STABLE MODELING OF CREDIT RISK

BY SVETLOZAR RACHEV, EDUARDO SCHWARTZ,
AND IRINA KHINDANOVA[1]

University of Karlsruhe, Germany,
University of California, Los Angeles,
and University of California, Santa Barbara,

We suggest the use of stable distributions in modeling credit risk. In this paper we investigate risk evaluation for credit instruments based on stable modeling of their returns. We develop a new technique for the correlation estimation, construct a new method for simulating portfolio values, and assess portfolio risk in various cases of the credit instruments' distributions: independent, symmetric dependent, and skewed dependent. We construct a one-factor model of credit risks, apply stable modeling to extracted credit risk spreads, and estimate portfolio credit risk. The suggested stable modeling can be employed for risk estimation of any financial assets, not only credit instruments, if their distributions are heavy-tailed and/or skewed.

Key words and phrases. Correlation estimation, Credit instruments, Credit returns, Credit risk, Portfolio risk, Skewed and Heavy-tailed distributions, Stable distributions, Risk evaluation.

[1] We thank C. Marinelli of University of California, Santa Barbara, and, especially, B. Racheva-Iotova of Bravo Consulting Group for computational assistance. We also thank Kristina Tetereva of University of Karlsruhe, Germany, for providing data.

Table of Contents

1. Introduction

We investigate risk evaluation for credit instruments. The risk exposure can be measured by changes in the portfolio value, or profits and losses. Studies show that distributions of value changes of credit instruments are skewed and heavy-tailed[2]. There is no consensus among analysts regarding a "standard" model for such distributions. We suggest the use of stable distributions for modeling credit value changes.

This work demonstrates advantages of stable modeling of credit risk, which are based on properties of stable laws. The stable distributions are described by four parameters: α - tail index, β - skewness, μ - location, and σ - scale. Modeling with such parameters will depict fat tails and skewness of distributions. Our empirical analysis confirms that, indeed, stable modeling captures asymmetry and heavy-tailedness of credit returns, and, therefore, produces accurate risk estimates. The stable distributions possess the additivity property: a linear combination of independent stable (or jointly stable) random variables with stability index α is again a stable random variable with the same α. The additivity property provides analytic formulas for parameters of portfolio returns. In the case of independent instruments, the formulas are simple and can be used for estimating portfolio risk without simulations. An analyst can employ "independent" risk measurements as lower bounds of portfolio risk estimates. A symmetric stable random variable can be interpreted as a transformation of a normal random variable. Based on this property, we develop a new technique for correlation estimation. A stable random variable can be decomposed into the "symmetry" and "skewness" parts. Building on this feature, we construct a new method for simulating a distribution of portfolio values. We utilize this method in portfolio risk evaluation.

A focus of this article is the risk management aspect of credit instruments. With such a motivation in mind, we analyze both total and credit risks of credit instruments. Total risks encompass interest rate, credit, liquidity, and other risks. Credit risks refer to potential losses, which might arise from the counterparty's credit quality changes, including rating migrations and default. Existing approaches to modeling credit risk can

[2] Gupton, Finger, and Bhatia (1997), Federal Reserve System Task Force on Internal Credit Risk Models (1998), Basle Committee on Banking Supervision (1999).

be divided into three groups: *structural, reduced-form,* and *hybrid*[3]. The models value credit risk following the same basic procedures: (i) assessment of the firm's financial position; (ii) derivation of default probabilities; (iii) pricing of a risky asset or estimation of credit losses. The structural models assume default is a predictable event and is determined by the dynamics of a firm's assets. Default probabilities are derived endogenously[4]. The reduced-form models suppose default is an unforeseeable event and do not relate directly default and the firm's assets. Default probabilities are described as an exogenous process or assumed to be given exogenously[5]. Hybrid models combine features of both structural and reduced-form approaches. An example of a hybrid model is the CreditMetrics™ product of J.P. Morgan[6]. It is based on the rating transition model of Jarrow, Lando, and Turnbull (1997) and assumptions that joint credit quality changes are driven by joint movements of firms' assets values. Our approach, stable modeling, is based on analysis of prices of credit instruments.

We apply the stable modeling for portfolio risk assessment in various cases of the credit instruments' distributions: independent, symmetric dependent, and skewed dependent. In evaluation of portfolio risk we employ our new correlation estimation and portfolio simulation methods, which are based on properties of stable distributions. We construct a *one-factor* model for estimating portfolio credit risk. The model is built on the two postulations: (i) constituent parts of the credit returns are the credit-risk-free part and the credit risk premium; (ii) the credit risk spread follows a stable law. Applying the one-factor model, we quantify credit risk for single instruments and then estimate portfolio credit risk as a cumulative result of stable distributed individual credit risks.

The paper is structured as follows. Section 2 investigates stable modeling of credit returns and discusses risk assessment for individual credit instruments. Section 3 considers portfolio risk estimation for independent portfolio assets and derives lower bounds for risk measurements. Sections 4 and 5 present, respectively, evaluation of

---

[3] Surveys of credit risk models can be found in Altman and Saunders (1998), Federal Reserve System Task Force on Internal Credit Risk Models (1998), Ammann (1999), Knoch and Rachev (1999), Basle Committee on Banking Supervision (1999).

[4] For references on structural models, see Merton (1974), Black and Cox (1976), Longstaff and Schwartz (1995), Zhou (1997), Crosbie (1998), and other papers.

[5] Reduced-form models are presented in Fons (1994), Madan and Unal (1994), Jarrow and Turnbull (1995), Nielsen and Ronn (1996), Jarrow, Lando, and Turnbull (1997), Duffee (1999), Duffie and Singleton (1999).

portfolio risk in two cases of dependent portfolio instruments': symmetric and skewed. Section 6 describes a main framework of the one-factor model. Section 7 discusses credit risk evaluation for portfolio assets. Section 8 explains portfolio credit risk estimation. Section 8 states conclusions.

2. Stable Modeling and Risk Assessment for Individual Credit Returns

We advocate modeling of credit returns by stable distributions. Recall that the stable distributions are characterized by four parameters: α-tail index, β-skewness, μ-location, and σ-scale. The modeling with such parameters will depict heavy tails and skewness of the distributions. Our empirical analysis confirms that, indeed: (i) credit returns exhibit asymmetry and heavy-tails; (ii) stable modeling captures these features of the returns.

We apply stable modeling to the Merrill Lynch indices of the US government and corporate bonds with maturities from one to 10 years and credit ratings from "BB" to "AAA". We suppose that returns on indices are stable-distributed: $R_i \sim S_{\alpha_i}(\sigma_i, \beta_i, \mu_i)$, where i=1,…,21. A description of the analyzed indices is provided in Table 1. Daily returns series of indices are illustrated on Figures 1 and in Appendix A. As a benchmark for assessment of the stable model properties, we chose a "normal" model, i.e. approximation of returns by normal distributions. By categorization of stable distributions, a normal distribution has a tail index α=2 and a symmetric distribution has a skewness parameter β=0. Values of α<2 indicate thicker tails than the tails of the normal distribution. In general, the smaller α is, the tails are heavier and the peak of the density is higher. If β<0, the distribution is skewed to the left. If β>0, the distribution is skewed to the right. Larger magnitudes of β point to stronger skewness. The stable and normal parameter estimates for the analyzed bond indices are presented in Table 1. It reports the following α and β values of the indices with a matutity band from one to three years: government bonds: α=1.696 and β=-0.160; AAA bonds: α=1.654 and β=-0.080; AA bonds: α=1.686 and β=-0.105; A bonds: α=1.688 and β=-0.135; BBB bonds: α= 1.653 and β=-0.113; BB bonds: α=1.686 and β=-0.252. For all 17 considered bond

[6] See Gupton, Finger, and Bhatia (1997).

indices, the tail index α is less than two, which reveals heavy-tailedness, and the skewness parameter β is below zero, which implies skewness to the left. We display fitted empirical, stable, and normal densities of indices on Figures 2 and in Appendix A.

Table 1. Normal and Stable Parameter Estimates of Bond Indices

Index*	Rating or Issuer	Maturity (years)	Normal		Stable			
			Mean	Standard deviation	Tail index α	Skew-ness β	Loca-tion μ	Scale σ
G102	US gov-t	1-3	0.026	0.096	1.696	-0.160	0.029	0.055
G202	US gov-t	3-5	0.030	0.204	1.739	-0.134	0.036	0.122
G302	US gov-t	5-7	0.032	0.275	1.781	-0.134	0.032	0.169
G402	US gov-t	7-10	0.033	0.352	1.808	-0.172	0.033	0.218
C1A1	AAA	1-3	0.027	0.096	1.654	-0.080	0.053	0.027
C2A1	AAA	3-5	0.029	0.175	1.695	-0.112	0.029	0.099
C3A1	AAA	5-7	0.032	0.249	1.710	-0.116	0.031	0.145
C4A1	AAA	7-10	0.032	0.319	1.739	-0.155	0.031	0.190
C1A2	AA	1-3	0.028	0.099	1.686	-0.105	0.027	0.056
C2A2	AA	3-5	0.029	0.177	1.722	-0.111	0.029	0.104
C3A2	AA	5-7	0.032	0.250	1.757	-0.121	0.032	0.150
C4A2	AA	7-10	0.033	0.325	1.778	-0.148	0.033	0.198
C1A3	A	1-3	0.028	0.098	1.688	-0.135	0.027	0.056
C2A3	A	3-5	0.030	0.180	1.702	-0.122	0.029	0.104
C3A3	A	5-7	0.032	0.255	1.743	-0.133	0.033	0.151
C4A3	A	7-10	0.033	0.333	1.753	-0.167	0.033	0.199
C1A4	BBB	1-3	0.029	0.112	1.653	-0.113	0.029	0.054
C2A4	BBB	3-5	0.032	0.183	1.662	-0.042	0.033	0.096

C3A4	BBB	5-7	0.034	0.249	1.690	-0.125	0.035	0.140
C4A4	BBB	7-10	0.035	0.316	1.694	-0.136	0.035	0.180
H0A1	BB	1-3	0.027	0.185	1.686	-0.252	0.042	0.104

*Each index set, except H0A1, includes 2418 daily observations from 3.13.90 to 7.29.99. Source of index series: Merrill Lynch, used with permission.

In order to assess riskiness of the individual credit series and properties of stable modeling in the risk evaluation, we estimated the 99% and 95% Value at Risk (VaR) measurements. Since credit returns have skewed and heavy-tailed distributions, VaR mesurements provide more adequate indication of risk than symmetric measurements (standard deviation or, in case of stable distributions, scale parameter) do. By definition, the c% VaR measurement, $VaR_{c\%}$, is a value such that

$Pr[R<-VaR_{c\%}]=1-c$,

where R is the returns and c is the VaR confidence level. In other words, the probability that losses will exceed $VaR_{c\%}$ is equal to 1-c. $VaR_{c\%}$ can be found as the negative of the (1-c)th quantile. The stable and normal VaR estimates are reported in Table 2. We obtain the normal VaR measurements for comparison purposes. The differences between empirical and modeled VaR, are given in Appendix B, Table B.1. The VaR evaluation for the bond indices is illustrated on Figures 3 and in Appendix A. Results of VaR estimations lead to the following conclusions[7]:

- the stable modeling produces conservative and accurate 99% VaR estimates, which is preferred by financial institutions and regulators. “Conservative” VaR estimates exceed empirical VaR, which implies that forecasts of losses were greater than observed losses,
- the stable modeling underestimates the 95% VaR,
- the normal modeling gives overly optimistic forecasts of losses in the 99% VaR estimation,
- the normal modeling is acceptable for the 95% VaR estimation.

[7] In this paper we compute “in-sample” VaR. Hence, our conclusions discuss in-sample VaR properties. Khindanova, Rachev, and Schwartz (1999) have examined forecasting VaR properties and reported excellent performance of stable modeling in forecast-evaluation of VaR, measurements of potential losses.

Table 2. Empirical, Normal, and Stable VaR Estimates for Bond Indices

Index	99% VaR estimates			95% VaR estimates		
	Empirical	Normal	Stable	Empirical	Normal	Stable
G102	0.242	0.198	0.275	0.127	0.132	0.119
G202	0.518	0.446	0.576	0.303	0.306	0.283
G302	0.739	0.609	0.747	0.412	0.421	0.399
G402	0.928	0.785	0.932	0.545	0.545	0.518
C1A1	0.238	0.196	0.284	0.129	0.130	0.119
C2A1	0.437	0.377	0.509	0.244	0.258	0.236
C3A1	0.687	0.548	0.734	0.369	0.378	0.353
C4A1	0.883	0.712	0.931	0.480	0.494	0.467
C1A2	0.237	0.201	0.285	0.132	0.134	0.125
C2A2	0.443	0.382	0.505	0.254	0.261	0.244
C3A2	0.663	0.550	0.689	0.373	0.380	0.355
C4A2	0.870	0.722	0.890	0.482	0.501	0.474
C1A3	0.237	0.207	0.286	0.135	0.134	0.125
C2A3	0.469	0.390	0.530	0.260	0.267	0.248
C3A3	0.705	0.560	0.719	0.376	0.386	0.361
C4A3	0.893	0.741	0.949	0.487	0.514	0.485
C1A4	0.262	0.231	0.290	0.124	0.155	0.119
C2A4	0.478	0.392	0.511	0.243	0.268	0.228
C3A4	0.711	0.545	0.741	0.361	0.375	0.343
C4A4	0.862	0.702	0.960	0.467	0.486	0.451
H0A1	0.557	0.403	0.570	0.258	0.277	0.245

We explain superiority of the stable modeling for high values of the VaR confidence level by the fact that it adequately describes heavy tails and skewness in the data. Our empirical analysis demonstrates advantages of stable modeling in evaluation of riskiness of single credit returns series. The next step is to examine properties of stable modeling in evaluation of portfolio risk.

Figure 1. H0A1 Daily Returns

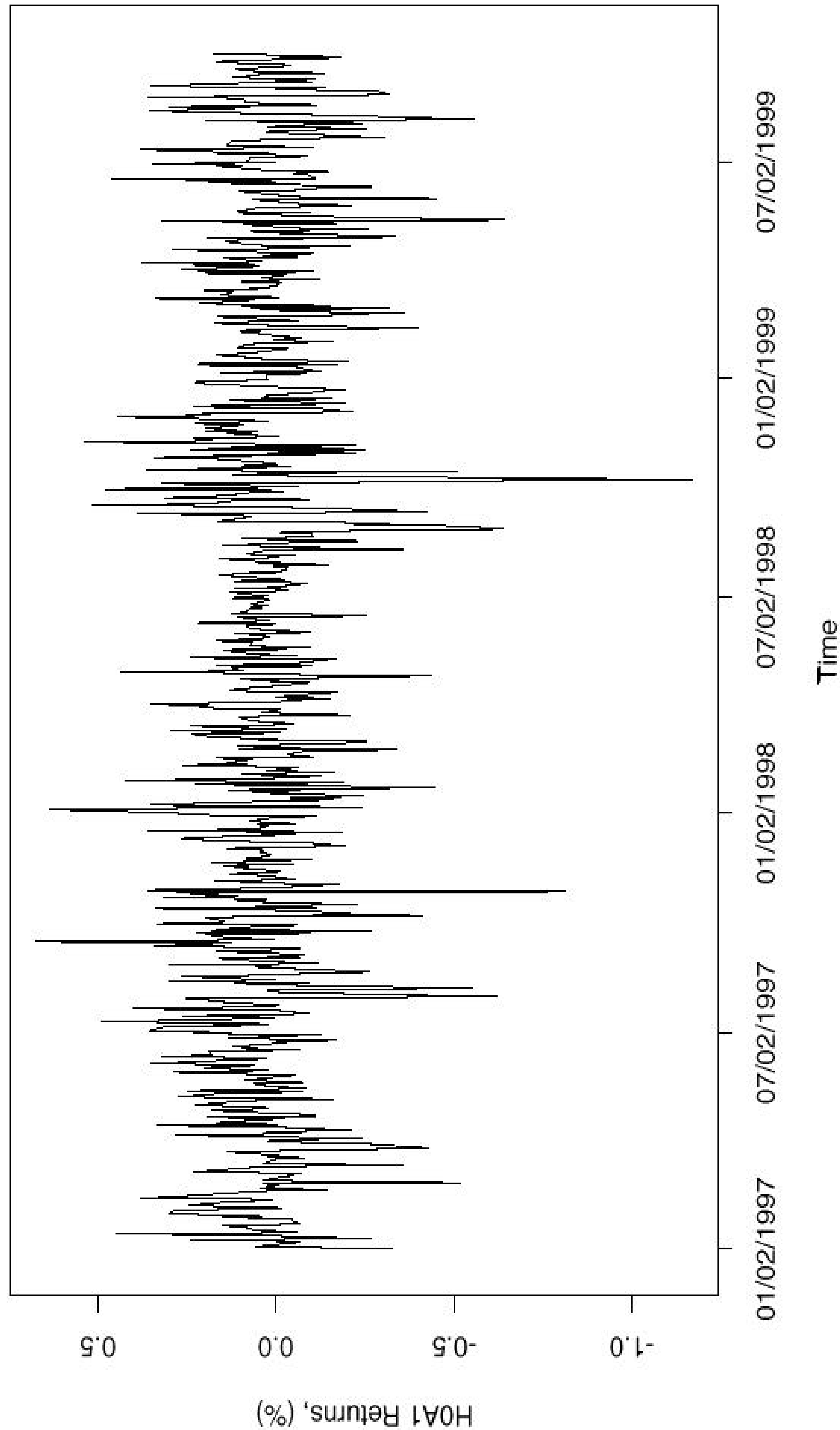


Figure 2. Stable and Normal Fitting of the H0A1 Index

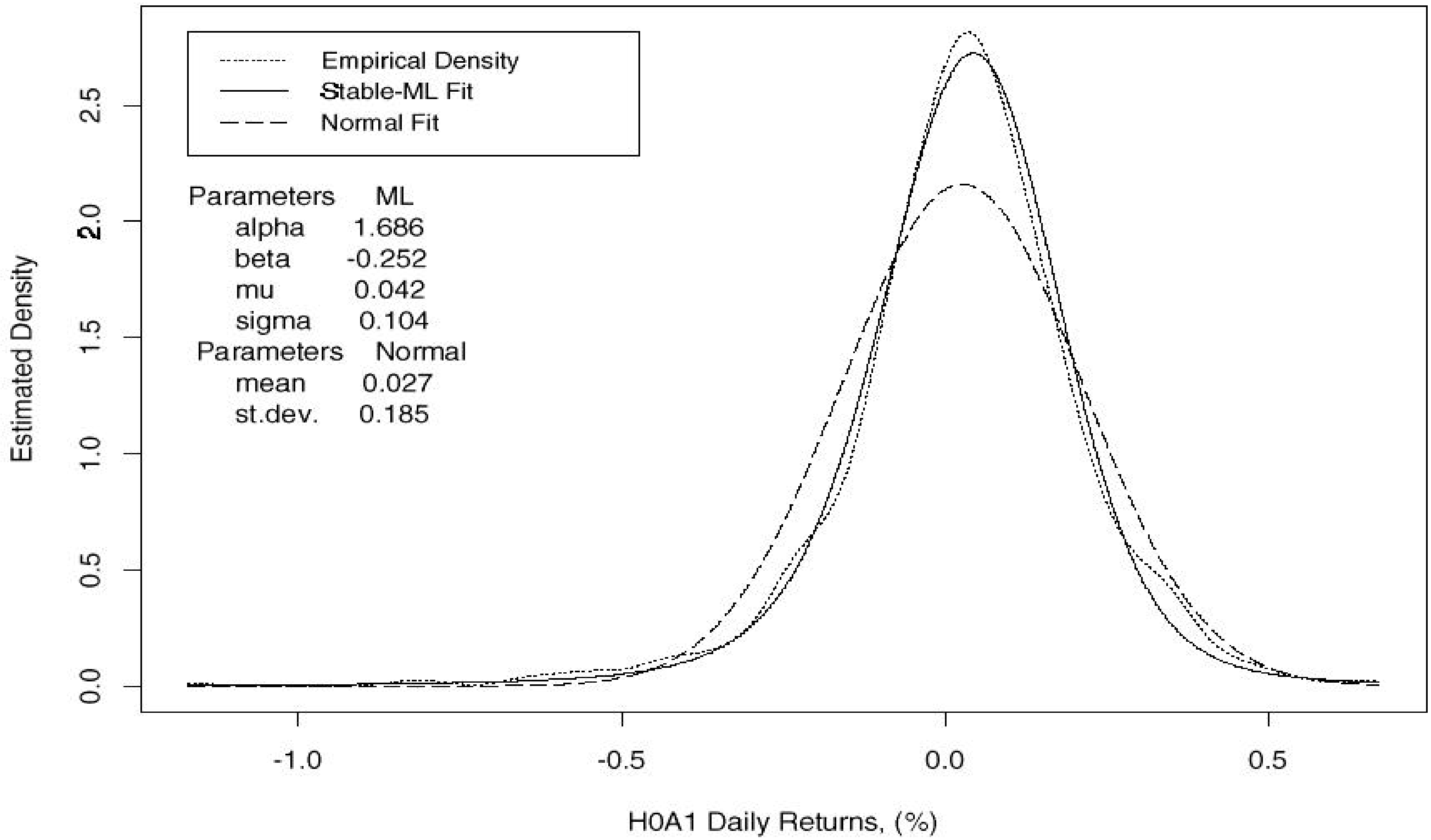
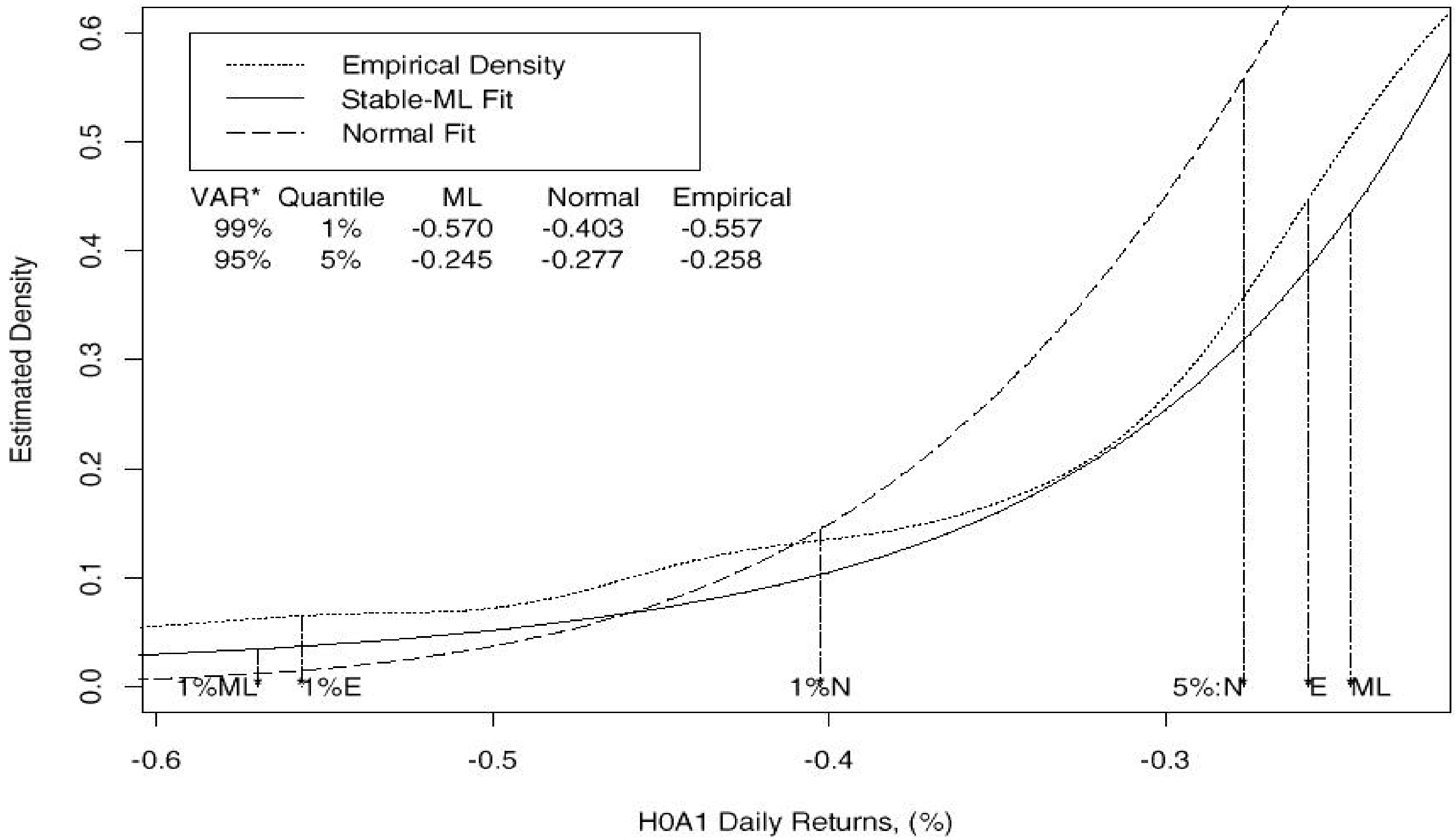


Figure 3. VAR Estimation for the H0A1 Index



3. Portfolio Credit Risk for Independent Credit Returns

Suppose that a portfolio includes n credit assets. Then, the portfolio return is given by $R_p = \sum_{i=1}^{n} w_i R_i$, where R_i is the return on the i-th asset, w_i is the weight of the i-th asset, i=1,…., n, $\sum_{i=1}^{n} w_i = 1$. In this section we outline stable modeling of portfolio risk assuming distributions of R_i are: (i) independent α-stable; (ii) characterized by the same index of stability, $R_i \sim S_\alpha(\sigma_{Ri}, \beta_{Ri}, 0)$[8], i=1,…., n. The additivity property of independent stable random variables provides analytic formulas for parameters of portfolio returns R_p. The use of formulas allows to obtain the estimates of portfolio parameters and risk without simulations. The obtained "independent" risk measurements can be employed as lower bounds of portfolio risk estimates.

By the additivity property of stable distributions, a linear combination of independent stable random variables is again a stable random variable. Therefore, $R_P = \sum_{i=1}^{n} w_i R_i$ follows a stable law:

$R_P \sim S_\alpha(\sigma_{Rp}, \beta_{Rp}, 0)$,

where α is the tail index, σ_{Rp} is the scale parameter, β_{Rp} is the skewness parameter,

$$\sigma_{R_P} = \left[\sum_{i=1}^{n} \left(|w_i| \sigma_{R_i} \right)^{\alpha} \right]^{\frac{1}{\alpha}},$$

$$\beta_{R_P} = \frac{\sum_{i=1}^{n} \left[sign(w_i) \beta_{R_i} \left(|w_i| \sigma_{R_i} \right)^{\alpha} \right]}{\sum_{i=1}^{n} \left(|w_i| \sigma_{R_i} \right)^{\alpha}}.$$

Thus, the distribution of the portfolio returns is characterized by three parameters: tail index (index of stability) α, skewness β_{Rp}, and scale σ_{Rp}. The parameter α is exogeneous. Estimation of β_{Rp} and σ_{Rp} can be implemented in three steps:

[8] We assume $\mu_{Ri} = 0$. If $\mu_{Ri} \neq 0$, we "center" the R_i observations: $R_i^* = R_i - \mu_{Ri}$.

Step 1: Find estimates of σ_{Ri} and β_{Ri} by stable fitting sets of R_{it}, t=1,…,T, i=1, …, n.

Step 2: Evaluate portfolio parameters σ_{Rp} and β_{Rp}:

$$\hat{\sigma}_{R_P} = \left[\sum_{i=1}^{n} \left(|w_i| \hat{\sigma}_{R_i} \right)^{\alpha} \right]^{\frac{1}{\alpha}} \tag{1}$$

$$\hat{\beta}_{R_P} = \frac{\sum_{i=1}^{n} \left[sign(w_i) \hat{\beta}_{R_i} \left(|w_i| \hat{\sigma}_{R_i} \right)^{\alpha} \right]}{\sum_{i=1}^{n} \left(|w_i| \hat{\sigma}_{R_i} \right)^{\alpha}}. \tag{2}$$

Having estimates of parameters of the portfolio credit risk, $\hat{\sigma}_{R_P}$ and $\hat{\beta}_{R_P}$, we can calculate the *portfolio VaR* as the negative of a certain quantile of the $\hat{R}_P$-distribution.

As an illustration of the method, we estimate portfolio risk for equally weighted returns on indices of the investment grade corporate bonds: C1A1, C2A1, C3A1, C4A1, C1A2, C2A2, C3A2, C4A2, C1A3,C2A3, C3A3, C4A3, C1A4, C2A4, C3A4, and C4A4[9]. Description of indices is given in Table 1 of Section 2. We assume the indices are: (i) characterized by the same tail index α; (ii) independent. We fix α at 1.708, the average of the α values for the bond return series (see Table 1), and recalculate other stable parameters: β_{Ri}, μ_{Ri}, and σ_{Ri}. New estimates are reported in Table 3. A condition of the same tail index α for all analyzed series does not appear to be very restrictive: new parameter estimates (given in Table 3) do not differ much from the previous parameter estimates (reported in Table 1); the new stable VaR estimates (see Table B.2 in Appendix B) are close to the initial stable VaR measurements (Table 2).

Table 3. Stable Fitting of the Bond Indices with Fixed α

Bond indices	Maturity	Stable parameters at α=1.708

[9] A digit after letter "C" denotes the maturity band: 1 – from 1 to 3 years, 2 – from 3 to 5 years, 3 – from 5 to 7 years, 4 – from 7 to 10 years; a digit after letter "A" denotes credit rating: 1 – "AAA", 2 – "AA", 3- "A", 4 – "BBB".

	(years)	β	μ	σ
C1A1	1-3	-0.084	0.027	0.054
C2A1	3-5	-0.111	0.029	0.099
C3A1	5-7	-0.116	0.031	0.144
C4A1	7-10	-0.146	0.031	0.188
C1A2	1-3	-0.107	0.027	0.057
C2A2	3-5	-0.105	0.029	0.103
C3A2	5-7	-0.098	0.033	0.148
C4A2	7-10	-0.128	0.032	0.194
C1A3	1-3	-0.144	0.027	0.057
C2A3	3-5	-0.120	0.030	0.104
C3A3	5-7	-0.125	0.032	0.149
C4A3	7-10	-0.151	0.032	0.196
C1A4	1-3	-0.118	0.029	0.054
C2A4	3-5	-0.045	0.033	0.098
C3A4	5-7	-0.128	0.035	0.141
C4A4	7-10	-0.143	0.035	0.180

We obtained small values of the μ estimates. Further on, we shall assume μ=0. We evaluate portfolio parameters following formulas (1)-(2): $\hat{s}_{U_P}$=0.659, $\hat{b}_{U_P}$=-0.125. Thus, $\hat{R}_P \sim S_{1.708}(0.659, -0.125, 0)$. The *portfolio* 99% (95%) *VaR* is calculated as the negative of the 1% (5%) quantile of the $\hat{R}_P$-distribution: the 99% VaR equals 3.518 and the 95% VaR equals 1.757. As credit returns typically have the non-negative dependence structure, the assumption of independence for single credit returns results in the lowest VaR measurement, the lower bound for portfolio VaR estimates. The upper bound of the portfolio VaR measurements will be given by the *non-diversified VaR*, the sum of the stand-alone VaR values. The non-diversified stable 99% (95%) VaR for our portfolio equals 9.813 (4.733). Analysis in Section 2 showed the 99% (95%) stable VaR estimates slightly exceed (underestimate) the true 99% (95%) VaR. Therefore, 9.813 (4.733) is a biased upwards (downwards) estimate of the portfolio non-diversified 99% (95%) VaR. Thus, we would expect that, in the case of dependent instruments, the portfolio 99% VaR will be in the range (3.518, 9.813) and the 95% VaR measurements will exceed 1.757.

4. Stable Modeling of Portfolio Risk for Symmetric Dependent Credit Returns

In this section we suppose that distributions of credit returns are symmetric α-stable and dependent. We interpret a symmetric random variable as a transformation of a normal random variable. Based on this interpretation, we develop a new methodology for correlation estimation. We apply the methodology for portfolio risk assessment.

We evaluate portfolio risk by determining portfolio VaR: (i) simulating a distribution of the $R_P = \sum_{i=1}^{n} w_i R_i$ values; (ii) finding a certain quantile of the R_P distribution, say, the 1% quantile, which corresponds to the 99% VaR confidence level. The aim of simulations is to project possible portfolio return values R_P at time T+1 given: (i) observations of individual returns over time: $R_{i1}, R_{i2}, \dots, R_{iT}$, i=1,…,n; (ii) weights of portfolio assets $w_1, \dots, w_n$. The simulations must account for dependence among individual credit returns R_i, i=1,…,n. A traditional approach of quantifying dependence is to calculate the covariance matrix. Under the α-stable assumption for distributions of R_i, computation of the covariance matrix is impossible.

We suggest a new method for deriving the dependence (association) structure. The method assumes that R_i are symmetric strictly stable: $R_i \sim S_{a_{Ri}}(s_{Ri}, 0, 0)$. A symmetric α-stable (SαS) random variable can be interpreted as a random rescaling transformation of a normal random variable (see Property 1 below). If a collection of SαS variables is obtained by applying a similar transformation to dependent normal variables, the dependence structure among variables will remain. Thus, the dependence among SαS random variables can be explained by the dependence among underlying normal random variables.

Property 1[10]. Assume that:

(i) G is a normal random variable with a zero mean: G ~ $S_2(\sigma_G, 0, 0) = N(0, 2\sigma_G^2)$,

(ii) Y is a symmetric α-stable random variable, α<2: Y ~ $S_\alpha(\sigma_Y, 0, 0)$,

(iii) S is a positive $\frac{\alpha}{2}$-stable random variable: S ~ $S_{\frac{\alpha}{2}}\left(\frac{\sigma_Y^2}{\sigma_G^2}\left(\cos\left(\frac{\pi\alpha}{4}\right)\right)^{\frac{2}{\alpha}}, 1, 0\right)$,

(iv) S and G are independent.

Then, the symmetric α-stable random variable Y can be represented as a random rescaling transformation of the normal random variable G:

$$Y = S^{\frac{1}{2}} G.$$

Simulations of the portfolio return values R_P can be divided into two fragments: (i) generating individual returns $\tilde{R}_i$ with the same dependence structure as the R_i's. We derive the dependence among R_i supposing that $R_i \sim S_{\alpha_{Ri}}(\sigma_{Ri}, 0, 0)$. Based on Property 1, R_i can be expressed as a transformation of a normal random variable:

$$R_i = S_i^{\frac{1}{2}} G_i, \quad (3)$$

where $G_i \sim S_2(\sigma_{Gi}, 0, 0) = N(0, 2\sigma_{Gi}^2)$, (4)

$$S_i \sim S_{\frac{\alpha_{Ri}}{2}}\left(\frac{\sigma_{Ri}^2}{\sigma_{Gi}^2}\left(\cos\left(\frac{\pi\alpha}{4}\right)\right)^{\frac{2}{\alpha_{Ri}}}, 1, 0\right), \quad (5)$$

S_i is independent of G_i, i=1,…, n.

Random rescaling transformations of normal variables G_i into R_i preserve the dependence structure. Hence, the dependence among R_i can be explained by the dependence among G_i, i=1,…, n. Based on this property, we generate dependent normal variables $\tilde{G}_i$,

[10] Property 1 is a slightly modified version of Proposition 1.3.1 in Samorodnitsky and Taqqu (1994).

maintaining the initial dependence[11], then, we generate $\tilde{R}_i = \tilde{S}_i^{\frac{1}{2}} \tilde{G}_i$, where $\tilde{S}_i$ is a simulated value of S_i;

(ii) computing $\tilde{R}_P = \sum_{i=1}^{n} w_i \tilde{R}_i$.

The simulations are performed according to the following algorithm[12]:

Step 1: Estimate stable parameters of R_i : α_{Ri}, σ_{Ri}, μ_{Ri}[13].

Step 2: "Center" the R_i observations: $R_i^* = R_i - \mu_{Ri}$. Further on, we shall assume $\mu_{Ri}=0$ and consider R_i^* as R_i: $R_i \sim S_{\alpha_{Ri}}(\sigma_{Ri},0,0)$, i=1,…,n.

Step 3: Assume: (i) R_i can be decomposed according to expressions (3)-(5); (ii) the covariance matrix of $(G_i)_{1\le i\le n}$ is equal to the covariance matrix of truncated $(R_i)_{1\le i\le n}$. Evaluate the covariance matrix of $(G_i)_{1\le i\le n}$ at time T+1, $\Sigma_{T+1} = \{c_{ij,T+1|T}\}$, i=1,…,n, j=1,…,n, using exponential weighting:

$$c^2_{i,T+1|T} = (1-\theta)\sum_{k=0}^{K} \theta^k R^2_{i,T-k} , \qquad (6)$$

$$c^2_{ij,T+1|T} = (1-\theta)\sum_{k=0}^{K} \theta^k R_{i,T-k} R_{j,T-k} , \qquad (7)$$

where T+1|T denotes a forecast for time T+1 conditional on information up to time T; θ is a decay factor, 0<θ<1; K is a number of observations' lags. Exponential weighting (6)-(7) allows to account for volatility and correlation clustering (GARCH effects)[14]. Formulas (4)-(5) can be expressed in recursive (GARCH-type) form[15]:

---

[11] Variables G_i, which enter formulas (1) and (2), are not observable. We suppose the dependence structure of Gaussian variables $(G_i)_{1\le i\le n}$ is "inherited" from the dependence structure of truncated values of stable variables $(R_i)_{1\le i\le n}$. Because we believe that the "outliers" are very important for the description of the dependence structure, we take the truncation value for R_i sufficiently large.

[12] The algorithm is implemented in the Mercury Software Package for Market Risk (VaR). See Rachev et al (1999).

[13] This section assumes $\beta_{Ri}=0$.

[14] An exponential weighting methodology follows the RiskMetrics' exponentially weighted moving average model. See Longerstaey and Zangari (1996).

[15] Formulas are adapted from Longerstaey and Zangari (1996).

$$c^2_{i,T+1|T} = qc^2_{i,T|T-1} + (1-q)R^2_{i,T}$$

$$c^2_{ij,T+1|T} = qc^2_{ij,T|T-1} + (1-q)R_{i,T}R_{j,T} \; .$$

<u>Step 4</u>: Generate a value of the multivariate normal random variable G=(G_1,G_2,…,G_n) with the covariance matrix Σ_{T+1}.

<u>Step 5</u>: Simulate values of stable random variables $S_i \sim S_{\frac{a_{Ri}}{2}}\left(\frac{2s^2_{Ri}}{c^2_i}\left(\cos\left(\frac{pa}{4}\right)\right)^{\frac{2}{a_{Ri}}},1,0\right)$, i=1,…,n.

<u>Step 6</u>: Compute $\tilde{R}_i = S_i^{\frac{1}{2}} G_i$, i=1,…,n.

<u>Step 7</u>: Calculate $\tilde{R}_P = \sum_{i=1}^{n} w_i \tilde{R}_i$.

<u>Step 8</u>: Repeat Steps 4-7 a large number of times to form an $\tilde{R}_P$-distribution.

Obtain a portfolio VaR measurement as the negative of a specified quantile of the $\tilde{R}_P$-distribution.

We evaluate portfolio risk for equally weighted returns on indices of the investment grade corporate bonds: C1A1, C2A1, C3A1, C4A1, C1A2, C2A2, C3A2, C4A2, C1A3,C2A3, C3A3, C4A3, C1A4, C2A4, C3A4, and C4A4. Description of indices is given in Table 1 of Section 2. We impose an assumption that returns on these indices are symmetric-α-stable. We compute the 99% and 95% VaR measurements in two procedures: (i) simulation of portfolio returns following the described above algorithm; (ii) calculation of the 99% (95%) VaR as the negative of the 1% (5%) quantile. In Step 3 of the portfolio returns simulations, derivation of the covariance matrix Σ_{T+1}, we used different truncation points and decay factor values. In order to estimate accuracy of simulations, we calculate the Kolmogorov Distance (KD) and Anderson-Darling (AD) statistics:

$$KD = \sup_x \left| F_e(x) - F_s(x) \right| ,$$

$$AD = \sup_x \left\{ \frac{|F_e(x) - F_s(x)|}{\sqrt{F_e(x)(1 - F_e(x))}} \right\},$$

where $F_e(x)$ is the empirical cumulative density function (cdf) and $F_s(x)$ is the simulated cdf. The computation results are summarized in Table 4.

Table 4. Portfolio VaR for Symmetric Individual Credit Returns

Decay factor θ	Truncation points, (%)	Portfolio VaR		Kolmo-gorov Distance	Anderson-Darling
		99% VaR	95% VaR		
0.85	10-90	7.508	4.886	3.880	0.086
	5-95	7.777	5.153	3.736	0.093
	No	8.286	5.346	4.859	0.111
0.94	10-90	7.793	5.147	3.556	0.081
	5-95	8.076	5.248	4.362	0.104
	1-99	8.389	5.434	5.650	0.128
	No	8.114	5.252	5.212	0.117
0.975	10-90	8.028	5.036	3.452	0.077
	5-95	8.166	5.318	9.085	0.234
	1-99	8.469	5.493	5.805	0.130
	No	8.516	5.470	7.274	0.167

The 99% VaR estimates in Table 4 are within the 99% VaR range (3.518, 9.813) derived in Section 3. At each truncation band, increasing the decay factor leads to higher values of the 99% VaR. Thus, as the decay factor grows, the 99% VaR generally rises. At each value of the decay factor, in general, reduction of truncated observations produced higher VaR numbers. We explain the latter observation by positive correlation in tails (concurrent extreme events). Consideration of a larger number of tail observations results in higher VaR. The KD and AD statistics, in general, decline with smaller decay factors.

Based on these observations, the decay factor of 0.85 and "no-truncation" method appear to be preferrable for VaR computations.

We examine how selection of the decay factor and the truncation method affects estimation of *marginal risks*. The marginal risk is a risk added by an asset to the portfolio risk. It is computed as the difference between the portfolio risk with an analyzed asset and the portfolio risk without the asset. We report the examination results in Table 5.

Table 5. Marginal Risk Examination

Decay Factor	Truncation, (%)	Cases: Marginal VaR > Stand-alone VaR	Cases: Higher Ratings Assets Contribute More Risk
0.85	10-90	0	0
	5-95	0	0
	No	0	0
0.94	10-90	0	0
	5-95	0	0
	1-99	3	2
	No	0	2
0.975	10-90	0	0
	5-95	0	4
	1-99	2	4
	No	3	4

The decay factor of 0.85 does not produce cases "Marginal VaR > Stand-alone VaR" and "Within one maturity band, higher ratings contribute more risk". In sum, the decay factor = 0.85 results in the lower KD and AD statistics and does not lead to irregular cases; the no-truncation method better accounts for correlation in tails. Hence, we would recommend the choice of the decay factor=0.85 and the no-truncation method.

In Table 6 we report marginal 99% VaR, stand-alone 99% VaR, and diversification effects at the decay factor of 0.85 and the no-truncation method. Marginal VaR estimates of Table 6 are consistent with the expectation that, for a given credit rating, bonds with longer maturities contribute more risk. Having marginal VaR numbers, we can identify concentration risks. We find that the C4A3 bond index makes the highest addition to the portfolio 99% VaR: the C4A3 marginal VaR of 0.920 exceeds all other marginal VaR. Marginal risks for all bond indices are smaller than stand-alone risks, which indicates that, indeed, diversification reduces risk. From Table 6, we notice that the C4A1 and C3A4 bond indices have highest diversification effects.

Table 6. Marginal VaR, Stand-alone 99% VaR, and Diversification Effects for Bond Indices (Decay factor = 0.85, No truncation)

Bond Indices	Marginal VaR	Stand-alone VaR	Diversification Effect
C1A1	0.199	0.284	0.085
C2A1	0.338	0.509	0.171
C3A1	0.572	0.734	0.162
C4A1	0.713	0.931	0.218
C1A2	0.245	0.285	0.040
C2A2	0.494	0.505	0.011
C3A2	0.575	0.689	0.114
C4A2	0.788	0.890	0.102
C1A3	0.190	0.286	0.096

C2A3	0.403	0.530	0.127
C3A3	0.592	0.719	0.127
C4A3	0.920	0.949	0.029
C1A4	0.185	0.290	0.105
C2A4	0.338	0.511	0.173
C3A4	0.522	0.741	0.219
C4A4	0.803	0.960	0.157

We studied stable modeling of portfolio risk under the assumptions of the independent and symmetric dependent instruments. In the next section we consider portfolio risk evaluation in the most general case – skewed dependent instruments.

5. Stable Modeling of Portfolio Risk for Skewed Dependent Credit Returns

We quantify portfolio risk R_P by generating a distribution of its possible values and deriving a portfolio VaR from the constructed distribution of R_P. In a case of a portfolio assets with skewed dependent credit returns, simulations of the R_P values should reflect the "cumulative" skewness and maintain the dependence (association) among them. In order to do that, we decompose single credit returns R_i into two independent parts: the first part accounts for dependence and the second – for skewness. Then, we obtain the portfolio dependence and skewness components separately aggregating the dependence and skewness parts of individual credit returns. Simulations of the portfolio credit returns values R_P can be divided into three portions: (i) generation of the portfolio dependence component maintaining the dependence structure among individual credit returns, (ii) generation of the portfolio skewness component, and (iii) computation of R_P as a sum of the two generated components. Explanations of our methodology are provided below.

A stable random variable $R \sim S_\alpha(\sigma, \beta, 0)$ can be decomposed (in distribution) into two independent stable random variables $R^{(1)}$ and $R^{(2)}$:

$$R \stackrel{d}{=} R^{(1)} + R^{(2)},$$

where

$R^{(1)} \sim S_\alpha(\sigma_1, \beta_1, 0)$,

$R^{(2)} \sim S_\alpha(\sigma_2, \beta_2, 0)$,

$$s = \left(s_1^{a} + s_2^{a}\right)^{\frac{1}{a}}, \tag{8}$$

$$b = \frac{b_1 s_1^{a} + b_2 s_2^{a}}{s_1^{a} + s_2^{a}}. \tag{9}$$

Suppose that: (i) $R^{(1)}$ is a symmetric stable variable: $\beta_1=0$; (ii) $\sigma_1=\sigma_2=\sigma^*$. Then, formulas (8) and (9) can be reduced to the following expressions:

$$s = 2^{\frac{1}{a}} s^*, \tag{10}$$

$$b = \frac{1}{2} b_2. \tag{11}$$

From equations (10) and (11), we have

$$s^* = 2^{-\frac{1}{a}} s \ ,$$

$$b_2 = 2b \ .$$

In sum, a stable random variable $R \sim S_\alpha(\sigma, \beta, 0)$ can be decomposed (in distribution) into two independent stable random variables: symmetric $R^{(1)}$ and skewed $R^{(2)}$:

$$R \stackrel{d}{=} R^{(1)} + R^{(2)}, \tag{12}$$

where

$$R^{(1)} \sim S_a\left(2^{-\frac{1}{a}} s, 0, 0\right), \tag{13}$$

$$R^{(2)} \sim S_a\left(2^{-\frac{1}{a}} s, 2b, 0\right). \tag{14}$$

Using methodology (12)-(14), we can divide individual credit returns $R_i \sim S_{a_{Ri}}(s_{Ri}, b_{Ri}, 0)$ into the "dependence" and "skewness" parts. First, we partition R_i into the "symmetry" and "skewness" fragments:

$$R_i \overset{d}{=} R_i^{(1)} + R_i^{(2)},$$

where

$$R_i^{(1)} \sim S_{a_{Ri}}\left(2^{-\frac{1}{a_{Ri}}} s_{Ri}, 0, 0\right),$$

$$R_i^{(2)} \sim S_{a_{Ri}}\left(2^{-\frac{1}{a_{Ri}}} s_{Ri}, 2b_{Ri}, 0\right),$$

parts $R_i^{(1)}$ and $R_i^{(2)}$ are independent, i=1,…,n. Second, we suppose: (i) $R_i^{(1)}$, i=1,…,n, are dependent and (ii) $R_i^{(2)}$, i=1,…,n, are independent. Consequently, symmetric terms $R_i^{(1)}$ explain dependence (association) among R_i's and terms $R_i^{(2)}$ account for skewness of R_i's.

Based on Property 1 (see Section 4), $R_i^{(1)} \sim S_{a_{Ri}}\left(2^{-\frac{1}{a_{Ri}}} s_{Ri}, 0, 0\right)$ can be written as a transformation of a normal random variable:

$$R_i^{(1)} = S_i^{\frac{1}{2}} G_i,$$

where $G_i \sim S_2(s_{Gi}, 0, 0) = N(0, 2s_{Gi}^2)$,

$$S_i \sim S_{\frac{a_{Ri}}{2}}\left(\frac{2^{-\frac{2}{a_{Ri}}} s_{Ri}^2}{s_{Gi}^2}\left(\cos\left(\frac{pa}{4}\right)\right)^{\frac{2}{a_{Ri}}}, 1, 0\right),$$

S_i is independent of G_i, i=1,…, n.

Random rescaling transformations of normal variables G_i into $R_i^{(1)}$ maintain the dependence structure. Therefore, from the dependence among G_i's we can determine the dependence among $R_i^{(1)}$, or the dependence among R_i.

Adding separately the dependence and skewness terms of R_i's, we obtain the two components of the portfolio returns R_P:

$$R_P = R_P^{(1)} + R_P^{(2)}, \tag{15}$$

where $R_P^{(1)} = \sum_{i=1}^{n} w_i R_i^{(1)} = \sum_{i=1}^{n} w_i S_i^{\frac{1}{2}} G_i$ is the "dependence" component and $R_P^{(2)} = \sum_{i=1}^{n} w_i R_i^{(2)}$ is the "skewness" component.

We simulate the R_P values based on decomposition (15): $\tilde{R}_P = \tilde{R}_P^{(1)} + \tilde{R}_P^{(2)}$. The simulations are executed according to the next algorithm[16]:

Step 1: Estimate stable parameters of R_i : α_{Ri}, β_{Ri}, σ_{Ri}, μ_{Ri}.

Step 2: "Center" the R_i observations: $R_i^* = R_i - \mu_{Ui}$. Further on, we shall assume $\mu_{Ri}=0$ and consider R_i^* as R_i: $R_i \sim S_{\alpha_{Ri}}(\sigma_{Ri}, \beta_{Ri}, 0)$, i=1,…,n.

Step 3: Evaluate the covariance matrix of normal random variables $(G_i)_{1\le i\le n}$ at time T+1, $\Sigma_{T+1} = \{c_{ij,T+1|T}\}$, i=1,…,n, j=1,…,n, using exponential weighting:

$$c_{i,T+1|T}^2 = (1-\theta)\sum_{k=0}^{K} \theta^k R_{i,T-k}^2 ,$$

$$c_{ij,T+1|T}^2 = (1-\theta)\sum_{k=0}^{K} \theta^k R_{i,T-k} R_{j,T-k} ,$$

where T+1|T denotes a forecast for time T+1 conditional on information up to time T; θ is a decay factor, 0<θ<1; K is a number of observations' lags.

Step 4: Generate a value of the multivariate normal random variable $G=(G_1,G_2,\ldots,G_n)$ with the covariance matrix Σ_{T+1}.

[16] This algorithm is an extended version of the algorithm in Section 4.

Step 5: Simulate values of stable random variables $S_i \sim S_{\frac{a_{Ri}}{2}}\left(\frac{2^{1-\frac{2}{a_{Ri}}} s_{Ri}^{2}}{c_i^{2}}\left(\cos\left(\frac{pa}{4}\right)\right)^{\frac{2}{a_{Ri}}},1,0\right)$, i=1,…,n.

Step 6: Compute $\tilde{R}_i^{(1)} = S_i^{\frac{1}{2}} G_i$, i=1,…,n.

Step 7: Generate $\tilde{R}_i^{(2)} \sim S_{a_{Ri}}\left(2^{-\frac{1}{a_{Ri}}} s_{Ri}, 2b_{Ri}, 0\right)$, i=1,…,n.

Step 8: Calculate $\tilde{R}_P = \sum_{i=1}^{n} w_i \tilde{R}_i^{(1)} + \sum_{i=1}^{n} w_i \tilde{R}_i^{(2)}$.

Step 9: Repeat Steps 4-8 a large number of times to form a $\tilde{R}_P$-distribution.

Obtain a portfolio VaR measurement as the negative of a specified quantile of the $\tilde{R}_P$ -distribution.

We have applied stable modeling to the total risk assessment of credit returns. Below we analyze stable modeling of isolated credit risk.

6. One-Factor Model of Potfolio Credit Risk

In this section we outline a *one-factor model* for quantifying portfolio credit risk. The model is built on two postulations: (i) constituent parts of the credit returns are the credit-risk-free part and the credit risk premium; (ii) the credit risk spread follows a stable law. Applying the one-factor model, in the following sections we quantify credit risk for single instruments and then estimate portfolio credit risk as a cumulative result of stably distributed individual credit risks.

Similar to the previous sections, we assume that a portfolio includes n assets. Then, the portfolio return is given by $R_p = \sum_{i=1}^{n} w_i R_i$, where R_i is the return on the i-th asset, w_i

is the weight of the i-th asset, i=1,…., n, $\sum_{i=1}^{n} w_i = 1$. We conjecture that individual returns R_i depend on one credit-risk-free factor Y_i:

$R_i = a_i + b_i Y_i + U_i$, (16)

where a_i and b_i are constants, U_i is the residual representing compensation for credit risk and random noise[17], i = 1,…., n.

Suppose the i-th portfolio instrument is a corporate bond of maturity τ with returns R_i. There are two possible choices for an underlying credit-risk-free factor Y_i: (i) returns on a Treasury bond of the same maturity τ; (ii) returns on a τ-year bond with a credit rating AAA. Then, the spread $U_i = R_i - a_i - b_i Y_i$ reflects charges for credit risk. If the j-th portfolio asset is a swap with a counterparty that has a low credit rating, say BBB, we can choose, as an underlying factor Y_j, returns on a similar swap with a company that has a credit rating AAA: $R_j = a_j + b_j Y_j + U_j$, the term U_j accounts for the credit risk of the BBB-swap.

We impose the following assumptions on the components of model (16):

(i) Credit risk spreads U_i are strictly stable, $U_i \sim S_{\alpha_{Ui}}(\sigma_{Ui}, \beta_{Ui}, 0)$ [18], α>1.

(ii) Default-free factors Y_i are strictly stable, $Y_i \sim S_{\alpha_{Yi}}(\sigma_{Yi}, \beta_{Yi}, 0)$ [19], γ>1.

(iii) U_i and Y_i are independent of each other, i = 1,…., n.

Then, the portfolio return R_P can be decomposed into three components:

$R_P = A + Y_P + U_P$,

where Y_P expresses aggregate effect of underlying factors, U_P represents *portfolio credit risk*,

$$A = \sum_{i=1}^{n} w_i a_i, \quad Y_P = \sum_{i=1}^{n} w_i b_i Y_i, \quad U_P = \sum_{i=1}^{n} w_i U_i .$$

[17] We interpret the yield spread as the credit risk premium and include the noise factor into the credit risk part. The noise factor could incorporate taxability, liquidity, and other premiums.
[18] The shift of U_i is, in fact, incorporated in a_i.
[19] Y_i is the centered return. If the returns of portfolio instruments, Z_i, are non-centered, then we take

We evaluate the portfolio credit risk U_P in two steps: (i) quantifying credit risk of each asset U_i; (ii) estimating U_P as a cumulative result of individual U_i, i = 1,….,n. Section 7 discusses credit risk evaluation for single portfolio assets. Section 8 examines portfolio credit risk estimation under the assumptions of independent, symmetric dependent, and skewed dependent credit risks.

7. Credit Risk Evaluation for Portfolio Assets

Approximations of the credit risk premium values U_i for portfolio assets can be obtained using model (16):

$$\hat{U}_{it} = R_{ti} - \hat{a}_i - \hat{b}_i Y_{ti}\,, \tag{17}$$

where $\hat{a}_i$ and $\hat{b}_i$ are the OLS estimates,

$$\hat{a}_i = \frac{\sum_{t=1}^{T} Y_{it}^2 \sum_{t=1}^{T} R_{it} - \sum_{t=1}^{T} Y_{it} \sum_{t=1}^{T} R_{it} Y_{it}}{T \sum_{t=1}^{T} Y_{it}^2 - \left(\sum_{t=1}^{T} Y_{it} \right)^2}\,, \tag{18}$$

$$\hat{b}_i = \frac{T \sum_{t=1}^{T} R_{it} Y_{it} - \sum_{t=1}^{T} Y_{it} \sum_{t=1}^{T} R_{it}}{T \sum_{t=1}^{T} Y_{it}^2 - \left(\sum_{t=1}^{T} Y_{it} \right)^2}\,, \tag{19}$$

i=1, …, n; t=1, …. T.

Estimators $\hat{a}_i$ and $\hat{b}_i$, given by expressions (18) and (19), are unbiased[20].

We analyze credit risk of corporate bonds applying one-factor model (16). Assume that returns on an index of the US corporate bonds, R_i, are described by returns on a credit-risk-free factor, Y_i, and a credit spread, U_i:

$R_i = a_i + b_i Y_i + U_i$,

$Y_{it} = Z_{it} - \bar{Z}_i$, t=1, …, T.

[20] For analysis of asymptotic properties of OLS estimators (18) and (19) under the stable distribution assumption for the disturbance term, see Götzenberger, Rachev, and Schwartz (1999).

where a_i and b_i are constants, i = 1,…., 16. We examine returns on the same 16 indices as in Section 2 (see Table 1): $R_i \in \{R_{C1A1}, R_{C2A1}, R_{C3A1}, R_{C4A1}, R_{C1A2}, R_{C2A2}, R_{C3A2}, R_{C4A2}, R_{C1A3}, R_{C2A3}, R_{C3A3}, R_{C4A3}, R_{C1A4}, R_{C2A4}, R_{C3A4}$, and $R_{C4A4}\}$. We choose, as corresponding credit-risk-free factors, returns on the indices of US government bonds in the same maturity band: $Y_i \in \{R_{G1O2}, R_{G2O2}, R_{G3O2}, R_{G4O2}\}$[21]. For example, if we consider returns on the index of bonds with maturity from one to three years, R_{C1A1}, then the returns on the index of the government bonds with maturity from one to three years, R_{G1O2}, serve as the underlying credit-risk-free factor. We approximate the percentage return values of the individual credit risks U_i, following approach (17): (i) run OLS regressions of model (16), (ii) compute the residuals' series $\hat{U}_i$. Coefficients of the OLS regressions are given in Appendix B, Table B.3. Obtained sets of OLS credit risk premiums $\hat{U}_i$ are plotted in Figures 4 and in Figures of Appendix C. Empirical densities of $\hat{U}_i$ are shown in Figures 5 and in Appendix C. We observe that the credit risk spread series $\hat{U}_i$ exhibit volatility clusters and heavy tails. Such behavior of the individual returns sets can be captured by stable and GARCH models.

Stable modeling of the credit risk premiums $\hat{U}_i$, entailed values of $\alpha<1.6$, $\beta\approx0$, and $\mu\approx0$ (see Table 7). These values of parameter estimates indicate that credit risk spreads of the corporate bonds' indices are fat-tailed and almost symmetric. Table 7 presents the following α and β values of the credit risks of the bond indices with a maturity band from one to three years: AAA bonds: α=1.333 and β=0.011; AA bonds: α=1.379 and β=0.030; A bonds: α=1.393 and β=-0.021; BBB bonds: α= 1.412 and β=0.004. Plots of the stable and normal fitting of the OLS credit risk spreads $\hat{U}_i$ are shown on Figures 5 and in Appendix C. Figures demonstrate that stable modeling well captures excess kurtosis and heavy tails of the credit risks $\hat{U}_i$.

Table 7. Stable and Normal Fitting of the OLS Credit Risk Premiums of Bond Indices

OLS credit risk	Maturity	Normal	Stable

[21] A digit after letter "G" denotes the maturity band: 1 – from 1 to 3 years, 2 – from 3 to 5 years, 3 – from 5 to 7 years, 4 – from 7 to 10 years.

of bond indices	(years)	Mean	Standard deviation	α	β	μ	σ
C1A1	1-3	0.0	0.045	1.333	0.011	0.000	0.017
C2A1	3-5	0.0	0.075	1.528	-0.089	-0.001	0.033
C3A1	5-7	0.0	0.096	1.590	-0.023	0.000	0.047
C4A1	7-10	0.0	0.116	1.456	-0.026	0.000	0.051
C1A2	1-3	0.0	0.037	1.379	0.030	0.001	0.015
C2A2	3-5	0.0	0.064	1.523	-0.074	0.000	0.029
C3A2	5-7	0.0	0.086	1.591	-0.060	0.000	0.044
C4A2	7-10	0.0	0.110	1.426	0.005	0.001	0.050
C1A3	1-3	0.0	0.038	1.393	-0.021	0.000	0.015
C2A3	3-5	0.0	0.069	1.483	-0.084	0.000	0.029
C3A3	5-7	0.0	0.098	1.519	-0.073	0.000	0.042
C4A3	7-10	0.0	0.124	1.366	-0.017	0.001	0.048
C1A4	1-3	0.0	0.074	1.412	0.004	0.001	0.018
C2A4	3-5	0.0	0.096	1.527	-0.024	0.001	0.033
C3A4	5-7	0.0	0.113	1.552	-0.077	0.000	0.048
C4A4	7-10	0.0	0.1424	1.480	-0.055	0.001	0.055

The GARCH approach models clustering of volatilities and fat tails, by expressing the conditional variance as an explicit function of past information:

$$R_{i,t} = a_i + b_i Y_{i,t} + U_{i,t} \,, \tag{20}$$

$$\text{where } U_{i,t} = \sigma_{i,t}\varepsilon_{i,t} \,, \tag{21}$$

$$\varepsilon_{i,t} \sim N(0,1), \tag{22}$$

$$s_{i,t}^{2} = c_i + \sum_{j=1}^{p} g_{i,j} s_{i,t-j}^{2} + \sum_{j=1}^{q} h_{i,j} U_{i,t-j}^{2} \,, \tag{23}$$

$i = 1,\ldots,n;\ t = 1,\ldots,T.$

We shall name model (20) - (23) as a *GARCH(p,q)-normal* model because it is based on the normality assumption for the disturbance term. In order to detect GARCH-dependencies, we examine sample autocorrelation and partial autocorrelation functions of the squared residuals $\hat{U}_i$. Visual inspection of the correlograms suggests values of p and q. Applying the Box-Jenkins methodology, we find that p=q=1 is adequate to capture temporal dependence of volatilities:

$$\sigma_{i,t}^2 = c_i + \gamma_i \sigma_{i,t-1}^2 + \eta_i U_{i,t-1}^2 . \qquad (24)$$

Coefficients of model (20)-(22) and (24) with $R_i \in \{R_{C1A1}, R_{C2A1}, R_{C3A1}, R_{C4A1}, R_{C1A2}, R_{C2A2}, R_{C3A2}, R_{C4A2}, R_{C1A3}, R_{C2A3}, R_{C3A3}, R_{C4A3}, R_{C1A4}, R_{C2A4}, R_{C3A4},$ and $R_{C4A4}\}$ and $Y_i \in \{R_{G1O2}, R_{G2O2}, R_{G3O2}, R_{G4O2}\}$ are reported in Appendix B, Table B.4. Densities of the GARCH(1,1)-normal residuals $U_{i,t} = \sqrt{c_i + \gamma_i \sigma_{i,t-1}^2 + \eta_i U_{i,t-1}^2} \times \varepsilon_{i,t}$ are displayed in Figures 6 and in Appendix D. Graphs demonstrate that the GARCH credit risk series have lower peaks.

In the portfolio context, implementation of the GARCH models is computationally complex because a number of parameters rapidly increases as the portfolio expands[22]. Hence, we evaluate portfolio credit risk U_P based on stable modeling of individual credit risks with accounting for GARCH effects by exponential weighting of observations[23]. In estimation of U_P, we separately investigate cases of independent, symmetric dependent, skewed dependent credit risks of portfolio instruments.

[22] For references on the multivariate GARCH, see Engle and Kroner (1995).
[23] An approach of modeling time-varying volatilities by exponential weighting follows the RiskMetrics' exponentially weighted moving average model described in J.P.Morgan (1996).

Figure 4. OLS Credit Risk Premium of the C1A1 bond index

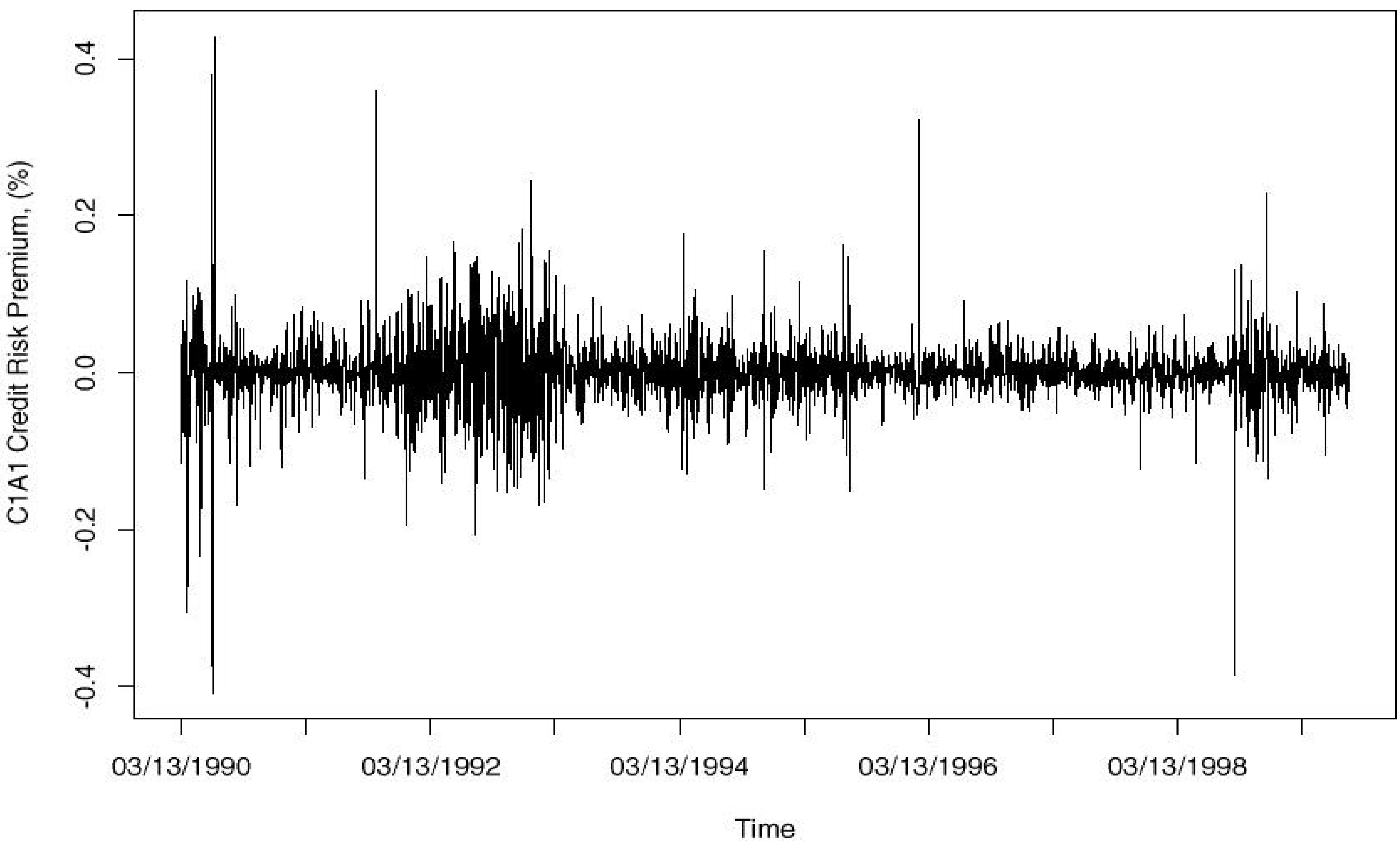


Figure 5. Stable and Normal Fitting of C1A1 OLS-Credit-Risks

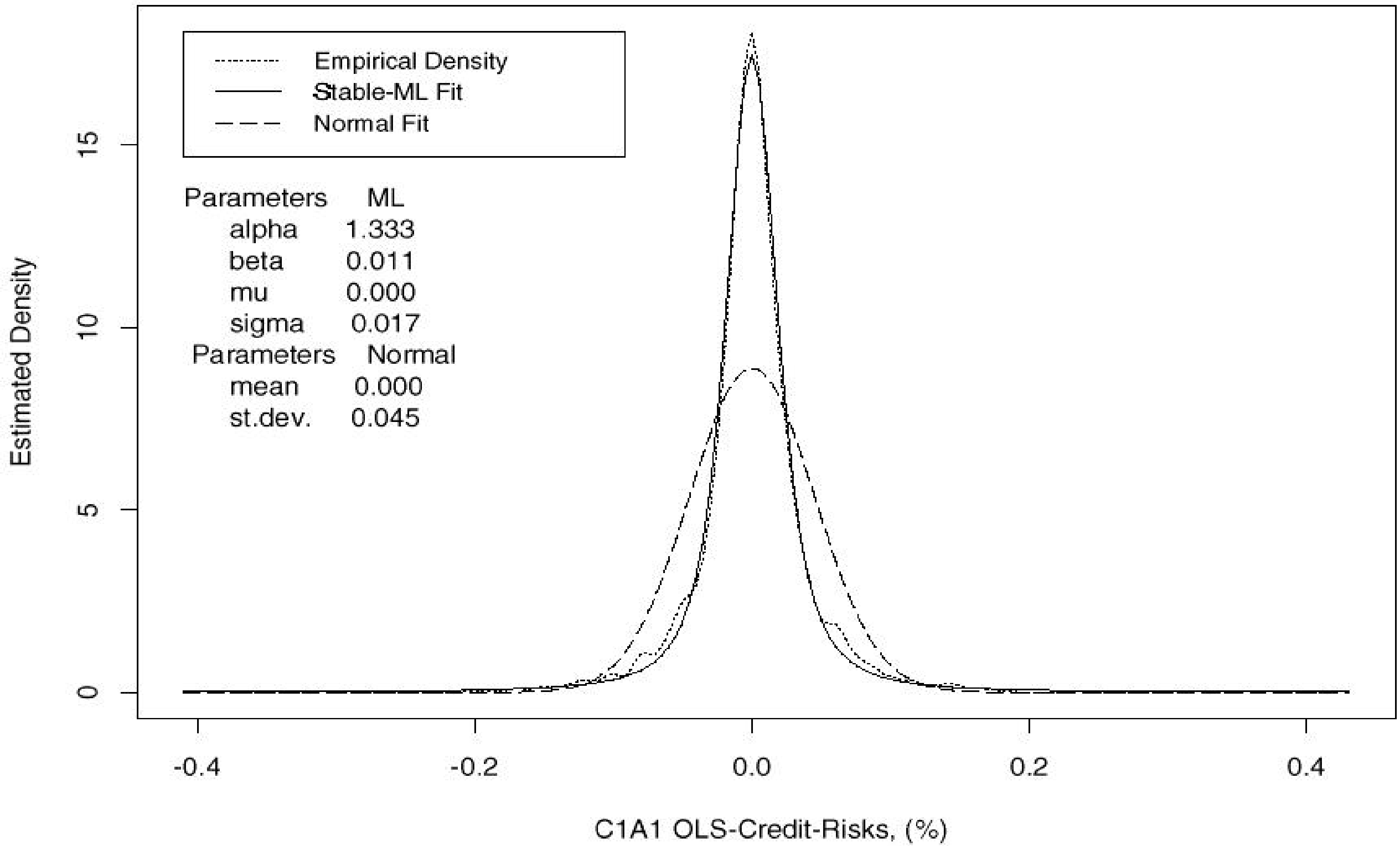
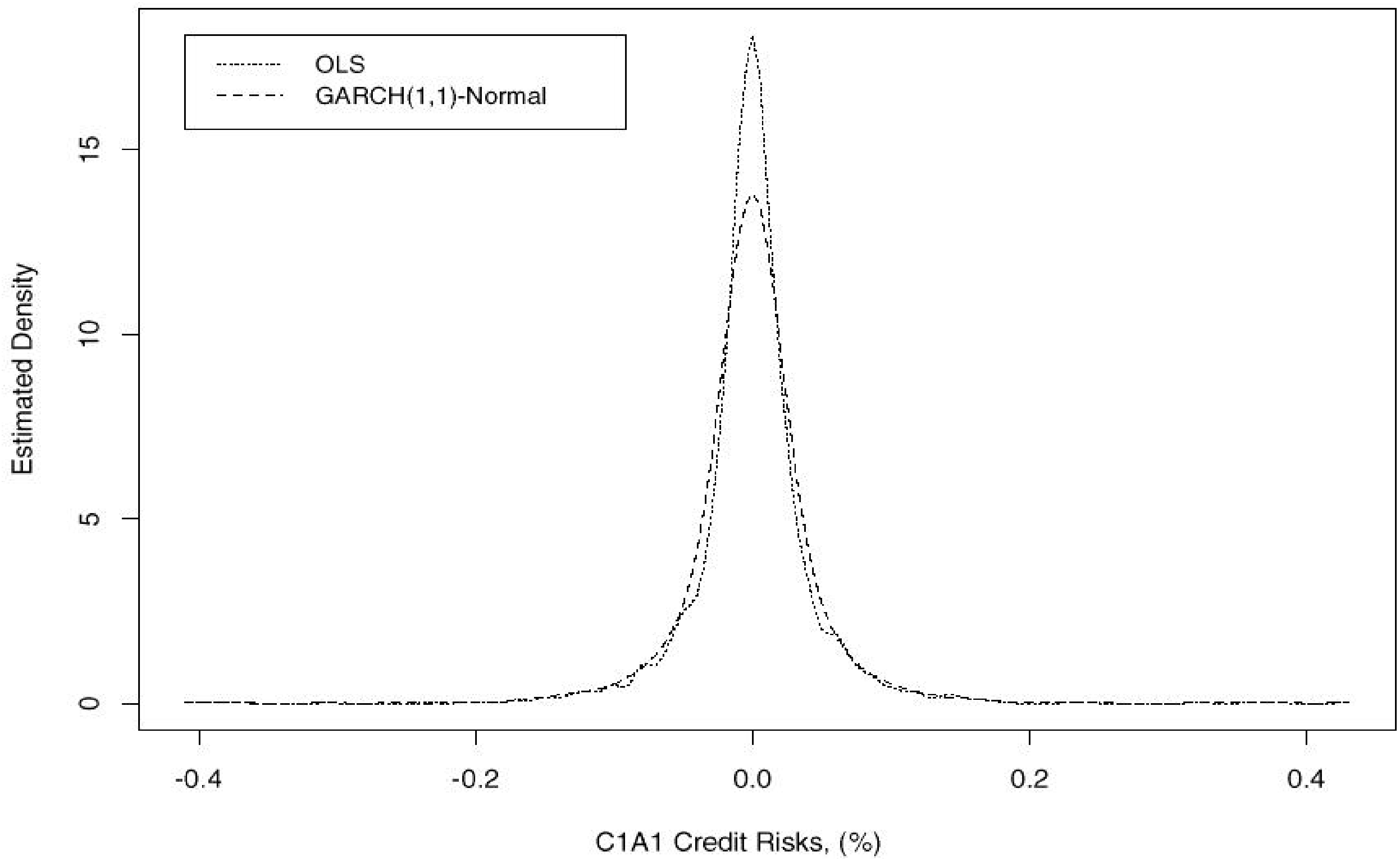


Figure 6. C1A1 Credit Risks: OLS and GARCH



8. Portfolio Credit Risk

In this section we follow the one-factor model of Section 6 and evaluate portfolio credit risk as a cumulative effect of stably distributed individual credit risks. We impose different assumptions on their distributions: independent, symmetric dependent, and skewed dependent. We show implementation of the approach on a portfolio of equally weighted OLS-credit-risk premimums from Section 7.

8.1. Independent Credit Risks

Suppose credit-risk-premiums are: (i) characterized by the same tail index α; (ii) independent. Then, by the additivity property of stable variables (see Section 3), the portfolio credit risk $U_P = \sum_{i=1}^{n} w_i U_i$ is stably distributed:

$U_P \sim S_\alpha(\sigma_{Up,}\ \beta_{Up},\ 0)$,

where α is the tail index, σ_{Up} is the scale parameter, β_{Up} is the skewness parameter,

$$\sigma_{U_P} = \left[\sum_{i=1}^{n} \left(|w_i| \sigma_{U_i} \right)^\alpha \right]^{\frac{1}{\alpha}}, \tag{25}$$

$$\beta_{U_P} = \frac{\sum_{i=1}^{n} \left[sign(w_i) \beta_{U_i} \left(|w_i| \sigma_{U_i} \right)^\alpha \right]}{\sum_{i=1}^{n} \left(|w_i| \sigma_{U_i} \right)^\alpha}. \tag{26}$$

Consider a portfolio of equally weighted OLS-credit-risk premimums from Section 7. Assume credit-risk-premiums are independent and have the same tail index α. We take α=1.472, the average of the α values for the credit-riks-premium series (see Table 7), and recompute other stable parameters: β_{Ui}, μ_{Ui}, and σ_{Ui}. New estimates are reported in Table 8. Similarly to returns on bond indices, a condition of the same tail index α for all analyzed credit risk series does not seem to be very restraining: new parameter estimates (Table 8) do not deviate much from the previous parameter estimates (Table 7).

Table 8. Stable Fitting of the OLS Credit Risk Premiums with Fixed α

OLS credit risk of bond indices	Maturity (years)	Stable parameters at α=1.472		
		β	μ	σ
C1A1	1-3	0.000	0.000	0.018
C2A1	3-5	-0.090	-0.001	0.032
C3A1	5-7	-0.019	0.000	0.045
C4A1	7-10	-0.019	0.001	0.052
C1A2	1-3	0.023	0.001	0.015
C2A2	3-5	-0.072	-0.001	0.029
C3A2	5-7	-0.039	0.000	0.042
C4A2	7-10	-0.004	0.000	0.051
C1A3	1-3	-0.040	0.000	0.015
C2A3	3-5	-0.084	0.000	0.029
C3A3	5-7	-0.067	0.000	0.041
C4A3	7-10	-0.032	0.001	0.049
C1A4	1-3	-0.010	0.001	0.019
C2A4	3-5	0.011	0.001	0.033
C3A4	5-7	-0.071	-0.001	0.046
C4A4	7-10	-0.053	0.001	0.055

Since obtained estimates of μ are very small, we assume μ=0. We evaluate portfolio parameters applying formulas (25)-(26): $\hat{s}_{U_P}$=0.015, $\hat{b}_{U_P}$=-0.038. Thus, $\hat{U}_P \sim S_{1.472}(0.015, -0.038, 0)$. The 99% (95%) *credit VaR* is derived as the negative of the 1% (5%) quantile of the $\hat{U}_P$-distribution : the 99% (95%) VaR equals 0.125 (0.046). Having analytic formulas for the U_p parameters, we obtained estimates of portfolio credit risk without simulations.

8.2. Symmetric Dependent Credit Risks

In order to assess portfolio credit risk, we obtain portfolio credit VaR. It is computed in two steps: (i) simulating a distribution of the $U_P = \sum_{i=1}^{n} w_i U_i$ values; (ii) inferring portfolio credit VaR from the simulated U_P distribution. This section examines the case of symmetric individual credit risks U_i: $U_i \sim S_{a_{Ui}}(s_{Ui},0,0)$, i=1,…,n.

We simulate U_P applying the methodology from Section 4:

(i) generate individual credit risks $\tilde{U}_i$ with the same dependence structure as the U_i's. We express U_i as a transformation of a normal random variable:

$$U_i = S_i^{\frac{1}{2}} G_i ,$$

where $G_i \sim S_2(s_{Gi},0,0) = N(0,2s_{Gi}^2)$,

$$S_i \sim S_{\frac{a_{Ui}}{2}}\left(\frac{s_{Ui}^2}{s_{Gi}^2}\left(\cos\left(\frac{pa}{4}\right)\right)^{\frac{2}{a_{Ui}}},1,0\right),$$

S_i is independent of G_i, i=1,…, n.

The dependence among U_i can be explained by the dependence among G_i, i=1,…, n. We form dependent normal variables $\tilde{G}_i$, preserving the initial dependence. Next, we generate $\tilde{U}_i = \tilde{S}_i^{\frac{1}{2}} \tilde{G}_i$, where $\tilde{S}_i$ is a simulated value of S_i;

(ii) calculate $\tilde{U}_P = \sum_{i=1}^{n} w_i \tilde{U}_i$.

A portfolio credit VaR can be measured from the $\tilde{U}_P$ -distribution.

As an illustration of the approach, we estimate credit risk for a portfolio of equally weighted OLS-credit-risk premimums of bond indices (see Section 7) assuming they are symmetric[24]. The estimation results are presented in Table 9. The portfolio credit VaR

[24] The symmetry proposition is plausible: the skewness parameters of credit risks premiums of bond indices

does not demonstrate a certain pattern of dependence on the decay factor. For each decay factor, reduction of the truncated observations does not seem to affect the portfolio credit VaR in a particular fashion. The no-truncation method approach led to the smallest VaR measurements. Possibly, the credit risk residuals of the investment grade indices have negative correlations in far tails. Taking into account more observations with negative correlations reduces the VaR estimates. Since the decay factor does not influence the VaR results in a specific way and the KD and AD statistics are smaller at the no-truncation approach, in further analysis, we consider the no-truncation method and arbitrarily select the decay factor of 0.85. Computation of the marginal VaR, stand-alone VaR, diversification effects for the no-truncation approach and the decay factor=0.85 is summarized in Table 10.

Table 9. Portfolio Credit VaR for Symmetric Credit Risks

Decay factor θ	Truncation points, (%)	Portfolio VaR		Kolmo-gorov Distance	Anderson-Darling
		99% VaR	95% VaR		
0.85	10-90	3.502	1.918	8.071	0.210
	5-95	3.710	1.896	8.898	0.228
	No	3.396	1.856	7.692	0.199
0.94	10-90	3.594	1.963	7.680	0.200
	5-95	3.643	1.941	8.162	0.209
	1-99	3.476	1.975	8.847	0.227
	No	3.321	1.792	6.736	0.164
0.975	10-90	3.623	1.877	7.578	0.194
	5-95	3.435	1.943	9.085	0.234
	1-99	3.578	2.004	9.665	0.254

are small (see Table 7).

	No	3.293	1.739	7.174	0.167

Table 10. Marginal VaR, Stand-alone 99% VaR, and Diversification Effects for Credit Risk Premiums of Bond Indices (Decay factor = 0.85, No truncation)

Bond Indices	Marginal VaR	Stand-alone VaR	Diversification Effect
C1A1	0.175	0.191	0.016
C2A1	0.203	0.251	0.048
C3A1	0.162	0.305	0.143
C4A1	0.145	0.441	0.296
C1A2	0.024	0.148	0.124
C2A2	0.153	0.222	0.069
C3A2	0.180	0.290	0.110
C4A2	0.296	0.453	0.157
C1A3	0.013	0.149	0.136
C2A3	0.097	0.244	0.147
C3A3	0.203	0.325	0.122
C4A3	0.295	0.507	0.212
C1A4	0.079	0.168	0.089
C2A4	0.091	0.243	0.152
C3A4	0.142	0.346	0.204

C4A4	0.366	0.457	0.091

From Table 10, highest contributions to portfolio credit risk are made by the C4A4, C4A3, and C4A2 bond indices: their marginal 99% VaR equal 0.366, 0.295, and 0.296. The credit risk premium of the C4A1 index displays the largest diversification effect.

8.3. Skewed Dependent Credit Risks

For estimation of portfolio risk for the skewed dependent credit risks, we propose to employ the approach of Section 5: (i) split individual credit risks U_i into the dependence and skewness parts; (ii) find the the portfolio dependence and skewness components by combining the dependence and skewness parts of single credit risks; (iii) evaluate the portfolio credit risk as a sum of the dependence and skewness fragments. Details are given below.

We divide individual credit risks $U_i \sim S_{a_{Ui}}(s_{Ui}, b_{Ui}, 0)$ into the "dependence" and "skewness" parts, applying methodology (12)-(14) (see Section 5):

$$U_i \overset{d}{=} U_i^{(1)} + U_i^{(2)},$$

where

$$U_i^{(1)} \sim S_{a_{Ui}}\left(2^{-\frac{1}{a_{Ui}}} s_{Ui}, 0, 0\right),$$

$$U_i^{(2)} \sim S_{a_{Ui}}\left(2^{-\frac{1}{a_{Ui}}} s_{Ui}, 2b_{Ui}, 0\right),$$

parts $U_i^{(1)}$ and $U_i^{(2)}$ are independent, i=1,…,n. We assume: (i) $U_i^{(1)}$, i=1,…,n, are dependent and (ii) $U_i^{(2)}$, i=1,…,n, are independent. Then, symmetric components $U_i^{(1)}$ explain dependence (association) among U_i's and components $U_i^{(2)}$ depict skewness of U_i's.

By Property 1 (see Section 4), $U_i^{(1)} \sim S_{\alpha_{Ui}}\left(2^{-\frac{1}{\alpha_{Ui}}}\sigma_{Ui},0,0\right)$ can be interpreted as a transformation of a normal random variable:

$$U_i^{(1)} = S_i^{\frac{1}{2}} G_i \,,$$

where $G_i \sim S_2(\sigma_{Gi},0,0) = N(0,2\sigma_{Gi}^2)$,

$$S_i \sim S_{\frac{\alpha_{Ui}}{2}}\left(\frac{2^{-\frac{2}{\alpha_{Ui}}}\sigma_{Ui}^2}{\sigma_{Gi}^2}\left(\cos\left(\frac{\pi\alpha}{4}\right)\right)^{\frac{2}{\alpha_{Ui}}},1,0\right),$$

S_i is independent of G_i, i=1,…, n.

Random rescaling transformations of normal variables G_i into $U_i^{(1)}$ maintain the dependence structure. Hence, we can derive the dependence among $U_i^{(1)}$, or the dependence among U_i, from the dependence among G_i's.

Combining separately the dependence and skewness terms of U_i's, we obtain the two components of the portfolio credit risk U_P:

$$U_P = U_P^{(1)} + U_P^{(2)} \,,$$

$$U_P^{(1)} = \sum_{i=1}^{n} w_i U_i^{(1)} = \sum_{i=1}^{n} w_i S_i^{\frac{1}{2}} G_i$$

$$U_P^{(2)} = \sum_{i=1}^{n} w_i U_i^{(2)} \,,$$

where $U_P^{(1)}$ is the "dependence" component and $U_P^{(2)}$ is the "skewness" component. The portfolio credit risk can be evaluated as a sum of the dependence and skewness fragments.

We suggested methodologies for portfolio credit risk assessment and demonstrated their applications on analysis of returns on bond indices. The methodologies can be employed for risk evaluation of any financial instruments if they have fat-tailed and/or skewed distributions.

9. Conclusions

This work proposes the application of stable distributions in modeling value changes of credit instruments. Our empirical analysis verifies that stable modeling well captures skewness and heavy-tails of returns on credit instruments and isolated credit risks. The superior fit allows to derive accurate risk estimates. Based on the properties of stable distributions, we design new methods for the correlation estimation and simulating portfolio values. We employ the methods in evaluation of portfolio and marginal VaR for three cases of the credit returns: independent, symmetric dependent, and skewed dependent. We suggest a one-factor model of credit risks. Applying the one-factor model, we quantify credit risk for individual assets and then assess portfolio credit risk as an aggregate effect of stable distributed individual credit risks. The suggested stable modeling can be applied for risk estimation of any financial assets, not only credit instruments, if their distributions are heavy-tailed and/or skewed.

References

1. Ammann, M., 1999, “Pricing Derivative Credit Risk”, Berlin: Springer-Verlag.
2. Altman, E.I. and A. Saunders, 1998, “Credit Risk Measurement: Developments Over the Last 20 years”, *Journal of Banking and Finance*, 21, 1721-1742.
3. Basle Committee on Banking Supervision, 1999, “Credit Risk Modelling: Current Practices and Applications”.
4. Black, F. and J.C. Cox, 1976, “Valuing Corporate Securities: Some Effects of Bond Indenture Provisions”, *Journal of Finance*, 31(2), 351-367.
5. Crosbie, P., 1998, “Modeling Default Risk”, in “Credit Derivatives: Trading & Management of Credit & Default Risk”, S. Das, ed., Singapore: Jon Wiley and Sons, 298-315.
6. Duffee, G.R., 1999, “Estimating the Price of Default Risk”, *The Review of Financial Studies*, Vol. 12, No. 1, 197-226.
7. Duffie, D. and K. Singleton, 1999, “Modeling Term Structures of Defaultable Bonds”, *The Review of Financial Studies*, Vol. 12, No. 4, 687-720.
8. Engle, R.F. and K. Kroner, 1995, “Multivariate Simultaneous Generalized GARCH”, *Econometric Theory* (1996), 122-150.
9. Federal Reserve System Task Force on Internal Credit Risk Models, 1998, “Credit Risk Models at Major U.S. Banking Institutions: Current State of the Art and Implications for Assessment of Capital Adequacy”.
10. Fons, J., 1994, “Using Default Rates to Model the Term Structure of Credit Risk”, *Financial Analysts Journal*, September-October, 25-32.
11. Götzenberger, G., S.T. Rachev, and E.Schwartz, 1999, “Performance Measurements: The Stable Paretian Approach”, Working Paper, University of Karlsruhe, Germany.
12. Gupton, G.M., C.C. Finger, and M. Bhatia, 1997, “CreditMetrics™ - Technical Document”, New York: J.P. Morgan.
13. Jarrow, R.A., D. Lando, and S.M. Turnbull, 1997, “A Markov Model for the Term Structure of Credit Risk Spreads”, *Review of Financial Studies*, 10(2), 481-523.
14. Jarrow, R.A. and S.M. Turnbull, 1995, “Pricing Derivatives on Financial Securities Subject to Credit Risk”, *Journal of Finance*, 50(1), 53-85.
15. Khindanova, I., S.T. Rachev, and E. Schwartz, 1999, “Stable Modeling of Value at Risk”, to appear in “Stable Models in Finance” Issue of *Mathematical and Computer Modeling*, Pergamon Press.
16. Knoch, M. and S.T. Rachev, 1999, “Credit Risk: Recent Advances”, Working Paper, University of Karlsruhe, Germany.
17. Longerstaey, J. and P. Zangari, 1996, “RiskMetrics™ - Technical Document”, Fourth edition, New York: J.P. Morgan.

18. Longstaff, F.A. and E.S. Schwartz, 1995, "A Simple Approach to Valuing Risky Fixed and Floating Rate Debt", *Journal of Finance*, 50(3), 789-819.

19. Madan, D.B. and H. Unal, 1994, "Pricing the Risks of Default", Working Paper 94-16, Wharton School, University of Pennsylvania.

20. Merton, R.C., 1974, "On the Pricing of Corporate Debt: The Risk Structure of Interest Rates", *Journal of Finance*, 2(2), 449-470.

21. Nielsen, S. S. and E.I. Ronn, 1996, "The Valuation of Default Risk in Corporate Bonds and Interest Rate Swaps", Wharton Financial Institutions Center, Working Paper 96-23.

22. Rachev, S.T., B. Racheva-Jotova, B. Hristov, and I. Mandev, 1999, "Technical Documentation of Mercury 1.0, Software Package for Market Risk (VaR) Modeling of Stable Distributed Financial Returns".

23. Samorognitsky, G. anf M.S. Taqqu, 1994, "Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance", New York: Chapman & Hall.

24. Zhou, C., 1997, "A Jump-diffusion Approach to Modeling Credit Risk and Valuing Defaultable Securities", Working Paper 1997-15, Federal Reserve Board, Washington D.C.