RATINGS- VERSUS EQUITY-BASED CREDIT RISK MODELING: AN EMPIRICAL ANALYSIS

Pamela Nickell
Bank of England

William Perraudin
Birkbeck College,
Bank of England
and CEPR

Simone Varotto
Bank of England

July 1999*

Abstract

Banks have recently developed new techniques for gauging the credit risk associated with portfolios of illiquid, defaultable instruments. These techniques could revolutionize banks' management of credit risk and could in the longer term serve as a more risk-sensitive basis for calculating regulatory capital on banks' loan books than the current 8% capital charge. In this paper, we implement examples of the two main types of credit risk models so far developed, ratings-based and equity-based approaches. Using price data on large Eurobond portfolios, we assess, on an out-of-sample basis, how well these models track the risks they claim to measure.

*We thank Angus Guyatt for outstanding research support. We are grateful to Patricia Jackson, Victoria Saporta, other Bank of England colleagues, and participants in the September 1998 Bank of England-FSA conference on Credit Risk Modeling for their comments, and to Peter Bryant of Reuters for his kind assistance. Correspondence should be addressed to the Bank of England, Threadneedle Street, London EC2. The views expressed in the paper are those of the authors and not necessarily those of the Bank of England.

1 Introduction

The systematic application of Value at Risk (VaR) models[1] by large international banks has significantly enhanced their ability to measure and hedge their trading book risks. A valuable side-effect of the new emphasis on VaR modeling is that regulators have been able to reduce the distortionary impact of prudential capital requirements for banks' trading portfolios by basing such requirements on VaRs generated by banks' internal risk management models.[2]

Recently, regulators have begun to consider the use of internal models for measuring credit risk to increase the effectiveness of the capital requirement regime. As with VaR applied to trading books, the possibility has arisen because banks themselves are exploring the use of credit risk models for measuring the riskiness of their portfolios. A crucial question preoccupying both firms and regulators has been whether internal models could be used to assess required capital for banking books.[3]

The fundamental difficulty in assessing credit risk is that most credit exposures have no easily observable market price. The lack of price information obliges one to base credit risk estimates on other kinds of data.[4] The two feasible approaches in current use are (i) ratings-based methods (exemplified by JP Morgan's Creditmetrics approach), and (ii) equity-price-based techniques (advocated by, for example, the consulting firm, KMV).

Ratings-based techniques attribute a rating to each defaultable investment in a portfolio and then estimate the probability of upward or downward moves in ratings using historical data on ratings transitions for different traded bond issues. The probabilities are collectively termed the ratings transition matrix. The average spreads for bonds from different ratings categories are then combined with the transition

[1] Such models estimate the VaR on a portfolio, i.e., the loss which will be exceeded on some given fraction of occasions if the portfolio in question is held for a particular period.

[2] See Basel Committee on Banking Supervision (1996).

[3] The conclusion of the Basel Committee has been that this would be premature although it accepts that model-based capital calculation may be feasible in the longer term.

[4] If prices were observed, one might attempt to assess risk through a relatively simple VaR calculation.

probabilities to derive mean and volatility estimates for the return on each credit exposure.

Correlations between different rating changes may be estimated in a variety of ways and then combined with the estimated volatilities to obtain a measure of the total volatility of the portfolio as a whole. Assuming approximate joint normality of returns, one may derive a VaR for the total credit risk using the portfolio volatility and the expected return.

The alternative equity-price-based approach starts from the observation that, under limited liability, a firm's equity value is a call option written on the firm's underlying assets. Using standard option pricing formulae, one may therefore infer from the equity and liability values of a firm the level and distribution of the firm's underlying assets. Assuming some trigger level for bankruptcy one may estimate the probability of default.

Integrating numerically over the estimated distribution of changes in the underlying assets, one may calculate (as with the ratings-based approach) the means, variances and covariances of pairs of bonds. As a last step, just as in the ratings-based approach, one may suppose approximate normality of the portfolio value and infer a VaR from the portfolio mean and variance.

In this paper, we perform a direct comparison of representative ratings-based and equity-price-based methodologies when applied to a portfolio of credit exposures. In effect, we conduct a "horse race" between standard implementations of ratings- and equity-based credit risk models, comparing on an out-of-sample basis the VaRs generated by each model with the out-turns.

The credit exposures we examine are large portfolios of dollar-denominated Eurobonds. The unusually rich dataset we employ includes 1,430 bond price histories observed from 1988 to 1998. All the bonds are straight bonds with no call or put features. To implement ratings- and equity-based credit risk models, we also constructed datasets of equity and liability values for the bond obligors in our sample and their ratings histories.

We implement ratings- and equity-based models month by month, calculating in each period a credit risk VaR for the following year. We are careful only to employ lagged data which would have been available to an analyst implementing the model in the given period. To assess the models' performance, we then compare the estimated VaRs with the actual out-turn for the portfolio in question one year later.[5] If the models supply unbiased VaR estimates, the fraction of occasions on which losses exceed the VaRs will roughly equal the VaR confidence level.

Our main finding is that the models under-estimate the VaRs. Most of the portfolios we examine experience distinctly more losses in excess of the VaR estimates than one might expect. In the case of the rating-based model, the problem originates with the bonds of non-US and non-industrial obligors which are riskier than the models predict. Bench-marking the equity-based model from bond spreads rather than default probabilities yields conservative VaR estimates which are not obviously biased.

Although credit risk models are a very recent development, they are the subject of a rapidly growing literature. Surveys of the techniques employed may be found in Basel Committee (1999) and Crouhy, Galai, and Mark (1999). Discussion of issues related to the regulatory use of credit risk models is provided by Mingo (1999), Jones (1999), and Jackson and Perraudin (1999).

Empirical investigations of these models has so far been limited. Lopez and Saidenberg (1999) suggest techniques for assessing models through cross-sectional evaluation of their risk measures but do not implement their suggestions on actual data. Gordy (1999)and Kiesel, Perraudin, and Taylor (1999) implement ratings-based models on stylized portfolios, studying how the risk measures vary across different types of portfolio. Gordy (1999) and Crouhy, Galai, and Mark (1999) also compare the VaRs implied for the similar portfolios at one point in time by different models. Nickell, Perraudin, and Varotto (1999) discuss the stability of rating transition matrices, the central component of ratings-based credit risk models. The current study is the only out-of-sample assessment of models using time series data.

[5]It is possible to do such comparisons only because we apply credit risk models to liquid defaultable instruments like bonds.

The structure of our paper is as follows. In Sections 2 and 3, we describe respectively the ratings-based and the equity-based credit risk modeling approaches we wish to assess. In Section 4, we detail the substantial datasets that we created in order to perform the model 'horse races'. Section 5 gives the results of our investigation and Section 6 concludes, drawing out the lessons and discussing further work that is needed to investigate aspects of credit risk modeling methodologies.

2 A Ratings-Based Credit Risk Model

2.1 The Creditmetrics Methodology

The best-known rating-based approach to credit risk modeling is that of Creditmetrics, see JP Morgan (1997). Another ratings-based approach (in which correlations are modeled differently) has been implemented by McKinsey Inc. (see Wilson (1997)). Credit Suisse Financial Products (1997) have proposed a model similar to ratings-based approaches in which obligors occupy just two "ratings categories", default or non-default. Among rating-based approaches, we focus in this study on the Creditmetrics model.

The Creditmetrics manual (see JP Morgan (1997)) describes a wide range of techniques of varying technical sophistication and practical use. We base our analysis on what appears to us to represent the core set of techniques. These involve the analytical derivation of the means and variances of bond portfolios from data on (i) rating transition matrices, (ii) bond spreads, (iii) equity index correlations and (iv) factor weights of individual obligor's equity values on the equity indices.

Given the mean and variance of a portfolio, assuming that bond portfolio distributions are approximately normal, one may directly infer the VaR as defined in footnote 1. Below, we examine in detail the assumption of approximate normality, and show that bond portfolio returns over a one-year investment horizon, adjusted to remove the effects of fluctuations in default-free interest rates, are close to normality.

To understand how Creditmetrics works, suppose that the probability that a firm will default is fully described by its current rating, say j. Consider a discount function $D_j(t,t')$ corresponding to the market price at date t of a promise to deliver \$1 at date t' in the event that a given j-rated bond issue does not default before t'. Let $D_0(t,t')$ denote the price of a default-free Treasury strip. We can implicitly define the credit spread for a j-rated obligor from t to t', denoted $S_j(t,t')$ as:

$$D_j(t,t') \equiv D_0(t,t') \exp[-S_j(t,t')(t'-t)] \quad . \tag{1}$$

If there are $J+1$ rating categories of which the $J+1$st represents default, the price at t of a j-rated, defaultable, coupon bond may be expressed as:

$$B_t^{(j)} \quad = \quad \begin{cases} \sum_{i=1}^N \; D_0(t,t_i)\; \exp[-S_j(t,t_i)(t_i-t)]\, c_i & j=1,2,\ldots,J \ , \\ \sum_{i=1}^N \; \xi\, D_0(t,t_i)\, c_i & j=J+1 \ . \end{cases} \tag{2}$$

where $c_1, c_2, \ldots, c_N$ are the promised cash flows (including the repayment of principal) subsequent to t at respective payment dates $t_1, t_2, \ldots, t_N$, and the recovery rate, ξ, is assumed fixed.

Let $\pi_{ij}(t,T)$ denote the probability that a bond issue rated i at t will be rated j at T. The matrix of such probabilities, $[\pi_{ij}(t,T)]$, is termed the rating transition matrix. Estimates of transition matrices covering different sample periods and for different types of obligor may be obtained from rating agency publications (see for example, Lucas and Lonski (1992) and Carty (1997)) or from academic articles (see Altman and Kao (1992) and Nickell, Perraudin, and Varotto (1999)).

Suppose that at date t, the rating-contingent discount functions which will pertain at $T > t$, namely $D_j(T,s)$, are known. From (2), one may deduce the value that the bond will have at T conditional on knowing its rating at T. Using transition probabilities, $\pi_{ij}(t,T)$, it is then straightforward to deduce moments of the bond price conditional on informations at t:

$$\text{Mean}_{it} \quad = \quad \sum_{j=1}^{J} \pi_{ij} B_T^{(j)} \quad \equiv \quad \mu_p^{(i)} \tag{3}$$

$$\text{Variance}_{it} \quad = \quad \sum_{j=1}^{J} \pi_{ij} (B_T^{(j)} - \mu_p^{(i)})^2 \quad . \tag{4}$$

The assumption made in Creditmetrics that the future rating-contingent discount factors are known is a strong assumption. Kiesel, Perraudin, and Taylor (1999) show how it may be relaxed and argue that allowing for randomly varying spreads is important if one is modeling the risk of portfolios comprising high credit quality exposures.

2.2 Modeling Correlations

Of course, to derive the mean and variance of a credit portfolio, one must calculate the *covariances* of the exposures which make up the portfolio, as well as their means and variances. For a portfolio which contains many exposures, the covariances will dominate in the calculation of portfolio volatility and the main subtleties in the Creditmetrics approach concern their calculation.

The way in which Creditmetrics allows for correlated rating transitions consists of assuming that each obligor's rating transitions is driven by a normally-distributed latent variable. More formally, suppose that for an i-rated bond issue, there exists a normally-distributed variable, R_i, such that, for a set of cut-off points, $(Z_{i,0}, Z_{i,2}, \ldots, Z_{i,J})$, if R_{ik} falls in the interval $(Z_{i,j-1}, Z_{i,j})$, the bond is rated j at date T.

Given an estimate of a rating transition matrix, $[\pi_{i,j}(t,T)]$, the cutoff points $Z_{i,j}$ may be deduced directly using the recursive equations:

$$\left\{ \begin{array}{rcll} \pi_{i,J}(t,T) & = & 1 - \Phi(Z_{i,J-1}) & \\ \pi_{i,j}(t,T) & = & \Phi(Z_{i,j}) - \Phi(Z_{i,j-1}) & \quad j = 2, \ldots, J \\ \pi_{i,1}(t,T) & = & \Phi(Z_{i,1}) \quad . & \end{array} \right. \tag{5}$$

The approach of allowing multinomial transitions to be driven by an underlying latent variable with a continuous distribution is widely applied in the discrete choice econometrics literature. When the latent variable is normally distributed it corresponds to an ordered probit approach. The major benefit of this approach for credit risk modeling is that it permits one to model correlation between different rating transitions in a straightforward fashion.

Creditmetrics makes the simple assumption that correlations between the latent variables driving transitions for different bond issues, say R and R', equal those of the firms' respective log equity values. Without loss of generality, one may standardize the variances and means of the latent variables to unity and zero, respectively. The joint distribution of rating transitions for a pair of obligors initially rated i and l is then:

$$\text{Prob}\left\{Z_{i,j-1} < R < Z_{i,j}\ ,\ Z_{l,m-1} < R' < Z_{l,m}\right\} \ = \ \int_{Z(i,j-1)}^{Z(i,j)} \int_{Z(l,m-1)}^{Z(l,m)} \phi(s, s'|\sigma) ds ds' \ . \tag{6}$$

Here, ϕ is a standard bivariate normal density with a correlation coefficient, ζ. Rather than estimate ζ using time series data on equity values, the Creditmetrics manual suggests that one construct proxies consisting of weighted averages of industry and country equity indices, assuming that the given equity return also contains some additional amount of idiosyncratic risk.

This approach is particularly simple when there is a single index for each obligor since one may then express the standardized (i.e., unit variance) log return of the nth firm's equity value as:

$$r_n = \omega_{1n} r_M + \omega_{2n} \hat{r}_n \tag{7}$$

where r_M and $\hat{r}_n$ denote the standardized (unit variance) return on the index and the idiosyncratic component of the firm's equity return. If ω_{1n} equals α, then the standardization implies that $\omega_{2n} = \sqrt{1 - \alpha^2}$.

Thus, given the correlation matrix for the standardized indices, $[\zeta_{mn}]$, and an assumption about the fraction of volatility that is idiosyncratic for each obligor (i.e., a choice of α for each obligor), one may deduce the correlation of the obligors' latent variables as

$$\alpha_n \alpha_m \zeta_{nm} \ . \tag{8}$$

Note that what we have just described is simpler than the approach set out in Creditmetrics since we have supposed that there exists a single index for each obligor rather than several national and industry indices with known weights, α_{mh}.

3 An Equity-Based Credit Risk Model

3.1 Basic Assumptions

The first step in implementing an equity-based is to deduce functional relations between a firm's underlying assets and its equity and bond values. Suppose that a firm has an earnings flow:

$$\delta(V_t - D_t) \ , \tag{9}$$

where V_t is the underlying asset value, D_t is the firm's liabilities, and δ is a dividend payout rate. Assume that asset and liability values and the market portfolio, M_t, follow [6]

$$\begin{aligned} dV_t &= \mu_v V_t dt + \sigma_v V_t dW_{1t} && (10) \\ dD_t &= \mu_D D_t dt && (11) \\ dM_t &= M_t \mu_m dt + M_t \sigma_m dW_{2t} \ . && (12) \end{aligned}$$

where the market and the firm's asset value are correlated in that $dW_{1t} dW_{2t} = \rho dt$.

Suppose for simplicity that the short-interest rate is deterministic and equal to r. By standard arguments, the risk-adjusted drift terms for V_t and M_t are respectively: $\mu_v^* = r - \delta$ and $\mu_m^* = r$. Assuming that there exists a representative agent with log utility who derives utility from the level of M_τ at some future date, τ, one may obtain the actual, equilibrium drift terms for our processes as: $\mu_v = r - \delta + \sigma_v \sigma_m \rho$ and $\mu_m = r + \sigma_m^2$.

We suppose that the firm is declared bankrupt when the ratio of assets to liabilities V_t/D_t first hits some low level, γ, and that equity-holders receive nothing in the bankruptcy settlement. Then, the value of the firm's equity, $X_t = X(V_t, D_t)$, satisfies the differential equation:

$$rX = \delta(V - D) + \mu_v^* V \frac{\partial X}{\partial V} + \mu_D D \frac{\partial X}{\partial D} + \frac{\sigma_v^2}{2} V^2 \frac{\partial^2 X}{\partial V^2} \ , \tag{13}$$

subject to the value-matching and no-bubbles conditions: $X(V, D)|_{V/D=\gamma} = 0$ and $\lim_{V/D \to \infty} X(V, D) = V - \frac{\delta}{r} D$. As we show in the Appendix, defining $k \equiv V/D$, one

[6] It is easy to generalize this model by allowing the liabilities to vary stochastically.

may exploit the homogeneity of the equation to obtain:

$$X(V,D) \;=\; D\left[k-\frac{\delta}{r-\mu_D}-\left(\gamma-\frac{\delta}{r-\mu_D}\right)\left(\frac{k}{\gamma}\right)^{\lambda}\right] \quad , \tag{14}$$

$$\lambda \;\equiv\; \frac{1}{\sigma_v^2}\left[-(r-\sigma_v^2/2-\delta-\mu_D)-\sqrt{(r-\sigma_v^2/2-\delta-\mu_D)^2+2\sigma_v^2(r-\mu_D)}\right] \; . \tag{15}$$

The trigger level for bankruptcy, γ is chosen by equity-holders in this model since the firm will continue to operate until equity-holders are unwilling to absorb more losses. γ is therefore determined by the first order condition, $\partial X/\partial\gamma = 0$, which after rearrangment yields:

$$\gamma \;=\; \frac{\lambda}{\lambda-1}\frac{\delta}{r-\mu_D} \quad . \tag{16}$$

3.2 Implementation of the Equity-Based Model

To implement an equity-based credit risk model for portfolios of bonds, we first need to estimate the parameters of the k_t processes for each obligor. Given time series data on each obligor's equity-to-liabilities ratio, and given that this ratio is a non-linear function of k_t (see equation (14), we may estimate the k_t parameters by applying Maximum Likelihood (ML) estimation, including a Jacobian term in the likelihood to allow for the transformation in variables.[7]

Two facts complicate the ML estimation. First, since our pricing expressions depend on the correlation parameter, ρ, we need to estimate a joint model of changes in k_t and M_t . Second, if the firm survives from t to $t+\Delta$, k_t must have remained above the bankruptcy trigger, γ, in this interval of time. The correct density to employ for the log processes is, therefore, that of a bivariate Brownian motion when one of the processes has an absorbing barrier. We solve for this density in the Appendix.

Once we have estimated the parameters of the k_t and M_t processes, we invert the functional relation between X/D and k to extract from our equity and liability data a time series of k_t estimates for each obligor. For each pair of obligors, we calculate correlations for these asset-to-liability ratios.

[7]Note that the ML estimation will involve inverting the functional relation between $X(k)$ and k every time that the likelihood is evaluated and, hence, is computationally demanding.

We performed the above estimations and inversions for data running up to each January in our sample. (Performing the estimations monthly would have been very demanding computationally.) We then supposed that the parameters of the k_t process remained constant until the following December.

3.3 Bond and Portfolio Moments

The last step in implementing the model was to calculate, for each month in our sample, conditional means, variances and covariances of future bond prices. To do this, we assumed that changes in interest rates and credit standing are independently distributed, in which case the price at date T of a bond which yields cash payments $c_1, c_2, \ldots, c_N$ at dates $t_1, t_2, \ldots, t_N$ may be written as

$$B_T \;=\; \sum_{i=1}^{N} c_i\, D_0(T, t_i)\ (\xi + (1-\xi)\,\mathrm{Prob}_T\,\{\text{no default by } t_i\}) \tag{17}$$

where ξ is the recovery rate in the event of default.[8]

Since bankruptcy is triggered when the asset-to-liabilities ratio, k_t, first hits γ, we can infer the conditional probability at t that no default has occurred prior to t_i by integrating the marginal density of the $\log(k_t)$ process (derived in the Appendix) from γ to infinity. This yields

$$\begin{aligned}\mathrm{Prob}_T\left\{\begin{array}{c}\text{no default}\\ \text{by } t_i\end{array}\right\} \;=\; & \int_{\log(\gamma)}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_v\sqrt{t_i - t}}\left(\exp\left[-\frac{(\log(k_{t_i}/k_T) - \mu_{lk})^2}{2\sigma_v^2(t_i - T)}\right] \;-\right.\\ & \exp\left[-\frac{(\log(k_{t_i}/k_T) - \mu_{lk} - 2\log(\gamma/k_T))^2}{2\sigma_v^2(t_i - T)}\right]\times\\ & \left.\exp\left[\frac{2(\log(\gamma/k_T)\mu_{lk}}{\sigma_v^2}(t_i - T)\right]\right)\, d\log(k_{t_i}/k_T)\end{aligned} \tag{18}$$

where $\mu_{lk} = (\mu_k - \sigma_v^2/2)(t_i - T)$.

[8]Changes in interest rates and credit spreads are probably not independent but the sign and magnitude of the correlations are not entirely clear. Duffee (1999) finds negative correlation over short horizons whereas Morris, Neal, and Rolph (1999) find the changes are positively correlated over long horizons.

The probability in equation (18) depends on k_T and hence so does the bond price formula given in equation (17). Our ML estimate of the k processes gives us the density, conditional on information at t, of k_T for different firms. Since we also estimated the correlation between changes in k for different firms, we have the joint density of k_T for each *pair* of obligors. Numerically integrating powers of bond price formulae like (17) over these conditional joint densities, we obtain means, variances and covariances of bond prices as desired.

As in the case of the ratings-based model, to calculate VaRs, we assume approximate normality, calculate the total mean and variance of portfolios using the moments of pairs of bonds obtained as described above, and then infer the portfolio VaR using the standard formula for VaRs when payoffs are Gaussian.

3.4 Bench-Marking Equity-Based Models

There is one final complication which, as we shall see, significantly affects the results for the equity-based model. In their equity-based credit risk model, KMV assume, in estimating asset volatility, that firms become bankrupt when their assets equal an exogenously weighted sum of their short- and long-term liabilities. However, when they use their model to price credit exposures and calculate moments of loan and bond values, they shift the default trigger so that the probability of default matches the historically-observed default probability for firms which are the same number of asset standard deviations from the exogenously given trigger.

In our implementation of an equity-based credit risk model, we similarly "bench-mark" the default trigger when we use the model in pricing. The two approaches we employ are to shift the default trigger so that (i) the probability of default equals the default probability of obligors with the same rating,[9] or (ii) the bond in question is correctly priced at the start of period for which we are performing the credit risk calculation. Of these two approaches, the first resembles that followed by KMV. The second is more conservative, especially for high quality debt since it attributes the

[9]In this, we use the default probabilities in the transition matrix based on Moody's data from 1970 to 1997 reported in Nickell, Perraudin, and Varotto (1999).

entire spread embodied in bond prices to credit risk whereas in fact some of the spread is almost certainly due to liquidity premia.

4 Data

4.1 The Bond Price Data

The data requirements of our study are quite considerable since we calculate risk measures month by month using two quite different methodologies and then compare them with out-turns for large portfolios of bonds. Let us start by describing the substantial bond price dataset we created.

This comprised 1,430 US dollar-denominated bonds[10] selected from the much larger number of bonds listed on the Reuters 3000 price service. Our criteria in selecting the bonds were (i) that they were straight bonds (not floaters), (ii) that they were neither callable nor convertible, (iii) that a rating history was available, (iv) that the coupons were constant with a fixed frequency, (v) that repayment was at par, and (vi) that the bond did not possess a sinking fund. To arrive at the 1,430, we further eliminated bonds for which the price and rating histories did not overlap for more than a year, and very illiquid bonds with price histories which contained at least one gap of more than 100 days.

The prices we used were Reuters composite bids. The Reuters composite is the best bid reported at close of trading by a market-maker from which Reuters has a data feed. Our data included comprehensive information about the cash flows, ratings and price histories of the bonds, and the name, domicile and industry code of the obligor.

To conduct our various analyses, we created a series of different portfolios. We use the term "total portfolio" to denote a portfolio comprising one unit of every bond available on a given day. (The composition of the portfolio therefore evolves over

[10] Of these, 90% were Eurobonds, the remainder being national bonds from several countries.

time.) The "restricted portfolio" consists of one unit of every bond for which we have equity and liability data available on a given day and hence can implement the equity-based credit risk model. We subdivided the total portfolio into bonds issued by US and non-US obligors, and into bonds issued by banks and financials or by other obligors. Lastly, we examined "quartile samples" made up of four randomly-selected sub-samples of the total sample each containing a quarter of the bonds available.

The composition of the total portfolio is shown in Table 1. 45% of bonds had issuers domiciled in the US. A further 31% had issuers domiciled in Japan, the Netherlands, Germany, France or the UK. A large fraction of bonds, namely 66%, were issued by firms in the financial services or banking industries. 58% of bonds were unsecured.

4.2 Data Requirements for the Credit Risk Models

To implement Creditmetrics, we needed: (i) transition matrices, (ii) default spreads and default free yield curves over time, (iii) equity index data, and (iv) a set of weights linking individual obligors to the equity indices, and (v) an assumption about the fraction of equity volatility that is idiosyncratic for each obligor. The transition matrix we employed was the unconditional Standard and Poors transition matrix provided by the Creditmetrics software Credit Manager. Default-free interest rates and spreads for different ratings categories were taken from Bloomberg.[11]

Obtaining equity indices for the obligors required a substantial effort. We created a time series dataset going back to 1985 comprising 243 country- and industry-specific indices. For each obligor, we then chose one of these indices as the source of non-idiosyncratic risk. We took the fraction of idiosyncratic risk in each obligor's equity to be a constant amount, choosing the weight ω_1 described in Section 2 to be 0.95. This was slightly higher than the example given in the Creditmetrics manual but it seemed appropriate since our obligors (being Eurobond issuers) were mostly large firms.

[11]We used spreads for US industrials since these had the longest series and the fewest missing observations.

To implement an equity-based approach required (i) default free term structures, (ii) equity market capitalization data for the obligors in question, (iii) liability data for the obligors. Once again, the term structure data came from Bloomberg. The equity and liability data were obtained from Datastream.[12]

5 Results

5.1 Value-at-Risk and Pricing Errors

To assess the performance of different credit risk models, we compare VaR measures for a one-year holding period with the actual out-turns of different portfolios. These comparisons are complicated, however, by the fact that both the ratings-based model and the equity-based approaches described above abstract from interest volatility in calculating risk measures. To see how well the models measure credit risk, one must, therefore, remove from the portfolio value realization that part of the value change which is attributable to changes in the default-free term structure.

To explain the adjustments we made, we adopt the following notation. Let P_t denote the value of a bond portfolio and let $P_{t,T}$ denote the expectation conditional on information at t of the portfolio value at T. Furthermore, let Q_t represent the price at t of a portfolio of default-free bonds having the same contractual payments as the defaultable bonds in our portfolio. Let $Q_{t,T}$ equal the forward price of the default-free portfolio at t.

To remove the effects of default-free interest rate changes on our portfolio return, we work with the following adjusted return:

$$\frac{P_T - P_{t,T} - (Q_T - Q_{t,T})}{P_{t,T}} \quad . \tag{19}$$

This quantity represents the net return one would have obtained by investing a unit

[12] Before collecting data on equity market capitalizations, we had to identify which equity issuer could be regarded as the parent obligor. In some cases, this entailed checking data on inter-firm ownership.

amount in the portfolio of defaultable bonds if value changes due to changes in general interest rates had been hedged. Suppose that one of our credit risk models predicts that for some confidence level, α, $\text{Prob}_t(P_T < \gamma) = \alpha$ for a cut-off point or "VaR quantile", γ, then we can compare the return in equation (19) with the quantity:

$$\frac{\gamma - P_{t,T}}{P_{t,T}} \quad . \tag{20}$$

If (19) falls below (20), then the loss on the position has exceeded the VaR.

A second problem that we face in gauging the accuracy of the models' risk predictions is that the models are effectively being used to *price* portfolios as well as to measure their risk. Let $P_t^{(r)}$ and $P_t^{(e)}$ denote the values that the ratings- and equity-based models attribute to the portfolio at time t. To correct for pricing errors in the expected price, $P_{t,T}$, we add the pricing discrepancy at the initial date, t. In (19) and (20), we, therefore, replace $P_{t,T}$ with,

$$\tilde{P}_{t,T}^{(i)} \equiv P_{t,T}^{(i)} + P_t - P_t^{(i)} \qquad \text{for} \quad i = r, e \ . \tag{21}$$

Both the ratings- and the equity-based methods described above provide estimates of the mean, μ_p, and variance, σ_p^2, of the portfolio value at T conditional on information dated t. If we assume that portfolio values over reasonably long holding periods such as 1 year are approximately normally distributed then the Value-at-Risk or loss which will be exceeded with some probability, say 1%, may be calculated by inverting the probability statement:

$$\text{Prob}\left\{P_T < \gamma\right\} = \Phi\left(\frac{\gamma - \mu_p}{\sigma_p}\right) = 0.01 \quad , \tag{22}$$

to obtain:

$$\gamma = \mu_p + \Phi^{-1}(0.01)\ \sigma_p \quad . \tag{23}$$

5.2 Return Normality

The VaRs we calculate are predicated on the assumption that our portfolio returns are approximately Gaussian. To justify this, in Table 2 we report estimates of the skewness and kurtosis for the returns over a one-year period for several portfolios. We report these statistics both for the raw returns (including interest rate risk) and for returns adjusted for interest rate risk and pricing discrepancies as described in the last subsection.[13]

As one may see, the returns are less negatively skewed when risks associated with changes in default-free interest rates are excluded. This is intuitively reasonable since monetary authorities typically adjust interest rates up sharply while downward movements tend to be smoothed and gradual. Also, the kurtosis is reduced after we remove interest rate risk and adjust for pricing errors.[14]

To show the implications of our moment calculations, we plot in Figure 1 members of the Pearson system of distributions (a widely-used flexible family of distributions) which have the same first four moments as the return series for our "total portfolio".[15] For comparison's sake, we also plot in Figure 1 normal distributions with the same mean and variance as our return series. As one may see, VaRs based on assumptions of normality are lower than the true VaRs when interest rate risk is included. However, after adjustments are made for interest rate risk, VaR calculations which presume normality appear to be slightly too conservative.

5.3 Equity-Based and Ratings-Based VaRs

We initially focus on the bonds for which we possessed equity and liability data and hence for which we could implement *both* ratings-based *and* equity-based models. Figure 2 shows ratings-based and equity-based VaRs and profit out-turns for

[13]The standard errors reported in Table 2 are asymptotically valid, Newey-West standard errors which fully allow for the fact that out 110 observations are heavily overlapping. To obtain them, we estimated the asymptotic covariance matrix of the second, third and fourth moments of the returns and used the delta method to infer standard errors for skewness and kurtosis.

[14]As a check, we also calculated the kurtosis removing interest rate risk but leaving out our pricing error adjustments. The kurtosis was similar or in some cases even lower than that reported in the lower half of Table 2.

[15]The technique of fitting data to the Pearson system of distributions by matching moments is discussed by Stuart and Ord (1994).

this restricted portion of the total portfolio. The estimates are in units of percent (multiplied by 100) and represent returns on the bond position over one year, calculated month-by-month through the sample period. The VaRs are based on a one-year holding period and a 1% confidence level.

In the terminology of Basel Committee on Banking Supervision (1996), an exception occurs when the out-turn loss on a portfolio exceeds the VaR measure supplied by a VaR model. In Figure 2, such an exception takes place when the solid line representing year-on-year returns[16] falls below one of the VaR quantiles which appear in the Figure as dashed and dotted lines. If the credit risk models were correctly measuring risk, and we had non-overlapping observations, the returns would cross the VaR quantiles approximately once every one hundred years. (With overlapping observations, the crossing will be slightly more frequent.)

The conclusion that emerges from Figure 2 is that the ratings-based model yields more exceptions (three over the ten years covered by the sample) than is likely to be the result of chance. In other words, the risk measures are biased downwards. When the equity-based model is bench-marked off default probabilities (as is the practice of KMV in their implementation of equity-based modeling), the number of exceptions is even greater. When our equity-based model is bench-marked off bond spreads, it appears rather conservative.

The large differences between equity-based VaRs based on default-probability or bond-spread bench-marking reflects the fact, upon which past empirical studies of structural bond pricing models have commented, that it is hard to explain the observed size of spreads on high quality bonds solely on the grounds of credit risk. See, most notably, Jones, Mason, and Rosenfeld (1984). It seems likely, therefore, that there is also a significant liquidity spread.

In Figure 3, we calculated VaRs for a portfolio consisting of the fifty lowest-rated bonds available at any point in time in the restricted total portfolio. As one might

[16] As a check of our returns data, we compared our return series with changes in Bloomberg spreads and found they were broadly consistent. We also noted that the two most marked losses narrowly preceded the quarters in which defaults as measured by Moody's peaked within our sample period.

expect (both because it involves lower quality debt and because it is less diversified), the return series in Figure 3 is much more volatile than that shown in Figure 2. Once again, the ratings-based VaRs and the equity-based VaRs using default probability bench-marking are much too small. The equity-based VaRs with bond-spread bench-marking now appear reasonably sensible in that there is one period when an exception almost occurs.

5.4 Ratings-Based VaRs for the Total Portfolio and Sub-Portfolios

Figures 4, 5 and 6 show results based on the total portfolio of bonds for which we can implement the ratings-based model. Figure 4 shows the one-year returns for the total portfolio and the ratings-based VaR quantiles. The plot confirms the results in Figure 2 in that there are more exceptions (three in total) than is consistent with unbiased VaR estimates.

In Figures 5 and 6, we split up the total portfolio into different sub-portfolios in order to see which categories of bonds are generating the exceptions. The results in Figure 5 suggest that banks and financials and non-US obligors are much riskier than the ratings-based model suggests. The returns for non-US obligors exhibit a striking eight exceptions in the decade covered by the sample period.

One reason why the ratings-based model under-estimates the riskiness of non-US borrowers (and hence generates excessive numbers of exceptions) might be that it exaggerates the degree to which one can diversify by investing in obligors with different domiciles. We calculated the contributions of covariances and variances to US and non-US portfolio variance as measured by the ratings-based approach. Covariances contributed distinctly less to the ratings-based estimate of the non-US portfolio variance.

When we calculated the standard deviation of returns for the banks and financials portfolio, we found it to be 40% lower than that of the portfolio comprising obligors from other industries. Our finding that the former portfolio yields too many exceptions suggests that the ratings-based model yields volatilities for banks and financials which in relative terms are even lower, however.

Figure 6 shows returns and ratings-based VaR quantiles for four equal-sized sub-portfolios of the total portfolio.[17] Since each of the sub-portfolios exhibits several exceptions, we conclude that the total portfolio results are not the result of a very small number of bonds.

As a final exercise, for each period, we randomly selected (with replacement) 50 portfolios of 10 bonds. For each portfolio, we then calculated the difference between the return and VaR quantiles. Ordering the differences by magnitude, we took the fifth, twenty-fifth and forty-fifth as estimates of the 90% quantile, median and 10% quantile. For each period, these quantities are plotted in Figure 7. To understand how one should interpret the figure, note that when the 10% quantile crosses zero, this means that five or more of the fifty portfolios exhibited exceptions in that period.

It is quite noticeable that the median return-less-VaR is close to zero in the 1998 exception, suggesting that this episode represented a general weakening in the market. Some of the other exceptions such as in 1990 only occurred for a small fraction of the portfolio suggesting that appropriate diversification would reduce the likelihood that the VaR be breached.

6 Conclusion

The literature on the performance of VaR models applied to market risk (see Jackson, Maude, and Perraudin (1997), Pritsker (1996), and Hendricks (1996)) suggests that it is important to explore the sensitivity of the estimates obtained to changes in the techniques employed. In this paper, we conduct the first out-of-sample evaluation of the new class of credit risk models.

Our approach consists of implementing over a ten year period representative examples of ratings- and equity-based credit risk models on large portfolios of Eurobonds.

[17] Bonds are allocated randomly between the four sub-portfolios.

Month-by-month, we calculate the risk measures implied by the models and compare them with the actual out-comes as credit spreads move around. We are careful in each period only to employ lagged data so that the evaluation is genuinely out-of-sample.

Our main findings are that both rating-based models and equity-based models bench-marked to default probabilities tend to under-estimate the riskiness of the bond portfolios. However, if the equity-based model is bench-marked using bond spreads rather than default probabilities, it yields much more conservative and possibly unbiased risk estimates. One might note that current industry practice (see the approach of KMV) is to bench-mark against default probability and that bench-marking against spreads is simply not feasible in the case of loan portfolios as there are no mark-to-market values available.

More detailed examination of the ratings-based model results suggested that the excessive numbers of exceptions came from the non-US and banking and financial obligors in the sample. It seems that the ratings-based model exaggerates the benefits of diversification across obligors from different domiciles.

Our conclusions should of course be interpreted with caution since our sample covers just ten years and the Eurobond portfolios we study may behave differently from other kinds of credit-sensitive portfolios. Nevertheless, our results suggest that wide safety margins should be built into capital allocation decisions and regulatory capital calculations if at a future date they were based on output from the current generation of credit risk models.

Appendix

Proof of Equation (14)

Define $Y \equiv X/D$ and $k \equiv V/D$. The homogeneity of the differential equation for X implies that $X = Y(k)D$. Taking derivatives and substituting in equation (13) yields:

$$rY = \delta(k-1) + (r-\delta)k\frac{\partial Y}{\partial k} + \mu_D\left(Y - \frac{\partial Y}{\partial k}k\right) + \frac{\sigma_v^2}{2}k^2\frac{\partial^2 Y}{\partial k^2} \quad . \tag{24}$$

Solving the differential equation (24) subject to the boundary conditions: $Y(\gamma) = 0$ and $\lim_{k\to\infty} = k - \delta/(r - \mu_D)$, one obtains the expression in equation (14).

The Likelihood for a Bivariate Brownian Motion with Absorbing Barrier

Recall that the logs of the asset to liabilities ratio and the market portfolio comprise a bivariate arithmetic Brownian motion:

$$d\log(k_t) = (r - \delta - \mu_D + \sigma_v\sigma_m\rho - \sigma_v^2/2)dt + \sigma_v dW_{1t} \tag{25}$$

$$d\log(M_t) = (r + \sigma_m^2/2)dt + \sigma_m dW_{2t} \tag{26}$$

where recall that $dW_{1t}dW_{2t} = \rho dt$.

Let the vector process be denoted $x_t \equiv (x_{1t}, x_{2,t}, \ldots, x_{nt})'$ and suppose that:

$$dx_{it} = \mu_i\, dt + \sigma_i\, dB'_{it} \qquad i = 1, 2, \ldots, n \tag{27}$$

B'_{it}, $i = 1, 2, \ldots, n$ are standard Brownian motions and $dB'_{it}dB'_{jt} = \xi_{ij}dt$. Also suppose that x_{1t} is absorbed at a. Let $\psi(x_t, t|x_{t_0})$ be the conditional density of x_t given x_{t_0}. ψ satisfies the following Kolmogorov forward equation:

$$\frac{\partial \psi}{\partial t} = -\sum_{i=1}^{n}\mu_i\frac{\partial \psi}{\partial x_{it}} + \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\sigma_i\sigma_j\xi_{ij}}{2}\frac{\partial^2\psi}{\partial x_{it}\partial x_{jt}} \tag{28}$$

subject to the two boundary conditions are (i) $\psi(a, x_{2t}, \ldots, x_{nt}, t|x_{t_0}) = 0$ for all $(x_{2t}, \ldots x_{nt})$ and t, and (ii) $\psi(x_{t_0}, t_0|x_{t_0}) = \delta(x_{t_0})$, where δ is a Dirac delta function (for discussions of such forward equations, see Cox and Miller (1973)). Henceforth, for simplicity of exposition, we normalize so that $x_{it_0} = 0$ for all i.

Notes on the solution to equation (28) are available on request from the authors. The solution for the bivariate case is

$$\begin{aligned} \psi(x_t, t|0_n) \quad = \quad & |\det(\Sigma)|^{-1} \Big(\exp\{ -0.5\,(x_t - \mu t)'\,(1/t)\,\Sigma^{-1}\,(x_t - \mu t) \} \quad - \\ & \exp(\zeta)\exp\{ -1/2\,(x_t - \phi - \mu t)'\,(1/t)\ \Sigma^{-1}\,(x_t - \phi - \mu t) \} \Big) \end{aligned} \tag{29}$$

where 0_n is an n-vector of zeros, $\phi_1 = 2a$, and $\phi_2 = -2a\sigma_v\rho/\sigma_m$ and $\zeta \equiv -\phi'\Sigma^{-1}\mu$, $\mu \equiv (\mu_1, \mu_2)'$ and

$$\Sigma \quad \equiv \quad \begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2\xi \\ \sigma_1\sigma_2\xi & \sigma_2^2 \end{bmatrix} \qquad . \tag{30}$$

References

ALTMAN, E. I., AND D. L. KAO (1992): "Rating Drift of High Yield Bonds," *Journal of Fixed Income*, pp. 15–20.

BASEL COMMITTEE (1999): "Credit Risk Modelling: Current Practices and Applications," Report 49, Bank for International Settlements, Basel, Switzerland.

BASEL COMMITTEE ON BANKING SUPERVISION (1996): "Overview of the Amendment to the Capital Accord to Incorporate Market Risk," unpublished mimeo, Bank for International Settlements, Basel.

CARTY, L. V. (1997): "Moody's Rating Migration and Credit Quality Correlation, 1920-1996," Special comment, Moody's Investors Service, New York.

COX, D., AND MILLER (1973): *The Theory of Stochastic Processes.*

CREDIT SUISSE FINANCIAL PRODUCTS (1997): "Credit Risk+ : Technical Manual," Discussion paper, CSFP.

CROUHY, M., D. GALAI, AND R. MARK (1999): "A Comparative Analysis of Current Credit Risk Models," *Journal of Banking and Finance*, forthcoming.

DUFFEE, G. R. (1999): "The Relationship Between Treasury Yields and Corporate Bond Yield Spreads," *Journal of Finance*, forthcoming.

GORDY, M. (1999): "A Comparative Anatomy of Credit Risk Models," *Journal of Banking and Finance*, forthcoming.

HENDRICKS, D. (1996): "Evaluation of Value-at-Risk Models Using Historical Data," *Federal Reserve Bank of New York Economic Policy Review*, 2(1), 39–69.

JACKSON, P. D., D. MAUDE, AND W. R. PERRAUDIN (1997): "Bank Capital and Value at Risk," *Journal of Derivatives*, 4(3), 73–89.

JACKSON, P. D., AND W. R. PERRAUDIN (1999): "Regulatory Implications of Credit Risk Modeling," *Journal of Banking and Finance*, forthcoming.

JONES, D. (1999): "Emerging Problems with the Accord: Regulatory Capital Arbitrage and Related Issues," *Journal of Banking and Finance*, forthcoming.

JONES, E. P., S. P. MASON, AND E. ROSENFELD (1984): "Contingent Claims Analysis of Corporate Capital Structures: An Empirical Investigation," *Journal of Finance*, 39(3), 611–627.

JP MORGAN (1997): *Creditmetrics-Technical Document.* JP Morgan, New York.

KIESEL, R., W. R. PERRAUDIN, AND A. TAYLOR (1999): "The Structure of Credit Risk," Working paper, Birkbeck College, London.

LOPEZ, J. A., AND M. R. SAIDENBERG (1999): "Evaluating Credit Risk Models," *Journal of Banking and Finance*, forthcoming.

LUCAS, D. J., AND J. G. LONSKI (1992): "Changes In Corporate Credit Quality 1970-1990," *Journal of Fixed Income*, pp. 7–14.

MINGO, J. (1999): "Policy Implications of the Federal Reserve Study of Credit Risk Models at Major U.S. Banking Institutions," *Journal of Banking and Finance*, forthcoming.

MORRIS, C., R. NEAL, AND D. ROLPH (1999): "Credit Spreads and Interest Rates," mimeo, Indiana University, Indianapolis.

NICKELL, P., W. R. PERRAUDIN, AND S. VAROTTO (1999): "The Stability of Ratings Transitions," *Journal of Banking and Finance*, forthcoming.

PRITSKER, M. (1996): "Evaluating Value at Risk Methodologies: Accuracy versus Computational Time," Working paper, Federal Reserve Board, Washington, DC.

STUART, A., AND K. ORD (1994): *Kendall's Advanced Theory of Statistics: Distribution Theory*, vol. 1. Edward Arnold, London, sixth edn.

WILSON, T. (1997): "Credit Risk Modelling: A New Approach," unpublished mimeo, McKinsey Inc., New York.

Table 1: TOTAL PORTFOLIO CHARACTERISTICS

Domicile	No.	%	Sector	No.	%	Seniority	No.	%
US	647	45.24	Financial Services	603	42.17	Unsecured	836	58.46
Japan	128	8.95	Banking	347	24.27	Guaranteed	274	19.16
Netherlands	106	7.41	Utilities, Elect.+Gas	70	4.90	Senior	201	14.06
Germany	77	5.38	Energy Sources	49	3.43	Government	58	4.06
France	75	5.24	Telecomm.	49	3.43	Secured	23	1.61
UK	66	4.62	Beverage+Tobacco	33	2.31	Subordinated	21	1.47
Neth. Antilles	41	2.87	Health+Personal Care	30	2.10	Mortgaged	13	0.91
Brazil	33	2.31	Bus.+Public Services	28	1.96	Collateralized	4	0.28
Cayman Is.	25	1.75	Merchandizing	27	1.89	–	–	–
Mexico	25	1.75	Food+Hshld Product	20	1.40	–	–	–
Australia	23	1.61	Multi-Industry	19	1.33	–	–	–
Others	184	12.87	Other	155	10.84	–	–	–

Note: The whole sample consists of all, liquid dollar-denominated Eurobonds available in each period.

Table 2: HIGHER MOMENTS OF BOND PORTFOLIO RETURNS

With Interest Rate Risk				
Portfolio†	Skewness	S.E.	Kurtosis	S.E.
Total Portfolio	-0.714	(0.669)	3.583	(1.211)
Banks and Financials	-0.772	(0.631)	3.647	(1.215)
Non Banks and Financials	-0.568	(0.736)	3.448	(1.193)
US Obligors	-0.630	(0.689)	3.428	(1.164)
Non US Obligors	-0.759	(0.650)	3.692	(1.250)
1st Quartile	-0.640	(0.687)	3.576	(1.199)
2nd Quartile	-0.687	(0.670)	3.510	(1.181)
3rd Quartile	-0.733	(0.653)	3.580	(1.199)
4th Quartile	-0.724	(0.676)	3.630	(1.245)
Adjusted to Remove Interest Rate Risk				
Portfolio†	Skewness	S.E.	Kurtosis	S.E.
Total Portfolio	-0.151	(0.463)	2.546	(0.749)
Banks and Financials	-0.185	(0.461)	2.883	(0.924)
Non Banks and Financials	-0.306	(0.533)	2.624	(0.784)
US Obligors	-0.612	(0.672)	3.722	(1.374)
Non US Obligors	0.138	(0.699)	3.504	(1.222)
1st Quartile	-0.053	(0.465)	2.843	(0.845)
2nd Quartile	-0.330	(0.470)	2.922	(0.911)
3rd Quartile	-0.008	(0.367)	2.403	(0.680)
4th Quartile	-0.475	(0.494)	2.836	(0.863)

† For portfolio definitions, see notes to Table 1.

Notes:

Returns are over-lapping one year returns measured monthly from January 1988 to January 1987.

Skewness and leptokurtosis are measured as 3rd and 4th moments divided by variances to the powers 3/2 and 2.

Figure 1: Normal and Actual Return Distributions

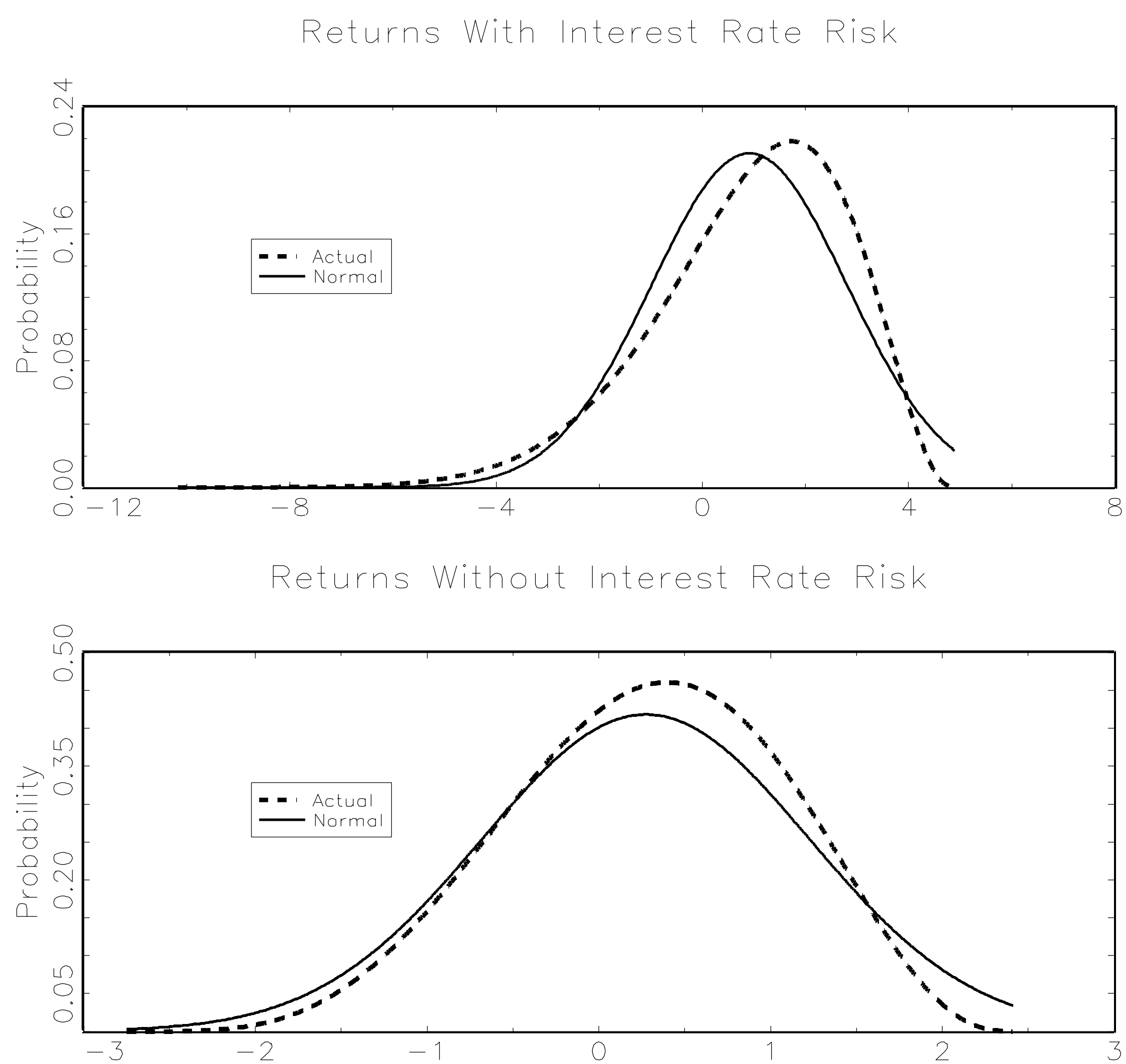


Figure 2: Returns and VaRs for Total Portfolio

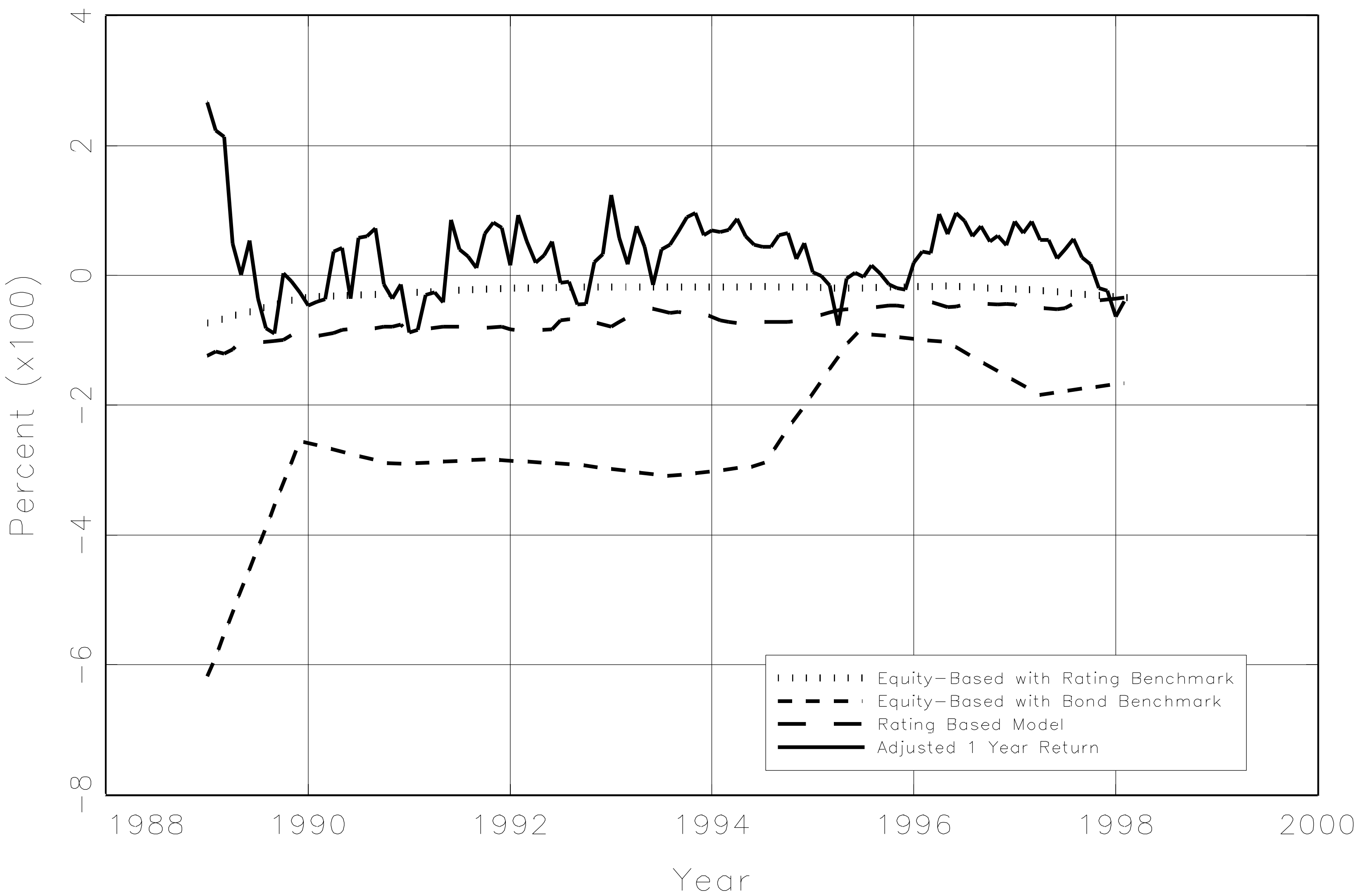


Figure 3: Returns and VaRs for Fifty Lowest Rated Bonds

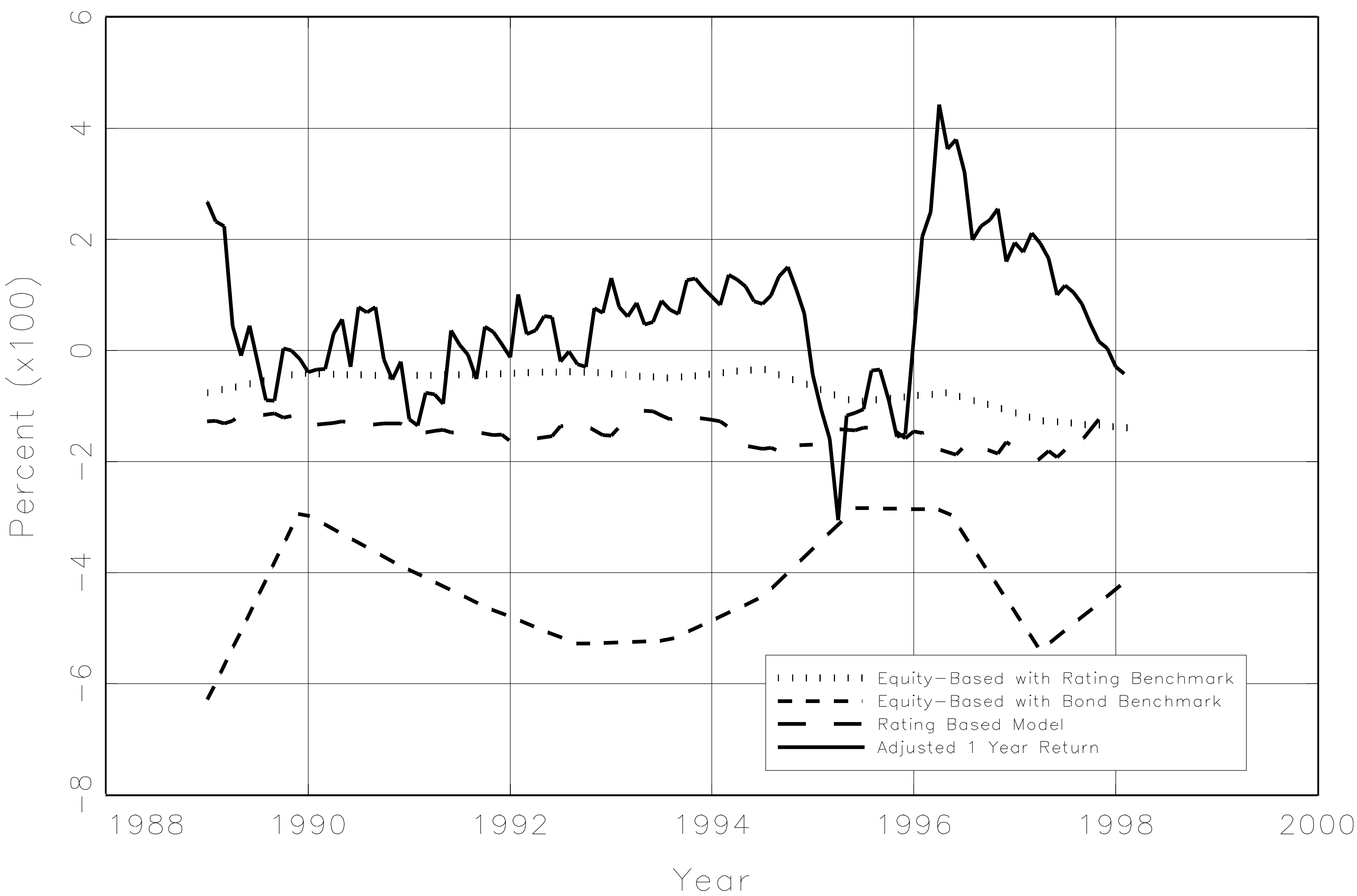


Figure 4: Total Portfolio Returns and Ratings-Based VaR

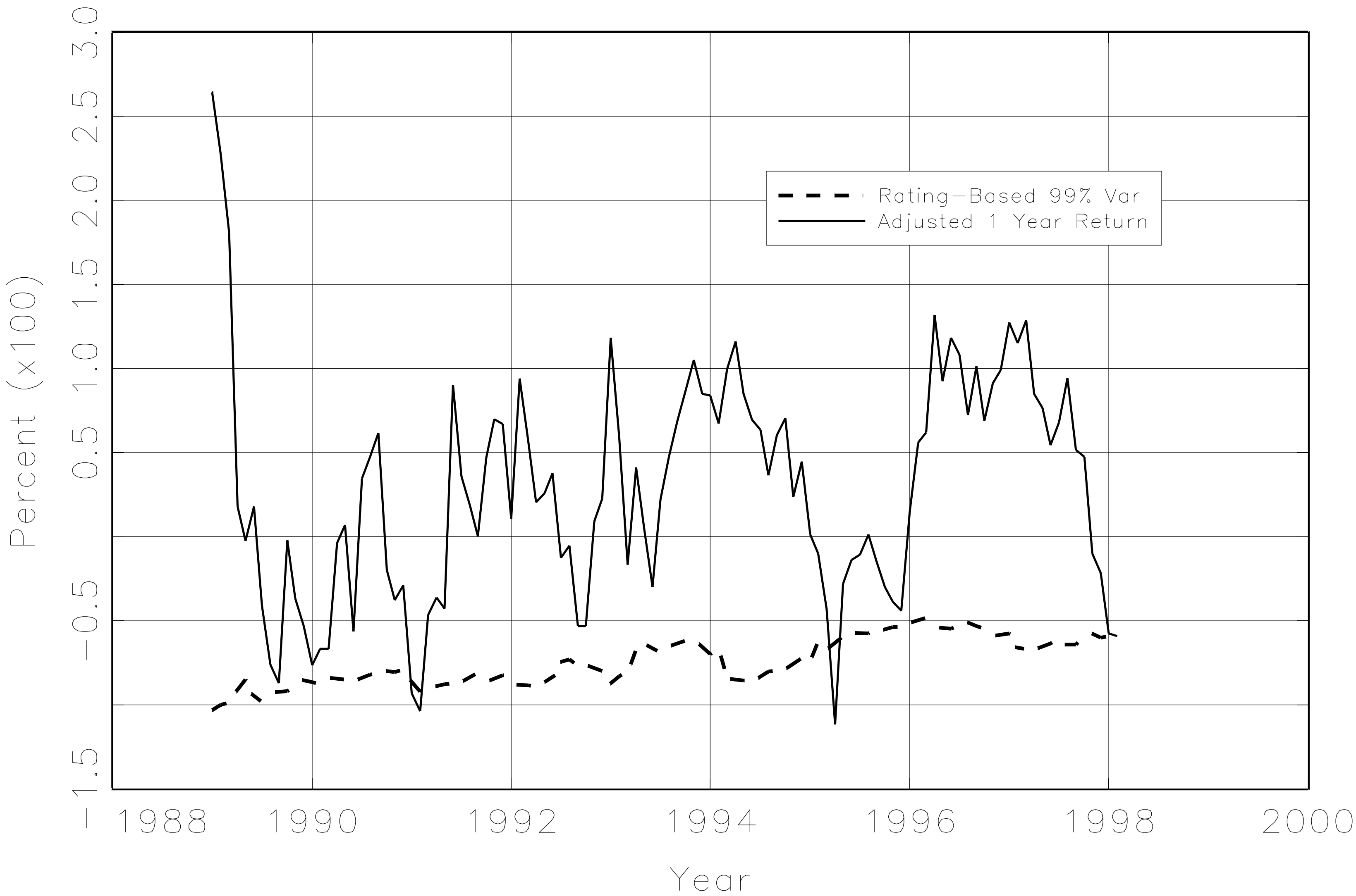


Figure 5: Industry and Domicile: Ratings-Based VaRs

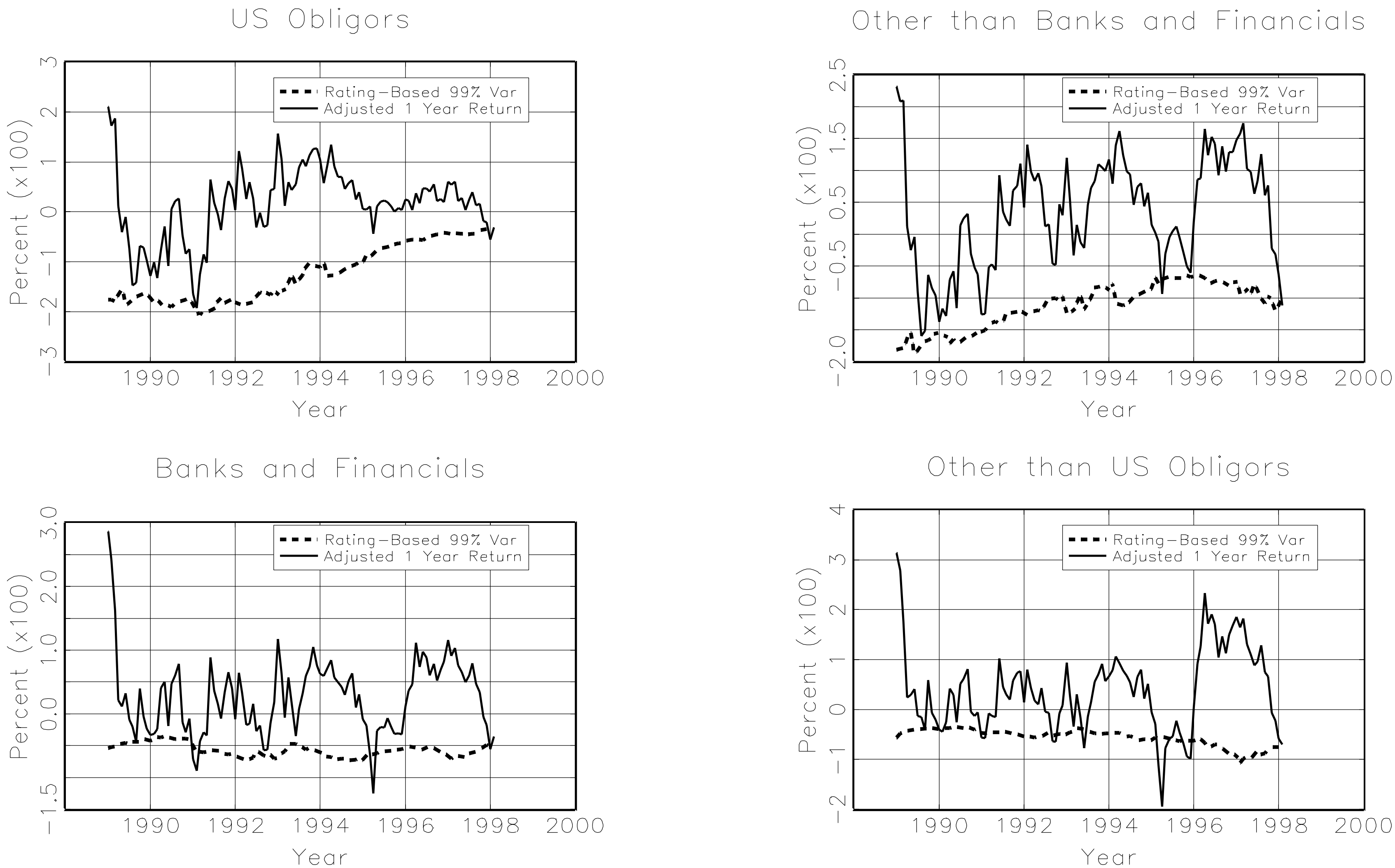
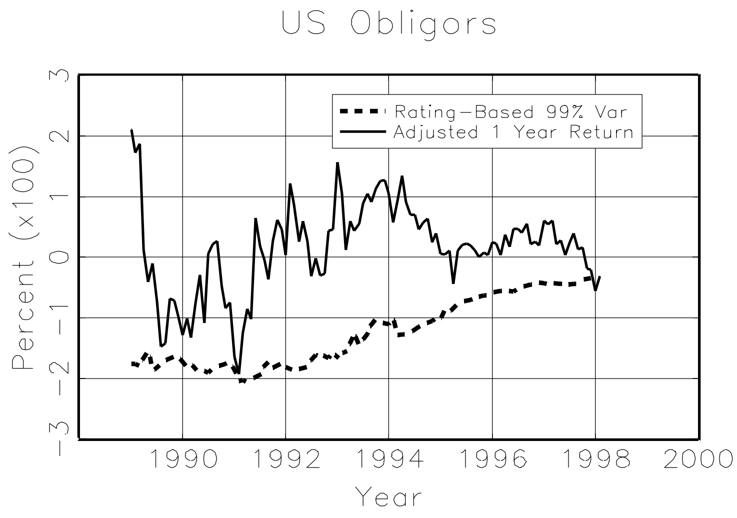
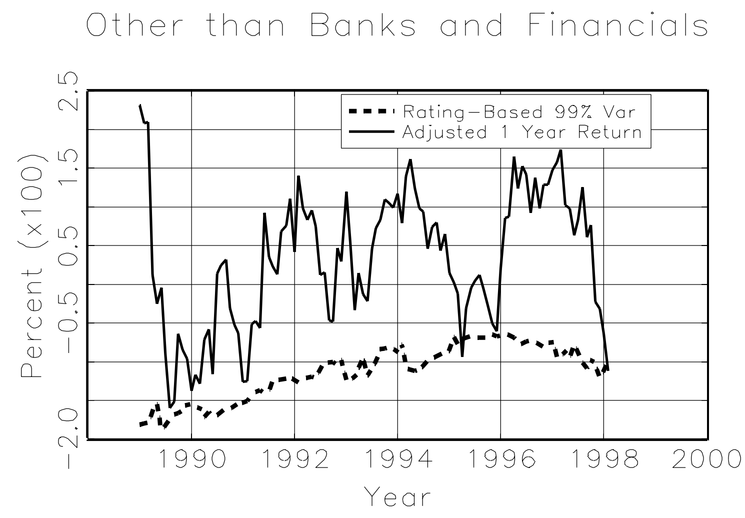
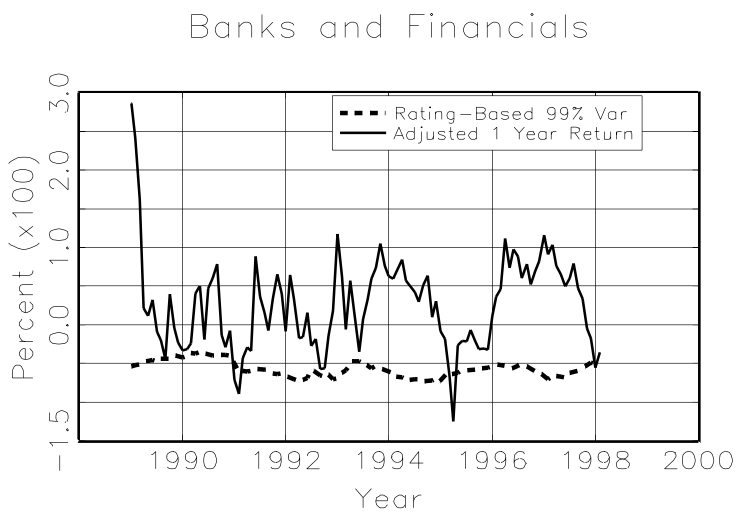


Figure 5: Industry and Domicile: Ratings-Based VaRs







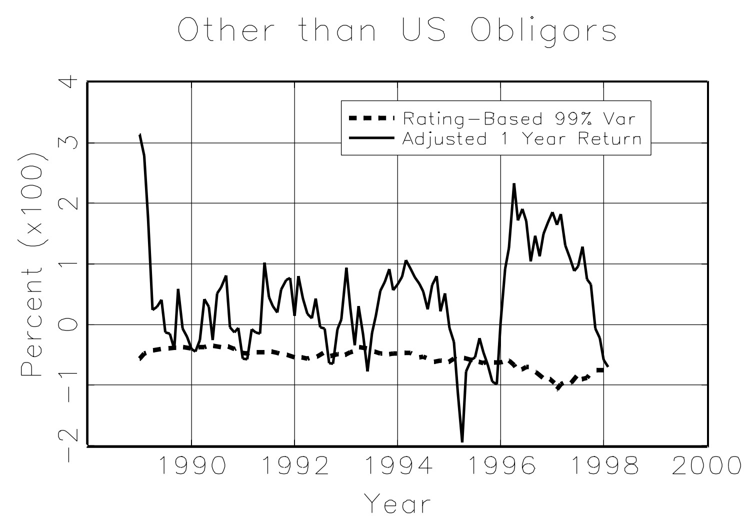
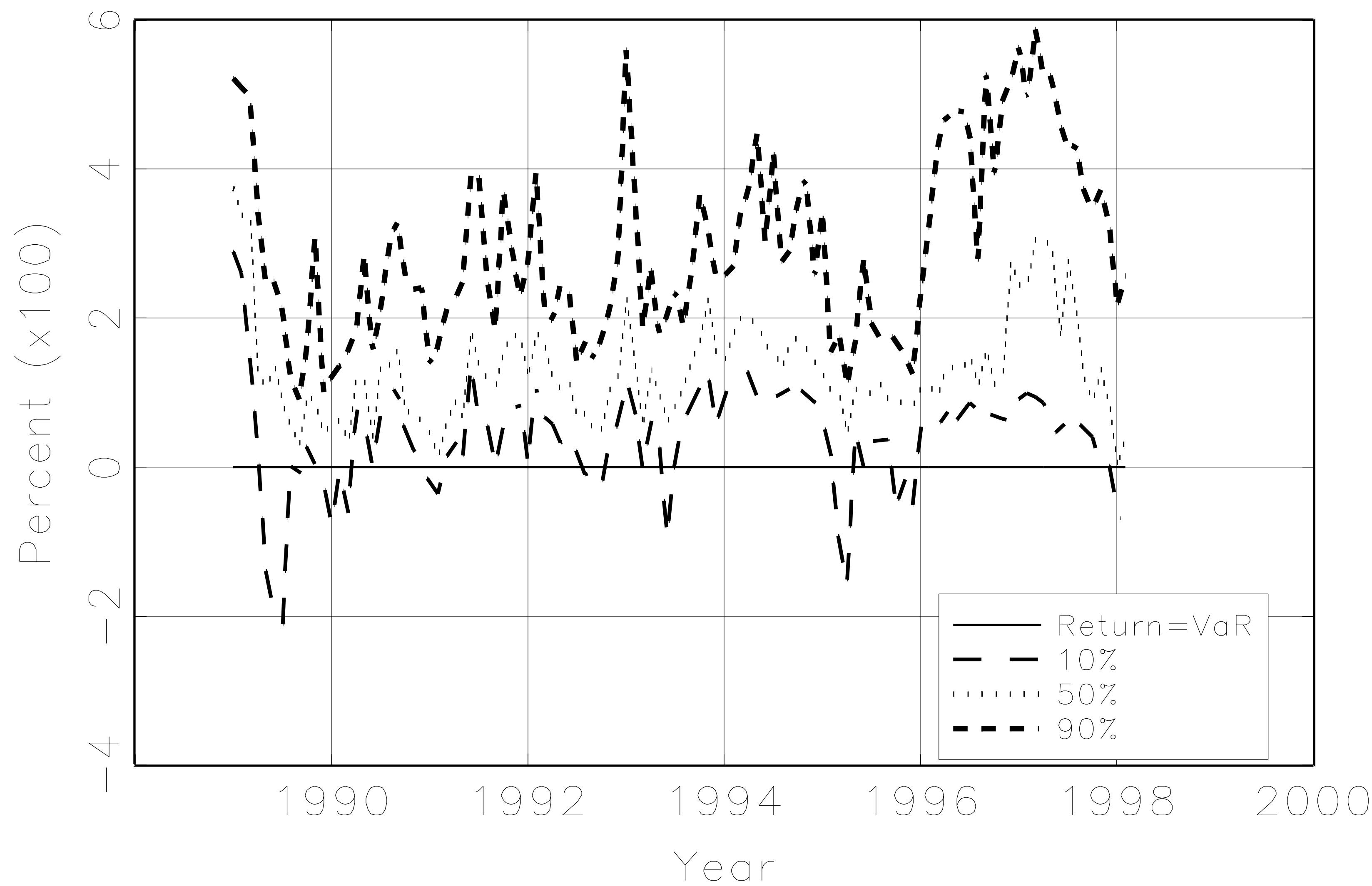


Figure 7: Quantiles of Return-less-Var-Loss



Notes for Convenience of the Referees

We now derive the exact conditional density of a vector Brownian motion when one of the processes is absorbed at a given level. Our derivation employs the Method of Images for solving partial differential equations. Let the vector process be denoted $x_t \equiv (x_{1t}, x_{2,t}, \ldots, x_{nt})'$ and suppose that:

$$dx_{it} \quad = \quad \mu_i \, dt \quad + \quad \sigma_i \, dB'_{it} \qquad i = 1, 2, \ldots, n \tag{31}$$

B'_{it}, $i = 1, 2, \ldots, n$ are standard Brownian motions and $dB'_{it} dB'_{jt} = \xi_{ij} dt$. Also suppose that x_{1t} is absorbed at a. Let $\psi(x_t, t | x_{t_0})$ be the conditional density of x_t given x_{t_0}. ψ satisfies the following Kolmogorov forward equation:

$$\frac{\partial \psi}{\partial t} \quad = \quad - \sum_{i=1}^{n} \mu_i \frac{\partial \psi}{\partial x_{it}} \quad + \quad \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\sigma_i \sigma_j \xi_{ij}}{2} \frac{\partial^2 \psi}{\partial x_{it} \partial x_{jt}} \tag{32}$$

subject to the two boundary conditions are (i) $\psi(a, x_{2t}, \ldots, x_{nt}, t | x_{t_0}) = 0$ for all $(x_{2t}, \ldots x_{nt})$ and t, and (ii) $\psi(x_{t_0}, t_0 | x_{t_0}) = \delta(x_{t_0})$, where δ is a Dirac delta function (for discussions of such forward equations, see Cox and Miller (1973)). Henceforth, for simplicity of exposition, we normalize so that $x_{it_0} = 0$ for all i.

The intuition behind the first boundary condition is as follows. The infinite variability of the process implies that if one is extremely close to a barrier then one will hit that barrier in the immediate future with probability one. Hence, for any fixed t, as x_{1t} approaches a, it is increasingly unlikely that the process x_1 has not hit a before t. In other words, the probability of being close to a for $t > 0$ goes to zero.

Note that the above partial differential equation (PDE) is homogeneous so, for any two solutions to the PDE, linear combinations will also solve the equation. It is straightforward to show that the PDE is satisfied by a standard multivariate normal density function with means $\mu_i t$ and covariance matrix, Σ.

$$\psi^{(0)} \quad \equiv \quad |\det(\Sigma)|^{-1} \; \exp\{ \; -0.5 \, (x_t - x_{t_0} - \mu t)' \, \frac{1}{t} \, \Sigma^{-1} \, (x_t - x_{t_0} - \mu t) \; \} \tag{33}$$

Here, Σ has diagonal elements $\sigma_i^2 t$ and off-diagonal elements $\xi_{ij} \sigma_i \sigma_j t$. But, multivariate normal densities with the vector mean $\mu t + \phi$ for some vector of constants ϕ will *also* satisfy the PDE. Let $\psi^{(\phi)}$ denote such a multivariate density.

We hypothesize that there exists a particular vector of constants, termed sources, ϕ, such that, for some constant weight ζ, $\psi^{(0)} - \exp(\zeta)\psi^{(\phi)}$ solves the PDE subject to *both* boundary conditions.

Now, for $\psi^{(0)} - \exp(\zeta)\psi^{(\phi)}$ to satisfy (i), it must be the case that, evaluated at $x_t = a$, the difference between the quadratic forms in the normal densities, $\psi^{(0)}$ and $\psi^{(\phi)}$ is a constant, independent of t and $x_{2t}, x_{3t}, \ldots, x_{nt}$. To pursue this idea, consider the difference:

$$(x-\phi-\mu t)'t^{-1}\Sigma^{-1}(x-\phi-\mu t)-(x-\mu t)'t^{-1}\Sigma^{-1}(x-\mu t) = \phi' t^{-1}\Sigma^{-1}\phi-2\phi t^{-1}\Sigma^{-1}(x-\mu t) \tag{34}$$

If we can choose ϕ so that, when $x_{1t} = a$, the right hand side is a constant, ζ, independent of t and $x_2, \ldots, x_n$, then $\psi^{(0)} - \exp(\zeta)\psi^{(\phi)}$ will satisfy (i) as required. One way to think about the derivation is that we are choosing the sources so as to eliminate terms in x_2 and $1/t$ in the difference between the quadratic forms. The remaining constant will give us the weight, ζ. Let $\sigma_1^*, \sigma_2^*, \ldots \sigma_n^*$ denote the column vectors of the inverse covariance matrix Σ^{-1}. First, to eliminate the x_2 terms, one must choose ϕ so that:

$$\phi' \; [\sigma_2^*|\sigma_3^*|\ldots|\sigma_n^*] \quad = \quad 0_{n-1} \tag{35}$$

where 0_{n-1} is a 1x$(n-1)$ vector of zeros. To eliminate terms in $1/t$, one must have:

$$\phi' \, \Sigma \, \phi \quad = \quad 2\,\phi' \sigma_1^* \, a \tag{36}$$

The above equations imply:

$$\phi' \, \sigma_1^* \, \phi_1 \quad = \quad 2\,\phi' \sigma_1^* \, a \tag{37}$$

Hence, $\phi_1 = 2a$. Substituting in the above and rearranging yields:

$$\begin{bmatrix} \phi_2 \\ \vdots \\ \phi_n \end{bmatrix} \quad = \quad -2a \begin{bmatrix} \sigma_{22}^* & \ldots & \sigma_{2n}^* \\ \vdots & \ddots & \vdots \\ \sigma_{n2} & \ldots & \sigma_{nn}^* \end{bmatrix}^{-1} \begin{bmatrix} \sigma_{12}^* \\ \vdots \\ \sigma_{1n}^* \end{bmatrix} \tag{38}$$

The only remaining terms in the difference between the quadratic forms is the constant, i.e., the scalar: $2\phi'\Sigma^{-1}\mu$. Setting: $\zeta \equiv -\phi'\Sigma^{-1}\mu$, apart from an unimportant scaling constant, we then have the bivariate density:

$$\begin{aligned}\psi(x_t, t|0_n) \quad = \quad & |\det(\Sigma)|^{-1} \Big(\exp\{ -0.5\,(x_t - \mu t)'\,(1/t)\,\Sigma^{-1}\,(x_t - \mu t) \} \quad - \\ & \exp(\zeta)\exp\{ -1/2\,(x_t - \phi - \mu t)'\,(1/t)\;\Sigma^{-1}\,(x_t - \phi - \mu t) \} \Big) \qquad (39)\end{aligned}$$

where $\phi_1 = 2a$, and $\phi_2 = -2a\sigma_1\xi/\sigma_2$ and $\zeta \equiv -\phi'\Sigma^{-1}\mu$.