November 22, 2002

Credit Risk Contributions to Value-at-Risk and Expected Shortfall

Alexandre Kurth, *UBS AG* [1]

Dirk Tasche, *Deutsche Bundesbank* [2]

Abstract

This paper presents analytical solutions to the problem of how to calculate sensible *VaR* (Value-at-Risk) and *ES* (Expected Shortfall) contributions in the CreditRisk$^+$ methodology. Via the *ES* contributions, *ES* itself can be exactly computed in finitely many steps. The methods are illustrated by numerical examples.

1 Introduction

From a theoretical point of view, there is no need to assign risk contributions to the parts of a portfolio. The reason is that no rational investor would be happy with a sub-optimal portfolio in the sense of best possible return-risk ratios. Therefore the assumption of already optimized portfolios predominates in the scientific literature. Of course, in case of an optimal portfolio surveillance of sub-portfolios is dispensable since by definition such a portfolio cannot be improved.

Nevertheless, in practice, and in particular in credit practice, optimal portfolios are quite rare. Even if the causes of bad performance in a credit portfolio have been located, as a consequence of limited liquidity it will in general be impossible to optimize the portfolio in one step. Indeed, credit portfolio management is rather a sequence of small steps than a one strike business. However, steering a portfolio step by step requires detailed risk diagnoses. The key to such diagnoses is assigning appropriate risk contributions to the sub-portfolios.

In a recent paper, Koyluoglu and Stoker (2002) have reviewed several approaches to the problem of how to define sensible risk contributions. We investigate here the "continuous marginal contribution" approach for the CreditRisk$^+$ (CSFP, 1997) credit portfolio model.

[1] Credit Risk Control, UBS Wealth Management & Business Banking, P.O. Box, 8098 Zurich, Switzerland. email: alexandre.kurth@ubs.com

[2] Deutsche Bundesbank, Postfach 10 06 02, 60006 Frankfurt a. M., Germany. email: tasche@ma.tum.de

More exactly, we determine analytically the contributions of each obligor to *VaR* (Value-at-Risk) and *ES* (Expected Shortfall). For continuous distributions *ES* is the conditional tail expectation. Moreover, *ES* is coherent (cf. Artzner et al., 1999), i.e., in contrast to *VaR*, expected shortfall always respects diversification as expressed by the subadditivity axiom.

This paper shows that the composition of tail losses may significantly differ when measured by *ES*, or *VaR*, or (a multiple of) the standard deviation. This fact has implications on the performance measurement for an active portfolio management. Apart from this, the contribution to *ES* is a useful indicator for concentrations, as shown in the examples (Section 6). A decreasing list of *ES*-contributors, even for large portfolios, may reveal that not just the largest exposures contribute the most, but also certain low quality loans in particularly volatile segments. This is a valuable input in analyzing the composition of tail losses, especially in terms of single name stress losses.

It turns out that the calculation of the *VaR*- and *ES*-contributions requires per segment only one extra run in the CreditRisk$^+$ model. As a byproduct, we show that in the CreditRisk$^+$ model *ES* can be calculated by only making use of the loss distribution up to the corresponding quantile loss.

Yamai and Yoshiba (2001a,b) observed that due to estimation errors simulation-based computations of *VaR*, *ES*, and of their respective risk decompositions must be rather extensive in order to arrive at acceptable precision. The analytical results presented in this paper show that CreditRisk$^+$ offers a fast alternative not only with respect to *VaR* and *ES* themselves but also for their risk decompositions.

This article first outlines the main features of the CreditRisk$^+$ model. In Section 3 we introduce some aspects of risk measures. The formulas for the *VaR*- and *ES*-contributions are derived in Sections 4 and 5 for the one segment case, and are then extended to several segments in Section 6. The paper finishes with an example where the different risk measures are compared based on independent, uncorrelated segments, as well as correlated segments. The appendix sketches the proof of the main statement given by Equation (13).

2 Short Overview of CreditRisk$^+$

We will briefly outline the main aspects and relations of the CreditRisk$^+$ model (CSFP, 1997) in the one segment case. The starting point of this credit portfolio model is the following equation for the random portfolio loss L over all obligors A:

$$L = \sum_A I_A \nu_A, \tag{1}$$

where I_A is interpreted as an indicator variable describing the default event of A, i.e. $I_A = 1$ if A defaults and $I_A = 0$ otherwise, and ν_A is the exposure net of recovery[3]. Let us denote the (unconditional) default probability of obligor A by p_A, i.e. p_A is the mean

[3] In this paper, the loss severities are assumed to be constant for each obligor. For an extension to variable severities confer Bürgisser et al. (2001).

of I_A. The dependence between obligors is incorporated by a common risk factor S which will later be assumed to be gamma-distributed. The mean of the variable S is equal to $\mu = \sum_A p_A$, and its volatility is denoted by σ. Then conditional on S, the expectation[4] of I_A is $S\,p_A/\mu$. This default scaling factor S reflects the intensity of the number of default events in the economy; cf. Bürgisser et al. (1999). The expected loss EL and the standard deviation UL of the loss variable L can now be derived (cf. Bürgisser et al., 2001):

$$EL = \sum_A p_A\,\nu_A, \quad UL^2 = \Big(\frac{\sigma}{\mu}\Big)^2 EL^2 + \sum_A \Big[p_A - \Big(1 + \Big(\frac{\sigma}{\mu}\Big)^2\Big)p_A^2\Big]\,\nu_A^2 \tag{2}$$

There is a slight difference to the formula in CSFP (1997, Eq. 118). This is caused by the assumption made there that the default is modelled by a conditionally Poisson distributed random variable while here defaults are Bernoulli events[5]. However, in order to get a hand on the loss distribution, and not just the first and second moments, it is convenient to adopt the Poisson approximation and to make use of the probability generating function[6] $G(z)$ which is defined as a power series of the form

$$G(z) = \sum_{n=0}^{\infty} p(n)\,z^n. \tag{3}$$

Here $p(n)$ is the probability of losing the amount n and z is a formal variable. In order to compute the coefficients $p(n)$ we make the following assumptions: First, the exposures (net of recovery) are banded[7]. Second, the default events I_A, conditional on S, are approximated by independent Poisson variables, and third, the common risk factor S is gamma-distributed with mean $\mu = \sum_A p_A$ and volatility σ.

With these assumptions, it turns out that the probability generating function of the portfolio loss satisfies the following relation (cf. CSFP, 1997, Eq. 68, and Bürgisser et al., 2001, Eq. 13).

$$G(z) = \left(1 - \frac{\sigma^2}{\mu}\,(Q(z) - 1)\right)^{-\alpha}, \tag{4}$$

where $\alpha = \mu^2/\sigma^2$, and $Q(z) = \mu^{-1}\sum_A p_A z^{\nu_A}$ denotes the so-called *portfolio polynomial*. Equation (4) can be used to derive a recursive condition on the coefficients $p(n)$ which is due to Panjer (1980, 1981) (cf. also CSFP, 1997, Eq. 77, and Bürgisser et al., 2001, Eq. 14):

$$p(n) = \frac{1}{n\,(1+\sigma^2/\mu)} \sum_{j=1}^{\min(m,n)} \Big(\sum_{A:\,\nu_A=j} p_A\Big)\Big\{\alpha^{-1}n + \Big(1-\alpha^{-1}\Big)\,j\Big\}\,p(n-j), \tag{5}$$

where $m = \deg(Q)$ is the largest exposure in the portfolio, and the initial value is given by $p(0) = (1+\sigma^2/\mu)^{-\alpha}$.

[4] $S\,p_A/\mu$ may also be regarded as the conditional default probability of obligor A given state S of the economy. However, in the model under consideration, $S\,p_A/\mu$ may take values greater than one. Thus, the interpretation as probability has to be understood in an approximative sense.

[5] The derivation of the UL-formula in CSFP (1997) makes use of probability generating functions as the exposures are banded. However, this is not necessary as shown in Bürgisser et al. (2001, Theorem 1).

[6] For an introduction of probability generating functions and their properties see Fisz (2000).

[7] I.e., the exposures (net of recovery) ν_A can be assumed to be integer values.

These equations can be extended to the case of several segments, i.e. to the case where the economy is described by a set of systematic risk factors $S_1, \ldots, S_N$. More specifically, under the assumption of independent segments with gamma-distributed factors, the extension for (5) is given in CSFP (1997, Eq. 77).

3 How to Measure Risk

A risk measure is a metric measuring the uncertainty of the portfolio loss. If a portfolio is given by an element of the set $P := \{\nu = (\nu_A \mid A \text{ obligor})\}$ where ν_A represents the exposure net of recovery of A, then the portfolio loss is given by (1), i.e. $L = L(\nu) = \sum_A I_A \nu_A$. Formally, a risk measure is described by a function $\rho : P \to \mathbb{R}$, and for every portfolio $\nu \in P$, the number $\rho(\nu)$ is the risk of ν.

There are two particularly popular examples of risk measures. One is the *standard deviation* (or *unexpected loss*, *UL*),

$$UL(\nu) \;=\; \sqrt{\mathrm{E}\Big[\big(L(\nu) - \mathrm{E}[L(\nu)]\big)^2\Big]}, \tag{6}$$

where $L(\nu)$ is defined by (1). The other is the *value-at-risk* (*VaR*) at level δ, defined as the δ-quantile of the portfolio loss $L(\nu)$:

$$VaR_\delta(\nu) \;=\; q_\delta(L(\nu)) \;=\; \min\Big\{l \in \mathbb{R} : \Pr\Big[L(\nu) \le l\Big] \ge \delta\Big\}. \tag{7}$$

UL is popular mainly for its computational simplicity. The level δ of *VaR* has an immediate interpretation as probability of solvency of the lender. However, both of these risk measures exhibit certain counter-intuitive properties (cf. Artzner et al., 1999). *UL* is not monotonous, i.e. it may happen that $L(\nu') \le L(\nu)$ with probability 1 but $UL(\nu') > UL(\nu)$. Moreover, *UL* does not distinguish the upside and downside potentials of portfolios.

The problem with *VaR* is that it is in general not sub-additive. There may occur portfolios ν and ν' such that $VaR(\nu + \nu') > VaR(\nu) + VaR(\nu')$. Translated into economic terms, this means that a *VaR*-investor may face situations where diversification does not pay.

The most promising alternative to *VaR* seems to be *Expected Shortfall* (*ES*) (or *Conditional Value-at-Risk (CVaR)*, see Acerbi and Tasche, 2002, and Rockafellar and Uryasev, 2001). From its definition

$$ES_\delta(\nu) \;=\; (1-\delta)^{-1} \int_\delta^1 q_u(L(\nu))\, du \tag{8}$$

it is clear that *ES* dominates *VaR* and hence preserves the interpretation in terms of solvency probability. More important, *ES* is fully coherent in the sense of Artzner et al. (1999). In particular, *ES* is sub-additive.

It can be shown (Acerbi and Tasche, 2002) that ES_δ is a convex combination of $VaR_\delta(\nu)$ and the conditional expectation of $L(\nu)$ given that $L(\nu) > q_\delta(L(\nu))$. As a consequence, ES_δ will be very well approximated by $\mathrm{E}[L(\nu) \,|\, L(\nu) > q_\delta(L(\nu))]$ if the probability $\Pr[L(\nu) =$

$q_\delta(L(\nu))]$ is small. This will be the case in particular in situations like those considered in the paper at hand. Here $L(\nu)$ represents the loss in a big loan portfolio. Therefore, in the sequel, we will make use of the conceptually simpler but slightly unprecise respresentation

$$ES_\delta(\nu) \;=\; \mathrm{E}[L(\nu)\,|\,L(\nu) > q_\delta(L(\nu))]. \tag{9}$$

4 Contributions to Value at Risk

For risk measurement and management it is crucial to allocate the risk to individual obligors or groups of obligors. So, after the decision for a risk measure, a further decision concerning the risk allocation is necessary. CreditRisk$^+$ defines the contribution $C_A^{(UL)}$ of obligor A to the portfolio UL by

$$C_A^{(UL)} = \nu_A \frac{\partial\, UL(\nu)}{\partial \nu_A} = \frac{\nu_A}{2\,UL}\frac{\partial\, UL^2(\nu)}{\partial \nu_A} = \frac{\nu_A \operatorname{cov}[I_A, L]}{UL}. \tag{10}$$

This is a transparent and fast way of allocating risk to individual obligors. Note that the UL contributions sum up to UL, i.e. $\sum_A C_A^{(UL)} = UL$. Actually, to define the contributions to portfolio risk by partial derivatives is the only way which is compatible to portfolio optimization (Tasche, 1999, Theorem 4.4).

Unfortunately, VaR is not in general differentiable with respect to the exposures. However, as several authors (Hallerbach, 2003; Gouriéroux et al., 2000; Lemus, 1999; Tasche, 1999) noticed, in case when VaR is differentiable, its derivative coincides with an expression that exists always under the assumptions of the CreditRisk$^+$ model, namely

$$\frac{\partial\, VaR_\delta(\nu)}{\partial \nu_A} \;=\; \mathrm{E}[I_A \;|\; L(\nu) = q_\delta(L(\nu))]. \tag{11}$$

$\mathrm{E}[I_A \;|\; L(\nu) = q_\delta(L(\nu))]$ is the conditional expectation of the Poisson variable I_A given that the portfolio loss $L(\nu)$ assumes the worst case value $q_\delta(L(\nu)) = VaR_\delta(\nu)$. This observation suggests the definition of

$$C_A^{(VaR_\delta)} \;=\; \nu_A\, \mathrm{E}[I_A \;|\; L(\nu) = q_\delta(L(\nu))] \tag{12}$$

as the contribution of obligor A to portfolio VaR in the general case. As with UL, also with Definition (12) of the VaR contributions the additivity property holds, i.e. $\sum_A C_A^{(VaR_\delta)} = VaR_\delta$.

Under the assumptions of Section 2 that have led to (4), one can show (see Appendix for a proof in the one segment case and Haaf and Tasche, 2002, for a different, general proof) that the expectation on the right-hand side of (12) is given by

$$\mathrm{E}_\alpha[I_A \;|\; L = q_\delta(L)] \;=\; p_A \frac{\Pr_{\alpha+1}[L = q_\delta(L) - \nu_A]}{\Pr_\alpha[L = q_\delta(L)]}, \tag{13}$$

where subscript α means that expectation and probability stem from the original generating function (4) whereas subscript $\alpha + 1$ means that the probability has to be derived from (4) with $\alpha = \mu^2/\sigma^2$ replaced by $\alpha + 1 = \mu^2/\sigma^2 + 1$.

Recall that $\mathrm{E}_\alpha[I_A \mid L = q_\delta(L)]$ in (13) is an approximation for the probability of "obligor A has defaulted" conditional on the event that the sum of losses equals $q_\delta(L)$. By the definition of conditional probability, $\Pr_{\alpha+1}[L = q_\delta(L) - \nu_A]$ therefore can be interpreted as an approximation of the probability of "sum of losses equals $q_\delta(L)$" conditional on "obligor A has defaulted". Note that $\mathrm{E}_\alpha[I_A \mid L = q_\delta(L)] = 0$ can happen, see Haaf and Tasche (2002) for a detailed discussion of this case. In Section 6, the formula for the multi-segment case is provided.

Equations (12) and (13) show that in CreditRisk$^+$ the *VaR* contributions can be calculated by running the Panjer algorithm (see (5)) twice: once with $\alpha = \mu^2/\sigma^2$ and again with $\alpha + 1 = \mu^2/\sigma^2 + 1$. The set up of this second run is given by scaling all p_A with the factor $1 + \alpha^{-1}$, and the default volatility σ with $\sqrt{1 + \alpha^{-1}}$.

Martin et al. (2001) took (11) as point of departure for another approach to VaR contributions by saddlepoint approximation. In the CreditRisk$^+$ setting, the technique used to arrive at (13) yields simple formulas also for this approach (Haaf and Tasche, 2002).

5 Contributions to Expected Shortfall

Via the decomposition

$$\mathrm{E}[L(\nu) \,|\, L(\nu) > q_\delta(L(\nu))] \;=\; \sum_A \nu_A \, \mathrm{E}[I_A \,|\, L(\nu) > q_\delta(L(\nu))] \tag{14}$$

Equation (9) suggests the definition of *ES* contributions as

$$C_A^{(ES_\delta)} \;=\; \nu_A \, \mathrm{E}[I_A \,|\, L(\nu) > q_\delta(L(\nu))]. \tag{15}$$

This definition can be justified in a similar way as the definition of $C_A^{(VaR_\delta)}$ in (12).

From (13) we know how to calculate $\mathrm{E}[I_A \mid L(\nu) = q_\delta(L(\nu))]$ in the CreditRisk$^+$ model. It is not hard to see that a similar formula obtains for $\mathrm{E}[I_A \mid L(\nu) > q_\delta(L(\nu))]$, namely

$$\mathrm{E}_\alpha[I_A \mid L > q_\delta(L)] \;=\; p_A \frac{\Pr_{\alpha+1}[L > q_\delta(L) - \nu_A]}{\Pr_\alpha[L > q_\delta(L)]}, \tag{16}$$

where the subscripts α and $\alpha + 1$ have the same meanings as in (13). Note that (16) can be calculated in finitely many steps since $\Pr_{\alpha+1}[L > q_\delta(L) - \nu_A] = 1 - \Pr_{\alpha+1}[L \le q_\delta(L) - \nu_A]$ and $\Pr_\alpha[L > q_\delta(L)] = 1 - \Pr_\alpha[L \le q_\delta(L)]$. This observation can be transferred to *ES* itself: The determination of the actual expected shortfall by the recursion-equation (5) requires that infinitely many elements are calculated. But applying the sum of all contributions $\sum_A C_A^{(ES_\delta)}$ using Equation (16) yields the expected shortfall precisely without knowing the tail beyond the corresponding δ-quantile loss.

6 Extension to several segments

There are two possibilities to extend the concept of the contributions to expected shortfall (resp. value-at-risk) to several segments. First, one can make use of N independent segments described by an N-tuple of independent systematic risk factors $S_1, \ldots, S_N$ which are gamma distributed. In this case one has to introduce factor loadings ω_{Aj} to be the portions of the default probability p_A allocated to segment j. This means that $\omega_{A0} = 1 - \sum_{j=1}^{N} \omega_{Aj}$ is the specific weight of A accounting for the idiosyncratic default risk. And, conditional on $S_1, \ldots, S_N$, the expectation of I_A is given by

$$\mathrm{E}[I_A \,|\, S_1, \ldots, S_N] \;=\; p_A \, \frac{\omega_{A0}\, \mu_0 + \sum_{j=1}^{N} \omega_{Aj}\, S_j}{\sum_{j=0}^{N} \omega_{Aj}\, \mu_j}, \tag{17}$$

with $\mu_j > 0$, $j = 1, \ldots, N$, being the unconditional expectation of S_j and $\mu_0 \geq 0$ being constant.

The extension of the *VaR*-contribution, as in Equation (13), to N independent segments has the following form:

$$\mathrm{E}_\alpha[I_A \,|\, L = q_\delta(L)] \;=\; p_A \, \frac{\displaystyle\sum_{j=1}^{N} \omega_{Aj}\ \mathrm{Pr}_{\alpha(j)}[L = q_\delta(L) - \nu_A] + \omega_{A0}\ \mathrm{Pr}_\alpha[L = q_\delta(L) - \nu_A]}{\mathrm{Pr}_\alpha[L = q_\delta(L)]}, \tag{18}$$

where $\alpha(j) = (\alpha_1, \ldots, \alpha_j + 1, \ldots, \alpha_N)$. The expression $\mathrm{Pr}_{\alpha(j)}[L = q_\delta(L) - \nu_A]$ is obtained by evaluating the corresponding element of the multivariate recursion of Equation (5) (cf. CSFP, 1997, Eq. 77 for N independent segments) where the j-th gamma distribution is specified by the pair $(\alpha_j + 1, \beta)$.

Similarly, by replacing the equality signs in $\mathrm{P}_{\alpha(j)}$ with inequalities $>$ one obtains the corresponding formula for the contribution to *ES* for N independent segments.

However, a segmentation with independent segments is rather restricting. A dependence structure between the obligors is achieved by apportioning the default probabilities to the orthogonal factors as done with the ω_{Aj}. This is rather arbitrary, and therefore creates an additional source of uncertainty. These factors typically represent the default behavior of industrial sectors or geographical areas etc., and as such mostly incorporate significant correlations between them.

To get around the restriction of independent segments one can introduce the correlation matrix of the N gamma variables $S_1, \ldots, S_N$, and can match the *UL*-formula of the loss distribution for these N segments with the corresponding one segment *UL*-formula (see Equation (2)). This procedure is introduced in Bürgisser et al. (1999, Eq. (13)). Now, the portfolio is unified to one segment, and the contributions to *VaR* and *ES* are determined as in the previous sections following Equations (13) resp. (16). This method is simple and fast. However, it destroys information about dependence and the contributions tend to the middle since the matched dependence structure is given by one volatility which turns out to be some average of the segment volatilities.

Orthogonalization of segment correlations does not represent an alternative because the process of orthogonalization is not unique. It can be shown by examples that significant differences in *VaR*, *ES* and their contributions show up.

7 Numerical Example

We want to show that the risk contributions according to the expected shortfall *ES* and value-at-risk *VaR* may significantly differ from the risk contributions according to the standard deviation. We will do so by using a wholesale bank portfolio that includes a retail subportfolio as well as commercial loans of various sizes. The following table describes the portfolio which is characterized by the loan sizes given in the first line.

	segment 1		**segment 2**					
Exposure per obligor [in Mio CHF]	1	1	10	20	100	500	1'000	2'000
# obligors	10'000	10'000	1'000	500	100	10	2	1
PD for each obligor	0.5%	1%	1%	1.75%	1.75%	1.25%	0.70%	0.30%
Exposure in class [in Mio CHF]	10'000	10'000	10'000	10'000	10'000	5'000	2'000	2'000
Exposure in % of total	16.9%	16.9%	16.9%	16.9%	16.9%	8.5%	3.4%	3.4%

Table 1: Specification of the sample portfolio. PD means "probability of default".

We assume that all loan exposures are given net of recovery. As indicated in the table we further assume a segmentation with two segments: The loans of exposure 1 Mio CHF form segment 1 (i.e. the retail segment), and all others build the second segment (commercial segment).

Using the portfolio in Table 1, we are going to present three different cases for the computation of *ES*, *VaR*, *UL* and the corresponding contributions. The first case is characterized by the segment volatilities σ_1, σ_2 of default rates and independence of segments 1 and 2, where we use Equation (18) for determining the *VaR*- (resp. *ES*-) contributions. In the second run we take the same segment volatilities as in the first case, but now we use the method of moment matching[8] as described in the last paragraph of Section 6. In the last case we even introduce correlations between both segments and use again the moment matching method to calculate the contributions (see Table 2). Note that the covariance of 0.21 in the last case corresponds to a 70% correlation between the segments. The segment volatilities $\tilde{\sigma}_i$ are given in terms of their relative means; e.g., the volatility σ_1 of defaults in segment 1 for all three cases is $\tilde{\sigma}_1 = \sqrt{0.16}$ times the expected number of defaults in segment 1, hence $\sigma_1 = \sqrt{0.16} \times 0.75\% \times 10'000$. The following table gives the exact description for these three cases.

Case name	dependence structure	Total exposure	*EL*	*EL* in %	*UL*	*UL*/*EL*	*VaR*	*ES*
independence	$\tilde{\sigma}_1 = \sqrt{0.16}$, $\tilde{\sigma}_2 = \sqrt{0.56}$	59'000	682	1.16%	490	72%	2'434	2'915
zero-correlation	$\begin{pmatrix} 0.16 & 0 \\ 0 & 0.56 \end{pmatrix}$	59'000	682	1.16%	490	72%	2'357	2'805
correlation	$\begin{pmatrix} 0.16 & 0.21 \\ 0.21 & 0.56 \end{pmatrix}$	59'000	682	1.16%	523	77%	2'481	2'954

Table 2: Portfolio figures for three cases of dependence. All absolute numbers are given in Mio CHF.

Observe that independence does not necessarily lead to the lowest possible risk measure. The following three tables list the segment contributions to the measures of risk for the

[8] Here we assume zero correlation whereas in case 1 we assume independence. Note that uncorrelatedness of two gamma distributions does not imply their stochastic independence.

three cases described above: standard deviation UL, 99-percentile VaR, and ES at 99%. In order to compare the contributions we give the ratios of the relative contributions per exposure class for the three risk measures.

independence	**segment 1**		**segment 2**						**portfolio totals**
Exposure per obligor	1	1	10	20	100	500	1'000	2'000	59'000
PD for each obligor	0.5%	1%	1%	1.75%	1.75%	1.25%	0.70%	0.30%	
$C^{(UL)}$	3	5	63	113	141	101	37	28	490
$C^{(VaR)}$	52	105	282	503	581	434	229	247	2'434
$C^{(ES)}$	53	105	312	555	643	478	264	504	2'915
$C^{(VaR)}\%/C^{(UL)}\%$	4.15	4.15	0.91	0.89	0.83	0.87	1.26	1.78	
$C^{(ES)}\%/C^{(VaR)}\%$	0.84	0.84	0.92	0.92	0.92	0.92	0.96	1.71	
$C^{(ES)}\%/C^{(UL)}\%$	3.47	3.48	0.83	0.82	0.77	0.80	1.21	3.03	

Table 3: Contributions to various risk measures in case 'independence'. All absolute numbers are given in Mio CHF.

zero-correlation	**segment 1**		**segment 2**						**portfolio totals**
Exposure per obligor	1	1	10	20	100	500	1'000	2'000	59'000
PD for each obligor	0.5%	1%	1%	1.75%	1.75%	1.25%	0.70%	0.30%	
$C^{(UL)}$	3	5	63	113	141	101	37	28	490
$C^{(VaR)}$	116	233	237	423	499	410	234	205	2'357
$C^{(ES)}$	123	245	250	447	526	428	262	524	2'805
$C^{(VaR)}\%/C^{(UL)}\%$	9.49	9.49	0.78	0.78	0.74	0.85	1.33	1.53	
$C^{(ES)}\%/C^{(VaR)}\%$	0.89	0.89	0.89	0.89	0.89	0.88	0.94	2.15	
$C^{(ES)}\%/C^{(UL)}\%$	8.41	8.42	0.70	0.69	0.65	0.74	1.25	3.27	

Table 4: Contributions to various risk measures in case 'zero-correlation'. All absolute numbers are given in Mio CHF.

correlation	**segment 1**		**segment 2**						**portfolio totals**
Exposure per obligor	1	1	10	20	100	500	1'000	2'000	59'000
PD for each obligor	0.5%	1%	1%	1.75%	1.75%	1.25%	0.70%	0.30%	
$C^{(UL)}$	13	26	65	117	143	98	35	27	523
$C^{(VaR)}$	128	255	259	462	536	400	211	230	2'481
$C^{(ES)}$	139	279	284	506	588	444	247	466	2'954
$C^{(VaR)}\%/C^{(UL)}\%$	2.06	2.06	0.84	0.84	0.79	0.86	1.26	1.83	
$C^{(ES)}\%/C^{(VaR)}\%$	0.92	0.92	0.92	0.92	0.92	0.93	0.99	1.70	
$C^{(ES)}\%/C^{(UL)}\%$	1.89	1.89	0.78	0.77	0.73	0.80	1.24	3.10	

Table 5: Contributions to various risk measures in case 'correlation'. All absolute numbers are given in Mio CHF.

Note that in all cases the retail loans (CHF 1 Mio), and particularly the large lumpy loans contribute more to Expected Shortfall ES (and to VaR) than they do to UL, which shows that ES measures concentration more sensitively. For the large loans this is quite intuitive, since once a tail loss above VaR is reached, the (lumpy) large loans more often contribute by their defaults. But apparently, a significantly large amount of small loans also suffices to produce a large loss. In case 1 the contributions for VaR and ES almost coincide for loans in segment 1, whereas they increasingly diverge for increasing loan amounts.

Note that in the 'zero-correlation' case (Table 4) the UL-contributions are calculated based on the corresponding covariance matrix (cf. Bürgisser et al., 1999, Equation (15)) whereas the ES- and VaR-contributions are calculated based on the matched one segment approach. This causes the large differences in the relative contributions of segment 1. Calculating the UL-contributions based on the matched one segment approach would lead to a ratio in the range of one for $C^{(ES)}\%/C^{(UL)}\%$ in segment 1. Applying one volatility for the entire portfolio causes a shift of the contributions to the 'average'. The exposure classes 1 and 10, both with $p_A = 1\%$ have the same total exposure and very similar contributions for the cases 'zero-correlation' and 'correlation'. But in the 'independence' case their contributions differ very much (e.g. 105 vs. 312 for *ES*).

Introducing correlations we see a large increase in the *UL*-contributions in segment 1 whereas they remain quite stable in the other segment. The *ES*-contributions increase by approximately 10%, except for both largest exposure classes, where a decrease of up to more than 10% is realized. The changes of *VaR*-contributions are similar but smaller, with the exception of the *VaR*-contribution reduction in exposure class 500.

8 Conclusion

We have introduced an analytical approach to calculate contributions to *VaR* and *ES* of a credit loss distribution in the CreditRisk$^+$ framework. The formulas we have derived are easy to implement. The time consumption of calculating these contributions for all obligors is the same as for calculating the loss distribution in the moment matching case of CreditRisk$^+$.

The results show that *VaR*- and *ES*-contributions may differ significantly from *UL*-contributions, in particular for large exposures. Moreover, especially *ES* is sensitive to concentrations coming from large loans which represent a source for stress losses in a loan portfolio. Hence, if a bank wants to actively manage its loan portfolio whose performance is measured (among others) by tail losses, it should take into account that such tail events are composed quite differently when measured by expected shortfall instead by standard deviation (or a multiple thereof).

Acknowledgements. The authors thank Isa Cakir, Bernd Engelmann, and Armin Wagner for helpful suggestions and valuable remarks. Alexandre Kurth works in the unit responsible for modeling credit risk at "UBS Wealth Managment & Business Banking Division" of UBS AG. Dirk Tasche has a position in the banking supervision department of Deutsche Bundesbank. Opinions expressed in this article are those of the authors and do not necessarily reflect the opinions of UBS or Deutsche Bundesbank.

References

Acerbi, C., and D. Tasche (2002): "On the coherence of Expected Shortfall," *Journal of Banking & Finance,* **26**(7), 1487-1503.

Artzner, P., Delbaen, F., Eber, J.-M., and D. Heath (1999): "Coherent measures of risk," *Mathematical Finance* **9**(3), 203-228.

Bürgisser, P., Kurth, A. and A. Wagner (2001): "Incorporating severity variations into credit risk," *Journal of Risk*, **3**(4), 5-31.

Bürgisser, P., Kurth, A., Wagner, A. and M. Wolf (1999): "Integrating Correlations," *Risk*, **12**(7), 57-60.

Credit Suisse Financial Products (CSFP) (1997): "CreditRisk$^+$: A Credit Risk Management Framework" *Technical document.* http://www.csfb.com/creditrisk/

Fisz, M. (1963): "Probability Theory and Mathematical Statistics," *John Wiley & Sons, Inc.*, New York, London.

Gouriéroux, C., Laurent, J. P. and O. Scaillet (2000): "Sensitivity analysis of Values at Risk," *Journal of Empirical Finance*, **7**, 225-245.

Haaf, H., and D. Tasche (2002): "Credit portfolio measurements," *GARP Risk Review* issue 07 Jul/Aug, 43-47.

Hallerbach, W. (2003): "Decomposing Portfolio Value-at-Risk: a General Analysis," *To appear in Journal of Risk.*

Lemus, G. (1999): "Portfolio Optimization with Quantile-based Risk Measures," *PhD thesis, Sloan School of Management, MIT.* http://citeseer.nj.nec.com/lemus99portfolio.html

Koyluoglu, H., and J. Stoker (2002): "Honour your contributions," *Risk,* **15**(4), 90-94.

Martin, R., Thompson, K., and Browne, C. (2001) VAR: who contributes and how much? *Risk* **14**(8), 99-102.

Panjer, H.H. (1980): "The aggregate claims distribution and stop-loss reinsurance," *Transactions of the Society of Actuaries,* **32**, 523-545.

Panjer, H.H. (1981): "Recursive evaluation of a family of compound distributions," *ASTIN Bulletin,* **12**, 22-26.

Rockafellar, R.T., and S. Uryasev (2001): "Conditional value-at-risk for general loss distributions," *Journal of Banking & Finance,* **26**(7), 1443-1471.

Tasche, D. (1999): "Risk contributions and performance measurement," *Working paper, Technische Universität München.* http://citeseer.nj.nec.com/tasche99risk.html

Tasche, D. (2002): "Expected Shortfall and Beyond," *Journal of Banking & Finance,* **26**(7), 1519-1533.

Yamai, Y., and Yoshiba, T. (2001a) "On the Validity of Value-at-Risk: Comparative Analyses with Expected Shortfall," *IMES Discussion Paper No. 2001-E-4, Bank of Japan.*

Yamai, Y., and Yoshiba, T. (2001b) " Comparative Analyses of Expected Shortfall and VaR: their estimation error, decomposition, and optimization," *IMES Discussion Paper No. 2001-E-12, Bank of Japan.*

Appendix

We sketch here the proof of Equation (13). Denote as usual by $I_{\{L=t\}}$ the indicator function of the event $\{L = t\}$, i.e. $I_{L=t} = 1$ if $L = t$ and $I_{L=t} = 0$ otherwise. By definition, we have

$$\mathrm{E}_\alpha[I_A \mid L = q_\delta(L)] \;=\; \frac{\mathrm{E}_\alpha[I_A\, I_{\{L=q_\delta(L)\}}]}{\mathrm{Pr}_\alpha[L = q_\delta(L)]}. \tag{19}$$

We will now compute the generating function of the sequence $t \mapsto \mathrm{E}_\alpha[I_A\, I_{\{L=t\}}]$ which is defined as the function

$$z \mapsto \mathrm{E}_\alpha[I_A\, z^L] \;=\; \sum_{t=0}^{\infty} \mathrm{E}_\alpha[I_A\, I_{\{L=t\}}]\, z^t.$$

Since in the CreditRisk$^+$ model, conditional on the intensity S, I_A is approximated by a Poisson variable with intensity $\frac{p_A}{\mu}\, S$ and the default events are assumed to be conditionally independent, we obtain

$$\begin{aligned}
\mathrm{E}_\alpha[I_A\, I_{\{L=t\}}] &= \sum_{k=0}^{\infty} k\; \mathrm{Pr}_\alpha[I_A = k, \sum_{B \neq A} \nu_B\, I_B = t - \nu_A\, k] \\
&= \sum_{k=1}^{\infty} k\, \mathrm{E}_\alpha\Big[(k!)^{-1}\Big(\frac{p_A}{\mu} S\Big)^k e^{-\frac{p_A}{\mu} S}\; \mathrm{Pr}_\alpha[\sum_{B \neq A} \nu_B\, I_B = t - \nu_A\, k \,|\, S]\Big] \\
&= \frac{p_A}{\mu} \sum_{k=0}^{\infty} \mathrm{E}_\alpha\Big[S\; \mathrm{Pr}_\alpha[I_A = k \,|\, S]\; \mathrm{Pr}_\alpha[\sum_{B \neq A} \nu_B\, I_B = t - \nu_A\, (k+1) \,|\, S]\Big] \\
&= \frac{p_A}{\mu}\, \mathrm{E}_\alpha[S\, I_{\{L=t-\nu_A\}}].
\end{aligned} \tag{20}$$

Equation (20) implies

$$\mathrm{E}_\alpha[I_A\, z^L] \;=\; \frac{p_A}{\mu}\, z^{\nu_A}\, \mathrm{E}_\alpha[S\, z^L]. \tag{21}$$

Under the assumption of a gamma-distributed factor S and the conditional Poisson distribution and independence of the variables I_A, it turns out that the expectation $\mathrm{E}_\alpha[S\, z^L]$ can be calculated the same way as $\mathrm{E}_\alpha[z^L] = G(z)$, the generating function of the loss L. The result of the calculation is quite similar, namely (with the same notation as in (4))

$$\mathrm{E}_\alpha[S\, z^L] \;=\; \mu \left(1 - \frac{\sigma^2}{\mu}\,(Q(z) - 1)\right)^{-(\alpha+1)}. \tag{22}$$

From (22) and (21) we conclude

$$\begin{aligned}
\mathrm{E}_\alpha[I_A\, z^L] &= p_A\, z^{\nu_A} \Big(1 - (\sigma^2/\mu)\,(Q(z) - 1)\Big)^{-(\alpha+1)} \\
&= p_A\, \mathrm{E}_{\alpha+1}[z^{L+\nu_A}].
\end{aligned} \tag{23}$$

Recall that both sides of (23) can be written as power series. Hence we can conclude that

$$\mathrm{E}_\alpha[I_A\, I_{\{L=t\}}] \;=\; p_A\; \mathrm{Pr}_{\alpha+1}[L = t - \nu_A] \tag{24}$$

for $t = 0, 1, 2, \ldots$. Choosing $t = q_\delta(L)$ now proves (13).