drift.** In our model, we propose to add the risk-neutralizing drift to the underlying diffusion process prior to the stochastic time change. In the examples in this article, we do not account for the market price of jump risk and use the same variance rate to define both the real-world and the risk-neutral measures. Introducing a market price for jump risk by resizing the variance rate doesn't introduce mathematical complexities but would require using other data such as derivative prices in addition to bond prices and migration rates to estimate the implied jump size.

One can add a drift term while leaving the volatility invariant and while retaining integrability by means of a non-linear transformation of the form  $g_t = G(f_t, t)$ . The equation satisfied by the process  $g_t$  is

$$(7) \quad dg_t = \left[ G'(f_t, t) \mu(f_t) + \dot{G}(f_t, t) + \frac{1}{2} G''(f_t, t) \sigma^2(f_t) \right] dt + G'(f_t, t) \sigma(f_t) dW_t.$$

The condition of invariance of the volatility function is

$$(8) \quad \sigma(g) = G'(f, t) \sigma(f).$$

and integrates to

$$(9) \quad \int_0^{G(f,t)} \frac{d\xi}{\sigma(\xi)} = \int_0^f \frac{d\xi}{\sigma(\xi)} + a(t).$$

for some function of time  $a(t)$ . Hence, our transformations are specified by a time dependent function  $a(t)$ .

As explained in Appendix III, the pricing kernel  $U_g(g, g_0, t)$  can be computed by solving a boundary value problem, where the location of the boundary is related to the function  $a(t)$ . This has a simple financial interpretation: under the risk-neutral measure defaults occur whenever a barrier located above the real-world default line at 0 is hit.

We introduce jumps by applying a stochastic time change with the historical variance rate, so that the time transformed process  $\tilde{g}_t$  is given by

$$(10) \quad \tilde{g}_t = g_{\gamma(t, 1, \nu)}$$

Also under the risk-neutral measure, we assume that a virtual excursion of the underlying diffusive variable across the barrier in financial time results in a default event. Hence the pricing kernel is

$$(11) \quad \tilde{U}_g(g, t; g_0) = \int_0^\infty U_g(g, s; g_0) \tilde{\Gamma}(t, s) ds.$$

Original Rating	Recovery Rate(%)
Aaa	68.34
Aa2	59.59
A2	60.63
Baa2	49.24
Ba2	39.05
B2	37.54
Caa	38.02

TABLE 1. Recovery rate for each credit rating. From [4].

**2.4. Model Estimation.** We estimated our model under the real-world measure using credit migration rates over a one year time horizon and default rates over a one, three and five year time horizon. The quality of fit is depicted in Figures 1 and 2.

We notice that the fit is quite good across all ratings and time horizons. The state-dependent volatility is shown in Figure 3. Notice that the volatility function depends on the credit rating and is higher for lower quality ratings.

The variance rate we estimate out of credit migration data is  $\nu = 8.1$  years. This is a large value as compared to the relevant time horizons, indicating that large jump amplitudes are required to justify the observed transition probabilities. Since the jump amplitudes are affected by the volatility, we conclude that the intensity of jumps is not homogeneously distributed between credit ratings. Specifically, larger jumps are associated with the lower ratings.

With the real-world model determined, we can add the risk-neutral drift through specification of a default boundary in  $x$ -space. The *credit spread* is the the portion of the total yield spread due to credit risk. It is shown in Huang and Huang [13] that the credit spread is a small part of the yield spread in higher ratings and is a larger fraction in lower ratings. The boundary in Figure 4 was chosen such that this qualitative behaviour is reflected in the spread rates computed by the model, as shown in Figure 5. Note that we are taking the simple case of zero-coupon bonds and assuming a constant interest rate, so that cumulative probabilities of default and recovery rates imply a term structure for spread rates through the pricing formula:

$$(12) \quad e^{-s(t)t} = 1 - P(t)(1 - R)$$

where  $s(t)$  is the yield spread for maturity  $t$ ,  $P(t)$  is the probability of defaulting before  $t$  and  $R$  is the recovery rate. For the recovery rates, we use historical recovery rates that are rating dependent. These are displayed in Table 1.

Elton et al. [11] find that the mid-point of effective state tax rates in the United States corporate bond market is 4.875%. Assuming zero-coupon bonds, this gives a tax-adjusted spread rate of:

$$(13) \quad s(t) - 4.875\%[s(t) + r(t)]$$

where  $r(t)$  is the treasury yield of maturity  $t$ . Figures 6 and 7 compare the tax-adjusted spread curves with the model derived spread curves assuming constant implied recovery rates. One can notice that the qualitative behaviour of the term structure of credit spreads is correctly reproduced by our model, with lower ratings corresponding to downward sloping curves and higher ratings corresponding to upward sloping and flatter profiles. A precise match can then be achieved by adjusting implied recovery rates.

The implied recovery rates shown in Figure 8 are the time dependent recovery rates needed in equation (12) in order for the market and model spread curves in Figures 6 and 7 to match exactly. The spread between implied and historical recovery rate term structures can be interpreted as a direct measurement of liquidity spreads (according to our earlier definition), that is, of the difference between yield spreads and credit spreads.

### 3. CONCLUSION

We introduce an integrable model for the credit quality process which gives a unified picture of credit migrations and default arrival rates under both the real-world and risk-neutral measures.

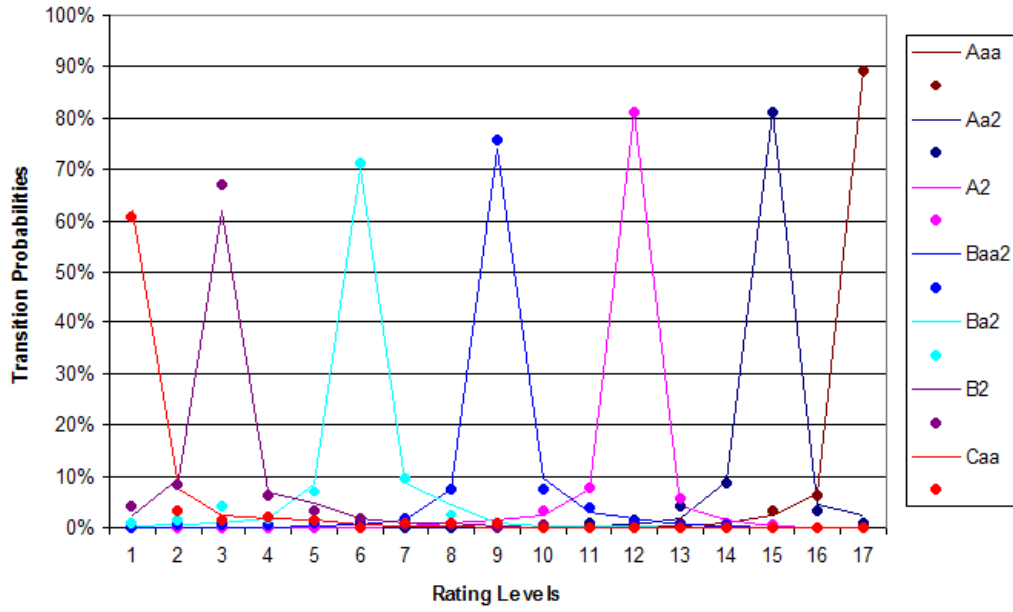


FIGURE 1. Comparison of model (lines) and historical (dots) one year transition probabilities. Historical transition probabilities are taken from [6] and are "Withdrawn Rating" adjusted. The parameters used for the model are:  $q_2/q_1 = 30$ ,  $\rho = 0.00125$ ,  $\theta = 0.3$  and  $\nu = 8.1$ .

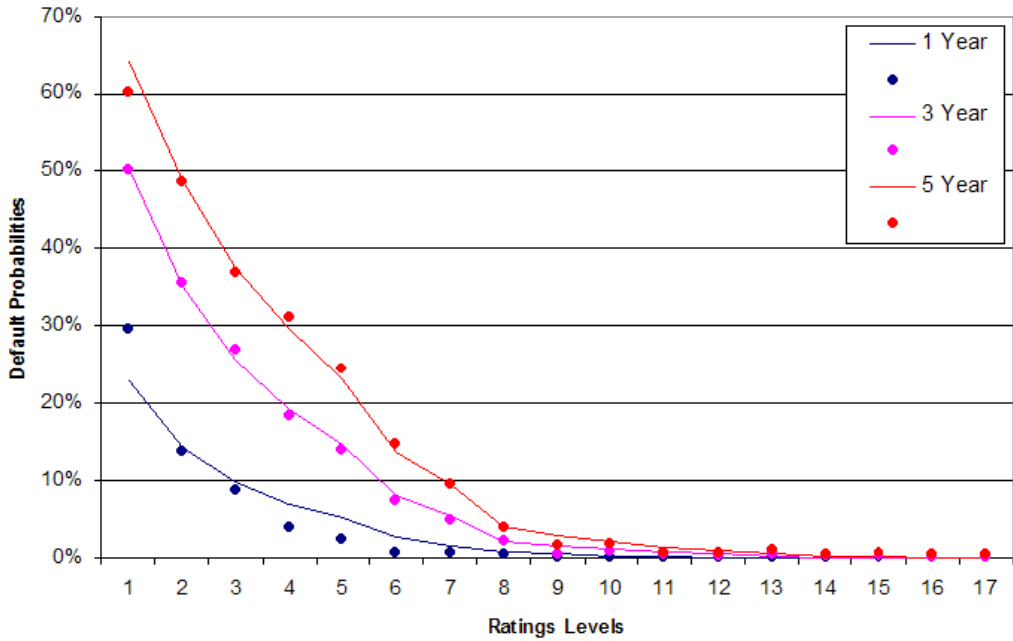


FIGURE 2. Comparison of model (lines) and historical (dots) default probabilities. Historical default probabilities are taken from [6] and are "Withdrawn Rating" adjusted.

Historical credit migration and default rates are reproduced with high accuracy. The process is pure-jump with state dependent volatility, an absorbing boundary corresponding to default events and a unattainable upper boundary. The model can also be calibrated to agree with spread curve data. Under the pricing measure, a risk-neutralizing drift biases downwards the credit process and is mathematically equivalent to a moving default barrier.

The model is analytically integrable, a crucial property that allows one to estimate it with high precision. The calibrated risk-neutral process provides a way of generating scenarios for spread curves under both the risk-neutral and the real-world measure. Monte-Carlo scenario generation algorithms are efficient due to the knowledge of probability kernels. Possible applications include pricing of structures such as basket default swaps and collateralized debt obligations, as well as credit risk management.

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## Appendix I. DIFFUSION PROCESS

The credit quality variable  $f_t$  follows a diffusion process in financial time. Following [3], we specialize to models which are integrable by reduction by changes of variable and of measure to an underlying  $x_t$  which follows a special case of a CIR process, i.e.

$$(A-1) \quad dx_t = (\theta + 2)dt + 2\sqrt{x_t}dW_t$$

It is simpler to work with a process  $\phi_t$  which we shall define and which is a non-linear transformation of  $f_t$ , where the ratings levels need not be evenly spaced. Under the transformation

$$(A-2) \quad \Phi(x) = \frac{2\sigma_0 I_\theta(\sqrt{2\rho x})}{q_2 \hat{u}(\sqrt{2\rho x})}$$

where  $\hat{u}(x) = q_1 I_\theta(x) + q_2 K_\theta(x)$ , and with numeraire

$$(A-3) \quad \frac{e^{\rho t} x^{\theta/2}}{\hat{u}(\sqrt{2\rho x})},$$

the process  $\phi_t = \Phi(x_t)$  satisfies the equation

$$(A-4) \quad d\phi_t = \sigma(\phi_t) dW_t.$$

Here,  $I_\theta$  and  $K_\theta$  are the modified Bessel functions, the variables  $q_1, q_2, \theta$  are estimation parameters. and the volatility function is given by:

$$(A-5) \quad \sigma(\Phi(x)) = \frac{2\sigma_0}{\sqrt{x} \hat{u}^2(\sqrt{2\rho x})}.$$

The transition probability density function (pdf),  $U_\phi(\phi, \phi_0, t)$ , in the space of the variable  $\phi$  solves the forward-time Kolmogorov equation:

$$(A-6) \quad \frac{\partial U_\phi}{\partial t} = \frac{1}{2} \frac{\partial^2}{\partial \phi^2} (\sigma(\phi)^2 U_\phi).$$

Notice that the function  $\Phi(x)$  is monotonic with positive derivative

$$(A-7) \quad \frac{d\Phi(x)}{dx} = \frac{\sigma_0}{x \hat{u}^2(\sqrt{2\rho x})}$$

so that we can define a function  $X$  to be the inverse transformation of  $\Phi$ . That is,

$$(A-8) \quad x = X(\phi) = \Phi^{-1}(\phi)$$

It can be shown from results in [1] (see also Appendix IV) that the pricing kernel for  $\phi_t$  is related to the pricing kernel  $U_x(x, t; x_0)$  for  $x_t$  (with initial value  $x_0$ ) by:

$$(A-9) \quad U_\phi(\Phi(x), t; \Phi(x_0)) = e^{-\rho t} \frac{x}{\sigma_0} \left( \frac{x_0}{x} \right)^{\theta/2} \frac{\hat{u}^3(\sqrt{2\rho x})}{\hat{u}(\sqrt{2\rho x_0})} U_x(x, t; x_0)$$

This is a general formula that holds for generic boundary conditions. In the case of the real-world measure, there is an absorbing boundary at  $x = 0$  corresponding to defaults. In this case we have that

$$(A-10) \quad U_\phi(\Phi(x), t; \Phi(x_0)) = e^{-\rho t - \frac{x+x_0}{2t}} \frac{x}{2\sigma_0 t} \frac{\hat{u}^3(\sqrt{2\rho x})}{\hat{u}(\sqrt{2\rho x_0})} I_\theta \left( \frac{\sqrt{xx_0}}{t} \right)$$

## Appendix II. DEFAULT PROBABILITIES

The default probabilities  $p_i^D(t)$  are among the most important characteristic of the transition matrix. By making use of equation (A-6) together with (A-10) into the equivalent of (4) while differentiating with respect to  $t$ , one readily shows that

$$(A-11) \quad \frac{dp_i^D(t)}{dt} = \frac{1}{2} \frac{\partial}{\partial \phi} \left( \sigma(\phi)^2 U_\phi \right) \Big|_{\phi=0} = \frac{q_2 e^{-\rho t - x_i/2t}}{2\hat{u}(\sqrt{2\rho x_i}) t^{\theta+1}} \left( \frac{x_i}{2\rho} \right)^{\theta/2}.$$

Here we have used the zero density boundary condition at  $\phi = \phi_{max}$  (corresponding to  $x \rightarrow \infty$ ) and the Taylor expansion of the modified Bessel function for small arguments. By integrating this expression we find a simple integral representation for  $p_i^D(t)$ :

$$(A-12) \quad p_i^D(t) = \frac{q_2}{2\hat{u}(\sqrt{2\rho x_i})} \left( \frac{x_i}{2\rho} \right)^{\theta/2} \int_0^t \frac{e^{-\rho s - \frac{x_i}{2s}}}{s^{1+\theta}} ds.$$

From this result one can see that if  $q_2 = 0$  the probability density function  $U_\phi$  integrates to unity for any time  $t \geq 0$  as is shown in [2].

It is instructive to investigate the long-time limit  $t \rightarrow \infty$  for  $p_i^D(t)$ . In this limit, the above integral tends to

$$(A-13) \quad p_i^D(\infty) = \frac{q_2 K_\theta(x_i)}{q_1 I_\theta(x_i) + q_2 K_\theta(x_i)}.$$

If  $q_1 \neq 0$ , indices will not attain the default state with probability one, even at arbitrarily large time. If  $q_1 = 0$ , the transition and default probabilities do not depend on  $q_2$ , as seen by using equations (A-7) and (A-10) in the integral expressions for the various probabilities. In this case the transition and default probabilities depend only on the parameters  $\rho$  and  $\theta$ .

Within the framework of the VG model, the default probabilities take the form can be obtained by integrating  $p_i^D(t)$  against  $\tilde{\Gamma}(s, t)$  to yield

$$(A-14) \quad \tilde{p}_i^D(t) = \frac{q_2}{2\hat{u}(\sqrt{2\rho x_i})} \left(\frac{x_i}{2\rho}\right)^{\theta/2} \int_0^\infty \Gamma\left(\frac{z}{\nu}, \frac{t}{\nu}\right) \frac{e^{-z - \frac{x_i}{2z}}}{z^{1+\theta}} dz,$$

where  $\Gamma(x, a)$  is the incomplete Gamma function

$$(A-15) \quad \Gamma(x, a) = \frac{1}{\Gamma(a)} \int_x^\infty y^{a-1} e^{-y} dy.$$

If the variance rate  $\nu$  approaches zero (i.e. no jumps), the incomplete Gamma-function  $\Gamma(z/\rho\nu, t/\nu)$  becomes the theta-function  $\theta(\rho t - z)$  and we get equation (A-12), as expected.

### Appendix III. BARRIERS AND DRIFTS

Here, we describe the pricing kernel under the risk-neutral measure.

Let  $b(t)$  be a positive function in  $x$ -space where we impose a Dirichlet boundary condition. That is, the pricing kernel  $U_x(x, t; x_0)$  in  $x$ -space satisfies the boundary conditions

$$(A-16) \quad \begin{aligned} U_x(x, 0; x_0) &= \delta(x - x_0) \\ U_x(b(t), t; x_0) &= 0 \\ \lim_{x \rightarrow \infty} U_x(x, t; x_0) &= 0 \end{aligned}$$

and within this domain is a solution to the forward-time Kolmogorov equation:

$$(A-17) \quad \frac{1}{2} \frac{\partial U_x}{\partial t} = x \frac{\partial^2 U_x}{\partial x^2} + (1 - \theta) \frac{\partial U_x}{\partial x}$$

For non-zero boundaries  $b(t)$ , the solution must be found numerically.

It is convenient to apply the iso-volatility transformation in equation (9) to the process  $\phi_t$ . In this coordinate, we have

$$(A-18) \quad \int_0^c \frac{d\xi}{\sigma(\xi)} = \sqrt{X(c)}.$$

With (A-18) we can derive from equation (9) that

$$(A-19) \quad G(f, t) = \Phi \left[ \left( \sqrt{X(\phi)} + a(t) \right)^2 \right].$$

It is possible to change variables to map the time-dependent boundary  $b(t)$  to 0. By equation (A-19), this is done by setting

$$(A-20) \quad a(t) = -\sqrt{b(t)}$$

Let  $\Phi_g(g, t)$  be the function such that  $G(\Phi_g(g, t), t) = g$  for all times  $t$ . Its derivative is found by equation (8) and we can obtain an expression for the pricing kernel  $U_g(g, g_0, t)$  simply by a change of variables so that

$$(A-21) \quad U_g(G(\Phi(x), t), t; G(\Phi(x_0), t)) = U_\phi(\Phi(x), t; \Phi(x_0)) \frac{\sigma(\Phi(x))}{\sigma(G(\Phi(x), t))}$$

We make the assumption that we can expand  $U_x$  in powers of  $\bar{x} = x - b(t)$  at the boundary  $b(t)$ :

$$(A-22) \quad U_x(x, x_0, t) = \sum_{m=1} \alpha_m(x_0, t; [b(\cdot)]) \bar{x}^m,$$

where  $\{\alpha_m\}_{m=1}$  is a sequence of functions dependent on  $x_0$  and  $t$  and functionally dependent on  $b(\cdot)$ . Notice that the constant term is zero since  $U_x$  vanishes at the boundary. Then it can be shown that the probability of default is given by:

$$(A-23) \quad p^D(t) = \rho \int_0^t \alpha_1(x_0, s; [b(\cdot)]) ds$$

Here, the  $\alpha_1(x_0, s; [b(\cdot)])$  term is just the gradient of  $U_x$  at the boundary. In the case  $b(t) = 0$ , this is seen to reduce to the formula for the default probability given in (A-12), as expected.

#### Appendix IV. EXACT SOLUTIONS

In this appendix we derive the formula A-9 in Appendix I. An intuitive financial derivation of this formula follows from the Fundamental Theorem of Finance upon properly interpreting the processes.

Consider a multi-currency financial model where domestic interest rates are zero, the process  $x_t$  in A-1 is interpreted as a price process for an asset denominated in a foreign currency and  $\phi_t = \Phi(x_t)$  is the price process for a contingent claim (a quanto option) in the domestic currency. Assume that under the risk neutral measure where  $\phi_t$  is driftless, the underlying foreign price process  $x_t$  obeys the equation

$$(A-24) \quad dx_t = \mu(x_t)dt + \nu(x_t)dW_t$$

for some drift function  $\mu(x)$ . Assume also that the volatility  $\nu(x_t)$  is such that, for some choice of the drift function  $\lambda(x)$ , one can solve the stochastic differential equation

$$(A-25) \quad dx_t = \lambda(x_t)dt + \nu(x_t)dW_t$$

in analytically closed form. By solving, we mean that it is possible to find the *pricing kernel*  $u(x, t; x_0)$ , which can be interpreted as the price of a butterfly spread option of maturity  $t$  for which the area under the payoff function is one and in the limit as the spread goes to zero around the point  $x$ , where  $x_0$  is the spot price of the underlying foreign asset.

Our objective is to show that if the volatility function for the quanto option  $\phi_t$  is defined as follows:

$$(A-26) \quad \sigma(\phi) = \frac{\sigma_0 \nu(X(\phi)) e^{-2 \int^{X(\phi)} \frac{\lambda(y) dy}{\nu(y)^2}}}{\hat{u}(X(\phi), \rho)^2},$$

then it is possible to find the pricing kernel for the quanto option  $\phi_t$  in analytically closed form. Here,  $\rho$  is a real valued parameter and the function  $\hat{v}(x)$  is defined as the solution of the equation:

$$(A-27) \quad \frac{\nu^2}{2} \hat{v}_{xx} = \rho \hat{v} - \lambda \hat{v}_x.$$

Notice that this equation can be interpreted as the Laplace transform of the Black-Scholes equation with drift  $\lambda(x)$ . Finally, the function  $X(\phi)$  in A-26 and its inverse  $\Phi(x)$  are defined as the solutions of the equation

$$(A-28) \quad \frac{dX(\phi)}{d\phi} = \frac{\nu(x)}{\sigma(\phi)}.$$

The key mathematical derivation involves a change of numeraire asset given by a process  $\gamma_t$  defined as follows:

$$(A-29) \quad \gamma_t = \frac{e^{\rho t}}{\hat{v}(x_t, \rho)}$$

and satisfies the equation

$$(A-30) \quad d\gamma = \left( \rho - \mu \frac{\hat{v}_x}{\hat{u}} + \nu^2 \left[ \left( \frac{\hat{v}_x}{\hat{u}} \right)^2 - \frac{1}{2} \frac{\hat{v}_{xx}}{\hat{u}} \right] \right) \gamma dt + \sigma \gamma dW,$$

where the log-normal volatility of  $\gamma_t$  is given by

$$(A-31) \quad \sigma^\gamma = -\frac{\hat{u}_x \nu}{\hat{u}}.$$

Substituting equation (A-27), we find that

$$(A-32) \quad \frac{d\gamma}{\gamma} = \left( \frac{\mu - \lambda}{\nu} \sigma^\gamma + (\sigma^\gamma)^2 \right) dt + \sigma^\gamma dW.$$

To demonstrate that  $\gamma_t$  defines a domestic asset price process, consider this equation in the original risk neutral measure where the domestic quanto option price process  $\phi_t$  is driftless. In this case, using Ito's Lemma on the inverse mapping  $x_t = X(\phi_t)$ , we arrive at an SDE of the form in equation (A-25) with drift given by

$$(A-33) \quad \mu(x) = \frac{\sigma(\Phi)^2}{2} \frac{d}{d\Phi} \frac{dX(\Phi)}{d\Phi} = \frac{\sigma(\Phi)^2}{2} \frac{d}{d\Phi} \frac{\nu(x)}{\sigma(\Phi)}.$$

where  $\Phi = \Phi(x)$ . Using the chain rule for differentiation, and expressing all functions in terms of  $x$ , we then have

$$(A-34) \quad \mu(x) = \frac{\sigma \nu}{2} \frac{d}{dx} \left( \frac{\nu}{\sigma} \right) = \frac{\nu}{2} \left[ \nu_x - \frac{\nu}{\sigma} \sigma_x \right],$$

where  $\sigma \equiv \sigma(\Phi(x))$  is the volatility function for the quanto option of price  $\phi_t$ . Hence, by substituting into the expression for the risk neutral drift of  $\gamma_t$  in A-32 we find

$$(A-35) \quad \frac{\mu - \lambda}{\nu} \sigma^g + (\sigma^g)^2 = \left[ \lambda + \frac{\nu^2}{2} \frac{\sigma_x}{\sigma} - \frac{1}{2} \nu \nu_x \right] \frac{\hat{u}_x}{\hat{u}} + \nu^2 \left( \frac{\hat{u}_x}{\hat{u}} \right)^2.$$

Using the expression A-26 for the volatility of the quanto option  $\phi_t$ , we find that

$$(A-36) \quad \frac{\sigma_x}{\sigma} = \frac{\nu_x}{\nu} - \frac{2\lambda}{\nu^2} - \frac{2\hat{u}_x}{\hat{u}}.$$

Substituting into (A-35), we find that the drift of  $\gamma_t$  under the risk neutral measure vanishes, as it ought to be for a domestic asset. Hence,  $\gamma_t$  can be interpreted as the process for a numeraire asset.

Next, consider equation (A-32) again, but this time under the measure having  $\gamma_t$  as numeraire. Under this pricing measure the price of risk is  $\sigma^\gamma$  and

$$(A-37) \quad d\gamma_t = (\sigma^\gamma)^2 \gamma_t dt + \sigma^\gamma \gamma_t dW.$$

Comparison with equation (A-32) shows that under this measure the drift of the underlying process  $x_t$  is  $\lambda$ , as stated. This implies that the risk neutral pricing kernel for the quanto option of volatility A-26 is given by

$$(A-38) \quad U(\phi, t; \phi_0, 0) = \frac{\nu(X(\phi))}{\sigma(\phi)} \frac{\hat{v}(X(\phi), \rho)}{\hat{v}(X(\phi_0), \rho)} e^{-\rho t} u(X(\Phi), t; x(\phi_0), 0).$$

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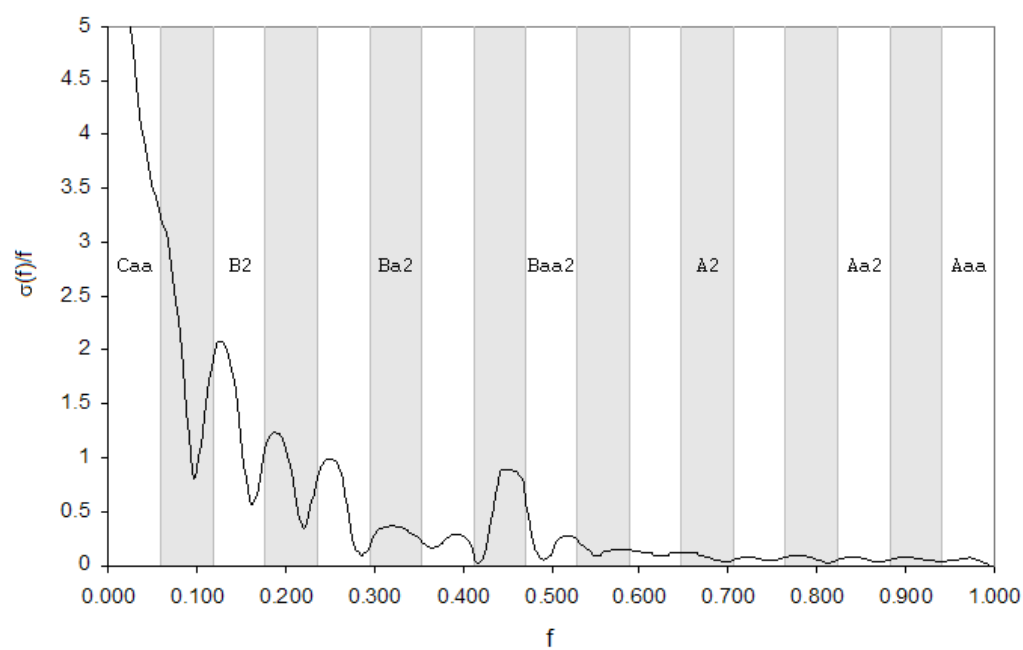
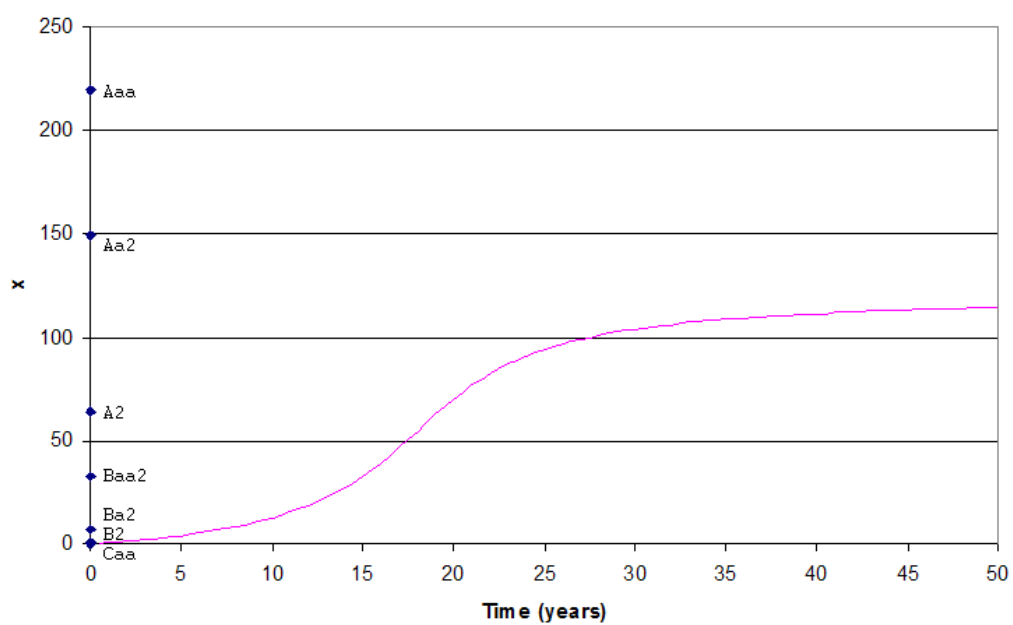
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FIGURE 3. The local volatility as a function of  $f$ .FIGURE 4. Initial rating levels (dots) and boundary (line) in  $x$ -space.

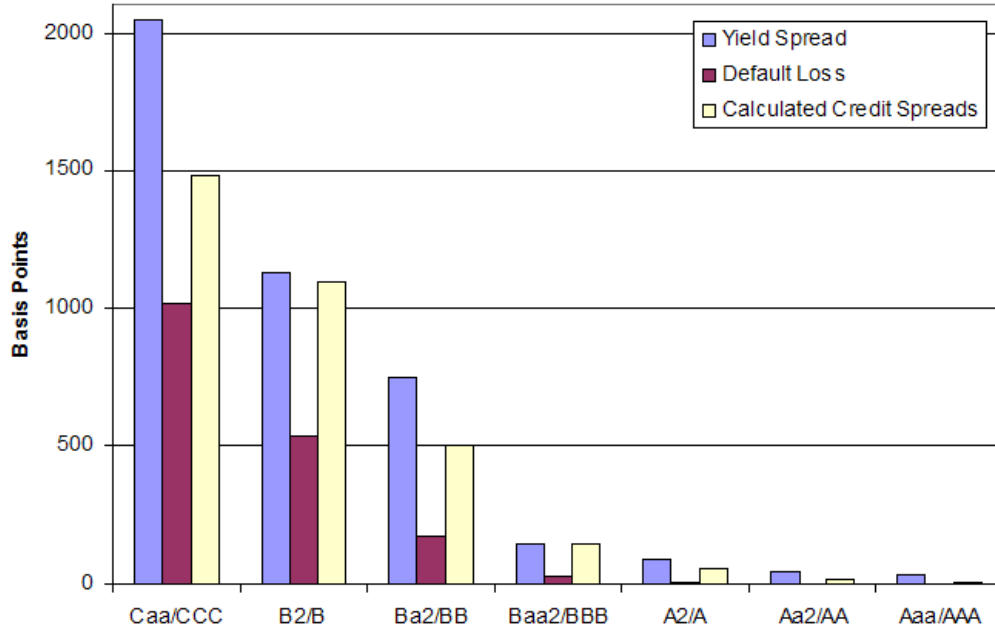


FIGURE 5. Yield spreads, default loss rates and calculated credit spreads for 5-Year bonds. Yield spreads are Bridge Evaluator Corporate Spreads for Industrials taken from <http://www.bondsonline.com> on Dec. 3, 2002.

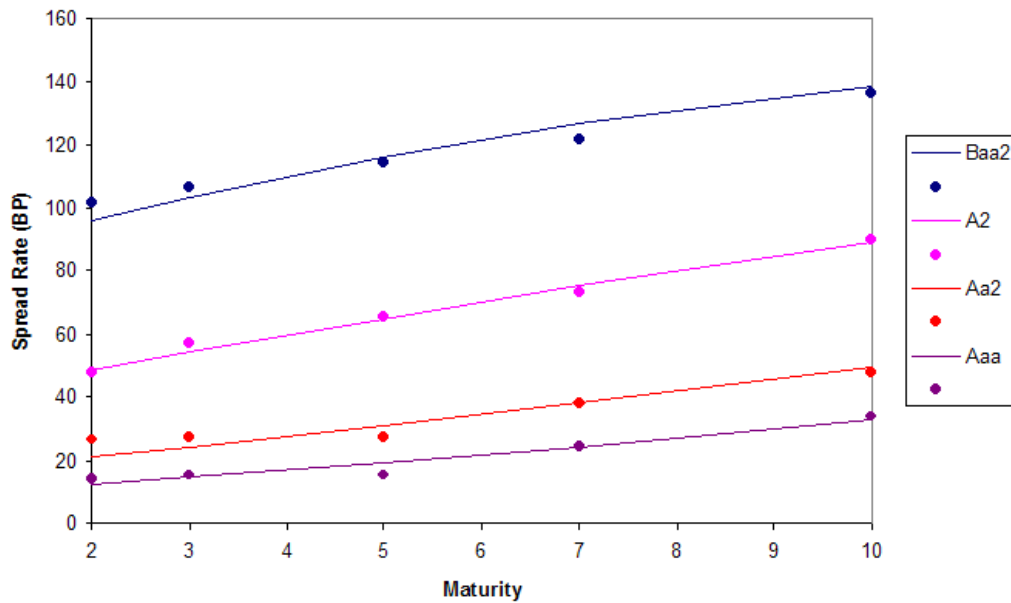


FIGURE 6. Comparison of the term structures for theoretical and tax-adjusted credit spreads for investment grade rated bonds assuming constant implied recovery rates.

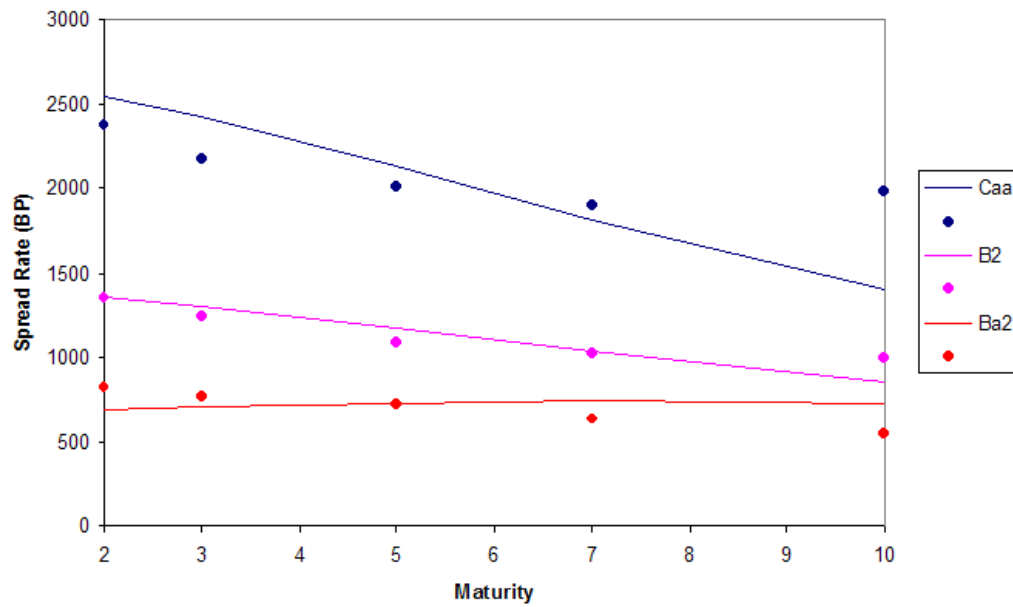


FIGURE 7. Comparison of the term structures for theoretical and tax-adjusted credit spreads for speculative grade rated bonds assuming constant implied recovery rates.

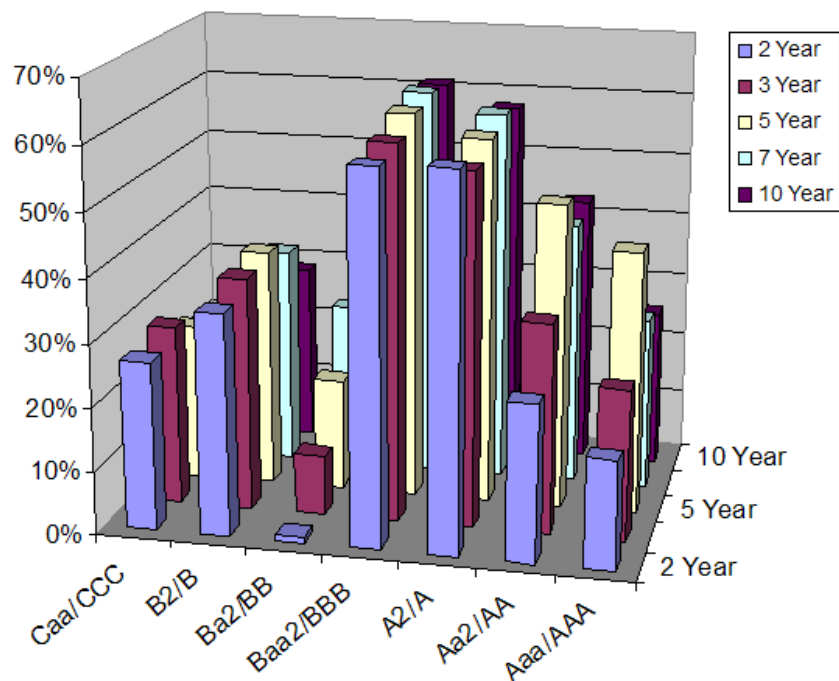


FIGURE 8. Implied Recovery Rates needed to match tax-adjusted market spread rates exactly.