CREDIT RISK MODELING AND VALUATION: AN INTRODUCTION

Kay Giesecke*

Cornell University

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Abstract

Credit risk refers to the risk of incurring losses due to changes in the credit quality of a counterparty. In this paper we give an introduction to the modeling of credit risks and the valuation of credit-risky securities. We consider individual as well as correlated credit risks.

Key words: credit risk; default risk; structural approach; reduced form approach; intensity; compensator.

JEL Classification: G12; G13

*Address: Department of Operations Research and Industrial Engineering, Cornell University, Ithaca, NY 14853-3801, USA, Phone (607) 255 9140, Fax (607) 255 9129, email: giesecke@orie.cornell.edu. I thank Matthias Fengler and Wolfgang Härdle for comments. Financial support by Deutsche Forschungsgemeinschaft, SFB 373, is gratefully acknowledged.

1 Introduction

Financial risks refer to adverse changes in the market value of financial positions, such as bonds, stocks, options, or other derivatives. With respect to the cause of price changes we may distinguish between market and credit risk. *Market risk* results from variations in underlying market prices or rates. For a bond for example, market risk arises from the variability of riskless interest rates. *Credit risk* is due to changes in the credit quality of the counterparty in the contract. For a bond, the creditworthiness of its issuer, which can be a (sovereign) government or a corporation, affects the bond price, since it is uncertain whether the issuer will be able to fulfil its obligations (coupon, principal) or not. Another example is a long option position, which exposes its holder to the credit risk of the option seller if the option is in the money.

Financial institutions hold typically thousands of financial positions. From a risk measurement and management point of view, individual risks play here only a minor role; of significance is only the *aggregated* risk associated with an institution's portfolio. This aggregated risk is closely related to the *dependence* between individual risks.

In this paper we give an introduction to the modeling of individual and dependent credit risks, as well as the valuation of defaultable securities. Our recurrent example will be a corporate bond.

At first glance, credit risk might be viewed as part of market risk and one could argue to apply methods well known in market risk modeling also to credit risks. This seems however difficult; there is a number of reasons which call for a distinction between market and credit risk. One reason is that credit-risky positions may be illiquid (think of bank loans, for example), so that market prices are not readily determined. A related point is that, due to the fact that bankruptcy is a rare event, historical default data is relatively sparse, compared to data related to market prices and rates, which are abound. Also the information needed for credit risk analysis is mainly contract-specific, as opposed to prices and rates which apply market-wide. The problems arising from the lack of credit risk-related data are indeed challenging; they call for methods fundamentally different from those specific to market risk analysis.

A crucial step in credit risk analysis is the modeling of a credit event with respect to some issuer or counterparty. Examples of credit events include failure to pay a due obligation, bankruptcy, repudiation/moratorium, or credit rating change. In the first three cases we also speak of a default, and the terms credit risk and default risk can be used interchangeably.

In the literature, two distinct approaches to model a default have evolved.

In the so-called *structural* approach, one makes explicit assumptions about the dynamics of a firm's assets, its capital structure, as well as its debt and share holders. It is then supposed that the firm defaults if its assets are not sufficient to pay off the due debt. In this situation corporate liabilities can be considered as contingent claims on the firm's assets. Recognizing that a firm may default well before the maturity of the debt, one may alternatively assume that the firm goes bankrupt when the value of its assets falls below some lower threshold. While this structural approach is economically appealing, some implied credit spread properties (the credit spread is the excess yield demanded by bond investors for bearing the risk of borrower default) hardly match empirical observations. This is due to the fact that in the structural framework the default can be anticipated by bond investors.

In the *reduced form* approach, the default is not causally modeled in terms of a firm's assets and liabilities, but is typically given exogenously. In this ad-hoc approach, the default occurs completely unexpectedly, by surprise so to speak. The stochastic structure of default is directly prescribed by an intensity or compensator process. Defaultable bond prices can be represented in terms of the intensity or the compensator, leading to tractable valuation formulas very similar to those arising in ordinary default-free term structure modeling. Due to the unpredictability of defaults, the implied credit spread properties are empirically quite plausible.

2 Structural Modeling of Credit Risk

2.1 Classic Option-theoretic Approach

The philosophy of the structural approach, which goes back to Black & Scholes (1973) and Merton (1974), is to consider corporate liabilities as contingent claims on the assets of the firm. Here the firm's market value (that is the total market value of the firm's assets) is the fundamental state variable.

Let us consider an economy with a financial market, where uncertainty is modeled by some probability space $(\Omega, \mathcal{F}, P)$. Consider a firm with market value $V = (V_t)_{t \geq 0}$, which represents the expected discounted future cash flows of the firm. The firm is financed by equity and a zero coupon bond with face value K and maturity date T. The firm's contractual obligation is to pay the amount K back to the bond investors at time T. Suppose debt covenants grant bond investors absolute priority: if the firm cannot fulfil its obligation, then bond holders will immediately take over the firm. In this case we say the firm defaults on its debt; the random default time τ is accordingly given by

$$\tau = \begin{cases} T & \text{if} \quad V_T < K \\ \infty & \text{if} \quad \text{else.} \end{cases}$$

We define the default indicator function

$$1_{\{\tau=T\}} = \begin{cases} 1 & \text{if} \quad \tau = T \Leftrightarrow V_T < K \\ 0 & \text{if} \quad \text{else.} \end{cases}$$

Assuming that the firm is neither allowed to repurchase shares nor to issue any new senior or equivalent claims on the firm, at the debt's maturity T we have the following payoffs to the firm's liabilities:

	Assets	Bonds	Equity
No Default	$V_T \geq K$	K	$V_T - K$
Default	$V_T < K$	V_T	0

Table 1: Payoffs to the firm's liabilities at maturity

If at T the asset value V_T exceeds or equals the face value K of the bonds, the bond holders will receive their promised payment K and the shareholders will get the remaining $V_T - K$. However, if the value of assets V_T is less than K, the ownership of the firm will be transferred to the bondholders. Equity is then worthless (because of limited liability of equity, the shareholders cannot be forced to make up the amount $K - V_T$). Summarizing, the value of the bond issue B_T^T at time T is given by

$$B_T^T = \min(K, V_T) = K - \max(0, K - V_T) \tag{1}$$

which is equivalent to that of a portfolio composed of a default-free loan with face value K maturing at T and a short European put position on the assets of the firm V with strike K and maturity T. The value of the equity E_T at time T is given by

$$E_T = \max(0, V_T - K), \tag{2}$$

which is equivalent to the payoff of a European call option on the assets of the firm V with strike K and maturity T.

With the payoff specifications just described, we are able to value corporate liabilities as contingent claims on the firm's assets. Taking as given some riskless short rate process $(r_t)_{t\geq 0}$, we suppose that there is a security with value

$\beta_t = \exp(\int_0^t r_s ds)$ at time t, which provides a riskless investment opportunity. For expositional purposes, in some situations discussed below we shall assume that $r_t = r$ is constant. Assuming that there are no arbitrage opportunities in the financial market, there exists a probability measure $\tilde{P}$, equivalent to the physical measure P, such that the processes of security prices, discounted with respect to β, are $\tilde{P}$-martingales [Harrison & Kreps (1979) and Harrison & Pliska (1981)]. $\tilde{P}$ is called equivalent martingale measure, and we let $\tilde{E}[\cdot]$ denote the corresponding expectation operator. Putting $Z_t = \beta_t^{-1}$, the value of a riskless (i.e. non-defaultable) zero bond maturing at T is at time $t = 0$ given by

$$\bar{B}_0^T = \tilde{E}[Z_T]. \tag{3}$$

The value B_0^T of the defaultable zero bond at time $t = 0$ can be written as

$$\begin{aligned} B_0^T = \tilde{E}[Z_T B_T^T] &= \tilde{E}[Z_T(K - \max(0, K - V_T))] \\ &= K\bar{B}_0^T - \tilde{E}[Z_T(K - V_T)1_{\{\tau=T\}}], \end{aligned} \tag{4}$$

which is the value of a riskless loan with face K less the risk-neutral expected discounted default loss. If the bond were default-free, then this expected loss would be zero. The equity value E_0 at time $t = 0$ is given by

$$E_0 = \tilde{E}[Z_T E_T] = \tilde{E}[Z_T \max(0, V_T - K)]. \tag{5}$$

The computation of the involved expectations requires an assumption about the dynamics of the assets. Following the classic route, let us model the evolution of the asset value V through time by a geometric Brownian motion:

$$\frac{dV_t}{V_t} = \mu dt + \sigma dW_t, \quad V_0 > 0, \tag{6}$$

where $\mu \in \mathbb{R}$ is a drift parameter, $\sigma > 0$ is a volatility parameter, and W is a standard Brownian motion. The SDE (6) has the unique solution

$$V_t = V_0 e^{mt + \sigma W_t} \tag{7}$$

where we put $m = \mu - \frac{1}{2}\sigma^2$. Supposing additionally that interest rates $r_t = r > 0$ are constant, we are in the classic Black-Scholes setting. Riskless bond prices are simply $\bar{B}_0^t = Z_t = e^{-rt}$ and the equity value is explicitly given by the Black-Scholes call option formula BS_C:

$$E_0 = BS_C(\sigma, T, K, r, V_0) = V_0\Phi(d_1) - K\bar{B}_0^T\Phi(d_2) \tag{8}$$

where Φ is the standard normal distribution function and

$$d_1 = \frac{\ln(\frac{V_0}{K}) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Similarly, the value B_0^T of the bonds at time $t = 0$ is given by

$$B_0^T = K\bar{B}_0^T - BS_P(\sigma, T, K, r, V_0) = V_0\Phi(-d_1) + K\bar{B}_0^T\Phi(d_2) \tag{9}$$

where BS_P is the Black-Scholes put option formula. Equivalently, we can find B_0^T as the difference between asset and equity value, i.e. $B_0^T = V_0 - E_0$. By comparison of (9) and (4), we see that the risk-neutral expected discounted default loss of the bond investors is equal to the value of the put position:

$$\tilde{E}[Z_T(K - V_T)1_{\{\tau=T\}}] = BS_P(\sigma, T, K, r, V_0).$$

With our specific assumption on asset dynamics, we can also write down default probabilities explicitly. From the definition of default,

$$\begin{aligned} P[\tau = T] = P[V_T < K] &= P[V_0 e^{mT+\sigma W_T} < K] \\ &= P[mT + \sigma W_T \leq \ln\frac{K}{V_0}] \\ &= P[W_T < \frac{\ln(\frac{K}{V_0}) - mT}{\sigma}] \\ &= \Phi(\frac{\ln(\frac{K}{V_0}) - mT}{\sigma\sqrt{T}}) \end{aligned} \tag{10}$$

since W_T is normally distributed with mean zero and variance T. Setting $\mu = r$, we find the risk-neutral default probability

$$\tilde{P}[\tau = T] = \Phi(-d_2) = 1 - \Phi(d_2).$$

We consider the following numerical example with results shown in Table 2:

$$V_0 = 100, \quad K = 75, \quad r = 5\%, \quad \sigma = 20\%, \quad \mu = 10\%, \quad T = 1 \text{ year}.$$

The *credit yield spread* or simply credit spread is the difference between the yield on a defaultable bond and the yield an otherwise equivalent default-free zero bond. This is the excess return demanded by bond investors to bear the potential losses due to a default of the bond issuer. Since the yield $y(t, T)$

Equity	$E_0 = 28.97$
Bonds	$B_0^T = 71.03$
Riskless Bonds	$\bar{B}_0^T = 71.34$
Default Probability	$P[\tau = T] = 3.3\%$
Risk-neutral Default Probability	$\tilde{P}[\tau = T] = 5.6\%$

Table 2: Numerical example in the Black-Scholes setting

on a bond with price $b(t,T)$ satisfies $b(t,T) = e^{-y(t,T)(T-t)}$, we have for the credit spread $S(t,T)$ prevailing at time t,

$$S(t,T) = -\frac{1}{T-t}(\ln B_t^T - \ln K\bar{B}_t^T), \quad T > t. \tag{11}$$

The *term structure of credit spreads* is the schedule of $S(t,T)$ against T, holding t fixed. In the Black-Scholes setting, we obtain

$$S(0,T) = -\frac{1}{T}\ln\left(\Phi(d_2) + \frac{1}{d}\Phi(-d_1)\right)$$

where $d = \frac{K}{V_0}e^{-rT}$ is the discounted debt-to-asset value ratio, which can be considered as a measure of the firm's leverage. We see that the spread is a function of maturity T, asset volatility σ (the firm's business risk), and leverage d. In Figure 1 we plot the term structure of credit spreads for varying d. We fix $r = 6\%$ per year and $\sigma = 20\%$ per year. Letting $d = 0.9$, in Figure 2 we plot spreads for varying asset volatilities σ.

Note finally that the assumption of constant interest rates was introduced for simplicity only. A generalization to stochastically varying interest rates, which are possibly correlated with the firm's asset value, is straightforward and can be found, for example, in Shimko, Tejima & van Deventer (1993).

2.2 First-Passage Model

In the last section we supposed that the firm may only default at debt maturity T, irrespective of the asset value path leading to the value V_T. There, the firm value was allowed to dwindle to nearly nothing without triggering a default; all that matters was its level at debt maturity. This is clearly not in the interest of the bond holders, as noted first by Black & Cox (1976). Bond indenture provisions therefore often include safety covenants providing the bond investors with the right to reorganize or foreclose on the firm if the asset value hits some lower threshold for the first time. This threshold can be pre-specified by

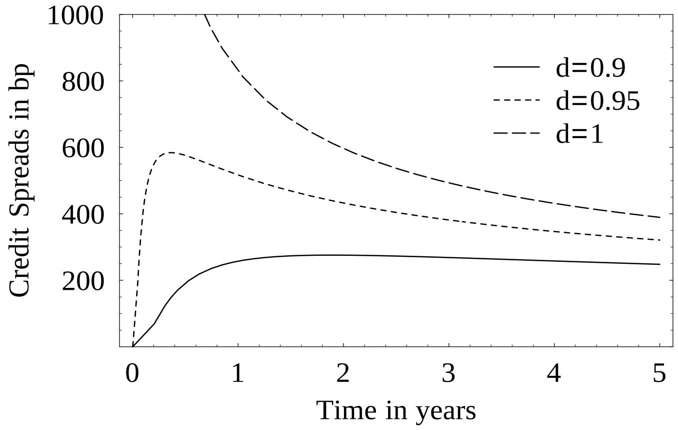

Figure 1: Term structure of credit spreads, varying firm leverage d (classic model).

the covenant, or endogenously determined by allowing the equity investors to optimally choose the time to default. For a given threshold process $(D_t, t \geq 0)$ with $0 < D_0 < V_0$, the default time τ is then given by

$$\tau = \inf\{t > 0 : V_t \leq D_t\} \tag{12}$$

so that τ is a random variable valued in $(0, \infty]$. For example, one could assume that the dynamics of both V and $(D_t, t \geq 0)$ are governed by an independent geometric Brownian motion, which would lead us to consider the (geometric Brownian motion) process $(V_t/D_t)_{t\geq 0}$ hitting the constant 1. A reasonable choice for a deterministically varying threshold is $D_t = Ke^{-k(T-t)}$ for some discount rate $k \in \mathbb{R}$, meaning that the threshold is given by the discounted liabilities K which are due at T.

As an example, let us calculate default probabilities for the case where V is governed by (6) and the threshold is constant through time, $D_t = D$ for $t \geq 0$. We define the *running minimum log-asset process* $M = (M_t)_{t\geq 0}$ by

$$M_t = \min_{s \leq t}(ms + \sigma W_s),$$

i.e. M keeps track of the historical low of the log-asset value. With (12) we

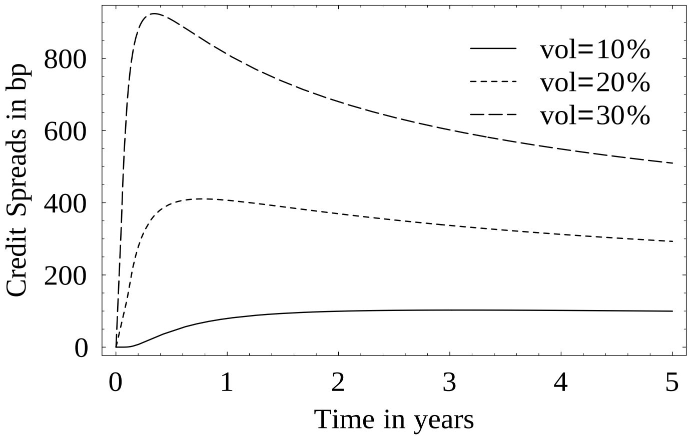

Figure 2: Term structure of credit spreads, varying asset volatility σ (classic model).

then find for the default probability

$$\begin{aligned} P[\tau \leq T] &= P[\min_{s \leq T} V_s \leq D] \\ &= P[\min_{s \leq T}(V_0 e^{ms+\sigma W_s}) \leq D] \\ &= P[M_T \leq \ln(D/V_0)]. \end{aligned}$$

That is, the event of default by time T is equivalent to the running minimum log-asset value at T being below the adjusted default threshold $\ln(D/V_0)$. Using the fact that the distribution of M_t is inverse Gaussian[1], we have

$$P[\tau \leq T] = 1 - \Phi\left(\frac{mT - \ln(D/V_0)}{\sigma\sqrt{T}}\right) + e^{\frac{2m\ln(D/V_0)}{\sigma^2}}\Phi\left(\frac{\ln(D/V_0) + mT}{\sigma\sqrt{T}}\right).$$

Clearly, to obtain risk-neutral default probabilities, we set the asset value drift equal to the riskless short rate in the above formula.

The calculation of bond prices is a straightforward exercise in the current framework. Let $R \in [0, 1]$ be a random variable expressing the *recovery rate* as a percentage of the bonds face value. That means, $(1 - R)K$ is the value bond investors lose in the event of a default before T. Suppose that the recovered

[1]To find that distribution, one first calculates the joint distribution of the pair (W_t, M_t^W), where M^W is the running minimum of W, by the reflection principle. Girsanov's theorem is then used to extend to the case of Brownian motion with drift.

amount is paid out at T. Then

$$\begin{aligned} B_0^T &= \tilde{E}[Z_T K(1_{\{\tau>T\}} + R1_{\{\tau\leq T\}})] \\ &= \tilde{E}[Z_T K(1-(1-R)1_{\{\tau\leq T\}})] \\ &= K\bar{B}_0^T - \tilde{E}[Z_T K(1-R)1_{\{\tau\leq T\}}], \end{aligned} \tag{13}$$

which is the value of a riskless loan with face value K and the risk-neutral expected discounted default loss. Note the analogy to the classic model (4). Of course, if we assume that default, riskless interest rate, and recovery rate are mutually independent,

$$B_0^T = K\bar{B}_0^T(1-\tilde{E}[1-R]\tilde{P}[\tau\leq T]), \tag{14}$$

so that we need riskless bond prices $\bar{B}_0^T = \tilde{E}[Z_T]$, the risk-neutral expected recovery rate $\tilde{E}[R]$, and risk-neutral default probabilities $\tilde{P}[\tau \leq T]$ to value the defaultable bond.

Structural default models share a common characteristic, namely the *predictability* of default events. Since investors can observe at any time the nearness of the assets to the default threshold, they are warned in advance when a default is imminent as long as assets and threshold follow some continuous process. Put another way, in structural models default is not a total surprise event. Let us mention three consequences of this property. First, bond prices do not jump downwards upon default, but converge continuously to their default-contingent value. Second, the financial market is complete: a default can always be (delta) hedged through positioning in bond or stock, respectively. Third and perhaps most importantly, short credit spreads, i.e. spreads for maturities going to zero, are zero:

$$\lim_{h\downarrow 0} S(t, t+h) = 0, \tag{15}$$

almost surely. That this is in fact implied by the predictability of defaults is proven by Giesecke (2001*b*), who shows that in any model with predictable defaults the short spread vanishes. Zero short spreads mean that bond investors do not demand a risk premium for assuming the default risk of an issuer, as long as the time to maturity is sufficiently short. For realistic parameterizations this can be up to a couple of months, as Figure 3 shows. There, assuming constant interest rates $r = 6\%$ and zero recovery $R = 0$, we plot credit spreads

$$S(0,T) = -\frac{1}{T}\ln\tilde{P}[\tau > T]$$

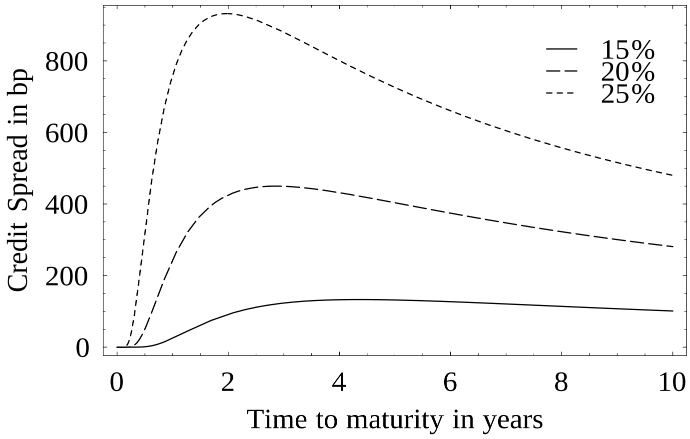

Figure 3: Term structure of credit spreads, varying asset volatility σ (first-passage model).

for varying asset volatilities σ (we set $V_0 = 1$, and $D = 0.7$). Irrespective of the riskiness of the firm, for small maturities the spread vanishes, which is hardly confirmed by empirical observations.

A way around this problem is to assume that the asset process is subject to jumps. Then there is always a chance that the firm value jumps unexpectedly below the default threshold. Zhou (2001*b*) studies spreads in case assets follow some jump-diffusion process, while Hilberink & Rogers (2002) analyze the general case where assets are modeled through a Lévy process. Another approach to avoid predictability is to drop the assumption of perfect information, which will be discussed later in more detail. Note that both jumps and incomplete information make the structural model in fact more realistic.

2.3 Dependent Defaults

A number of studies have investigated historical bond price and default data. They found, quite plausibly, that credit spreads as well as aggregate default rates are strongly related to general macro-economic factors such as the level of default-free interest rates, GDP growth rates, equity index returns and other business cycle indicators. Another observation from the latest Moody's report is that there are default clusters around times of economic downturn. This clustering refers to infection effects and cascading defaults, where the default of a firm immediately increases the default likelihood of another firm dramatically. In its extreme form, a default directly triggers the default of another firm. Such effects can for instance be induced through mutual capital holdings, financial guarantees, or parent-subsidiary relationships. In a recession, default rates increase and so does the likelihood of observing infectious defaults. Recent evidence of the default clustering phenomenon includes the banking crisis in Japan.

These empirical observations have an important consequence: defaults of firms are *stochastically dependent*. We can distinguish two mechanisms leading to default dependence. First, the financial health of any firm depends on common factors related to the state of the general economy. Second, firms are also directly linked and thus the health of a particular firm also depends on the default status of other firms. A thorough understanding of these mechanisms is of vital importance for corporate security valuation, design and analysis of default insurance contracts, default risk aggregation and management, regulation of financial institutions, and the counteraction of financial crises.

How can we accommodate this default correlation in the structural framework? To model firms' dependence on common economic factors in a natural way, we can assume that assets are correlated through time. Let us consider the simplest case with two firms, labelled 1 and 2, and asset dynamics

$$\frac{dV_t^i}{V_t^i} = \mu_i dt + \sigma_i dW_t^i, \quad V_0^i > 0, \quad i = 1, 2, \tag{16}$$

where $\mu_i \in \mathbb{R}$ is a drift parameter, $\sigma_i > 0$ is a volatility parameter, and W^i is a standard Brownian motion. We let $\mathrm{Cov}(W_t^1, W_t^2) = \rho\sigma_1\sigma_2 t$, i.e. the two asset value processes are correlated with correlation coefficient ρ. Fixing some time horizon $T > 0$, we then obtain for the joint default probability in the classic setting

$$\begin{aligned} P[\tau_1 = T, \tau_2 = T] &= P[V_T^1 < K_1, V_T^2 < K_2] \\ &= P[W_T^1 < L_1, W_T^2 < L_2] \\ &= \Phi_2(\rho, L_1, L_2) \end{aligned} \tag{17}$$

where $\Phi_2(\rho, \cdot, \cdot)$ is the bivariate standard normal distribution with correlation ρ, and L_i is called the standardized distance to default of firm i:

$$L_i = \frac{\ln(K_i/V_0^i) - mT}{\sigma_i\sqrt{T}}.$$

In the first-passage model with constant thresholds $D_t^i = D^i$, we get for the joint default probability

$$
\begin{aligned}
p(T_1, T_2) = P[\tau_1 \le T_1, \tau_2 \le T_2] &= P[\min_{s \le T_1} V_s^1 < D^1, \min_{s \le T_2} V_s^2 < D^2] \\
&= P[M_{T_1}^1 \le S_1, M_{T_2}^2 \le S_2] \\
&= \Psi_2(T_1, T_2; S_1, S_2; \rho)
\end{aligned}
$$

where M^i is the running minimum log-asset process of firm i, $\Psi_2(\cdot, \cdot; \cdot, \cdot; \rho)$ is the bivariate inverse Gaussian distribution function with correlation ρ, and $S_i = \ln(D^i/V_0^i)$ is the adjusted default threshold. This joint default probability is calculated in closed-form by Zhou (2001*a*).

Having derived the joint distribution of (τ_1, τ_2), we can separate the complete non-linear *default dependence structure* from the joint default behavior. In fact, there exists a function C^τ, called the *copula* of (τ_1, τ_2), such that joint default probabilities can be represented as

$$
p(T_1, T_2) = C^\tau(p_1(T_1), p_2(T_2)) \tag{18}
$$

where we denote $p_i(t) = P[\tau_i \le t]$ (for general facts on copula functions we refer to Nelsen (1999)). This representation is unique if the τ_i are continuous. Note that this is only the case in the first-passage model, in which C^τ is the inverse Gaussian dependence structure. In the classic model, the copula representation of the joint default probability is not unique and the Gaussian copula corresponding to (17) is only a *possible* default dependence structure. C^τ is a joint distribution function with standard uniform marginals and satisfies the Fréchet bound inequality

$$
\max(u + v - 1, 0) \le C^\tau(u, v) \le \min(u, v), \quad u, v \in [0, 1]. \tag{19}
$$

If C^τ takes on the lower bound, defaults are perfectly negatively correlated (we speak of countermonotone defaults); if it takes on the upper bound, defaults are perfectly positively correlated (and we speak of comonotone defaults). Clearly, if $C^\tau(u, v) = uv$ then defaults are independent.

(19) suggests a partial ordering on the set of copulas as function-valued default correlation measures. A scalar-valued measure such as rank correlation can perhaps provide more intuition about the degree of stochastic dependence between the defaults. Spearman's rank default correlation ρ^S is simply the linear correlation ρ of the copula C^τ given by

$$
\rho^S(\tau_1, \tau_2) = \rho(p_1(\tau_1), p_2(\tau_2)) = 12 \int_0^1 \int_0^1 C^\tau(u, v) du dv - 3 \tag{20}
$$

showing that ρ^S is a function of the copula C^τ only. While rank default correlation measures the degree of monotonic default dependence, linear default time correlation $\rho(\tau_1, \tau_2)$ measures the degree of linear default time dependence only. Another commonly used default correlation measure is the linear correlation of the default indicator variables $\rho(1_{\{\tau_1 \leq t\}}, 1_{\{\tau_2 \leq t\}})$. The conclusions drawn on the basis of $\rho(\tau_1, \tau_2)$ and $\rho(1_{\{\tau_1 \leq t\}}, 1_{\{\tau_2 \leq t\}})$ should however be taken with care. Both are covariance-based and hence are only the natural dependence measures for joint elliptical random variables, cf. Embrechts, McNeil & Straumann (2001). Neither default times nor indicators are joint elliptical, and hence these measures can lead to severe misinterpretations of the true default correlation structure. ρ^S is in contrast defined on the level of the copula C^τ and therefore does not share these deficiencies.

Asset correlation arises from the dependence of firms' on common economic factors. Capturing the contagion effects arising from direct firm linkages is more difficult. Giesecke (2001*a*) provides a model for such contagion mechanisms, where bond investors have incomplete information about the default thresholds of firms. In lack of complete information, investors form a prior distribution on firms' thresholds, which they update with the default status information arriving over time. This updating can be thought of as a re-assessment of a firm's financial health in light of the default of some other, closely linked firm. Such an immediate re-assessment leads to jumps in credit spreads of surviving firms upon default events in the market, which can be considered as information-based contagion effects. Both joint default probability and default copula can be established in terms of the prior and the asset correlation.

2.4 Parameter Estimation

To put a classical structural model to work, we have to estimate the following set of parameters (Black-Scholes setting):

$$(T, r, K, V_0, \sigma, \Sigma),$$

where Σ is the asset correlation matrix. The maturity date T as well as the face value K of the firm's bonds can be obtained from the terms of the bond issue or the firm's balance sheet. Riskless interest rates r can be estimated from default-free Treasury bond prices via standard procedures. However, the asset value V_0 and its volatility σ cannot be observed directly. If the firm is publicly traded, we can estimate V_0 and σ indirectly from observed equity prices E_0 and equity volatility σ_E. Given these quantities, we solve a system of two equations for V_0 and σ. The first is provided by the Black-Scholes formula $E_0 = BS_C(\sigma, T, K, r, V_0)$, cf. (8). The second is

$$\sigma_E E_0 = \Phi(d_1)\sigma V_0. \tag{21}$$

This relation is obtained from applying Itô's lemma to the equity value $E_t = f(V_t, t)$, yielding

$$dE_t = (\frac{\partial E_t}{\partial V}\mu V_t + \frac{1}{2}\frac{\partial^2 E_t}{\partial V^2}\sigma^2 V_t^2 + \frac{\partial E_t}{\partial t})dt + \frac{\partial E_t}{\partial V}\sigma V_t dW_t, \tag{22}$$

and comparing the diffusion coefficient to that of the equity dynamics

$$dE_t = \mu_E E_t dt + \sigma_E E_t dW_t^E, \tag{23}$$

which arise from the assumption that f is a C^2 function in its first argument (a C^2 function of an Ito process (6) is again an Ito process with dynamics (23)). μ_E and σ_E are equity drift and volatility parameters, respectively, and W^E is a standard Brownian motion.

For a classical structural portfolio model, it remains to estimate the asset correlation matrix Σ. In practice, such estimates are provided by KMV, for example. KMV's implementation assumes a factor model for asset returns, i.e.

$$\ln V_T^i = \sum_{j=1}^{n} w_{ij}\psi_j + \epsilon_i$$

where the ψ_j are (independently) normally distributed systematic factors, the w_{ij} are the factor loadings, and the ϵ_i are iid normal idiosyncratic factors. The pairwise asset correlation is now determined by the factors loadings, cf. Kealhofer (1998).

3 Reduced Form Approach

While the structural approach is based on solid economic arguments, the reduced form approach is quite ad-hoc. Here one does not argue why a firm defaults, but models a default as a Poisson-type event which occurs completely unexpectedly. The stochastic structure of default is prescribed by an exogenously given intensity process (*intensity based* reduced form approach), or, less restrictive, by the compensator process (*compensator based* reduced form approach).

3.1 Intensity Based Reduced Form Modeling

3.1.1 Default as Poisson Event

Let us first recall some well-known facts. Let $T_1, T_2, \ldots$ denote the arrival times of some physical event. We call the sequence (T_i) a (homogeneous) *Poisson process with intensity* λ if the inter-arrival times $T_{i+1} - T_i$ are independent and exponentially distributed with parameter λ. Equivalently, letting

$$N_t = \sum_i 1_{\{T_i \le t\}} \tag{24}$$

count the number of event arrivals in the time interval $[0, t]$, we say that $N = (N_t)_{t \ge 0}$ is a (homogeneous) Poisson process with intensity λ if the increments $N_t - N_s$ are independent and have a Poisson distribution with parameter $\lambda(t-s)$ for $s < t$, i.e.

$$P[N_t - N_s = k] = \frac{1}{k!}\lambda^k (t-s)^k e^{-\lambda(t-s)}. \tag{25}$$

The fundamental assumption of the intensity based approach consists of setting the default time equal to the first jump time of a Poisson process N with given intensity λ [Jarrow & Turnbull (1995)]. Thus $\tau = T_1$ is exponentially distributed with parameter λ and the default probability is given by

$$F(T) = P[\tau \le T] = 1 - e^{-\lambda T}. \tag{26}$$

The intensity, or hazard rate, is the conditional default arrival rate given no default:

$$\lim_{h \downarrow 0} \frac{1}{h} P[\tau \in (t, t+h] \,|\, \tau > t] = \lambda, \tag{27}$$

which may loosely be expressed as $P[\tau \in [t, t+dt) \,|\, \tau > t] \approx \lambda dt$. Letting f denote the density of F, from (27) we obtain

$$\lambda = \frac{f(t)}{1 - F(t)}. \tag{28}$$

The valuation of defaultable bonds zero bonds is straightforward in the intensity based approach. Let us assume that in the event of a default, bond investors recover some constant fraction $R \in [0, 1]$ of the unit face value of the bond with maturity T. This convention is called *recovery of face value* or simply constant recovery. Assume that interest rates $r > 0$ are constant; hence non-defaultable zero bond prices are given by $\bar{B}_0^T = e^{-rT}$. Supposing that the recovery is paid at T, we then get for the defaultable bond price at time zero

$$\begin{aligned} B_0^T &= \tilde{E}[e^{-rT}(1_{\{\tau>T\}} + R1_{\{\tau\leq T\}})] \\ &= e^{-rT}(e^{-\tilde{\lambda}T} + R(1-e^{-\tilde{\lambda}T})) \\ &= \bar{B}_0^T - \bar{B}_0^T(1-R)\tilde{P}[\tau \leq T], \end{aligned} \tag{29}$$

which is the value of a risk-free zero bond minus the value of the risk-neutral expected default loss. In case $R = 0$ we have that

$$B_0^T = \bar{B}_0^T \tilde{P}[\tau > T] = e^{-(r+\tilde{\lambda})T}, \tag{30}$$

i.e. we can value a defaultable bond as if it were default free by simply adjusting the discounting rate. Instead of discounting with the risk-free interest rate r, we discount with the default-adjusted rate $r+\tilde{\lambda}$, where $\tilde{\lambda}$ is the risk-neutral intensity. This is a central and important feature of the intensity based approach. We will later see that this holds true also for more complex defaultable (zero recovery) securities.

Another frequently applied recovery convention is called *equivalent recovery*. Here bond investors receive a fraction $R \in [0, 1]$ of an otherwise equivalent default-free zero bond. Supposing that R is constant and that the recovery is paid at default, we get

$$\begin{aligned} B_0^T &= \tilde{E}[e^{-rT}1_{\{\tau>T\}} + e^{-r\tau}R\bar{B}_\tau^T 1_{\{\tau\leq T\}}] \\ &= \tilde{E}[e^{-rT}1_{\{\tau>T\}}] + R\bar{B}_0^T - R\tilde{E}[e^{-rT}1_{\{\tau>T\}}] \\ &= (1-R)\tilde{E}[e^{-rT}1_{\{\tau>T\}}] + R\bar{B}_0^T \\ &= (1-R)\bar{B}_0^T\tilde{P}[\tau > T] + R\bar{B}_0^T, \end{aligned} \tag{31}$$

which is the value of $1-R$ zero recovery bonds plus the value of R risk-free zero bonds. While the latter amount is received by bond investors with certainty (irrespective of whether a default occurs or not), the former amount is only received when the firm survives until T.

Finally, the *fractional recovery* convention has investors receive a fraction $R \in [0, 1]$ of the pre-default market value $B_{\tau-}^T = \lim_{t\uparrow\tau} B_t^T$ of the bond. Assuming that R is constant and recovery is paid at default, we get

$$\begin{aligned} B_0^T &= \tilde{E}[e^{-rT}1_{\{\tau>T\}} + e^{-r\tau}RB_{\tau-}^T 1_{\{\tau\leq T\}}] \\ &= e^{-(r+(1-R)\tilde{\lambda})T}, \end{aligned} \tag{32}$$

which is the value of a zero recovery bond with *thinned* risk-neutral default intensity $\tilde{\lambda}(1-R)$. Put another way, we can value the fractional recovery defaultable bond as if it were default-free by simply using the adjusted discounting rate $r+(1-R)\tilde{\lambda}$. Expression (32) can be explained intuitively as follows. Suppose that the bond defaults with intensity $\tilde{\lambda}$. If default happens, the bond becomes worthless with probability $(1-R)$, and its value remains unchanged with probability R. Clearly, the pre-default value $B^T_{\tau-}$ of the bond is not changed by this way of looking at default. Consequently, for pricing the bond we can just ignore the harmless default (that in which the value remains unchanged), which occurs with intensity $\tilde{\lambda}R$. We then price the bond as if it had zero recovery and a default intensity $\tilde{\lambda}(1-R)$ (the intensity leading to a default with complete loss of value). The pricing formula (32) is then implied by (30).

Now suppose, as it is often the case in practice, that after a default a reorganization takes place and the bond issuing firm continues to operate. At any default before the bond's maturity, investors loose some fraction $(1-R)$, where $R \in [0,1]$, of the bond's pre-default market value $B^T_{\tau-}$. In that case, the payoff of the bond at maturity T is given by the random variable R^{N_T}, where $N_T \in \mathbb{Z}^+$ is the number of defaults by time T, cf. (24). Letting R be a constant, we obtain for the bond price

$$\begin{aligned} B^T_0 = \tilde{E}[e^{-rT}R^{N_T}] &= e^{-rT}\sum_{k=0}^{\infty} R^k \tilde{P}[N_T = k] \\ &= e^{-rT}\sum_{k=0}^{\infty} R^k \frac{(\tilde{\lambda}T)^k}{k!} e^{-\tilde{\lambda}T} \\ &= e^{-(r+(1-R)\tilde{\lambda})T} \qquad (33) \end{aligned}$$

where for the second line we have used the Poisson distribution function (25), and in the last line we have used the fact that $\sum_{k=0}^{\infty} \frac{1}{k!} z^k = e^z$. So from a pricing perspective, the possibility of multiple defaults with fractional recovery does not complicate the problem.

Having investigated bond prices, we can now look at the resulting credit yield spreads. From our general formula, with a zero recovery convention we get

$$S(0,T) = -\frac{1}{T}\ln\frac{e^{-(r+\tilde{\lambda})T}}{e^{-rT}} = \tilde{\lambda} \qquad (34)$$

so that the spread is in fact given by the risk-neutral intensity. With different recovery assumptions analogous results obtain. Obviously, even for maturities going to zero the spread is bounded away from zero. That is, for very short maturities bond investors still demand a premium for default risk. This empirically confirmed property is due to modeling default as a Poisson event, which implies that the default is totally unpredictable. That means default is a complete surprise event; there is no way to anticipate it as was the case in the structural approach. This has also consequences for bond price behavior: upon default, bond prices will jump to their recovery values. But from a pricing perspective, the unpredictability of defaults leads to markets being incomplete in the intensity based framework. As long as there is no asset having default contingent payoffs available for trading, defaultable bonds cannot be perfectly hedged. An intuitive explanation for this is that unpredictable jumps in bond prices cannot be duplicated with predictable trading strategies.

With a constant intensity (and recovery rates), the term structure of credit spreads is of course flat. For richer term structures, we need more sophisticated intensity models. An extension towards time variation and stochastic variation in intensities is considered in the following three sections.

3.1.2 Time-Varying Intensities

The simple (homogeneous) Poisson process can be generalized as follows. N is called an inhomogeneous Poisson process with deterministic intensity function $\lambda(t)$, if the increments $N_t - N_s$ are independent and for $s < t$ we have

$$P[N_t - N_s = k] = \frac{1}{k!}\left(\int_s^t \lambda(u)du\right)^k e^{-\int_s^t \lambda(u)du} \tag{35}$$

The default probability is then given by

$$P[\tau \leq T] = 1 - P[N_T = 0] = 1 - e^{-\int_0^T \lambda(u)du} \tag{36}$$

Assuming constant interest rates $r > 0$, zero recovery credit spreads are then given by

$$S(0,T) = -\frac{1}{T}\ln\frac{e^{-rT-\int_0^T \tilde{\lambda}(u)du}}{e^{-rT}} = \frac{1}{T}\int_0^T \tilde{\lambda}(u)du \tag{37}$$

and, analogously to the constant intensity case, we find for short spreads

$$\lim_{t\downarrow 0} S(0,t) = \tilde{\lambda}(0), \tag{38}$$

to be compared with (15) in the structural case.

For derivatives pricing purposes in practice, one often assumes that (risk-neutral) intensities are piecewise constant:

$$\tilde{\lambda}(t) = a_i, \quad t \in [T_{i-1}, T_i], \quad i \in 1, 2, \ldots, n \tag{39}$$

for some $n \geq 1$ and constants T_i, which represent the maturities of n traded default-contingent instruments such as bonds or default swaps of the same issuer. Given some estimate of riskless interest rates r and some recovery assumption, the prices of these instruments can then be used to back out the constants a_i, i.e. to calibrate the risk-neutral intensity $\tilde{\lambda}(t)$ from observed market prices.

3.1.3 Cox Process

A Cox process N with intensity $\lambda = (\lambda_t)_{t \geq 0}$ is a generalization of the inhomogeneous Poisson process in which the intensity is allowed to be random, with the restriction that conditional on the realization of λ, N is an inhomogeneous Poisson process. For this reason N is also called a conditional Poisson process or a doubly-stochastic Poisson process. The conditional default probability is therefore given by

$$P[\tau \leq T \,|\, (\lambda_t)_{0 \leq t \leq T}] = 1 - P[N_T = 0 \,|\, (\lambda_t)_{0 \leq t \leq T}] = 1 - e^{-\int_0^T \lambda_u du} \tag{40}$$

The law of iterated expectations then leads to the unconditional default probability:

$$P[\tau \leq T] = E[P[\tau \leq T \,|\, (\lambda_t)_{0 \leq t \leq T}]] = 1 - E[e^{-\int_0^T \lambda_u du}] \tag{41}$$

Lando (1998) applies the Cox process framework to model default as the first time a continuous-time Markov chain U with state space $\{1, \ldots, Y\}$ hits the absorbing state Y. The process U models the credit rating of some firm over time. State 1 is interpreted as the highest credit rating category ('AAA' in Moody's system), state $Y-1$ is interpreted as the lowest rating before default, and state Y is the default state. The evolution of the Markov chain is described by a generator matrix Λ with random transition intensities $\lambda_{i,j}(X_t)$, which are modeled as (continuous) functions of some state process X:

$$\Lambda_t = \begin{pmatrix} -\lambda_1(X_t) & \lambda_{1,2}(X_t) & \ldots & \lambda_{1,Y}(X_t) \\ \lambda_{2,1}(X_t) & -\lambda_2(X_t) & \ldots & \lambda_{2,Y}(X_t) \\ \vdots & & & \\ \lambda_{Y-1,1}(X_t) & -\lambda_{Y-1,2}(X_t) & \ldots & \lambda_{Y-1,Y}(X_t) \\ 0 & 0 & \ldots & 0 \end{pmatrix},$$

where

$$\lambda_i(X_t) = \sum_{j=1, j\neq i}^{Y} \lambda_{i,j}(X_t), \quad i = 1, \ldots, Y-1.$$

Intuitively, for small Δt we can think of $\lambda_{i,j}(X_t)\Delta t$ as the probability that the firm currently in rating class i will migrate to class j within the time interval Δt. Consequently, $\lambda_i(X_t)\Delta t$ is the probability that there will be any rating change in Δt for a firm currently in class i. This generalizes the model proposed earlier by Jarrow, Lando & Turnbull (1997), where the transition intensities $\lambda_{i,j}$ were assumed to be constant. With $\tau = \inf\{t \geq 0 : U_t = Y\}$, the default process is a Cox process with random default intensity $\lambda_t = \lambda_{U_t,Y}(X_t)$ at time t (note that the default intensity is represented by the last column in the above generator matrix Λ).

3.1.4 General Stochastic Intensities

Using the Doob-Meyer decomposition theorem, one can show that there exists an increasing and predictable process A with $A_0 = 0$ such that the difference process $1_{\{\tau \leq t\}} - A^\tau$, where $A_t^\tau = A_{t\wedge\tau}$ is the process A stopped at τ, is a martingale. The unique process A^τ is called the *compensator* of the default indicator process. If A is absolutely continuous, i.e.

$$A_t = \int_0^t \lambda_s ds \tag{42}$$

for some non-negative process $\lambda = (\lambda)_{t\geq 0}$, then λ is called an intensity of τ. From this we can show that for $t < \tau$ the intensity satisfies

$$\lambda_t = \lim_{h\downarrow 0} \frac{1}{h} P[\tau \in (t, t+h] \,|\, \mathcal{F}_t] \tag{43}$$

almost surely, where $\mathcal{F}_t$ can be thought of as the information available to bond investors at time t (including survivorship information and other state variables). The interpretation of λ_t as a conditional default arrival rate is analogous to the Poisson case, cf. (27), where the conditioning information consisted only of survivorship information. Under technical conditions, in this general case default probabilities are given by

$$P[\tau \leq T] = 1 - E[e^{-\int_0^T \lambda_u du}], \tag{44}$$

cf. Duffie & Singleton (1999) and Duffie, Schroder & Skiadas (1996). As for the valuation of defaultable securities, the results derived in the Poisson case can be generalized as follows. Consider a defaultable security (X, T) paying off a random variable X at T if no default occurs and zero otherwise (for $X = 1$ this is a defaultable zero bond with zero recovery). This security has under technical conditions a value given by

$$\tilde{E}\left[e^{-\int_0^T r_s ds} X 1_{\{\tau > T\}}\right] = \tilde{E}\left[X e^{-\int_0^T (r_s + \tilde{\lambda}_s) ds}\right] \tag{45}$$

where r is the risk-free short rate process and $\tilde{\lambda}$ is the risk-neutral intensity process for default. Also in the most general case, the defaultable claim (X, T) can be valued as if it were default-free by simply adjusting the rate used for discounting by the risk-neutral intensity. In the intensity based framework, defaultable term structure modeling exactly parallels ordinary non-defaultable term structure modeling.

Due to these modeling parallels, affine processes used for (risk-free) short rate modeling play a prominent role also for stochastic intensity models. The basic affine intensity model is given by

$$d\lambda_t = a(b - \lambda_t)dt + \sigma\sqrt{\lambda_t} dW_t + dJ_t,$$

which has a diffusive component W (a standard Brownian motion with volatility σ), and a jump component J (an independent jump process with Poisson arrival intensity c and exponential jump size distribution with mean d). While the diffusive component models the continuous changes of the issuer's credit quality over time, the jumps model abrupt and unexpected changes in the credit quality. The parameter a controls the mean-reversion rate and the long-run mean of λ is given by $b + cd/a$. The famous Cox-Ingersoll-Ross model is the special case where $c = 0$. For affine intensities default probabilities (44) are in closed-form. For the estimation of an affine intensity model without jumps see, for example, Duffee (1998) or Duffie, Pedersen & Singleton (2000).

In the general setting, under technical conditions on $\tilde{\lambda}$ short spreads satisfy

$$\lim_{T \downarrow t} S(t, T) = \tilde{\lambda}_t \tag{46}$$

almost surely, as expected. In the general intensity based approach, credit spreads are bounded away from zero, and short spreads are given by the risk-neutral intensity.

3.1.5 Default Correlation

To accommodate default correlation arising from the dependence of firms on common economic factors, a natural way is to introduce correlation between firms' intensity processes through time. However, defaults are then independent given the intensity paths. For example, one could set $\lambda_t^i = h_t + h_t^i$ for firm i, where h and h^i are independent processes. h would model the systematic default intensity from macro-economic shocks, while h^i models the idiosyncratic, or firm-specific, default intensity. With correlated default intensities, joint default probabilities are easily computed. In the two firm case,

$$P[\tau_1 > t, \tau_2 > t] = E[e^{-\int_0^t (\lambda_u^1 + \lambda_u^2) du}], \tag{47}$$

and we have that

$$P[\tau_1 \leq t, \tau_2 \leq t] = E[e^{-\int_0^t (\lambda_u^1 + \lambda_u^2) du} + e^{-\int_0^t \lambda_u^1 du} + e^{-\int_0^t \lambda_u^2 du}] - 1. \tag{48}$$

In order to induce a stronger type of correlation, one can let the intensity of a particular firm jump upon the default of some other firm(s), corresponding to the idea of contagion among defaults. In particular, this would allow to model direct inter firm linkages, cf. Jarrow & Yu (2001). Another idea is to admit common jumps in intensities, corresponding to joint credit events. For a discussion of such types of models, we refer to Duffie & Singleton (1998).

Giesecke (2002*a*) considers a simple model which is based on the assumption that a firm's default is driven by idiosyncratic as well as other regional, sectoral, industry, or economy-wide shocks, whose arrivals are modeled by independent Poisson processes. In this approach a default is governed again by a Poisson process and default times are jointly exponentially distributed. In the bivariate case, suppose there are Poisson processes N^1, N^2, and N with respective intensities λ_1, λ_2, and λ. We interpret λ_i as the idiosyncratic shock intensity of firm i, while we think of λ as the intensity of a macro-economic or economy-wide shock affecting both firms simultaneously. Firm i defaults if either an idiosyncratic or a systematic shock (or both) strike the firm for the first time (that is, firm i defaults with intensity $\lambda_i + \lambda$). The joint survival probability is found to be

$$\begin{aligned} s(t,u) = P[\tau_1 > t, \tau_2 > u] &= P[N_t^1 = 0, N_u^2 = 0, N_{t \vee u} = 0] \\ &= s_1(t) s_2(u) \min(e^{\lambda t}, e^{\lambda u}). \end{aligned} \tag{49}$$

Defining $\theta_i = \frac{\lambda}{\lambda_i + \lambda}$, the copula K^τ associated with s is given by

$$K^\tau(u,v) = s(s_1^{-1}(u), s_2^{-1}(v)) = \min(vu^{1-\theta_1}, uv^{1-\theta_2}),$$

and is called the exponential copula. The copula C^τ associated with the joint default probability p can be calculated from K^τ via

$$C^\tau(u,v) = K^\tau(1-u, 1-v) + u + v - 1$$

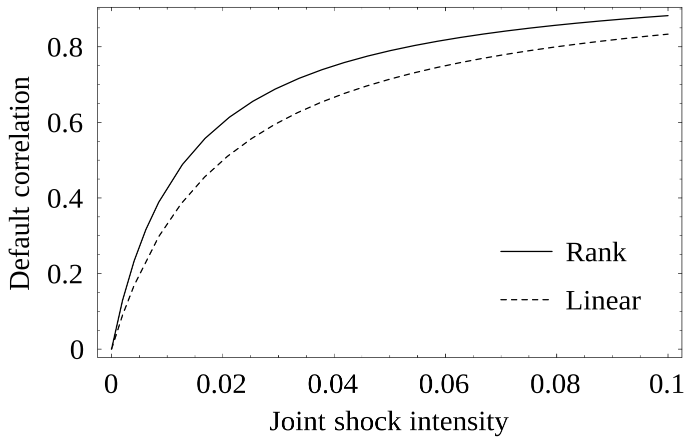

Figure 4: Rank and linear default correlation in the exponential model.

The parameter vector $\theta = (\theta_1, \theta_2)$ controls the degree of dependence between the default times. If the firms default independently of each other ($\lambda = 0$ or $\lambda_1, \lambda_2 \to \infty$), then $\theta_1 = \theta_2 = 0$ and we get $C^\tau(u, v) = uv$, the product copula. If the firms are perfectly positively correlated and the firms default simultaneously ($\lambda \to \infty$ or $\lambda_1 = \lambda_2 = 0$), then $\theta_1 = \theta_2 = 1$ and $C^\tau(u, v) = u \wedge v$, the Fréchet upper bound copula. Spearman's rank correlation ρ^S is simply

$$\rho^S(\tau_1, \tau_2) = \frac{3\lambda}{3\lambda + 2\lambda_1 + 2\lambda_2}, \tag{50}$$

while linear default time correlation is given by

$$\rho(\tau_1, \tau_2) = \frac{\lambda}{\lambda + \lambda_1 + \lambda_2}. \tag{51}$$

As discussed in Section 2.3, ρ is not an appropriate default correlation measure. This is obvious when we compare (50) and (51). Clearly $\rho \leq \rho^S$ and linear default correlation *underestimates* the true default dependence, cf. Figure 4, where we plot both measures as functions of the joint shock intensity λ. We fix $\lambda_1 = \lambda_2 = 0.01$, which corresponds to a one-year default probability of about 1% when firms are independent. If λ is zero, then firms default independently and $\rho = \rho^S = 0$. With increasing λ, the joint shock component of the default risk dominates the idiosyncratic component, and the default correlation increases.

3.2 Compensator Based Reduced Form Modeling

In the previous Section 3.1 we described the intensity based reduced form models, where the stochastic structure of default was described through an intensity process. In this section, we consider a more general reduced form modeling approach, in which the stochastics of default are described through the *default compensator* [Giesecke (2001*b*)]. While we still assume that the default is an unpredictable surprise event, we do not require that an intensity exists. From the technical point of view, this means we assume the default compensator to be merely continuous, but not necessarily absolutely continuous. If an intensity does in fact exist, intensity based and compensator based models are of course equivalent. The existence of an intensity is however not granted in general, as we will see in the next section when we integrate structural and reduced form approach.

Let us briefly re-consider the notion of the compensator. There exists an increasing and predictable process A with $A_0 = 0$ such that the difference process defined by $1_{\{t \geq \tau\}} - A^\tau_t$, where $A^\tau = A_{\cdot \wedge \tau}$, is a martingale. That is, the unique compensator A^τ counteracts the increasing default indicator process in a predictable way, such that the difference between default indicator and compensator forms a martingale. Thus the increment of the compensator can be viewed as the expected increment of the default indicator, given all available information $\mathcal{F}_t$ at time t: $dA^\tau_t =$ "$E[d1_{\{t \geq \tau\}} \,|\, \mathcal{F}_t]$".

Assuming that the default is unpredictable, under some technical conditions we can establish the default probability in terms of the process A:

$$P[\tau \leq T] = 1 - E[e^{-A_T}], \tag{52}$$

cf. Giesecke (2001*b*). Note the analogy to (44). If an intensity does in fact exist, then (52) is equivalent to (44). Now consider a defaultable security (X, T) paying off a random variable X at T if no default occurs and zero otherwise. If the default is unpredictable, under some additional technical conditions the value of (X, T) can be written as

$$\tilde{E}\big[e^{-\int_0^T r_s ds} X 1_{\{\tau > T\}}\big] = \tilde{E}\big[X e^{-\int_0^T r_s ds - A_T}\big]. \tag{53}$$

Again, if an intensity exists this price representation is equivalent to (45).

The credit spread term structure properties induced by the compensator based model are very similar to those in the intensity based model. Due to the unpredictability of defaults, the short spread does not necessarily vanish as was the case in the structural approach. Likewise, bond prices do not continuously converge but jump to their default contingent values. These properties are empirically quite plausible.

4 Integrating Both Approaches

While the structural approach is economically sound, it implies less plausible credit spread properties. The reduced form approach is ad hoc though, but tractable and implies plausible credit spread properties. A natural question arising then is whether and to what extent both approaches are consistent. This question can be answered by considering (42) together with the properties of the compensator. If the default is predictable, as in the structural approach, then the compensator is given by the default jump process itself: $A^\tau_t = 1_{\{t \geq \tau\}}$. In this case A^τ is discontinuous and admits no intensity. We can thus conclude that structural and reduced form approach are not consistent.

The key to unify both approaches lies in the probabilistic properties of the default event. To integrate structural and compensator based reduced form approach, the default must at least be unpredictable in the structural model. To achieve this, jumps in the asset process as in Zhou (2001*b*) are not sufficient, unless the asset value is modeled through a pure jump process. A more successful starting point is to drop the restrictive perfect information assumption commonly made in structural models [Duffie & Lando (2001)]. With imperfect information on assets and/or default threshold, investors are at any time uncertain about the nearness of the assets to the threshold. While then the default is an unpredictable event, it is not obvious whether an intensity exists. This depends in fact on the extent of the available information. If investors have imperfect *asset* information in form of noisy accounting reports which are received at discrete points in time, an intensity does exist. This was shown by Duffie & Lando (2001), who directly computed the intensity as the limit (43). If assets follow a geometric Brownian motion with volatility σ, they obtained

$$\lambda_t = \frac{1}{2}\sigma^2 a_z(t, D),$$

where $a(t, \cdot)$ is the conditional log-asset density given the noisy accounting reports and survivorship, and D is the a priori known default threshold. For general observation schemes, Giesecke (2001*b*) considers the default compensator. With incomplete asset and threshold observation (a realistic information assumption for a private firm) the compensator is under technical conditions given by $A^\tau = A_{\cdot \wedge \tau}$ with

$$A_t = -\ln\left(1 - \int_{-\infty}^{0} H(t, x)\, g(x)\, dx\right)$$

where $H(t, \cdot)$ is the conditional distribution of the running minimum asset value M_t given the imperfect asset information, and g is the density of the threshold D. Here, given again technical conditions, an intensity exists; it can be directly calculated from A^τ:

$$\lambda_t = \frac{\int_{-\infty}^{0} \dot{H}(t,x)\, g(x)\, dx}{1 - \int_{-\infty}^{0} H(t,x)\, g(x)\, dx}, \quad t \in (0,\tau),$$

where $\dot{H}(t,x) = \frac{\partial}{\partial t} H(t,x)$. In this situation compensator and intensity based reduced form modeling are in fact equivalent. But this changes if there is imperfect threshold observation only, which might be a reasonable assumption for a firm with listed stock. Then the compensator is given by $A^\tau = A_{\cdot \wedge \tau}$ with

$$A_t = -\ln G(M_t),$$

where G is investors' prior default threshold distribution. Here the compensator is *singular* and an intensity does not exist. While in this situation the compensator based price representation (53) still holds, the intensity based price representation (45) breaks down. Here the generality provided by the compensator pays off, as we need to presume only the unpredictability of defaults for pricing purposes, but not an intensity.

The idea of furnishing an intensity based model with structural interpretation is generalized in Giesecke (2002*b*) to the multi-firm case with correlated defaults. Based on the explicitly constructed compensators for successive default events, efficient algorithms for the simulation of correlated defaults can be formulated.

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