

# Modeling the Distance-to-Default Process of a Firm

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October 26, 2001

## 1 Introduction

Credit derivatives provide synthetic protection against bond and loan defaults. A simple example of a credit derivative is the credit default swap, in which one counterparty makes periodic payments to another in exchange for the right to be paid a notional amount if a credit event happens. Recently, we have seen the emergence of "wholesale" credit protection in the form of first-to-default swaps written on a basket of underlying credits. Pricing of credit derivatives requires quantifying the likelihood of default of the reference entity. Default probabilities can be estimated from the spreads of the bond issued by the reference entity <sup>1</sup>.

Two kinds of mathematical frameworks for pricing credit derivatives have been proposed to this date: the *structural models*, introduced by Merton [11] and others [3, 4, 13, 9, 12, 6], and the *reduced form models* of Duffie and Singleton [5]. Here, we will be concerned with the structural approach, and in particular with the "default

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<sup>1</sup>A more difficult problem, which often occurs in practice, is to estimate default probabilities of a firm that has not issued any debt.

barrier” methodology recently introduced by Hull and White [8]. This paper draws from Hull and White [8] and extends it in several directions.

First, we consider a continuous-time version of the Hull-White model in which the default index follows a general diffusion process. We show that, in this general framework, calibration of the default barrier leads to a new free boundary problem for the associated Fokker-Planck partial differential equation.

Second, based on these results, we propose a new interpretation of the default barrier model in terms of a *risk-neutral distance-to-default* process for the firm (or risk-neutral debt-to-value ratio, etc.). We show that finding a default boundary which is calibrated to a set of default probabilities is equivalent to specifying an appropriate “excess drift” for the distance-to-default process. This excess drift can be interpreted as a market price of risk for the value of the firm, *a la* Merton [11] which renders the process consistent with observed credit spreads.

Finally, we discuss the numerical implementation of the model, using a finite difference scheme for the Fokker-Planck equation, coupled with a Newton-Raphson scheme for determining the excess drift at each successive time step. The theory is applied to calculating the default boundaries and drifts for AAA and BAA1 credits under different assumptions about default probabilities and volatility functions.

## 2 The Barrier Diffusion Model

Following Hull and White [8], we define the function  $P(t)$  to be the probability that the firm has defaulted by time  $t$ , as seen today<sup>2</sup>. The default probability density is given by  $P'(t)$ . In particular,  $P'(t)\Delta t$  represents the probability of default between times  $t$  and  $t + \Delta t$ , as seen at time zero. Just as in [8], we consider a “default index” associated with the firm, represented by an Ito process  $\{X(t), X(0) = X_0\}$

$$dX(t) = a(X(t), t)dt + \sigma(X(t), t)dW(t), \quad (1)$$

where  $W(t)$  is the standard Wiener process. We pronounce that the firm defaults at time  $t$  if

$$X(t) = b(t), \quad \text{and} \quad X(s) > b(s), \quad s < t, \quad (2)$$

where  $b(t)$  describes a *barrier function*. A financial-economic interpretation of this barrier function will be given below in Section 4. The default time is, by definition, the first time that  $X(t)$  hits this barrier, i.e.

$$\tau = \inf \{t \geq 0 : X(t) \leq b(t)\}. \quad (3)$$

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<sup>2</sup>We assume henceforth this probability has been determined from bond spreads or other procedure [7].

Let  $f(x, t)$  be the *survival probability density function* of  $X(t)$ , given by

$$f(x, t)dx = \text{Prob}[x < X(t) < x + dx, \tau \geq t], \quad (4)$$

for  $x \geq b(t)$ .

From standard results in probability theory, the function  $f(x, t)$  satisfies the *forward Fokker-Planck equation*

$$f_t = \frac{1}{2} \left( \sigma^2(x, t) f \right)_{xx} - (a(x, t) f)_x, \quad t > 0, \quad x > b(t); \quad (5)$$

with initial and boundary conditions

$$f(x, t)|_{t=0} = \delta(x - X_0), \quad (6)$$

$$f(x, t)|_{x=b(t)} = 0. \quad (7)$$

We note that the integral

$$\int_{b(t)}^{\infty} f(x, t) dx \quad (8)$$

represents the survival probability up to time  $t$ ,  $1 - P(t)$ . Therefore, the default probability  $P(t)$  is related to the survival probability density  $f(x, t)$  and the barrier  $b(t)$  by the equation

$$P(t) = 1 - \int_{b(t)}^{\infty} f(x, t) dx. \quad (9)$$

We see from this equation that the barrier function  $b(t)$  must be chosen appropriately so that the pair  $\{f(x, t), b(t)\}$  is consistent with the given default probabilities  $\{P(t), t > 0\}$ . To see this in a more explicit way, we differentiate the default probability with respect to time in (9), and use the equations satisfied by  $f$ , whence

$$\begin{aligned} P'(t) &= - \int_{b(t)}^{\infty} \frac{\partial f}{\partial t} dx + f(b(t), t) b'(t) \\ &= - \frac{1}{2} \int_{b(t)}^{\infty} (\sigma^2 f)_{xx} dx + \int_{b(t)}^{\infty} (a f)_x dx \\ &= \frac{1}{2} \frac{\partial}{\partial x} (\sigma^2 f) \Big|_{x=b(t)}. \end{aligned} \quad (10)$$

Thus, aside from (7), the survival density function must satisfy an additional boundary condition at the barrier  $x = b(t)$ . This gives rise to a *free boundary problem* for the forward Fokker-Planck equation, since the boundary  $b(t)$  is unknown and must be determined consistently with the two boundary conditions (7) and (10).

### 3 Similarity Transformation

We observe that the model is invariant under a scaling, or similarity, transformation. In fact, let  $\sigma_0$  be a positive number, and consider the transformation

$$\tilde{x} = \frac{x}{\sigma_0}, \quad \tilde{b}(t) = \frac{b(t)}{\sigma_0}, \quad \tilde{\sigma}(\tilde{x}, t) = \frac{\sigma(x, t)}{\sigma_0}, \quad \tilde{a}(\tilde{x}, t) = \frac{a(x, t)}{\sigma_0}. \quad (11)$$

It follows immediately that the new function  $\tilde{f}(\tilde{x}, t)$  given by

$$\tilde{f}(\tilde{x}, t) = \sigma_0 f(x, t) \quad (12)$$

also satisfies Eqs.(5-7) and Eq.(10). In particular, if  $\sigma(x, t)$  is a constant, we can “scale out” the volatility parameter – solutions with arbitrary constant volatility  $\sigma$  can be derived from the solution with  $\sigma = 1$ . The latter case is, in essence, the Hull and White model [8], in which the default index is taken to be a standard Brownian motion.

If  $\sigma$  is not a constant, the proposed diffusion model allows us to incorporate volatility functions that depend on the index value as well as time. This observation can be useful if one expects the volatility of the credit default index to increase as the firm approaches default, for example.

### 4 Distance-to-Default

Let us make the following

**Definition:** Let  $P(t)$  denote the market-implied default probability of the firm. A risk-neutral distance-to-default process (RNDD) is a diffusion process satisfying

$$dY(t) = \tilde{a}(Y, t)dt + \tilde{\sigma}(Y, t)dW(t), \quad (13)$$

such that (i)  $Y(0) > 0$ , and (ii)  $\text{Prob}[\inf_{s < t} Y(s) \leq 0] = P(t)$ .

This definition is consistent with the notion that a firm defaults when the value of its assets falls below the value of the debt.<sup>3</sup>

We notice that if we set

$$Y(t) = X(t) - b(t), \quad (14)$$

where  $X(t)$  is the default index process discussed in the previous section, then  $Y(t)$  is a RNDD process with  $\tilde{a}(Y, t) = a(X, t) - b'(t)$  and  $\tilde{\sigma}(Y, t) = \sigma(X, t)$ . Furthermore, we have

$$dY(t) = dX(t) - b'(t)dt. \quad (15)$$

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<sup>3</sup>The RNDD  $Y(t)$  can be interpreted as the difference between the value of the assets and the debt, or the log of the debt-to-equity ratio, etc., seen in a “risk-neutral” world.

Therefore, the problem of finding the barrier in the continuous-time analog of the Hull-White model is equivalent to the problem of finding the *excess drift* in the distance-to-default process that makes the latter “risk-neutral” (calibrated to data on default probabilities).

The RNDD survival density  $u(y, t)$  is related to the default index survival density  $f(x, t)$  by

$$u(y, t) = f(y + b(t), t). \quad (16)$$

It follows that the Fokker-Planck equation for the survival probability of the RNDD process is given by

$$u_t = b' u_y - (au)_y + \frac{1}{2}(\sigma^2 u)_{yy}, \quad y > 0, \quad t > 0, \quad (17)$$

$$u|_{t=0} = \delta(y - Y_0), \quad (18)$$

$$u|_{y=0} = 0, \quad t > 0, \quad (19)$$

$$\frac{1}{2} \left[ \frac{\partial}{\partial y} (\sigma^2 u) \right]_{y=0} = P'(t), \quad t > 0. \quad (20)$$

Here,  $b'$  must be chosen, adaptively, in such a way that the second boundary condition (20) is satisfied at all times. We conclude that the free boundary problem for the default index is transformed into a control problem for the RNDD. In this reformulation,  $b'(t)/\sigma$  can be viewed as a “market price of risk” associated with the firm’s perceived creditworthiness, consistently with Merton [11].

## 5 Initial Layer and Matching of Solutions

We first consider the special case where the coefficients  $\sigma$  and  $a$  are constants and  $b(t)$  is an affine function. Without any loss of generality we set  $a = 0$ .<sup>4</sup> This special case will be used later to construct the general solution.

Assume accordingly that the default barrier is given by the equation

$$\bar{b}(t) = -\alpha - \beta t, \quad \alpha > 0, \quad 0 < t < t_0. \quad (21)$$

In this case, it can be shown that the corresponding default probability is given by

$$\bar{P}(t) = 1 - \int_{\bar{b}(t)}^{\infty} f(x, t) dx = N \left( \frac{-\alpha - \beta t - X_0}{\sigma \sqrt{t}} \right) + e^{\frac{-2(\alpha + X_0)\beta}{\sigma^2}} N \left( \frac{-\alpha + \beta t - X_0}{\sigma \sqrt{t}} \right), \quad (22)$$

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<sup>4</sup>The case of Brownian motion with drift is analogous, with the only difference being that the slope of the line defining the barrier must be modified.

where  $N(x)$  is the standard cumulative normal distribution, and the density is

$$\bar{P}'(t) = \frac{\alpha + X_0}{t\sqrt{2\pi t}\sigma} e^{-\frac{(\alpha + \beta t + X_0)^2}{2\sigma^2 t}}. \quad (23)$$

This follows from standard properties of Brownian motions. Under the same assumptions, the survival probability density is given by

$$f_{\alpha,\beta}(x, t) = \frac{1}{\sigma\sqrt{2\pi t}} e^{-\frac{(x - X_0)^2}{2\sigma^2 t}} \left[ 1 - e^{-\frac{2(\alpha + X_0)}{\sigma^2 t}(x + \alpha + \beta t)} \right] \quad (24)$$

for  $-\alpha - \beta t \leq x < \infty$ .

The RNDD density is given by

$$u_{\alpha,\beta}(y, t) = f_{\alpha,\beta}(y - \alpha - \beta t, t) = \frac{1}{\sigma\sqrt{2\pi t}} e^{-\frac{(y - \beta t - Y_0)^2}{2\sigma^2 t}} \left[ 1 - e^{-\frac{2Y_0}{\sigma^2 t}y} \right] \quad (25)$$

for  $y \geq 0$ , where  $Y_0 = X_0 + \alpha > 0$  is the initial distance-to-default.

In Figure 1, the default probability  $\bar{P}(t)$  and the default probability density  $\bar{P}'(t)$  as functions of  $t$  are shown for several sets of positive  $\alpha$  and  $\beta$  values. In all these cases, we take  $X(t)$  to be a standard Brownian motion with drift zero,  $\sigma = 1$  and  $X_0 = 0$ . In this case,  $\alpha > 0$  is the initial distance-to-default. We observe that  $\bar{P}'$  is unimodal and drops to zero exponentially after reaching its maximum. The reason is that the barrier, which is linear in time, outgrows the square root scaling in time of the Brownian motion, therefore as time increases and passes over a certain level, the default probability density will decrease towards zero.

We now consider the solution of the model for arbitrary data  $P(t)$ . The idea is to use the straight line model for a finite but small time  $t_0$ , and then to match this solution to a numerically computed  $b(t)$ . The need for an “initial layer” arises from the fact that the  $\delta$ -function initial data vanishes to all orders for  $t = 0$  and must be regularized consistently with the boundary conditions that we want to impose for small values of  $t$ . For a given default probability data  $P(t)$ , let us choose the parameters  $\alpha$  and  $\beta$  in such a way that

$$\bar{P}(t_0) = P(t_0), \quad (26)$$

$$\bar{P}'(t_0) = P'(t_0), \quad (27)$$

where  $P(t_0)$  and  $P'(t_0)$  are estimated from the market data. A simple Newton-Raphson solver can lead to a solution of  $\alpha$  and  $\beta$  for small values of  $P(t_0)$  and  $P'(t_0)$ . In Figure 2, we consider an example where  $t_0 = 0.5$  is chosen, and  $P(0.5) = 0.01$ ,  $P'(0.5) = 0.02$ , which lead to  $\alpha = 1.044$  and  $\beta = 1.949$ . The survival distribution at  $t = 0.5$  is plotted in the graph.

Once the initial survival distribution at  $t = t_0$  has been determined, we use it as an initial condition for the PDE problem (17-20). Since this distribution is derived from the default probability conditions (26) and (27), the compatibility condition (20) is automatically satisfied at  $t = t_0$ . A second-order finite difference algorithm is described below to solve the PDE for time beyond  $t_0$ .

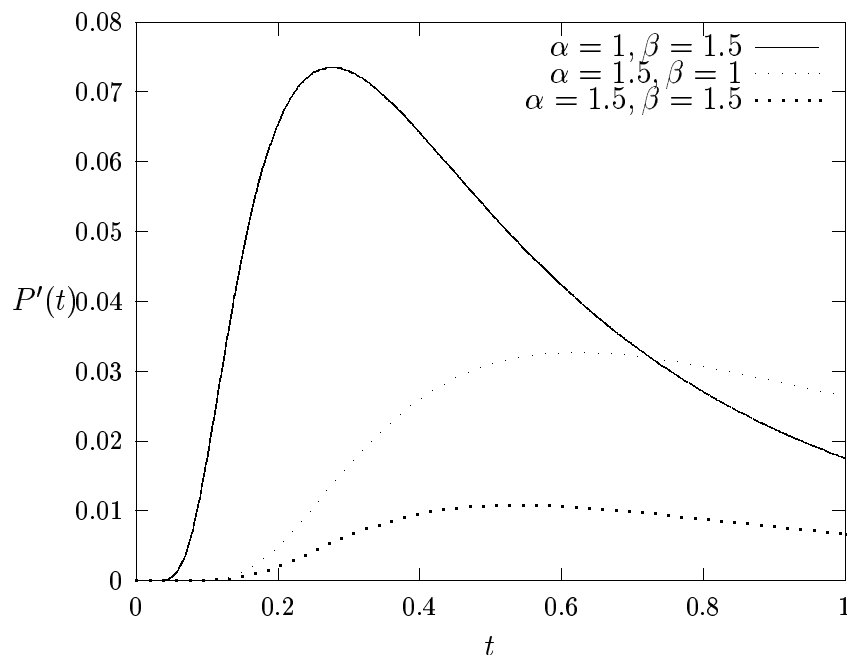
## 6 Numerical Algorithm for General Default Probabilities

The numerics are based on the “RNDD formulation”, i.e. on solving a control problem for the unknown drift coefficient  $b'(t)$ . For simplicity, we write down the scheme for the case  $\sigma = 1$ , and  $a = 0$ . The extension of the algorithm to variable coefficients is obvious.

We start from  $t = t_0$  with the initial condition from the initial layer solution, that is,

$$u(y, t_0) = u_{\alpha, \beta}(y, t_0), \quad y \geq 0. \quad (28)$$

Figure 1: Default Probability Density for Some  $\alpha$  and  $\beta$  Values



A second-order finite difference algorithm for Eq.(17-19) can be constructed as follows. Define  $y_j = (j - \frac{1}{2})h$ ,  $t^n = n\Delta t$ , and let  $u_j^n$  represent the numerical approximation to  $u(y_j, t^n)$ . We consider a Crank-Nicholson scheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \lambda^{n+\frac{1}{2}} \frac{u_{j+\frac{1}{2}}^{n+\frac{1}{2}} - u_{j-\frac{1}{2}}^{n+\frac{1}{2}}}{h} + \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{4h^2} + \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{4h^2} \quad (29)$$

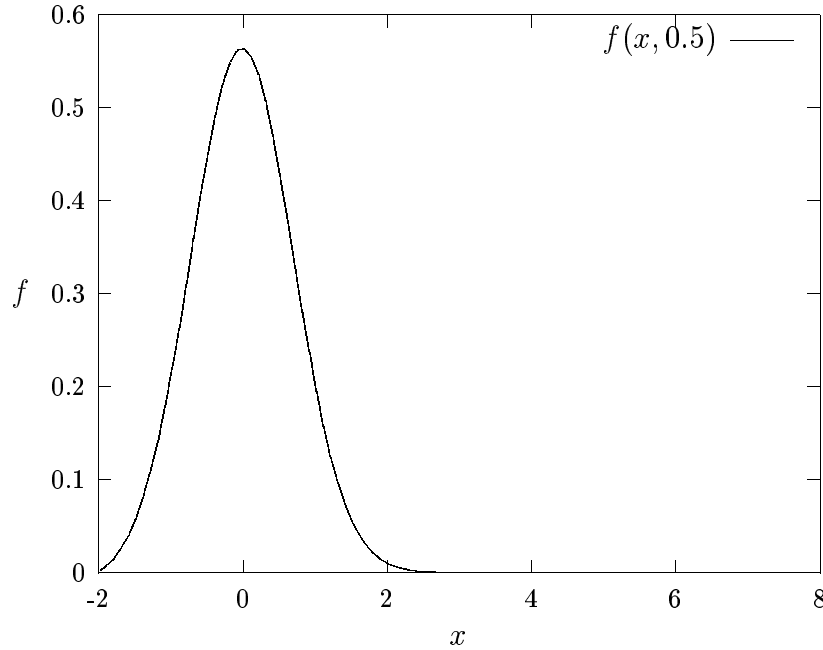
with boundary condition  $u_0^n = 0$ ,  $n \geq 0$ , where  $\lambda^{n+\frac{1}{2}}$  is an undetermined drift which depends on the time-step ( $n$ ) and which will be determined inductively. Here  $u_{j+\frac{1}{2}}^{n+\frac{1}{2}}$  is computed from a predictor step, which involves Taylor extrapolations in space and time with an upwind scheme approximation for the spatial derivative.<sup>5</sup>

For each value of  $\lambda^{n+\frac{1}{2}}$ , the resulting tridiagonal system is solved using a standard linear algebra package. The value of  $\lambda^{n+\frac{1}{2}}$  which matches the extra boundary condition (20),  $\lambda_*^{n+\frac{1}{2}}$ , is found by using the Newton-Raphson iteration method. Finally, we

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<sup>5</sup>We note that this numerical scheme is second-order accurate in both space and time, and all of our calculations are unconditionally stable with respect to the choices of  $h$ ,  $\Delta t$  and  $\lambda^{n+\frac{1}{2}}$ .

Figure 2: Survival Distribution at  $t = 0.5$



equate the computed  $\lambda_*^{n+\frac{1}{2}}$  with the drift, i.e.

$$b'(t^{n+\frac{1}{2}}) = \lambda_*^{n+\frac{1}{2}}, \quad (30)$$

and extend the barrier in one time step

$$b(t^{n+1}) = b(t^n) + \lambda_*^{n+\frac{1}{2}} \Delta t. \quad (31)$$

The numerical stability of the algorithm, *i.e.* the continuous dependence of the function  $b'(t)$  on the probability density  $P'(t)$ , is an important consideration, considering the fact that credit default data is discrete and that, consequently, the probability density needs to be constructed by interpolation. We mention here without further proof that the free-boundary problem and the algorithm admit a unique, stable solution on any interval where the probability density  $P'(t)$  is positive. Further comments on stability and the issue of interpolation of probabilities are made in the study of concrete examples in the next section.

## 7 Examples

For the numerical example, we introduce a finite domain ( $0 \leq x \leq 20$ ) which is large enough to cover the dynamics of the solutions studied in this section. Also, we use an initial layer with  $t_0 = 0.5$ . Unless explicitly noted, a constant volatility  $\sigma = 1$  is assumed. In the finite difference calculations, we use 400 points in the  $x$  direction and choose  $\Delta t = 0.05$ .

In the first example we consider default probabilities for the bank industry with Standard and Poor's AAA and BAA1 ratings <sup>6</sup>. The default probabilities for several recovery rates are shown in Table 1, where a bank's default probabilities in each of the next 10 years are listed. For instance, 0.0073 means that a AAA-rated bank would have a 0.73% probability to default within the next year, and likewise 0.0136 means that it would have a 1.36% probability to default within the second year. These default probabilities were estimated based on an expected recovery rate of 30%, 50% and 70%. As discussed in [7], different expected recovery rates cause very different default probability estimates. As a consequence, our default barriers will exhibit a strong dependence on the expected recovery rate assumed. In our calculations, the data from the table is expanded to generate a piecewise constant function of time  $P'(t)$ . In Figure 3, default barriers for AAA and BAA1 banks from our model are plotted for  $0 \leq t \leq 10$ , based on data sets in Table 1 with a 50% expected recovery rate. As we notice, the shapes of the curves for these two ratings are quite similar,

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<sup>6</sup>data and sources are available from the authors upon request.

since they both belong to the same industry and therefore bear similar characteristics as when the firm is more likely to default in the future. The barrier for the lower rating (BAA1) is always above the barrier with the higher rating (AAA), indicating that it is much more likely for the firm with a lower rating to default.

One of the advantages of the generalized model presented here is the ability to incorporate general volatility structures. Here we consider an example where the volatility is increased to a higher level once the Brownian path gets near to the default boundary, and compare the result to the result with a constant volatility ( $\sigma = 1$ ). In particular, we choose

$$\sigma(x) = \begin{cases} 1 & 0 \leq x \leq 2, \\ 1 - \frac{1}{4}(x - 2) & 2 < x \leq 4, \\ \frac{1}{2} & x > 4. \end{cases} \quad (32)$$

In Figure 4, default barriers from our model are plotted for  $0 \leq t \leq 10$ , based on data set with expected recovery rate 0.5 in Table 1. Two barrier curves represent the cases with the constant volatility and the variable volatility, respectively. We find that the barrier with variable volatility lies above the barrier with constant volatility. This is because the average volatility in the variable case is lower than the constant volatility level chosen for the problem. To achieve the same exit probability, the barrier has to move up to accommodate a lower volatility.

Next, we compare this PDE model with the original Hull-White model in this application. We implemented the Hull-White model according to [8] with the same

Table 1: Default Probability for Banks

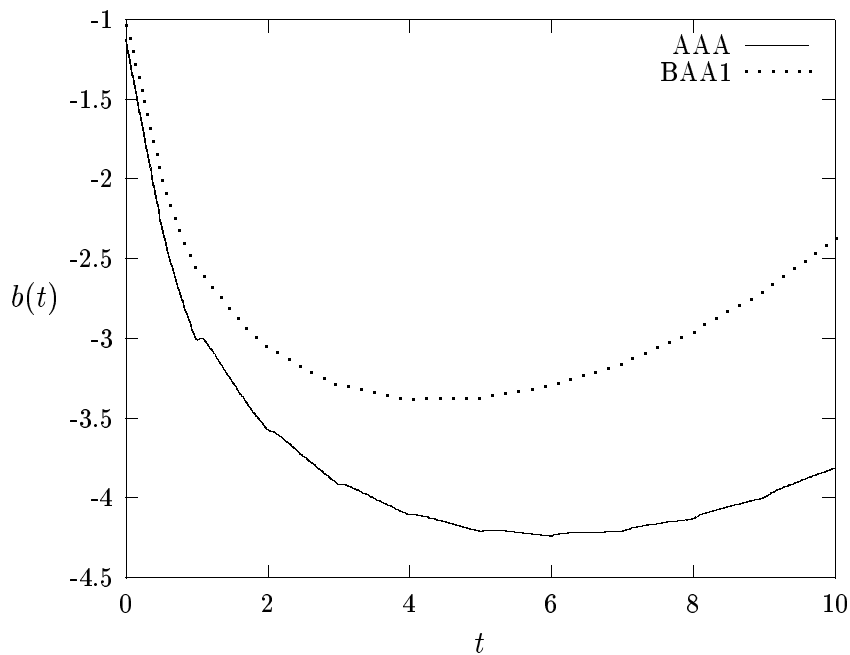
year	AAA			BAA1
	expected recovery rate			
	30%	50%	70%	50%
1	0.0052	0.0073	0.0122	0.0222
2	0.0097	0.0136	0.0227	0.0285
3	0.0119	0.0166	0.0277	0.0315
4	0.0135	0.0190	0.0316	0.0339
5	0.0150	0.0210	0.0351	0.0360
6	0.0164	0.0229	0.0382	0.0380
7	0.0176	0.0246	0.0410	0.0396
8	0.0188	0.0264	0.0439	0.0415
9	0.0203	0.0284	0.0473	0.0437
10	0.0220	0.0307	0.0512	0.0466

discretization and numerical parameters. The results are shown in Figure 5, where default probabilities assuming an expected recovery rate of 50% are used. In regard to the shape and location of the barriers, the main difference between the models is that in Hull-White paths are allowed to exit from the barrier only at a discrete times, whereas the paths can exit any time in the PDE model. This explains the fact that the barrier from the PDE model lies slightly below the Hull-White barrier.

Barriers for default probabilities with other expected recovery rates can also be computed. In Figure 6, barriers corresponding to default probabilities for AAA banks listed in Table 1 for recovery rates of 30%, 50% and 70% are plotted. As expected, since default probabilities for a lower recovery rate are smaller than the corresponding default probabilities for a higher recovery rate, the barrier for this low recovery rate will lie below a barrier with a higher recovery rate.

In general, default probability data is discrete, as shown in Table 1. Since the PDE method requires interpolation of the probabilities, it is important to verify that different interpolation methods for the default probability density function do not produce significant changes in the barriers generated by the model. In all of the above calculations, we used a piecewise constant default probability density  $P'(t)$  computed in a straightforward way from cumulative default probability data. To

Figure 3: Default Boundaries for AAA and BAA1 companies



study the sensitivity to different interpolation schemes, we considered a piecewise linear interpolation scheme for  $P'(t)$ , requiring that  $P(t)$  generated be consistent with the data at the original data points. In Figure 7, we plot the barriers resulted from these two default probability densities. The numerical results indicate that the scheme is stable with respect to small perturbations of the probability density function representing the data.

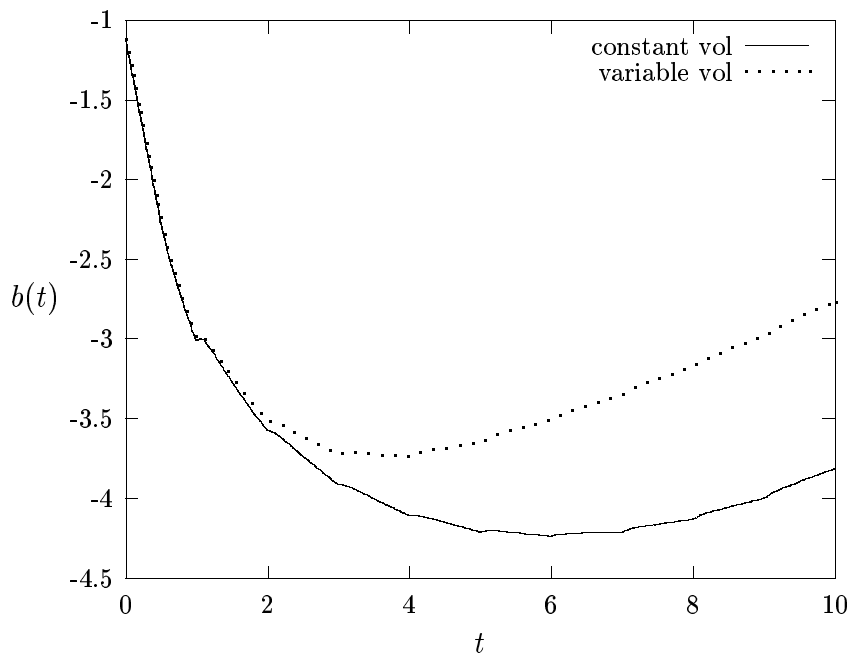
In Figure 8, we display the default probability density (input) and the drift function  $b'(t)$  (output) for the case of AAA-rated banks. This graph shows qualitatively the way in which the drift “responds” to the default probability density data: an increase in the default probability will certainly lead to an increase in the drift, which moves the barrier up, making the paths more likely to exit the barrier.

Once the default barrier for a firm has been computed for a period ( $0 \leq t \leq T$ ), it is possible to calculate *forward default probabilities* at certain future time  $T_0 > 0$  as well using this model. In fact, we just need to solve the PDE starting from  $T_0$  with the default barrier fixed, and start the survival density from  $T_0$

$$u(x, T_0) = \delta(x - X_0), \quad (33)$$

so the initial distance-to-default at  $T_0$  is the same as today. As we discussed the

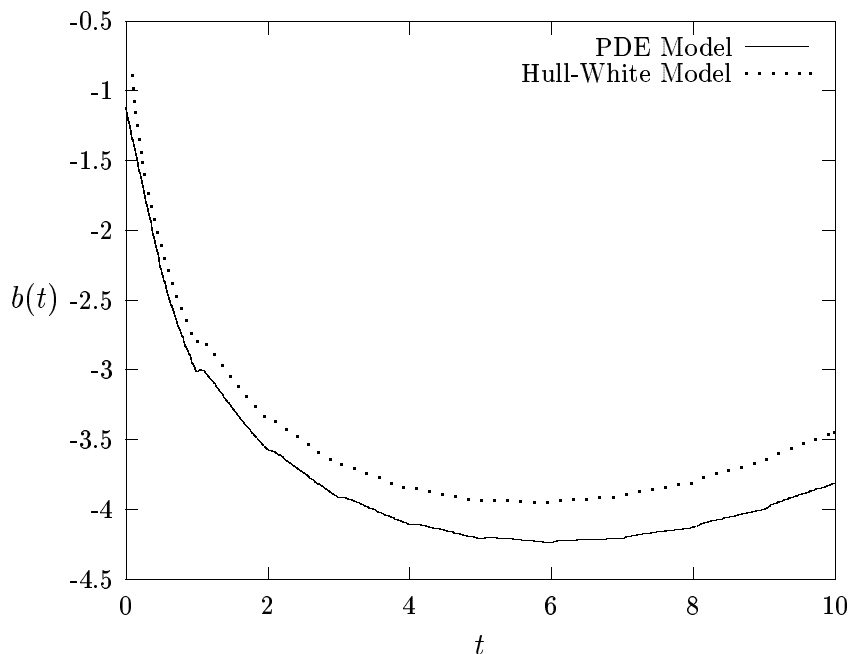
Figure 4: Barriers from Different Volatility Structures



equivalence between the drift  $a(t)$  and the default boundary  $b(t)$  in sections 3 and 4, here they will have different roles to play. If there is enough information available, it is possible to fit the drift to a forward default probability structure. In Table 2, we present the 5 year forward default probabilities from the results in Figure 3. Intuitively speaking, these are the default probabilities for the next 6 to 10 years, given that the firm has survived the first 5 years and the default probability for the next instant is the same as today. It is observed that the forward default probabilities are much larger than the spot probabilities, due to the fact that the shape of our barrier function is concave upward.

Finally, as a verification of the numerical scheme, it is mathematically interesting to study the case where the default probability reaches a level where a default is certain to happen by certain time  $T$ , as predicted by the input default probability. This should be reflected in the fact that the default boundary will be exited before this particular time  $T$  by virtually all Brownian paths  $X(t)$ , which can only happen when  $b'(t)$  blows up at this time and the curve  $b(t)$  becomes “vertical” as  $t$  approaches  $T$ . To verify this, we choose a uniform default probability density  $P'(t) = 0.1$ . In this case, the cumulative default probability  $P(10) = 1$ , which means that the firm will necessarily default before  $T = 10$  with probability one. In Figure 9, we see that the

Figure 5: Comparison with Hull-White Model



barrier function  $b(t)$  indeed becomes vertical as  $t$  approaches 10. Also shown in the same figure is the result of another experiment where the default probability density is increased to 0.2, where the blow-up time is pushed to approximately  $T = 5$ , as predicted from the fact that  $P(t)$  reaches 1 as  $t$  approaches 5.

Figure 6: Default Boundaries for Different Expected Recovery Rates

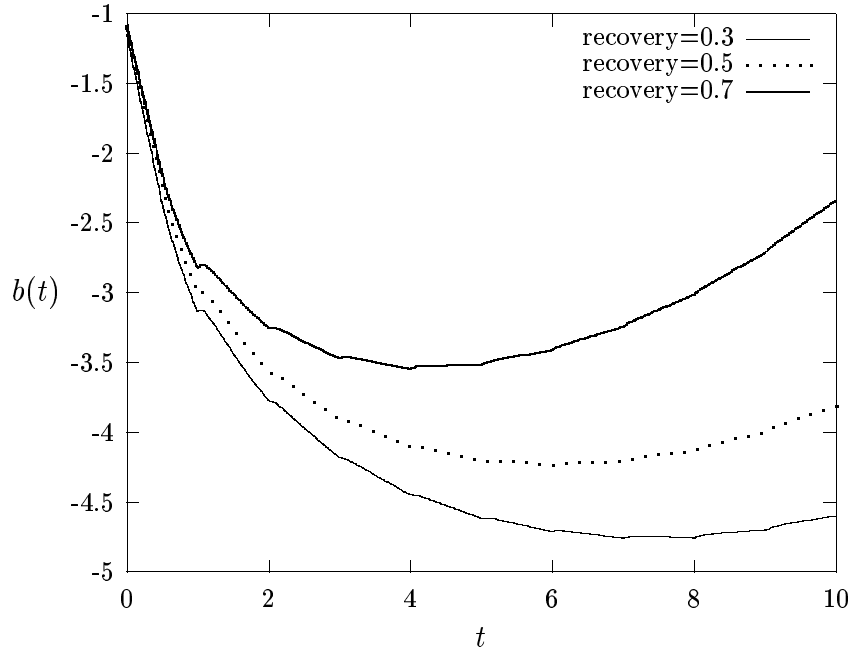


Table 2: Forward Default Probabilities

year	AAA	BAA1
1	0.244002	0.301495
2	0.165700	0.182113
3	0.096906	0.103299
4	0.068161	0.071312
5	0.053285	0.055123

Figure 7: Different Interpolation Methods for Default Probabilities

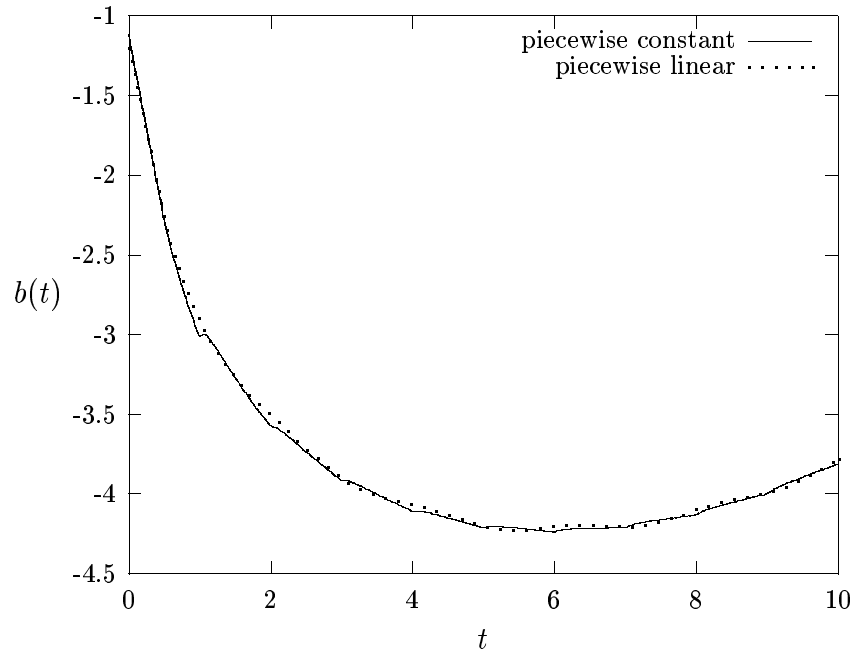
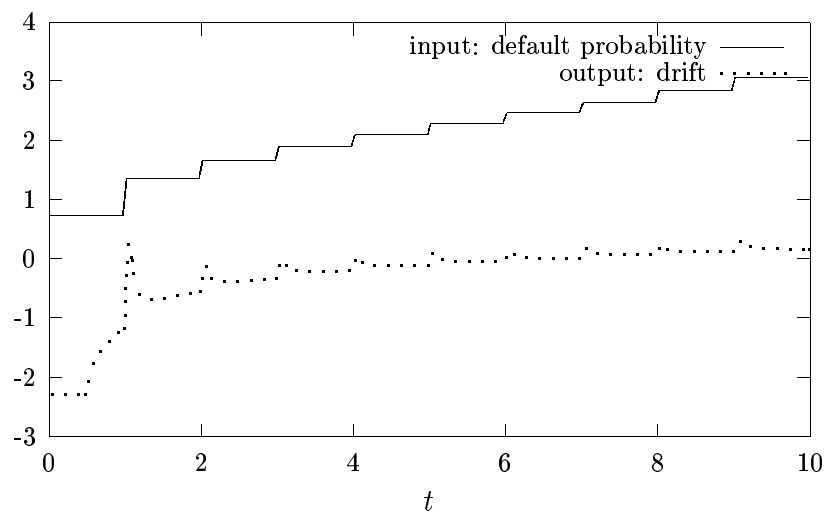


Figure 8: Default Probability and Corresponding Drift

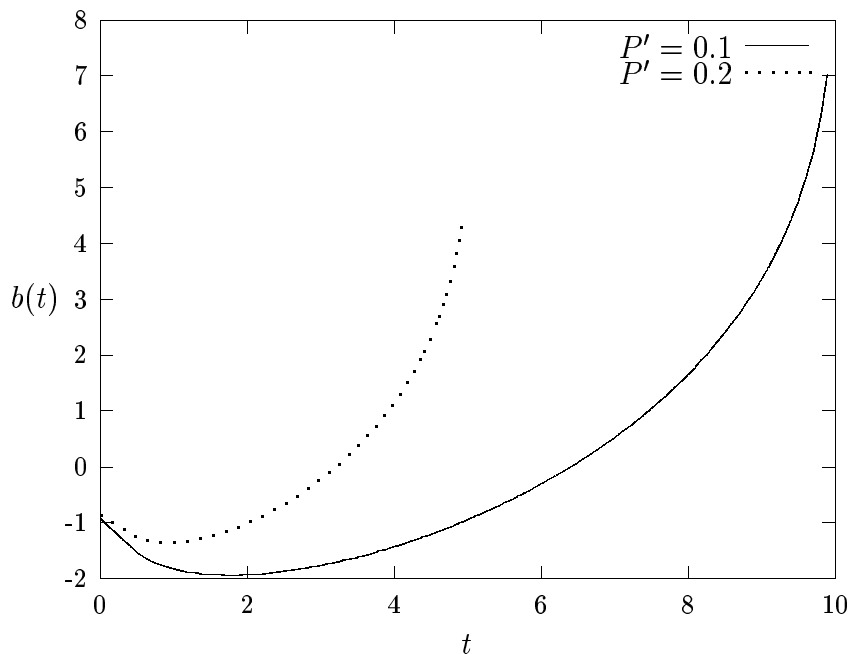


## 8 Conclusions

Generalizing the Hull-White model [8] to continuous-time default index models, we propose a general framework for modeling default indices as diffusions and default events as first-passages across barriers that generalizes the Hull-White discrete model based on a discrete random walk. We show that the calibration of such continuous-time default index models to default probability data leads to a free-boundary problem for the corresponding Fokker-Planck equation. We also established an isomorphism between the default index formulation of Hull and White and the concept of a risk-neutral distance-to-default (RNDD) index. This isomorphism allows us to reinterpret the derivative of the Hull-White default boundary as an “market price of risk” that has to be added to the distance-to-default process of the firm to make it consistent with observed default probabilities extracted from bond spreads or credit ratings. We proposed a simple numerical algorithm for finding the unknown drift, based on a discretization of a control problem. Several examples and tests were presented, indicating that the algorithm produces reasonable results and is stable with respect to small perturbations of the input probability densities.

Finally, we point out that it is also possible to construct “non-parametric” models

Figure 9: Blow Up of Default Boundary



that implement the concepts of risk-neutral default index and RNDD. These models would be based on fitting the first-passage times of random paths across a barrier to given default probabilities. For example, a Monte Carlo simulation of different scenarios for the distance-to-default of a firm can be generated using econometric data on the volatility of the firm. In a second step, the probabilities of the different scenarios can be appropriately re-calibrated so as to reflect contemporaneous data on cumulative default probabilities, as in the Weighted Monte Carlo method [2].

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