The University of Reading

THE BUSINESS SCHOOL
FOR FINANCIAL MARKETS

On Modelling Credit Risk Using Arbitrage Free Models*

Frank S. Skinner
Antonio Díaz

June 12, 1998
Revised November 1998, April 1999, March 2000
This Version, July 2001

The University of Reading • ISMA Centre • Whiteknights • PO Box 242 • Reading RG6 6BA • UK
Tel: +44 (0)118 931 8239 • Fax: +44 (0)118 931 4741
Email: research@ismacentre.rdg.ac.uk • Web: www.ismacentre.rdg.ac.uk

Abstract

By examining the distribution of state prices obtained from binomial versions of Jarrow and Turnbull (1995), Lando (1998) and Duffie and Singleton (1999), we are able to suggest which credit risk parameters are of critical interest. We find that it appears worthwhile to parameterize credit risk since even the simplest parameterized model obtains large changes in the distribution of state prices when compared to a non-parameterized model. Similarly we find large differences in the distribution of state prices as we add correlation and moderate changes as we add time varying recovery rates. Finally, the choice between the RM or RF recovery assumption appears innocuous, but the choice between RT and these two recovery assumptions is not.

JEL Classification: *G13*

Key Words: *Credit risk, credit derivatives, binomial lattice, Arbitrage free pricing*

[*] Associate Professor, (corresponding author) ISMA Centre, University of Reading, Whiteknights Park, Box 242, Reading, RG6 6BA, UK. E-mail f.skinner@ismacentre.reading.ac.uk. Tel 011 44 118 931-6407 and Associate Professor Departamento de Economía y Empresa, Universidad de Castilla - La Mancha, Spain respectively.

We would like to thank Peter Abken, Harold Bierman, Salih Neftci, Philipp Schönbucher, Michalis Ioannides, Nicolas Papageorgiou and Stavros Peristiani for many comments. We also thank Fundació Caixa Castelló-Bancaixa for financial support for Antonio's stay as visiting scholar at ISMA while he was Assistant Professor, Departamento de Finanzas y Contabilidad, Universitat Jaume I, Spain. The authors are responsible for any errors. Comments welcome.

On Modelling Credit Risk using Arbitrage Free Models

In recent years we have learned much about modelling term structures subject to credit risk. However, there is little empirical work regarding such basic issues as the relative importance of different parameters describing credit risk and whether correlation between credit risk and pure (default free) interest rates really matters. Indeed, we do not even know if we should not parameterize credit risk at all and instead apply pure interest rate modelling methods directly to interest rates subject to credit risk. In this paper, we attempt to address these basic questions by studying variations of the Jarrow and Turnbull (1995), Lando (1998) and Duffie and Singleton (1999) models.

We address these questions by implementing binomial stochastic processes for pure rates of interest and credit risk in an arbitrage free framework. The resulting models yield corporate state prices that can be used to price credit derivatives. In fact, *precisely* the same credit derivative price is obtained by either solving backwards using the binomial structure of corporate interest rates or multiplying promised cash flows by corporate state prices.

Since the whole point of modelling credit risk is to obtain accurate prices we obtain a diagnostic check concerning the behaviour of credit risk models. Specifically we know that the structure of state prices at each point in time are forced to agree with corporate zero prices that underlie the corporate term structure. That is, each corporate state price *promises* to pay €1 only if a particular corporate interest rate state at time t occurs, and pays nothing otherwise. Forming a portfolio of all possible corporate state securities at a date in the future we obtain a portfolio that promises to pay €1 no matter what corporate state occurs at time t, and pays nothing at all other dates. This is our replicating portfolio as it replicates the payoff of a corporate zero coupon bond.

All credit risk models are adjusted until the replicating portfolios of corporate state security prices agrees with the implied zero coupon prices exogenously supplied by estimates of the corporate zero coupon term structure. Therefore different models impose different parameterisation schemes that force different distributions of state prices. While all models will generate replicating portfolios of state security prices that will agree with an exogenously supplied corporate term structure, different *distributions* of state prices imply *different* prices for derivatives. Then we can suggest which parameters seem critical. That is if by including, say correlation between credit risk and pure rates of interest the distribution of state prices remains the same then we can suggest that correlation does not matter as both models will obtain the same derivative price. On the other hand we may observe that adding negative correlation changes the distribution of state prices such that at low corporate interest rate states, state prices are higher, but at high corporate interest rates states, they are lower. Then we would be able to suggest that not only can adding correlation in our models be important, but that by neglecting negative correlation we would under price credit risky call options and over price credit risky put options. In this way we can judge the relative importance of different parameters for credit risk and the consequences of different methods of modelling the default process.

We obtain five results. First we find that applying even the simplest parameterised default process, one with constant recovery and zero correlation with pure rates of interest, we obtain large changes in the distribution of state prices when compared to the distribution obtained when we model the short corporate interest rate without parameterizing the default process. This suggests that we can obtain large improvements in modelling credit risk if we can parameterize the default process correctly.

Second, we find that by adding correlation to the previously uncorrelated default process obtains large changes in the distribution of state prices. Furthermore, these differences are proportional to the size of correlation. This suggests that the correlation between credit risk and pure rates of interest should be important.

Third, while we agree with Duffie and Singleton (1999) that the return of market value (RM) and return of face value (RF) recovery assumptions makes little difference, we find that the return of Treasury (RT) recovery assumption obtains large differences in the distribution of state prices. Both the RM (and RF) value recovery assumption share the same strong theoretical basis, so the cost associated with using the analytically convenient RM (or RF) recovery assumption can only be assessed by further empirical study.

Fourth, we find that allowing for a time varying recovery rate, modest differences in the distribution of state prices are obtained. This suggests that allowing for different recovery rates at different future points in time may improve the accuracy of prices provided that we can obtain accurate estimates of future recovery rates. However the potential for improvement in accuracy of pricing appears to be modest. Nevertheless, for those credit derivatives whose payoffs are conditional upon default, for example credit default swaps, a time varying recovery rate may prove to be of vital interest.

Fifth, the importance of the above findings is directly related to the size of the credit spread. This suggests that if there is little credit risk, there is little point to parameterizing credit risk.

In Section I we manipulate the credit risk pricing framework introduced by Duffie and Singleton (1999) to obtain Jarrow and Turnbull (1995) and a binomial version of Lando (1998) by adding correlation and time varying recovery amounts to Jarrow and Turnbull (1995). By adjusting the recovery assumption to the RM and then the RF recovery assumption of Duffie and Singleton (1999) we obtain two binomial versions of their model. This allows us to compare Jarrow and Turnbull (1995), two binomial versions of Lando (1998) and two binomial variations of the Duffie and Singleton model. In section II we describe our empirical procedures and Section III reports our empirical results. Section IV summarises and concludes.

I Obtaining Nested Models

Under the risk-neutral probability measure Q conditional upon information available up to date t, Duffie and Singleton (1999) show that the price of a one period defaultable zero is written as,

$$V_t = E_t^Q\left[h_t e^{-r_t}\omega_{t+1} + (1-h_t)e^{-r_t}V_{t+1}\right] \quad (1)$$

Note that h_t is the conditional (upon no prior default) hazard probability and r_t is the pure interest rate at time t. Meanwhile ω_{t+1} is the recovery rate and V_{t+1} is the promised payoff of €1 at maturity t+1. In other words a defaultable zero promises to pay V_{t+1} at maturity t+1, but the promise may be broken at hazard rate h_t. If default occurs with hazard rate h_t at time t, an amount ω_{t+1} is paid at time t+1, conditional upon no prior default. Then, under the risk-neutral probability measure Q, these future expected cash flows are discounted by the pure rate of interest.

The above is a general expression for the value of a one period defaultable zero. Nested within it are the Jarrow and Turnbull (1995), Lando (1998) and Duffie and Singleton (1999) models. To highlight the differences among these models and the challenges confronted when modelling credit risk, we re-write (1) in state price format in the case of a two period corporate zero. We assume the existence of an equivalent martingale measure. [1] We also allow for time varying, but non-stochastic recovery rates.

$$\begin{aligned} V_t = \{&h(t,j)(\omega_{t+1})0.5[e^{r(t+1,i+1)} + e^{r(t+1,i)}] + 0.5[h(t+1,j+1) + h(t+1,j)][1-h(t,j)]\omega_{t+2} \\ &+ [1-0.5\{h(t+1,j+1)+h(t+1,j)\}][1-h(t,j)]V_{t+2}\}e^{-r(t,i)}0.5[e^{-r(t+1,i+1)} + e^{-r(t+1,i)}] \end{aligned} \quad (2)$$

The above expression says that a corporate zero may default during the first period with hazard rate h(t,j) and recover ω_{t+1} at the end of the first period. The amount is reinvested in a Treasury security to earn the evolving stochastic Treasury interest rate until promised maturity. If the corporate zero survives the first period with probability [1-h(t,j)], it may default at maturity in a high credit risk (high hazard rate) state with hazard rate h(t+1,j+1), or it may default at maturity in a low credit risk (low hazard rate) state with hazard rate h(t+1,j). If the corporate zero defaults during the second period, investors recover ω_{t+2} at maturity, which may be different than ω_{t+1}. The corporate zero pays the promised €1 (V_{t+2}) at maturity conditional upon survival for both periods. All potential cash flows, both the terminal payoff and recovery amounts, are discounted back to the present using binomial stochastic pure interest rates.

Note that (2) explicitly recognizes that prior to maturity recovery amounts ω are reinvested in a Treasury security for the remaining maturity of the zero. This is necessary as (1) and (2) implicitly assume that the investor is choosing (under the equivalent martingale measure) between a credit riskless and credit risky zero, and this reinvestment assumption ensures that the time horizon of the alternatives are consistent. In turn, this reinvestment assumption leads to the RT assumption, for now recovery amounts are expressed as a fraction of a two-period Treasury zero. In other words, the RT has a strong theoretic basis, for without it we would be modeling investor's choice between a Treasury and a corporate zero, where the corporate zero may have a different actual maturity depending upon the realization of default prior to scheduled maturity.

Two challenges are evident in (2). First, what is the relationship between hazard probabilities h(t,j) that evolve in credit risk state j and pure rates of interest r(t, i) that evolve in interest rate state i? Second, hazard probabilities are conditional probabilities in that in order to default at t_2, the bond must survive t_1. This means that in all possible credit risky states j and interest rate states i, one must measure expected conditional payoffs in the event of default under the equivalent martingale measure for not only the current period, but also all possible prior periods. This requires a considerable computing effort. While all three models discussed here deal with both challenges, Jarrow and Turnbull (1995) and Lando (1998) focus on the first challenge while Duffie and Singleton (1999) focus on the second.

It is tempting to solve the first challenge by "brute force", that is calculate all possible hazard and pure interest state prices for all time periods. However this would be computationally expensive. The number of pure interest rate states will equal t+1, and the number of hazard states will be t+1 and all possible combinations will be $(t+1)^2$. Consequently, Jarrow and Turnbull (1995) and Lando (1998) impose distributional assumptions regarding the relationship between hazard and pure interest rates.

We now obtain a binomial version of the Lando (1998) model from (2). Cases of this result will be Jarrow and Turnbull (1995) and versions of Duffie and Singleton (1999). By examining how the binomial version of Lando (1998) is transformed as we add assumptions will illustrate how Jarrow and Turnbull (1995) and Duffie and Singleton (1999) have met the above challenges. To obtain a binomial version of Lando (1998) we suggest the following binomial stochastic process for the pure rate of interest and the hazard rate.

$$r(t,i) = r(0,0)e^{\left(u_t \Delta T + (2i - t)\sigma_r \sqrt{\Delta T}\right)} \quad (3)$$

$$h(t,j) = h(t,i) = h(0,0)e^{\left(v_t \Delta T + \rho_{h,r} \frac{\sigma_h}{\sigma_r} r(t,i)\Delta T\right)} \quad (4)$$

The first binomial stochastic process is the familiar multiplicative model, where r(t,i) refers to the pure interest rate that evolves in state i and time t, r(0,0) is the current observable short term pure interest rate, u_t is a time dependent parameter that calibrates the interest rate tree by forward induction through use of state prices to the sovereign zero yield curve, ΔT is the time step and σ_r is an estimate of interest rate volatility. Note that when t = 0, then r(t,i) is defined to be r(0,0). This process was chosen since it prevents negative pure interest rates, and it is of simple form. Any other single factor pure interest rate arbitrage free model can be easily accommodated. Imposing a more realistic pure interest rate process will simply obscure the analysis of this section.[2]

The second binomial stochastic process describes the evolution of the one period hazard rate and the joint probability distribution between r (t, i) and h (t, j). Through covariance between the pure rate of interest and the hazard rate, correlation $\rho_{h,\,r}$ between these parameters is included. This covariance is scaled by the pure interest rate variance leading to a multiplicative term that models the volatility of hazard rates as the responsiveness of hazard rates to the current pure rate of interest.[3] Of course this means (4) generates a recombining hazard rate process since (3) is a recombining process. In (4) the time dependent parameter v_t calibrates the hazard rate tree by forward induction through use of corporate state prices to the corporate zero yield curve and σ_h is the constant hazard rate volatility parameter. Note that when t = 0, then h(t,j) is defined to be h(0,0).

Together the binomial processes (3) and (4) form a model similar to Das and Tuffano (1996) in that we assume a linear scaling of cash flows. By applying the law of iterated expectations, the two binomial trees (3) and (4) are combined to calculate corporate state prices which forms a single binomial tree. Procedurally we first calibrate the pure interest rate process at today's date t=0 to the sovereign zero yield curve by adjusting the calibration factor u_t for all future dates. This obtains the pure interest binomial tree the values of which [r (t, i)] are included in the hazard rate process (4). We then calibrate the hazard rate process, which is correlated with the pure interest rate process generated by the first calibration, to a corporate zero yield curve by adjusting the calibration factor v_t. Simultaneously this calibration adjusts the structure of corporate state securities until the yield implied by this replicating portfolio of corporate zeros agrees with our estimate of the corporate zero yield curve.

Substituting (3) and (4) into (2) and rewriting slightly to highlight the RT assumption we obtain a binomial version of Lando (1998). The result is as follows.

$$V_0 = e^{-r(0,0)}\,0.5[e^{-r(0,0)e(u_1\Delta T+\sigma_r\sqrt{T})} + e^{-r(0,0)e(u_1\Delta T-\sigma_r\sqrt{T})}]$$
$$\text{x}\left\{\begin{matrix}\left[1 - 0.5\left(h(0,0)e(v_1\Delta T+\rho_{h,r}\dfrac{\sigma_h}{\sigma_r}r(1,1)\Delta T + h(0,0)e(v_1\Delta T+\rho_{h,r}\dfrac{\sigma_h}{\sigma_r}r(1,0)\Delta T\right)\right] \\ \left[1-h(0,0)\right]V_2 + \delta_2\end{matrix}\right\} \quad (5)$$

where

$$\delta_2 = h(0,0)\omega_1\left[0.5\left(e^{r(0,0)(u_1\Delta T+\sigma_r\sqrt{T}} + e^{r(0,0)(u_1\Delta T-\sigma_r\sqrt{T}}\right)\right]+$$
$$0.5\left[h(0,0)e^{v_1\Delta T+\rho_{h,r}\frac{\sigma_h}{\sigma_r}r(1,1)\Delta T} + h(0,0)e^{v_1\Delta T+\rho_{h,r}\frac{\sigma_h}{\sigma_r}r(1,1)\Delta T}\right]\omega_2\left(1-h(0,0)\right) \quad (5a)$$

Equation (5) is a binomial version of Lando (1998). Like Lando (1998) this model allows for correlation of any type and for time varying recovery rates. Equation (5a) is the RT recovery assumption where conditional upon no prior default in any prior time and pure interest rate state, an (possibly time varying) amount ω is paid at the end of the current period. Should default occur at any prior time period, the recovery amount is reinvested in a Treasury security until promised maturity. The sum of these recovery amounts is then included in (5) and therefore these recovery amounts are expressed as a fraction of the value of a two-period Treasury zero.

By restricting parameters to particular forms we can obtain other models. If correlation is zero and recovery rates are constant, $\omega_1 = \omega_2 = \omega$, then (5) becomes Jarrow and Turnbull (1995). Specifically, (5) is transformed to

$$V_0 = e^{-r(0,0)}\,0.5[e^{-r(0,0)e(u_1\Delta T+\sigma_r\sqrt{T})} + e^{-r(0,0)e(u_1\Delta T-\sigma_r\sqrt{T})}]$$
$$\text{x}\,\{[1-(h(0,0)e(v_1\Delta T))][1-h(0,0)]V_2 + \delta_2\} \qquad (6)$$

Where

$$\delta_2 = h(0,0)\omega_1\left[0.5\left(e^{r(0,0)(u_1\Delta T+\sigma_r\sqrt{T}} + e^{r(0,0)(u_1\Delta T-\sigma_r\sqrt{T}}\right)\right] + \left[h(0,0)e^{v_1\Delta T}\right]\omega_2\left(1-h(0,0)\right) \qquad (6a)$$

Note that the RT assumption remains intact. Equation (6) is considerably easier to implement than (5). We calculate just one h(t,j) for each time period. Since h(t,j) is equally likely at each interest rate state r(t,i), we add the sum of expected promised values under h(t,j) at time t and then present value this sum at each possible interest rate r(t,i) at time t. We continue to do this rolling backwards through the corporate price tree.[4] In contrast, to implement (5) we need to calculate two binomial trees, one for r(t,i) and another for h(t,j), and then combine them to form h(t,i). We then present value each expected value under h(t,i) by the corresponding state contingent pure interest rate r(t,i) at date t. We continue to do this rolling backwards through the corporate price tree.

If we adjust (5a) to conform to the return of market or to the return of face value assumptions suggested by Duffie and Singleton (1999) we obtain two binomial variations of their model. Specifically, the Duffie and Singleton (1999) binomial version of (5) becomes

$$V_0 = e^{-r(0,0)}\,0.5[e^{-r(0,0)e(u_1\Delta T+\sigma_r\sqrt{T})} + e^{-r(0,0)e(u_1\Delta T-\sigma_r\sqrt{T})}]$$
$$\left\{1-\left(h(0,0)+0.5\begin{pmatrix} h(0,0)e(v_1\Delta T+\rho_{h,r}\dfrac{\sigma_h}{\sigma_r}r(1,1)\Delta T \\ +\,h(0,0)e(v_1\Delta T+\rho_{h,r}\dfrac{\sigma_h}{\sigma_r}r(1,0)\Delta T)\end{pmatrix}\right)L_{t+1}\right\}V_2 \qquad (7)$$

In (7), L_{t+1} is the end of period loss rate that is equal to the following expression under the RM recovery assumption.

$$L_{t+1} = 1 - \left\{ 1 - \left(h(0,0) + 0.5 \begin{pmatrix} h(0,0)e(v_1\Delta T + \rho_{h,r}\frac{\sigma_h}{\sigma_r} r(1,1)\Delta T) + \\ h(0,0)e(v_1\Delta T + \rho_{h,r}\frac{\sigma_h}{\sigma_r} r(1,0)\Delta T) \end{pmatrix} \right) \right\} \omega_{t+1} V_{t+1} \tag{7a}$$

In other words, (7a) says that upon default the investor loses an amount L_t. This amount is one minus the recovery amount. In turn this recovery amount is a fraction ω of the end of period survival contingent value of a €1 face value zero. Under the RF assumption the fractional loss L_t is simply $(1-\omega_t V_t)$. The advantage of this formulation is that if recoveries are fractions of survival contingent values (RM) or fractions of promised values (RF), then values associated with prior period defaults are included in (7) as a *multiplicative* term. The facilitates forward induction, allowing us to apply pure interest rate modelling techniques directly to a corporate short rate of interest without the extra computational complexity of adding values associated with prior period defaults to each corporate state price. In contrast, (5a) and (6a) show that to implement the RT recovery method, one needs to make a separate *additive* calculation of expected conditional payoffs in (5) and (6) respectively in the event of default for all possible prior periods at each pure interest rate state.

Notice that the RM recovery assumption models recoveries are fractions of survival contingent values in (7a), whereas the RT recovery assumption models recoveries in (5) as explicit fractions of Treasury zeros. Otherwise they both assume that recovery amounts are reinvested in Treasury securities. This means that whether we use the RM (and RF) or RT, we still assume investor's face the choice to invest in Treasury or corporate zeros and the time horizon of the alternatives is consistent. In other words, both the RM (and RF) and RT recovery assumptions share the same theoretical basis as they are all incorporated in the basic pricing equation (2).

II Empirical Procedures

We would rather use actual term structures based on accurate bond prices to implement the above models rather than generate arbitrary data. In this way we can generate information about how these models behave under market conditions.

We use the University of Houston's Fixed Income Data Base. This data base consists of monthly over the counter information on most publicly traded bonds since 1973. Each issue is identified by cusip number and includes information on industry sector, issue date, maturity date, flat price (noted as quote or matrix based), coupon, accrued interest, bond rating, call and put features.

We select all US Treasuries and AA rated financial industry bonds that were quoted rather than matrix priced on June 30, 1988. Sarig and Warga (1989) and Warga (1991) note that quoted over the counter bond prices are much more accurate than exchange prices and prices obtained by a matrix-based approach. We chose bonds that have no call or put features. We select Treasury and Banker Acceptance interest rates as our shorter term interest rates. Bankers Acceptance rates are discount yields applicable for high credit quality financial institutions. Finally, we chose June 30, 1988 because we know from Standard and Poors that the average recovery rate during the subsequent recession in 1990 was approximately 32.5%. Assuming rational expectations, we use this figure as our estimate of the recovery fraction.

We then apply Vasicek and Fong (1982) without call adjustments to estimate the Treasury and AA financial yield curves up to ten years maturity.[5] We chose to estimate an AA yield curve because we are confident that the Fixed Income Data Base contains a large pool of AA straight (no optionality) bonds with quoted prices. We chose only financial industry bonds since we wish to construct a credit risky yield curve that has comparable credit risk all along the yield curve. Hickman (1958) notes that in general we cannot expect that, say an AA utility bond has the same credit risk as say an AA financial, so we at least assure ourselves that our yield curve is constructed from bonds in the same industry. Since the financial industry is most directly involved in credit derivatives, we thought this would be the most interesting yield curve to use. We estimate these yield curves up to ten years maturity because like Gruber and Green (1998), we found that beyond ten years the lack of on the run Treasury bonds between ten and thirty years to maturity created a liquidity problem.

The June 30, 1988 estimated Treasury and AA Financial yield curves are shown in Figure 1. Note that the credit spread is quite wide, 138 basis points on average, and the credit spread is fairly constant. To generate information about how the credit spread may affect the influence of credit risk parameters, we estimated the Treasury and AA financial yield curves on January 31, 1989 and again on April 30, 1990. These yield curves are reported in Figures 2 and 3. Notice that while the January 31, 1989 credit spread widens with maturity, the average spread is now much smaller at 81 basis points. Finally Figure 3 shows that credit spreads have narrowed to 42 basis points, and the credit spread is again fairly constant.

[Figures 1, 2 and 3 about here]

We then calibrate (5) to these yield curves using the procedures as described in section I. To correspond to our monthly data observations, we use a monthly time step for a total of 120 steps for each ten-year yield curve. We did this calibration sixty times to estimate Jarrow and Turnbull (1995), binomial versions of Lando (1998) and versions of Duffie and Singleton (1999). The base case uses June 30, 1988 yield curves, a constant recovery rate of 32.5%, a constant pure interest rate volatility of 10% and a constant hazard volatility of 1%. As interest rate cap volatility is not available prior to 1990, our choices for pure interest and hazard rate volatility estimates are arbitrary. However, when employing a wide variety of alternative volatilities, essentially the same results as reported below are found.[6]

For Jarrow and Turnbull (1995) we set the correlation between pure interest and hazard rates to zero. We then estimate Lando (1998) four times assuming a constant hazard rate of 32.5%, but correlation of +1, +0.5, -0.5 and –1. We then re-estimate this set of correlated Lando (1998) models along with the zero correlation case using time varying recovery rates. Specifically we made up the following function.

$$\omega_t = \omega_1 e(-\beta t \Delta T)$$

In other words future recovery rates change with time as an exponential function of a parameter β. We first assume $\omega_1 = 0.7$ and then solve β such that the term structure of recovery rates average to 32.5%. This generates a downward sloping term structure of recovery rates. We then assume $\omega_1 = 0.115$ and then solve β such that the term structure of recovery rates average to 32.5%. This generates an upward sloping term structure of recovery rates that is basically symmetrical to the downward sloping term structure of recovery rates obtained earlier. These recovery term structures are reported in Figure 4.

[Figure 4 about here]

Next we impose the return of market and return of face value recovery assumptions suggested by Duffie and Singleton (1999) for +1, + 0.5, 0, -0.5 and –1 correlation and a constant recovery rate set. Finally we calibrate corporate interest rates to the corporate yield curve using (3) as our corporate interest rate process. This obtains estimates of corporate state prices in the same way we currently obtain pure interest state prices and so make no attempt to parameterize the credit risk process.

In summary we estimate Jarrow and Turnbull (1995) once, Lando (1998) fourteen times, Duffie and Singleton ten times and the non-parameterized credit risk model once, twenty-six estimates in all for June 3, 1988. We repeated the non-parameterised, Jarrow and Turnbull (1995) and the Lando (1998) constant and time varying recovery amount cases on January 31, 1989 and May 31, 1990 for a total of sixty estimates.

III Empirical Results

Now we plot differences in the corporate state prices generated by the various models at month 120. These state prices are today's value of hypothetical securities that *promises* pay €1 only if the corresponding corporate interest rate state occurs at month 120. Hence the y-axis report the differences in today's cash price per Euro for this hypothetical security. These corporate state prices are used to price credit derivatives so changes in the distribution of these corporate state prices caused by say, high correlation, imply that correlation may be important in modelling credit risk as adding correlation obtains different credit derivative prices.

To guide the reader through the next eleven figures, we summarise the results in Table 1. This table reports the maximum absolute differences in state prices found when comparing cases of the non-parameterized (NP), Jarrow and Turnbull (JT), Lando (L), the return of market (RM) and return of face value (RF) recovery versions of Duffie and Singleton (1999) and the return of Treasury (RT) recovery value case.

[Table one about here]

To aid comprehension of the following figures, we always plot corporate state price differences as the "strawman" model less the more complex model. This means that the distance from the x axis represent the bias created by using the strawman model rather than the more complex model, assuming of course that the more complex model is "better". Furthermore, since call options payoff in low corporate interest rate states, then graphs plotting above the x-axis at low corporate interest rate states means that calls are overpriced if we use the strawman model and it is incorrect. Similarly, graphs plotting above the x-axis at high interest rate states means that puts are overpriced.

Figure 5 shows that the non-parameterized corporate interest rate model generates corporate state prices that are substantially different than the corresponding Jarrow and Turnbull (1995) state prices. This suggests that even under the restrictive assumptions of zero correlation between credit risk and pure rates of interest and constant recovery rates, this parameterized credit risk model can achieve substantially more accurate derivative prices than those derived by a non-parameterized model. We observe that at low corporate interest rate states, the non-parameterized model leads to higher state prices. The opposite occurs at high interest rate states. In other words, if we believe that Jarrow and Turnbull (1995) is "more correct" than the non-parameterized model, then use of the non-parameterized model leads to overvalued call options and undervalued put options.

Notice that the bias in Figure 5 decreases as we move from January 31, 1988 when the credit spread averaged 138 basis points to April 30, 1990 when the credit spread averaged only 42 basis points. This suggests that the bias is related to the size of the credit spread rather than the shape of the credit spread. Furthermore, this suggests that if the credit spread is very narrow, there maybe little point to parameterizing credit risk since parameterized and non-parameterized credit risk models may well yield similar state prices.

[Figure 5 about here]

Figure 6 compares Jarrow and Turnbull (1995) as the strawman with a binomial version of Lando's (1998) extension of Jarrow and Turnbull (1995) for non-zero correlation but constant recovery rates. The bias is related to the sign and size of correlation. For positive correlation Jarrow and Turnbull (1995) generates higher corporate state prices at lower corporate interest rate states and lower corporate state prices at higher corporate interest rate states. In contrast, negative correlation will lead to the oppose result. Notice that the size of this bias is directly related to the size of correlation.

[Figure 6 about here]

Note that the size of the bias in Figure 6 is less than the size of the bias reported in Figure 5. However, the differences in Figure 6 are still large. Imagine we are pricing a vulnerable European put option when credit risk and pure rates of interest rate have a negative 0.5 correlation.[7] Since the put will pay off at high corporate interest rate states, then the maximum size of the price difference will be equal to the sum of price differences from corporate interest rate state 61 to state 120. This will be €0.0011 per dollar of future pay off or €0.11 per hundred. For perfect negative correlation, the difference will be roughly double. Now imagine we are pricing a ten-year vulnerable interest rate cap, which pays off semi-annually. As the size of the total difference will grow with more monthly time steps, we can roughly approximate the total price discrepancy using a linear approximation. This suggests that the cap may be over priced by €1.10 per hundred. While we may think that a 1.1-% bias is not very much, many investment banks hold positions in tens of millions, so the absolute numbers will have a substantial impact.

Figures 7 and 8 compares Lando (1998) with correlation but constant recovery rates as the strawman with a binomial version of Lando (1998) with correlation and time varying recovery rates. Figure 7 uses a recovery rate that increases with time and Figure 8 uses a recovery rate that decreases with time in the manner outlined earlier and reported in Figure 4. For the upward sloping term structure of recovery rates, we observe the same pattern to pricing bias as we observed in Figure 6. That is, for positive correlation call options are overpriced and put options are underpriced and for negative correlation calls are underpriced and puts are overpriced. Figure 7 reports that for decreasing recovery rates, the opposite occurs. Figures 7 and 8 report biases that are roughly 1/10th the size of Figure 6. This suggest that the potential improvement in pricing accuracy obtained by using a time varying recovery rate is more modest that the potential improvement obtained by adding correlation and a constant recovery rate to the Jarrow and Turnbull (1995) model. However, this improvement may still be important for large derivative positions in absolute cash value terms.

[Figures 7 and 8 about here]

Figure 9 compares Jarrow and Turnbull (1995) as the strawman with a binomial version of Lando (1998) that similarly assumes zero correlation, but with time varying recovery rates. The state price differences thereby obtained are trivial. This finding supports the view that time varying recovery rates are of secondary importance. That is, when a model that includes both non-zero correlation and time varying recovery rates are compared to a model that includes non-zero correlation but constant recovery rates (see Figures 7 and 8) we find modest changes in state prices. However Figure 9 shows that when both models have zero correlation, but different recovery assumptions only trivial differences in state prices are obtained.[8]

Nevertheless we note that this conclusion is based on changes in the distribution of state prices. For some securities, such as credit default swaps, a portion of the state price related to payments in the event of default forms a disproportionate part of the value of the security. The remainder (and usually much larger) portion of the state price related to survival contingent values have no direct influence on the value of default contingent payoffs. Furthermore this survival contingent value is more influenced by the correlation with pure rates of interest than the default contingent value simply because it forms a larger portion of the full state price. Hence it is possible that for default contingent credit derivatives like credit default swaps, the recovery assumption may prove to be important, yet correlation with pure rates of interest is less important.[9] This is precisely the opposite conclusion we reach when examining the distribution of the full state security price. This leads us to suggest that whether the correlation or recovery fraction is important for modelling credit derivatives really depends upon the task at hand.

[Figure 9 about here]

We replicated Figures 7 and 8 that includes the impact of time varying recovery rates and non-zero correlation, for the widening credit spread of January 31, 1989 and the narrow constant credit spread of April 30, 1990 in Figures 10, 11, 12 and 13 respectively. The main thing to notice here is that we obtain basically the same pattern of the price biases as reported in Figures 7 and 8, only that the size of these biases decrease with the size rather than the shape of the credit spread. In particular, notice that when the widening credit spread narrows to an average of 81 basis points on January 31, 1989, (see Figures 10 and 11) the price biases are smaller than those of Figures 7 and 8. Further, when the fairly constant credit spread further narrows to an average 42 basis points on April 30, 1990 (see Figures 12 and 13) the price biases decrease once again.

[Figures 10,11,12 and 13 about here]

Figure 14 compares the RM as the "strawman" with the RF recovery assumption when both versions have a constant recovery parameter. Notice that the RM and RF assumption obtains basically the same set of state prices no matter what level of correlation is used. The observed differences are approximately 1/100th of the modest differences in state prices found when we extend the binomial Lando (1998) model from constant to time varying recovery rates in Figure 7 and 8. In contrast, Figure 15 compares the RM as the "strawman" with the RT assumption when both versions have a constant recovery parameter. Now notice that the differences in state prices are much larger being in the same order of magnitude as found when we added correlation to the Jarrow and Turnbull model in Figure 6. In other words, the choice between the RT and RM (or RF) is not innocuous, as they will obtain different derivative prices.[10] As both sets of recovery assumptions share the same theoretical basis, it is not obvious which is the best to use.

[Figures 14 and 15 about here]

IV Summary and Conclusions

By examining the behaviour of state prices obtained from binomial versions of Jarrow and Turnbull (1995), Lando (1998) and Duffie and Singleton (1999), we are able to suggest which credit risk parameters are of critical interest and therefore which model may yield the most accurate prices. It appears worthwhile to parameterize credit risk since even the simplest parameterized model obtains large differences in state prices when compared to a non-parameterized model. While correlation between pure rates of interest and credit risk and time varying recovery rates both appear influential in determining state prices, correlation appears more influential than time varying recovery rates.

The latter conclusion is valid for all derivatives whose price is dependent upon both the survival and default contingent portions of the state price. However unlike, for example vulnerable options, credit default swap values depend more upon the default contingent portion of the state price so we may reach precisely the opposite conclusion, namely that correlation is less important than time varying recovery rates. This suggests that which of these parameters are the most important depends upon the task at hand.

The choice between the RM or RF recovery assumption appears innocuous, but the choice between RT and these two recovery assumptions is not. Since large differences in corporate state prices are obtained when using the RT as opposed to the RM (or RF) recovery assumption, but both share the same strong theoretic basis, it is not obvious which set of recovery assumptions is best. Only further empirical work can establish if it is worthwhile to sacrifice the analytical convenience of the RM or RF recovery assumption and instead use the less convenient RT recovery assumption. Finally, apparent differences in state prices obtained as we vary recovery assumptions and parameter estimates appear related to the size rather than the shape of the credit spread. This suggests that if there is little credit risk, say when examining US government agency yield curves, there is little point to parameterizing credit risk. But if credit risk is large, say that when examining emerging market sovereign yield curves, how we parameterize credit risk becomes a critical issue.

References

Das R.S. and P. Tufano, 1996, Pricing credit sensitive debt when interest rates, credit ratings and credit spreads are stochastic, *Journal of Financial Engineering* 5, 161-198

Duffie, D., and K. Singleton, 1999, Modeling term structures of defaultable bonds, *Review of Financial Studies*, 12, pp 687-720**.**

Duffee,G., 1999, Estimating the price of default risk, *Review of Financial Studies*, 12, 197-226.

Gruber, J. E. and C. Green, 1998 Tax and Liquidity Effects in Pricing Government Bonds, *The Journal of Finance* 53, 1533-1562.

Harrison, J. Michael and David M. Kreps, 1979 Martingales And Arbitrage In Multiperiod Securities Markets, *Journal of Economic Theory* 20, 381-408.

Hickman, W. B., 1958, *Corporate bond quality and investor experience.* Princeton University Press.

Jarrow, R, D. Lando and S. Turnbull, 1997, A Markov model for the term structure of credit spreads, *The Review of Financial Studies*, 10, 481-523.

Jarrow, R., and S. Turnbull, 1995, Pricing derivatives on financial securities subject to credit risk, *Journal of Finance* 50, 53-85.

Lando, D., 1998, “On Cox processes and credit risky securities, *Review of Derivatives Research* 2, 99-120.

McCulloch, J. Huston, 1975. The tax-adjusted yield curve, *Journal of Finance* 30, 811-830.

Nelson, Charles R. and Andrew F. Siegel, 1987. Parsimonious modeling of yield curves, *Journal of Business* 60, 473-490.

Sarig, Oded and Arthur Warga., 1989, Bond price data and bond market liquidity, *Journal of Financial and Quantitative Analysis*, v24(3), 367-378.

Vasicek, Oldrich A. and H. Gifford Fong, 1982. Term structure modeling using exponential splines, *Journal of Finance* 37, 339-348.

Warga, Arthur D., 1991. Corporate bond price discrepancies in the dealer and exchange markets, *Journal of Fixed Income* 1, 7-17.

Table 1

This table summarises Figures 5 though 15 by reporting the maximum absolute difference in corporate state prices found in each figure. NP refers to the non-parameterized, JT refers to Jarrow and Turnbull (1995) and L refers to our binomial version of the Lando (1998) model. RM, RF and RT refers to the use of the return of market value, return of face value and return of Treasury recovery assumptions.

Models:	Fig.	Maximum absolute values:					Cases:
NP – JT	#5	4.58e-04	2.22e-04	1.67e-04			♣, ♦, ♠
JT – L(ρ≠0,ω)♣	#6	2.52e-06	1.26e-06	1.26e-06	2.52e-06		ρ=1,0.5,-0.5,-1
L(ω) – L(↑ω_t) ♣	#7	2,62e-08	1.30e-08	6.31e-11	1.30e-08	2.59e-08	ρ=1,0.5,0,-0.5,-1
L(ω) – L(↓ω_t) ♣	#8	1,11e-07	5.56e-08	1.01e-10	5.56e-08	1.11e-07	ρ=1,0.5,0,-0.5,-1
JT – L(ρ=0,ω_t) ♣	#9	1.01e-10	6.31e-11				↑ω_t , ↓ω_t
L(ω) – L(↑ω_t)♦	#10	7.68e-09	3.73e-09	3.15e-10	3.84e-09	7.44e-09	ρ=1,0.5,0,-0.5,-1
L(ω) – L(↓ω_t)♦	#11	2.78e-08	1.39e-08	1.13e-11	1.39e-08	2.76e-08	ρ=1,0.5,0,-0.5,-1
L(ω) – L(↑ω_t) ♠	#12	4.38e-09	2.30e-09	1.79e-11	2.24e-09	4.57e-09	ρ=1,0.5,0,-0.5,-1
L(ω) – L(↓ω_t)♠	#13	2.16e-08	1.09e-08	3.83e-10	1.09e-08	2.18e-08	ρ=1,0.5,0,-0.5,-1
RM(ω) – RF(ω)♣	#14	5.68e-10	5.78e-10	2.72e-10	2.39e-10	3.37e-10	ρ=1,0.5,0,-0.5,-1
RM(ω) – RT(ω)♣	#15	2.17e-06	1.08e-06	6.20e-11	1.08e-06	2.16e-06	ρ=1,0.5,0,-0.5,-1

♣ June 30, 1988
♦ January 31, 1989
♠ April 30, 1990

Figure1
Term Structures of June 30, 1988

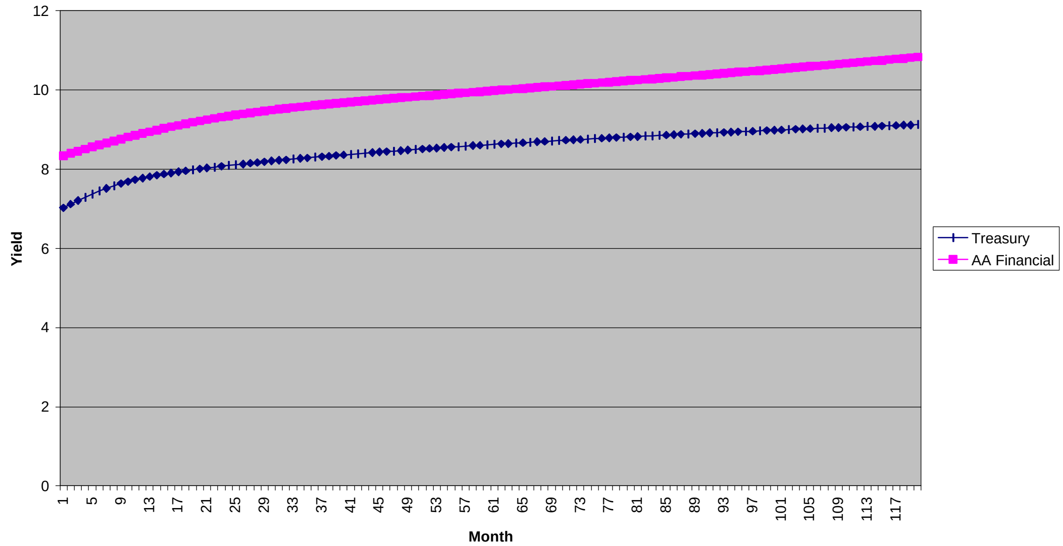


Figure 2
Term Structures of December 31, 1989

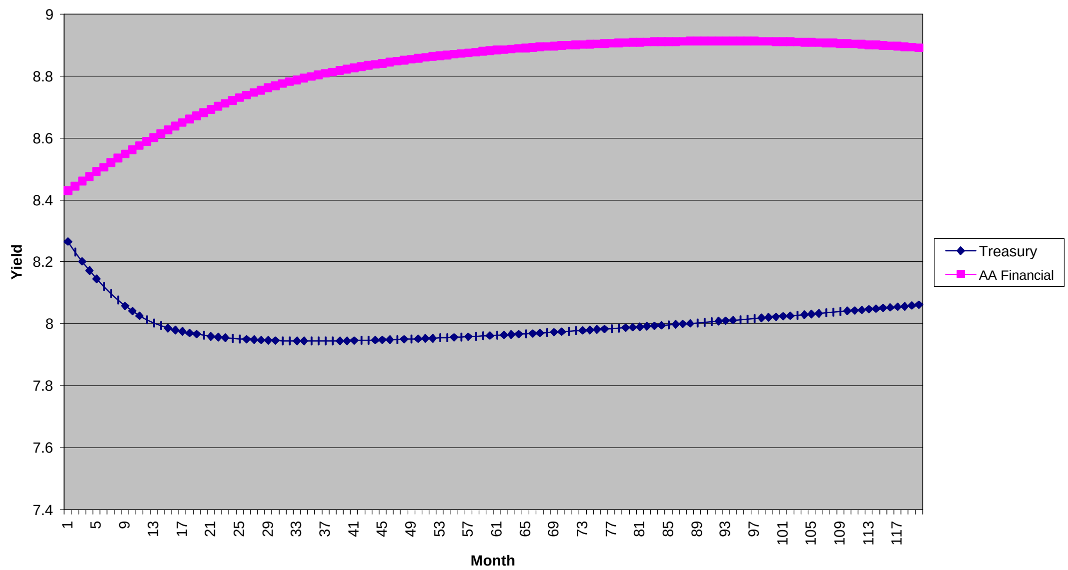


Figure 3
Term Structures of April 30, 1990

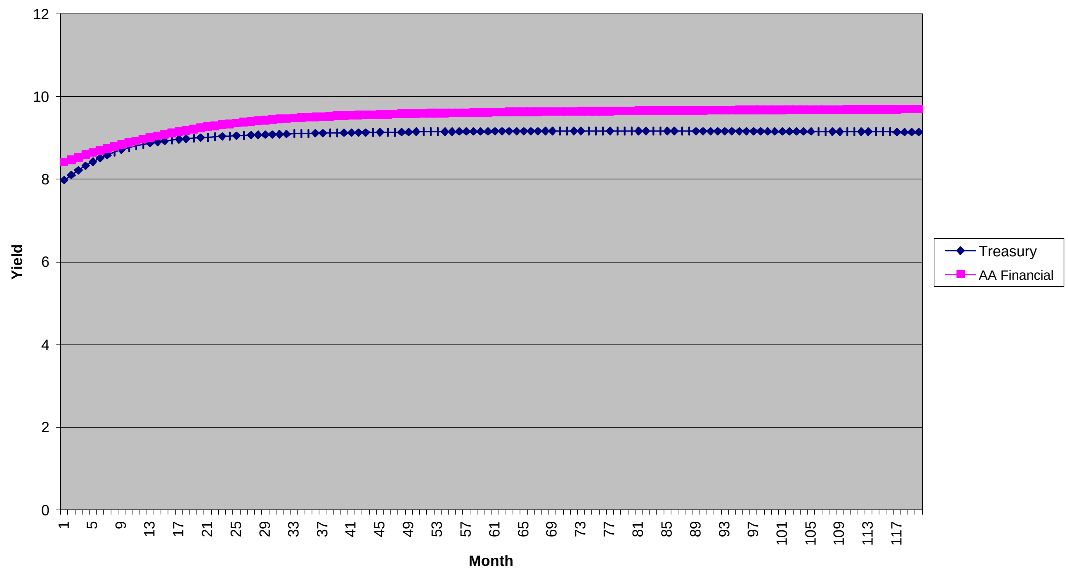


Figure 4
Term Structure of Recovery Rates

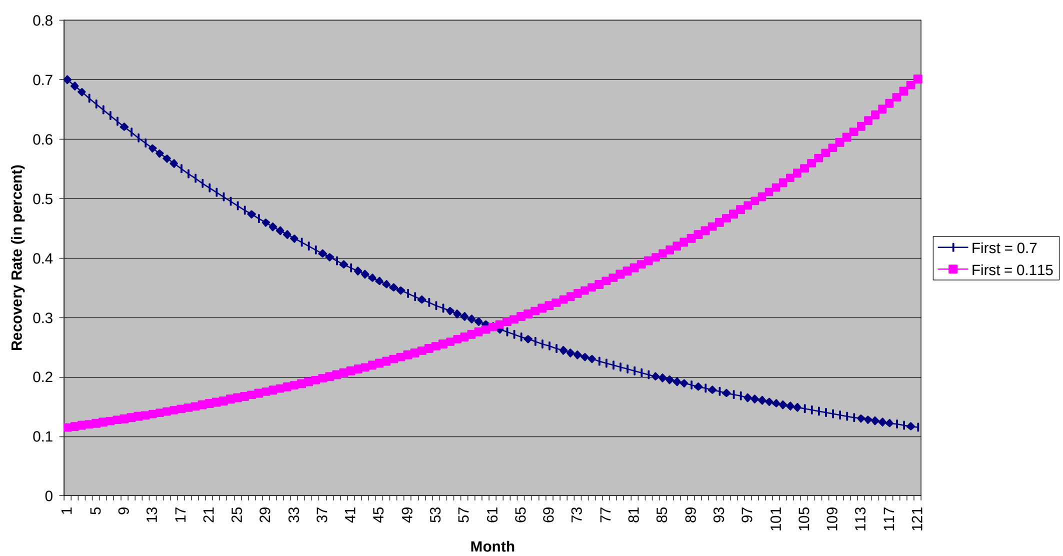


Figure 5
Nonparameterized Corporate Interest Rate Model Comparied to Jarrow and Turnbull (JT)

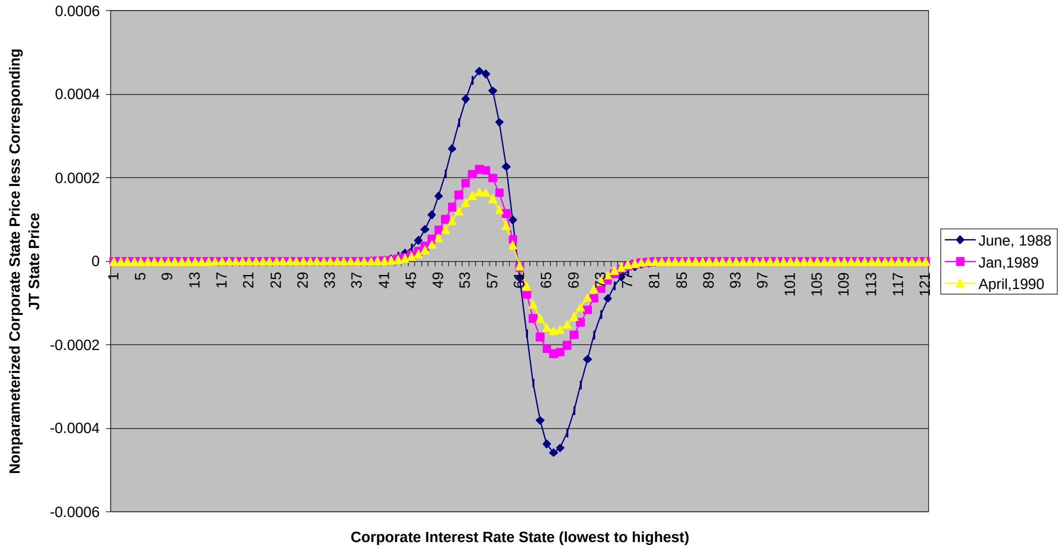


Figure 6
Corporate State Prices and Correlation with Treasury Interest Rates
(Upward Sloping Term Structure of June 30, 1988)

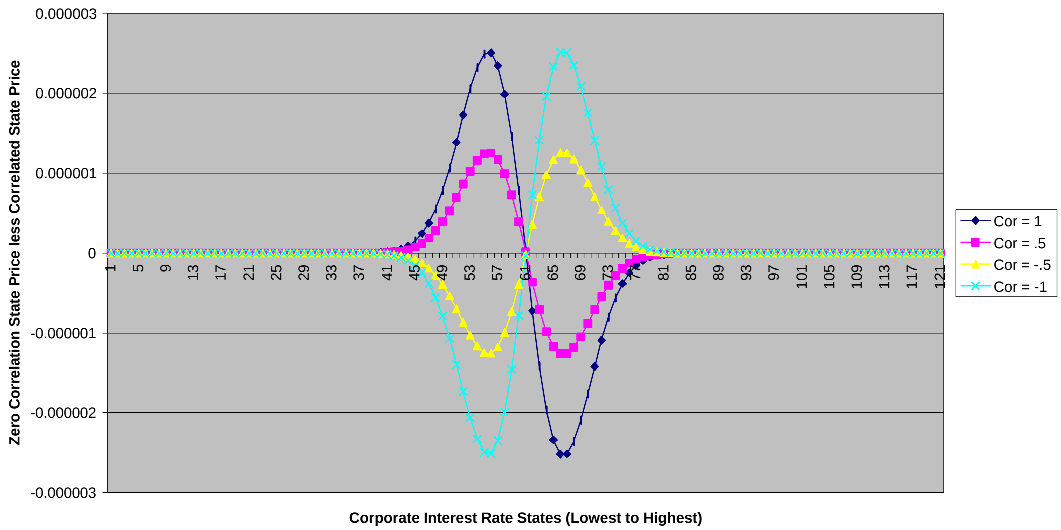


Figure 7
Increasing Recovery and State Prices
(Upward Sloping Term Structure of June 30, 1988)

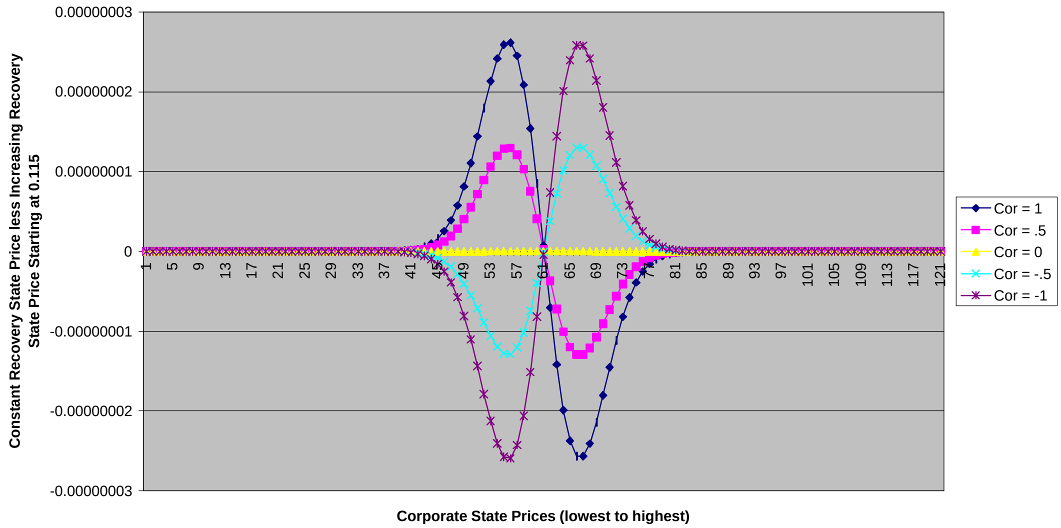


Figure 8
Decreasing Recovery and State Prices
(Upward Sloping Term Structure of June 30, 1988)

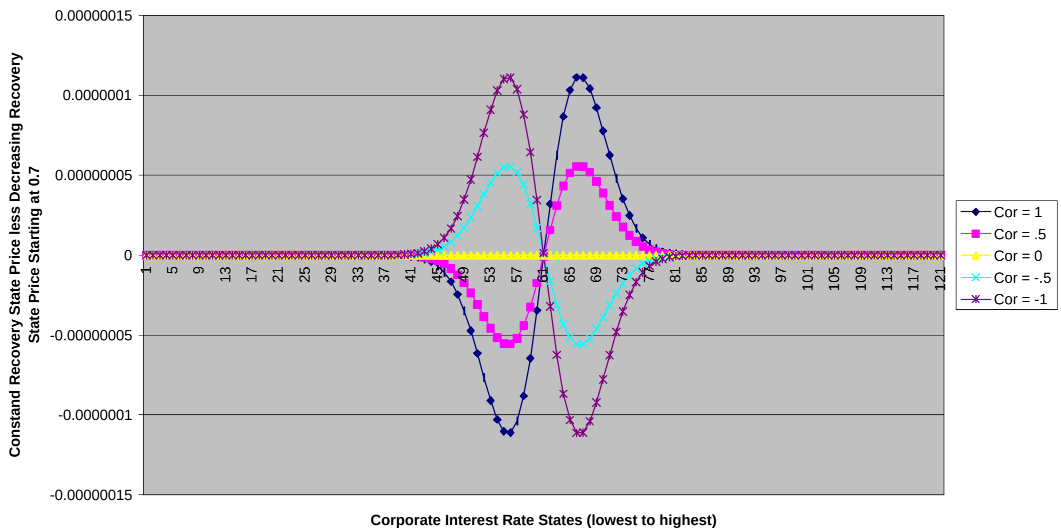


Figure 9
Time Varying Recovery Rates with Zero Correlation and State Prices
(Upward Sloping Term Structure of June 30, 1988)

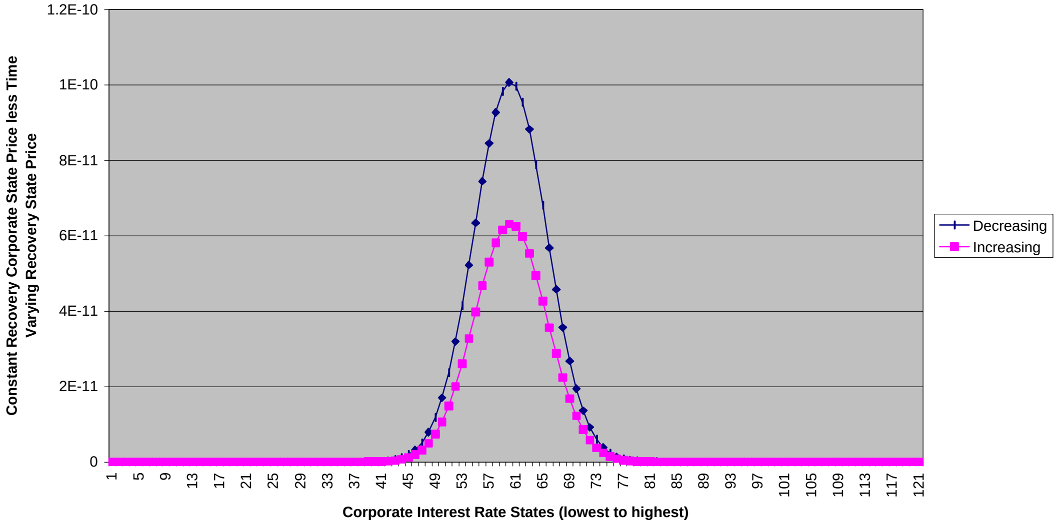


Figure 10
Increasing Recovery and State Prices
(Downward Sloping Term Structure of December, 1989)

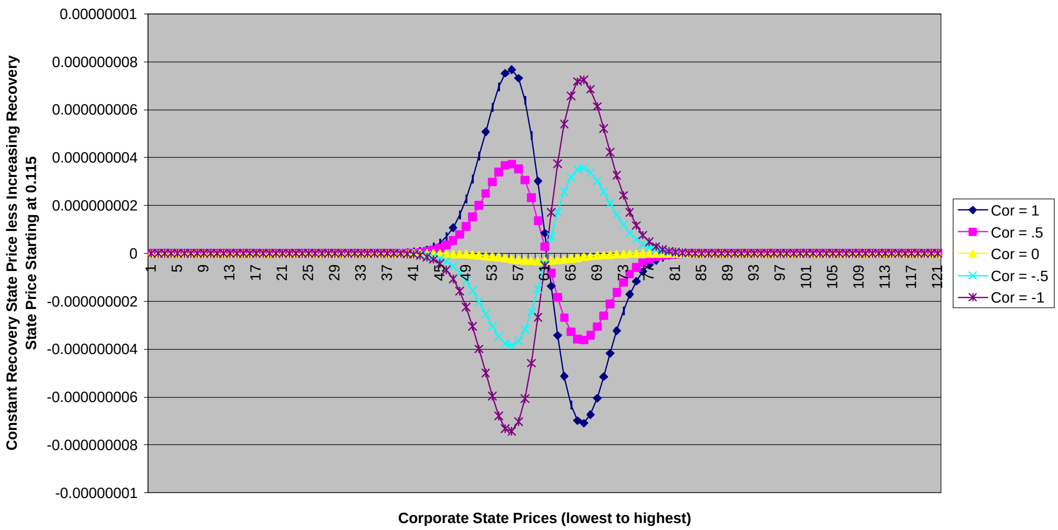


Figure 11
Decreasing Recovery and State Price
(Downward Sloping Term Structure of December, 1989)

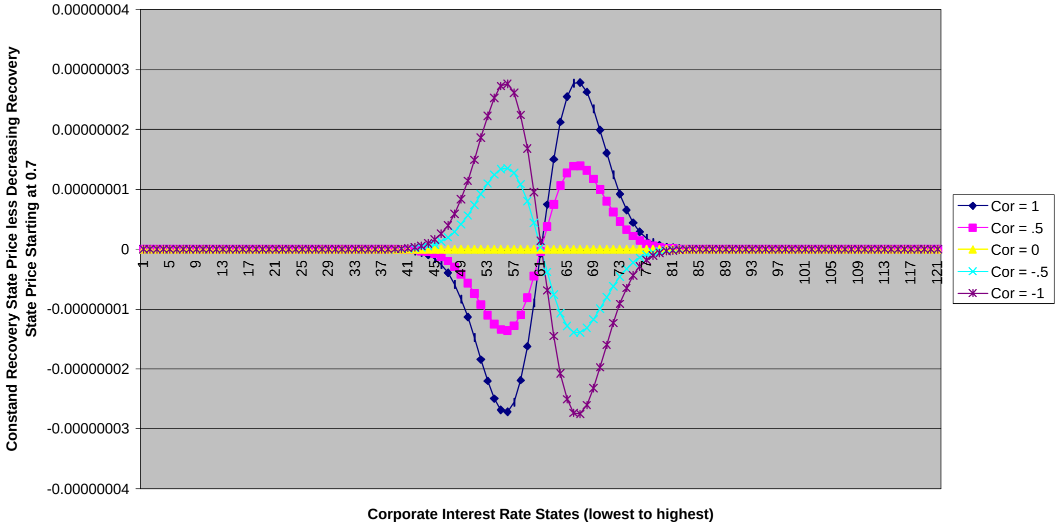


Figure 12
Increasing Recovery and State Prices
(Downward Sloping Term Structure of April, 1990)

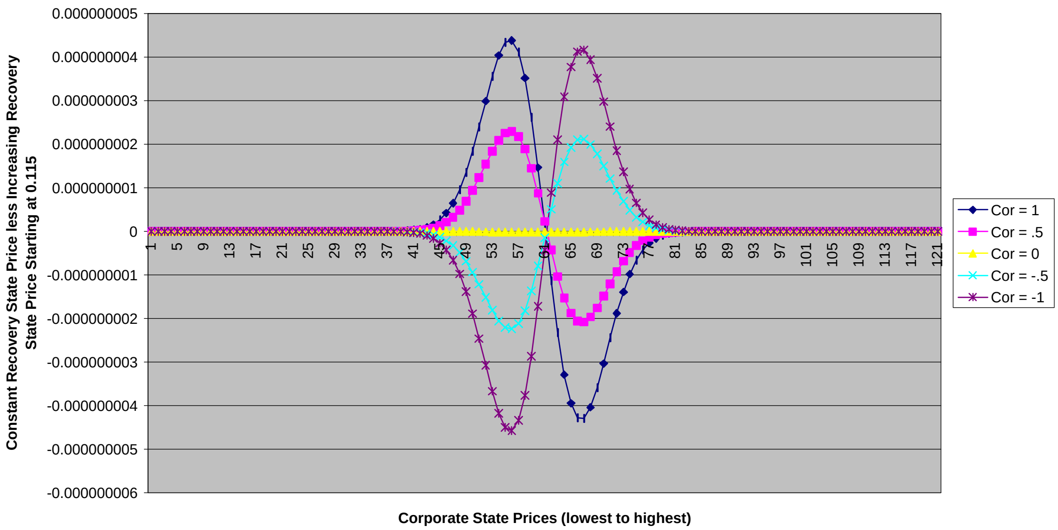
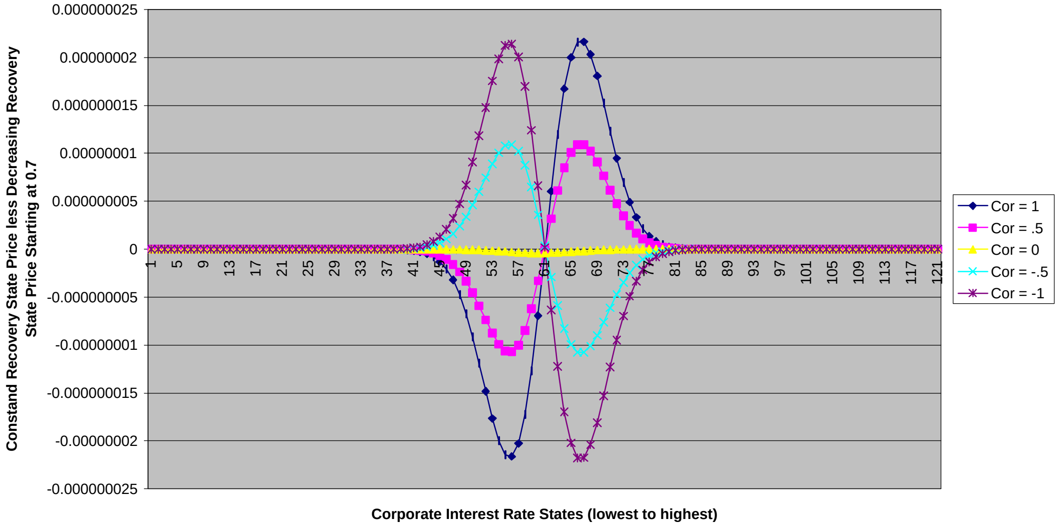


Figure 13
Decreasing Recovery and State Price
(Downward Sloping Term Structure of April, 1990)



The Effect of the Return of Market (RM) and the Return of Face (RF) Recovery Assumption on State Price (Upward Sloping Term Structure of June 30, 1988)

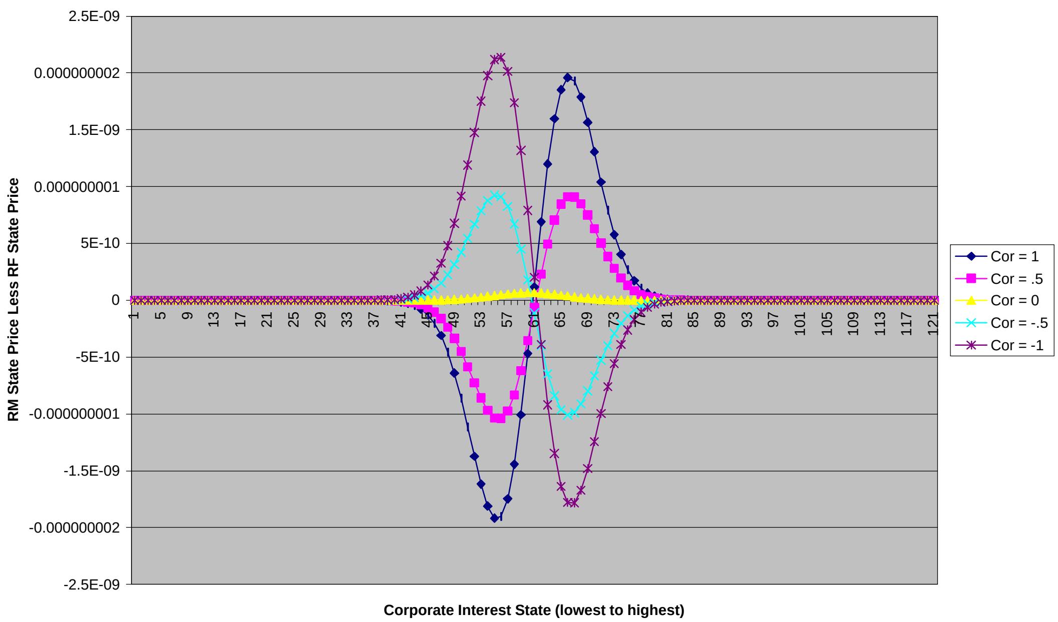
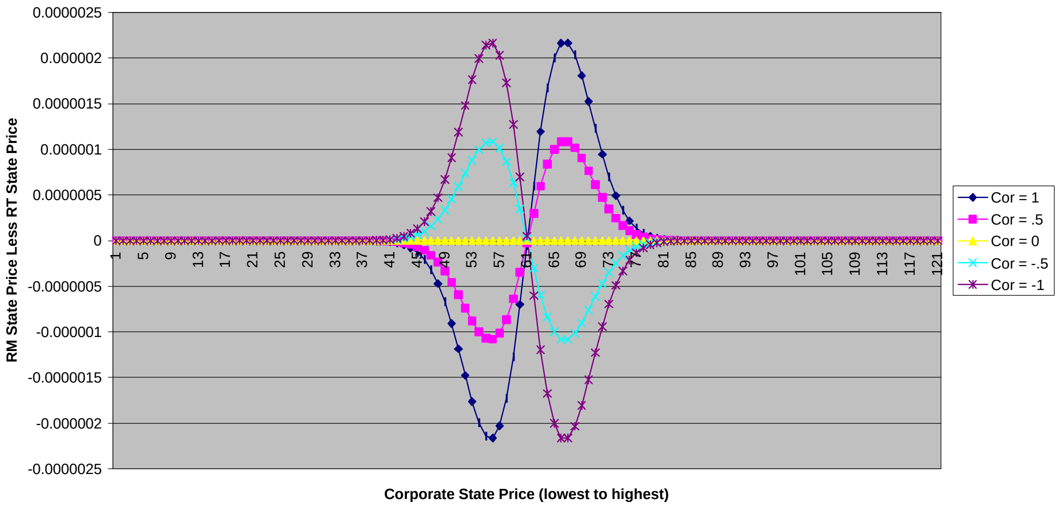


Figure 15
The Effect of the Return of Market (RM) and the Return of Treasury (RT) Recovery Assumption on State Prices
(Upward Sloping Term Structure of June 30, 1988)



---

[1] Harrison and Kreps (1979) show that equivalent martingale measures exist in the absence of arbitrage. These measures are unique if markets are complete.
[2] Note that our purpose is to explore the consequences of parameterizing the default process. A simple pure interest process places all the resulting models on a common basis suitable for comparative purposes.
[3] Scaling the covariance between to variables by the variance of the independent variable is very common in finance. Some examples are OLS hedge ratios and the CAPM model.
[4] Alternatively we can roll forwards through the corporate state price tree by multiplying expected values under hazard probabilities by pure interest state security prices. The same comment applies to binomial versions of Lando (1998) and Duffie and Singleton (1999).
[5] We also applied McCulloch (1975) and Nelson and Siegel (1987) finding that the results remained intact. We conclude that our results are robust with respect to alternative yield curve estimation schemes.
[6] How can we estimate unobservable hazard rate volatility? One strategy is to run a regression of corporate yields against Treasury yields. The slope coefficient will be an estimate of the scaled covariance expression included in (4), so there would be no need to separately identify the hazard volatility.
[7] By vulnerable we mean that the underlying asset is not subject to credit risk, but the writer is.
[8] Using the non-constant credit spreads of January 31, 1989 and the narrow credit spread of April 30, 1990 obtained basically the same result.
[9] We thank Philipp Schönbucher for this observation.
[10] We also replicated the RM, RF and RT results for time varying as opposed to constant recovery rates, for the non constant credit spread of January 31, 1989 and for the narrow credit spread of April 30, 1990. We did not find anything new, so for the sake of brevity we chose not to report these results.