Probability of Loss on Loan Portfolio

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Oldrich Vasicek, 2/12/87

Consider a portfolio consisting of n loans in equal dollar amounts. Let the probability of default on any one loan be p, and assume that the values of the borrowing companies' assets are correlated with a coefficient ρ for any two companies. We wish to calculate the probability distribution of the percentage gross loss L on the portfolio, that is,

$$P_k = \mathrm{P}\left[L = \frac{k}{n}\right], \ k = 0, 1, \dots, n$$

Let A_{it} be the value of the i-th company's assets, described by a logarithmic Wiener process

$$dA_i = rA_i dt + \sigma_i A_i dz_i$$

where z_{it} , i =1, 2, …, n are Wiener processes with

$$\mathrm{E}\left(dz_i\right)^2 = dt$$

$$\mathrm{E}\left(dz_i\right)\left(dz_j\right) = \rho\, dt, \quad i \neq j$$

The company defaults on its loan if the value of its assets drops below the contractual value of its obligations D_i payable at time T. We thus have

$$\begin{aligned} p &= \mathrm{P}\left[A_{iT} < D_i\right] \\ &= N\left(-c_i\right) \end{aligned}$$

where

$$c_i = \frac{1}{\sigma\sqrt{T}}\left(\log A_{i0} - \log D_i + rT - \tfrac{1}{2}\sigma^2 T\right)$$

and N is the cumulative normal distribution function.

Because of the joint normality and the equal correlations, the processes z_i can be represented as

$$z_i = bx + a\varepsilon_i\,, \quad i = 1, 2, \dots, n$$

where

$$b = \sqrt{\rho}\,,\ \ a = \sqrt{1-\rho}$$

and

$$\begin{aligned}
&\mathrm{E}\left(dx\right)^2 = dt\\
&\mathrm{E}\left(d\varepsilon_i\right)^2 = dt\\
&\mathrm{E}\left(dx\right)\left(d\varepsilon_i\right) = 0\\
&\mathrm{E}\left(d\varepsilon_i\right)\left(d\varepsilon_j\right) = 0, \quad i \neq j
\end{aligned}$$

The term bx can be interpreted as the i-th company exposure to a common factor x (such as the state of the economy) and the term $a\varepsilon_i$ represents the company's specific risks. Then

$$\begin{aligned}
P_k &= \mathrm{P}\left[L = \frac{k}{n} \right]\\
&= \binom{n}{k} \mathrm{P}\left[A_{1T} < D_1, \ldots, A_{kT} < D_k, A_{k+1T} \geq D_{k+1}, \ldots, A_{nT} \geq D_n\right]\\
&= \binom{n}{k} \int_{-\infty}^{\infty} \mathrm{P}\left[A_{1T} < D_1, \ldots, A_{kT} < D_k, A_{k+1T} \geq D_{k+1}, \ldots, A_{nT} \geq D_n \,|\, x_T = u\right] d\,\mathrm{P}\left[x_T < u\right]\\
&= \binom{n}{k} \int_{-\infty}^{\infty} \mathrm{P}\Big[c_1\sqrt{T} + bx_T + a\varepsilon_{1T} < 0, \ldots, c_k\sqrt{T} + bx_T + a\varepsilon_{kT} < 0, c_{k+1}\sqrt{T} + bx_T + a\varepsilon_{k+1T} \geq 0,\\
&\qquad\qquad \ldots, c_n\sqrt{T} + bx_T + a\varepsilon_{nT} \geq 0 \,|\, x_T = u \,\Big] d\,\mathrm{P}\left[x_T < u\right]\\
&= \binom{n}{k} \int_{-\infty}^{\infty} \left(N\left(-\frac{c + bu}{a} \right) \right)^k \left(1 - N\left(-\frac{c + bu}{a} \right) \right)^{n-k} dN(u)
\end{aligned}$$

In terms of the original parameters p and ρ, we have

$$P_k = \binom{n}{k} \int_{-\infty}^{\infty} \left(N\left(\frac{1}{\sqrt{1-\rho}} \left(N^{-1}(p) - \sqrt{\rho}u \right) \right) \right)^k \left(1 - N\left(\frac{1}{\sqrt{1-\rho}} \left(N^{-1}(p) - \sqrt{\rho}u \right) \right) \right)^{n-k} dN(u), \quad k = 0, 1, \ldots, n$$

Note that the integrand is the conditional probability distribution of the portfolio loss given the state of the economy, as measured by the market increase or decline in terms of its standard deviations.

LIMITING LOAN LOSS PROBABILITY DISTRIBUTION

Oldrich Vasicek, 8/9/91

The cumulative probability that the percentage loss on a portfolio of n loans does not exceed θ is

$$F_n(\theta) = \sum_{k=0}^{[n\theta]} P_k$$

where P_k are given by an integral expression in Oldrich Vasicek's memo, *Probability of Loss on Loan Portfolio*, 2/12/87. The substitution

$$s = N\left(\frac{1}{\sqrt{1-\rho}}\left(N^{-1}(p) - \sqrt{\rho}u\right)\right)$$

in the integral gives $F_n(\theta)$ as

$$F_n(\theta) = \sum_{k=0}^{[n\theta]} \binom{n}{k} \int_0^1 s^k (1-s)^{n-k}\, dW(s)$$

where

$$W(s) = N\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}N^{-1}(s) - N^{-1}(p)\right)\right)$$

By the law of large numbers,

$$\lim_{n\to\infty} \sum_{k=0}^{[n\theta]} \binom{n}{k} s^k (1-s)^{n-k} = 0 \quad \text{if} \quad \theta < s$$

$$= 1 \quad \text{if} \quad \theta > s$$

and therefore the cumulative distribution function of loan losses on a very large portfolio is

$$F_\infty(\theta) = W(\theta)$$

This is a highly skewed distribution. Its density is

$$f_{\infty}(\theta)=\sqrt{\frac{1-\rho}{\rho}}\exp\left(-\frac{1}{2\rho}\left(\sqrt{1-\rho}N^{-1}(\theta)-N^{-1}(p)\right)^{2}+\frac{1}{2}\left(N^{-1}(\theta)\right)^{2}\right)$$

Its mean, median and mode are given by

$$\bar{\theta}=p$$

$$\theta_{med}=N\left(\frac{1}{\sqrt{1-\rho}}N^{-1}(p)\right)$$

$$\theta_{mode}=N\left(\frac{\sqrt{1-\rho}}{1-2\rho}N^{-1}(p)\right)\quad\text{for}\quad\rho<\tfrac{1}{2}$$