The Extended Geske-Johnson Model and Its Consistency with Reduced Form Models

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ABSTRACT

In this paper, we extend the Geske-Johnson model (1984) to multiple periods ($n > 2$) and to incorporate random interest rates. We derive similar quasi closed form solutions to the ones in Geske (1977) and Geske and Johnson (1984). Furthermore, we show that an externally specified default barrier can potentially generate internal inconsistency. Finally, we show that the extended Geske-Johnson model can be compared with reduced form models in a discretized, binomial framework. This result makes comparison of various models empirically possible. We demonstrate, with a credit derivative example, how different recovery assumptions impact the derivative prices.

1 INTRODUCTION

There have been two well-known approaches, structural and reduced form, for credit risk modeling. Reduced form models, represented by Jarrow and Turnbull (1995) and Duffie and Singleton (1997, 1999) assume defaults (or credit events) occur exogenously (usually by a Poisson process) and a separately specified recovery is paid upon default.[1] Structural models, on the other hand, assume defaults occur when the value of the firm falls below a certain default barrier.[2]

Among various structural models that extend the seminal work of Black and Scholes (1973) and Merton (1974), the compound option model by Geske (1977) and Geske and Johnson (1984) maintains the most structure of the original Black-Scholes-Merton model. In particular, The Geske-Johnson model is the only structural model that allows both default and recovery to be endogenously specified, while the other structural models assume exogenously given barrier and recovery. However, despite of the desirably endogenous default and recovery assumptions, unlike the reduced form and other structural models, the Geske-Johnson model has not been widely used. We attribute the reasons to (i) lack of stochastic interest rates, (ii) lack of an efficient implementation algorithm, and (iii) lack of the intuition provided by reduced form and barrier structural models.

In this paper, we first extend the Geske-Johnson model to n periods and incorporate random interest rates.[3] This is necessary because corporate bonds are sensitive to both credit and interest rate risks.[4] We derive similar quasi closed form solutions to the ones in Geske (1977) and Geske and Johnson (1984). Then, we provide a simple algorithm for the implementation of the model. By recognizing that there are only two state variables (asset price and short rate), we replace the expensive high-dimension normal integrals by a fast bi-variate lattice. Lastly, in a discrete binomial framework, we show that the Geske-Johnson model carries the same intuition of reduced form models. We show that the extended Geske-Johnson model can be thought as a special case of the reduced form model.[5] Then it is clearly seen that the only difference between

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[1] Other reduced form models include, among others, Duffie and Lando (2001), Jarrow, Lando, and Turnbull (1997), Jarrow and Yu (2001), Lando (1998), Madan and Unal (2000), and Schönbucher (1998).

[2] Barrier structural models, pioneered by Black and Cox (1976), include, for example, Anderson and Sundaresan (1996), Leland and Toft (1996), Longstaff and Schwartz (1995), Briys and de Varenne (1997), Collin-Dufresne and Goldstein (2001), Zhou (2001), and Bélanger, Shreve, and Wong (2002) and many others. Readers can also find a more thorough survey by Uhrig-Homburg (2002).

[3] The formulas provided by Geske (1977) are incorrect and corrected by Geske and Johnson (1984). However, Geske and Johnson only present formulas for $n = 2$. Here, we generalize their formulas to an arbitrary n.

[4] High grade bonds contain more interest rate risk than credit risk.

[5] Bélanger, Shreve, and Wong (2002) unify reduced form and barrier structural models.

the two approaches is their different recovery assumptions. This result makes it possible to compare various models. We demonstrate, with a credit derivative example that is sensitive to the recovery assumption, how different recovery assumptions impact its prices.

In order to compare the Geske-Johnson model with other barrier structural models, we derive a discrete time barrier model that will converge in the limit to any chosen continuous time barrier model. We show that an externally specified barrier can potentially generate internal inconsistency.

The remaining of the paper is organized as follows. Section 2 lays out the basic valuation equations. Section 3 presents the binomial framework under which all models share the same valuation formula. In particular, we specify the forward measure technique to incorporate stochastic interest rates. In Section 4, we discuss calibration issues of various models and perform model comparison. Section 5 uses the models discussed to value the most popular credit derivative contract – default swaps. Finally, the paper is concluded in Section 6.

2 THE EXTENDED GESKE-JOHNSON MODEL

Before we extend the Geske-Johnson model to incorporate random interest rates, it is convenient if we derive the discrete barrier model first. The discrete barrier model serves as a link between the Geske-Johnson model and other barrier models. The discrete barrier model is consistent with Geske-Johnson model when the barrier is set to equal the endogenously solved default points and is consistent with any continuous barrier model when it approaches the continuous time limit with a specific barrier function.

A. The Discrete Barrier Model

The structural models can be traced back to Black and Scholes (1973) and Merton (1974) who observe that the company's equity is a European call option and hence the single-maturity-date debt contains default risk identical to a covered call. The recovery in the Black-Scholes-Merton model is therefore the firm value if default occurs at maturity. To extend the single period model of Black-Scholes and Merton, as mentioned earlier, there are two approaches. Barrier structural models assume an exogenous default barrier. Default is defined as the asset value crossing such a barrier.[6] The extension along this line is pioneered by Black and Cox (1976) and numerous others.[7] Another is the compound option model by Geske (1977) and Geske and Johnson (1984) who treat default as the inability of the company to fulfill its debt obligations (i.e. negative equity value).

[6] The barrier-based structural models are particularly popular in industry. See, for example, KMV (recently acquired by Moody's) and CreditGrades.
[7] See footnote 2.

In this sub-section, we include a multi-period debt structure in the Black-Scholes-Merton model. We assume default occurs if the firm fails to meet its cash obligations at any given time. Since the cash obligations are exogenously given, the "discrete barrier" model can be viewed as a discretized version of any existing barrier structural model.

Let $X_1, \cdots, X_n$ be the series of external barriers, crossing which by the firm value represents default. These discrete time barriers can be interpreted as cash obligations at each time and failure to make the obligation results in default of the firm. These cash obligations can be regarded as a series of zero coupon bonds issued by the firm. The values of assets and debts (zero coupon) are labeled as $A(t)$, and $D(t, T_n)$ respectively for an arbitrary t. Default is defined as $A(T_i) < X_i$ at time T_i. To obtain closed form solutions, we assume the continuity of $A(t)$, as by Geske (1977) and Geske and Johnson (1984). We also assume the same seniority structure of these debts as Geske (1977) and Geske and Johnson (1984) that earlier maturing debts are more senior than later debts. To arrive at closed form results, we need to assume a log normal process for the firm's asset value and a normal process for the instantaneous short rate:

$$(1) \qquad \begin{bmatrix} \dfrac{dA(t)}{A(t)} \\ dr(t) \end{bmatrix} = \begin{bmatrix} r(t) \\ \alpha(r,t) \end{bmatrix} dt + \begin{bmatrix} \sigma & 0 \\ 0 & \gamma \end{bmatrix} \times \begin{bmatrix} \sqrt{1-\rho_{Ar}^2} & \rho_{Ar} \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} dW_A^{\mathbb{Q}}(t) \\ dW_r^{\mathbb{Q}}(t) \end{bmatrix}$$

where σ, γ, and ρ_{Ar} are constants, and $dW_A^{\mathbb{Q}} dW_r^{\mathbb{Q}} = 0$. From (1), we know that the correlation between the (log) asset value and the interest rate is ρ_{Ar}. Also note that both $W_A^{\mathbb{Q}}(t)$ and $W_r^{\mathbb{Q}}(t)$ are independent Wiener processes under the risk-neutral, $\mathbb{Q}$, measure. Let a risk free discount factor be

$$(2) \qquad P(t,T) = E_t^{\mathbb{Q}}\left[\exp\left(-\int_t^T r(u)du\right)\right]$$

Then by Ito's lemma, we obtain that:

$$(3) \qquad \begin{aligned} \frac{dP(t,T)}{P(t,T)} &= \frac{1}{P(t,T)}\left[\frac{\partial P(t,T)}{\partial r}\alpha(r,t) + \frac{1}{2}\frac{\partial^2 P(t,T)}{\partial r^2}\gamma^2 + \frac{\partial P(t,T)}{\partial t}\right]dt + \frac{1}{P(t,T)}\frac{\partial P(t,T)}{\partial r}\gamma dW_r^{\mathbb{Q}}(t) \\ &\equiv r(t)dt + \sigma_P(r,t,T)dW_r^{\mathbb{Q}}(t) \end{aligned}$$

The Gaussian models that satisfy (3) are Vasicek (1977), Ho and Lee (1986), and Hull and White (1990), in all of which the diffusion term, $\sigma_P(r,t,T)$, is independent of the interest rate and can be written as $\sigma_P(t,T)$.

Then, we can derive the value of T_n-maturity, zero-coupon debt as follows:

$$(4)\qquad \begin{aligned} D_{\mathrm{DB}}(0,T_i) &= E_0^{\mathbb{Q}}\left[\exp\left(-\int_0^{T_i} r(u)du\right)\min\{X_i, A(T_i)\}\right] \\ &= P(0,T_i)X_i\mathsf{N}_i^- + A(0)(\mathsf{N}_{i-1}^+ - \mathsf{N}_i^+) \end{aligned} \qquad 1 \le i \le n$$

where

$$\begin{aligned} \mathsf{N}_i^{\pm} &= N_i(h_1^{\pm}(X_1),\cdots,h_i^{\pm}(X_i);\Sigma)\,, \\ h_i^{\pm}(X_i) &= \frac{\ln\frac{A(0)}{P(0,T_i)X_i} \pm \frac{1}{2}v^2(0,T_i)}{v(0,T_i)}\,, \\ v^2(0,T_i) &= \int_0^{T_i} \sigma_A^2 + \sigma_P^2(u,T_i) - 2\rho_{Ar}\sigma_A\sigma_P(u,T_i)du\,,\ \text{and} \end{aligned}$$

$N_i(\cdots;\Sigma)$ is an i-dimensional cumulative normal probability with the correlation matrix Σ.[8] Note that $\mathsf{N}_0^{\pm} \equiv 1$ and $\Sigma = [\rho_{ij}]$ where $\rho_{ij} = \rho[\ln A(T_j), \ln A(T_i)] = \sqrt{T_j / T_i}$ for all $0 < j < i \le n$. The derivation of (4) is given in an appendix. In (4), both N_i^+ and N_i^- are probabilities of survival (in the money), only defined under different probability measures.[9]

Equation (4) has an interesting interpretation. Note that N_i^- is the survival probability from now till T_i and $\mathsf{N}_{i-1}^- - \mathsf{N}_i^-$ (or $\mathsf{N}_{i-1}^+ - \mathsf{N}_i^+$ under a different measure) is the *unconditional* default probability between T_{i-1} and T_i. In other words, (4) implies that if default does not happen (with probability N_i^-), the bond receives X_i. If default does happen, it receives a random asset amount: $A(T_i)$. Note that any default prior to T_{i-1} should pay no recovery for such a bond since it is more junior than any of its preceding bonds and should receive recovery only all the prior bonds are paid in full. And this recovery is multiplied by the default probability and discounted to be $A(0)(\mathsf{N}_{i-1}^+ - \mathsf{N}_i^+)$ today.

It should be noted that (4) is not a closed form result unless $v(0,T_i)$ has a closed form expression. Rabinovitch (1989) shows that a closed form expression for $v(0,T_i)$ exists if the term structure model follows Vasicek (1977) and does not exist if it follows Cox-Ingersoll-Ross (1985). We restate the closed form result of Rabinovitch in an appendix.

A company that pays periodical payments as interests to its debt owners can be viewed as holding a string of zero coupon debts and therefore the total debt value is equal to the sum of the values of all zero coupon debts defined in (4):

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[8] Equations (3) and (4) are consistent with Vasicek (1977), but not Cox, Ingersoll, and Ross (1985).

[9] See Appendix C for the details.

$$V_{\mathrm{DB}}(0,T_n) = \sum_{i=1}^{n} D(0,T_i) \tag{5}$$
$$= A(0)[1-\mathsf{N}_n^+] + \sum_{i=1}^{n} P(0,T_i)X_i\mathsf{N}_i^-$$

Equation (5) can be easily generalized (even though the closed form solution may not be readily available) to compare against various barrier models. For example, the Longstaff-Schwartz model (1995) is directly comparable. In our notation, the Longstaff-Schwartz model for the zero-coupon bond can be written as:

$$D_{\mathrm{LS}}(0,T_n) = P(0,T_n)[1-w\Phi] \tag{6}$$

where w is loss given default (or 1 minus recovery) and Φ is the cumulative default probability for the period $[0,T_n]$. The formula is very intuitive and directly comparable with (4) if we re-arrange the Longstaff-Schwartz model as follows:

$$D_{\mathrm{LS}}(0,T_n) = P(0,T_n)[(1-\Phi)+(1-w)\Phi)] \tag{6a}$$

Note that the Longstaff-Schwartz model is for zero coupon bond and hence $X_n = 1$ and $X_i = 0$ for $i < n$. $P(0,T_n)(1-\Phi)$ resembles $P(0,T_n)\mathsf{N}_n^-$ in (4) as the discounted survival probability and $P(0,T_n)(1-w)\Phi$ resembles $A(0)(\mathsf{N}_{n-1}^+ - \mathsf{N}_n^+)$ in (4) as the discounted recovery value (note that Φ resembles $\mathsf{N}_{n-1}^+ - \mathsf{N}_n^+$ as the default probability and $P(0,T_n)(1-w)$ resembles $A(0)$ as the present value of the recovery amount).

B. The Extended Geske-Johnson Model

The Geske-Johnson model, first derived by Geske (1977) and then corrected by Geske and Johnson (1984), extends the Black-Scholes-Merton model in the most straightforward way in which internal strikes are solved to guarantee positive equity value. We extend the Geske-Johnson model to an n period risky debt and to incorporate random interest rates.[10] The T_n-maturity zero coupon bond can be written as follows.

$$D_{\mathrm{GJ}}(0,T_n) = \sum_{i=1}^{n} P(0,T_i)X_i[\Pi_i^-(K_{1n},\cdots,K_{in}) - \Pi_i^-(K_{1n-1},\cdots,K_{in-1})] \tag{7}$$
$$+A(0)[\Pi_{n-1}^+(K_{1n-1},\cdots,K_{n-1n-1}) - \Pi_n^+(K_{1n},\cdots,K_{nn})]$$

[10] Later on, for calibration, we also augment the model to include non-constant volatility.

where[11]

$$\Pi_i^{\pm}(K_{1k}, K_{2k}, \cdots, K_{ik}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} N_i(h_1^{\pm}(K_{1k}), h_2^{\pm}(K_{2k}), \cdots, h_i^{\pm}(K_{ik}))\varphi^{\pm}(r(T_1), r(T_2), \cdots, r(T_i)) \\ dr(T_1)dr(T_2), \cdots, dr(T_i)$$

for $n \geq k \geq i$ where φ is the joint density function of various interest rate levels observed at different times under the forward measure. The derivative of (7) is in an appendix. Note that K_{ij} is a function of r and $h_j^{\pm}(K_{ij})$, for $i < j$, is defined in (4). K_{ij} is the internal solution for $A(T_i)$ to the following equation:

(8) $$A(T_i) = \sum_{k=i}^{j} D(T_i, T_k) + X_i$$

which is certainly a function of the interest rate at time T_i, and $K_{ii} = X_i$. Note that K_{ij}'s must be solved recursively. For example, at the end of the period, T_n, $K_{nn} = X_n$. Then, at T_{n-1}, $K_{n-1n-1} = X_{n-1}$ and K_{n-1n} is the solution to:

(8a) $$A(T_{n-1}) = D(T_{n-1}, T_n) + X_{n-1}$$

or

(8b) $$\begin{aligned} X_{n-1} &= A(T_{n-1}) - D(T_{n-1}, T_n) \\ &= E(T_n) \\ &= A(T_{n-1})N(h^+) - P(T_{n-1}, T_n)X_n N(h^-) \end{aligned}$$

where

$$h^{\pm} = \frac{\ln \frac{A(T_{n-1})}{P(T_{n-1}, T_n)X_n} \pm \frac{1}{2} v^2(T_{n-1}, T_n)}{v(T_{n-1}, T_n)}$$

and $v(\cdot,\cdot)$ is defined in (4). If interest rates are deterministic, then $\Pi_i^{\pm} = N_i(\cdots;\Sigma)$ the standard multi-variate normal probability function with the correlation matrix Σ. If we further assume

[11] Note that the implementation of (7) does not need multi-variate integrals. The easiest way to implement it is to construct a bi-variate lattice. Eom, Helweg, and Huang (2002), for example, use a one-dimension binomial model to implement the deterministic Geske-Johnson model.

that default points are equal to cash obligations, i.e. $K_{ij} = X_i$ for all j, then $\Pi_i^{\pm} = \mathsf{N}_i^{\pm}$ defined in (4).

The value of the total debt is equal to the sum of all discount debts:

$$\begin{aligned} V_{\mathrm{GJ}}(0,T_n) &= \sum\nolimits_{i=1}^{n} D(0,T_i) \\ &= A(0)[1-\Pi_n^+(K_{1n},K_{2n},\cdots,K_{nn})] + \sum\nolimits_{i=1}^{n} P(0,T_i)X_i\Pi_i^-(K_{1n},\cdots,K_{in}) \end{aligned} \tag{9}$$

This equation is extremely similar to (5) except that the probabilities are defined differently due to different strikes. Hence, by observation, the recovery must be $R_n(0) = A(0)[1-\Pi_n^+(K_{1n},K_{2n},\cdots,K_{nn})]$.

Note again that $\Pi_n^-(K_{1n},K_{2n},\cdots,K_{nn})$ is the total survival probability (under the forward measure) because the asset value, $A(t)$ needs to stay above its default boundaries, K_{ij}, at all times. Hence, the total (cumulative) default probability is $1-\Pi_n^-(K_{1n},K_{2n},\cdots,K_{nn})$. To see it, we know that the total default probability can be derived directly from summing the default probability of each period:

$$\begin{aligned} &\int_0^{K_{1n}} \varphi^-(A_1)dA_1 + \int_{K_{1n}}^{\infty}\int_0^{K_{2n}} \varphi^-(A_1,A_2)dA_2dA_1 + \cdots + \int_{K_{1n}}^{\infty}\cdots\int_{K_{n-1n}}^{\infty}\int_0^{K_{nn}} \varphi^-(A_1,\cdots,A_n)dA_n\cdots dA_1 \\ &= 1-\int_{K_{1n}}^{\infty}\cdots\int_{K_{n-1n}}^{\infty}\int_0^{K_{nn}} \varphi^-(A_1,\cdots,A_n)dA_n\cdots dA_1 \\ &= 1-\Pi_n^-(K_{1n},\cdots K_{nn}) \end{aligned} \tag{10}$$

where $A_i = A(T_i)$ is a short-hand notation. The recovery is to consider the cash amount received upon default:

(11)

$$\begin{aligned} &\int_0^{K_{1n}} A_1\varphi^-(A_1)dA_1 + \int_{K_{1n}}^{\infty}\int_0^{K_{2n}} A_2\varphi^-(A_1,A_2)dA_2dA_1 + \cdots + \int_{K_{1n}}^{\infty}\cdots\int_{K_{n-1n}}^{\infty}\int_0^{K_{nn}} A_n\varphi^-(A_1,\cdots,A_n)dA_n\cdots dA_1 \\ &= A_0\left[1-\int_{K_{1n}}^{\infty}\cdots\int_{K_{n-1n}}^{\infty}\int_0^{K_{nn}} \varphi^+(A_1,\cdots,A_n)dA_n\cdots dA_1\right] \\ &= A_0[1-\Pi_n^+(K_{1n},\cdots K_{nn})] \end{aligned}$$

The change of measure can be found in an appendix.

C. Internal Inconsistency of an Arbitrarily Specified Barrier

The barrier model is usually categorized as a structural model, because of its use of the asset value. However, the barrier over which the default is defined is exogenous and arbitrary. This could cause negative equity value. To see that, we can look at the payoffs at time T_{n-1}:

$$
A(T_{n-1}) > X_{n-1} \quad \Rightarrow \begin{cases} D(T_{n-1}, T_{n-1}) = X_{n-1} \\ D(T_{n-1}, T_n) = E_{n-1}^{\mathbb{Q}}\left[\exp\left(-\int_{T_{n-1}}^{T_n} r(u)du\right)\min\{A(T_n), X_n\}\right] \\ E(T_{n-1}) = E_{n-1}^{\mathbb{Q}}\left[\exp\left(-\int_{T_{n-1}}^{T_n} r(u)du\right)\max\{A(T_n) - X_n, 0\}\right] - X_{n-1} \end{cases}
$$

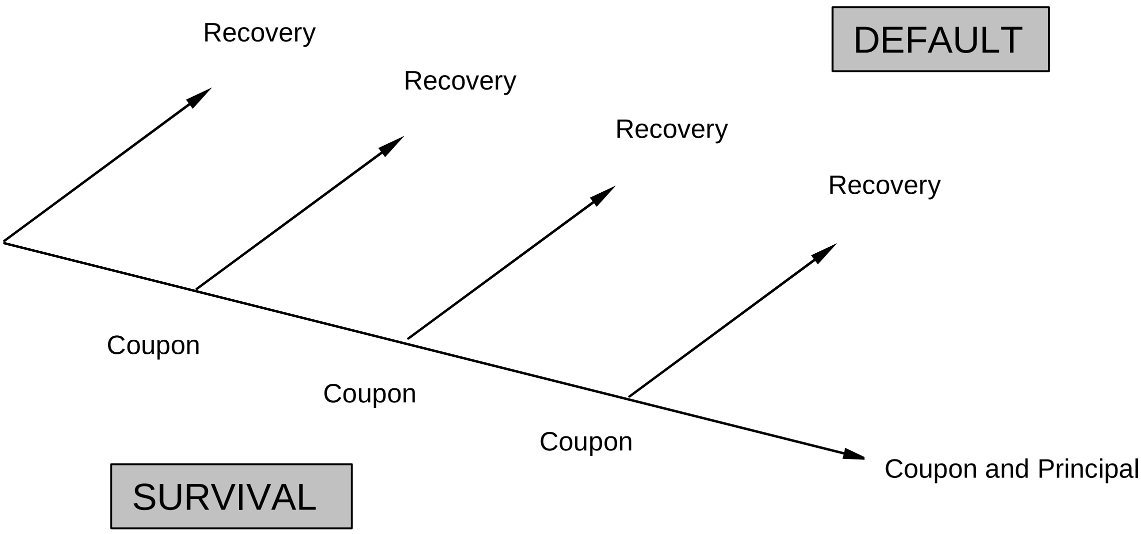
$$
A(T_{n-1}) \le X_{n-1} \quad \Rightarrow \begin{cases} D(T_{n-1}, T_{n-1}) = A(T_{n-1}) \\ D(T_{n-1}, T_n) = 0 \\ E(T_{n-1}) = 0 \end{cases}
$$

When $A(T_{n-1})$ is small, there is no guarantee that the equity value, $E(T_{n-1})$, can exceed X_{n-1} since X_{n-1} can be arbitrary. To keep the continuity assumption of the asset value at time T_{n-1}, we need to issue new equity when it is negative. In other words, we allow the company to raise new equity when it is already in bankruptcy. Too see it, let us have a numerical example. Suppose a company has two zero coupon debts, one and two years to maturity and each has \$100 face value. The asset is worth \$400 and the debts are together worth \$170. A year later, the asset grows to \$450. Geske (1977) argues that the firm at this time should raise equity to pay for the debt so that the asset value can avoid a sudden drop. So the asset value after paying off the first debt is still \$450. The second debt at this time has a value of \$90. The equity is \$360 which includes \$100 new equity (to pay for the face value of the first debt) and \$260 old equity. Now, instead of \$450, the asset value drops to \$150. The firm, as in the previous case, would like to raise equity to pay off the first debt. The second debt now is assumed to be priced at \$75 (lower than \$90 due to higher risk). Hence, the equity is \$75. But the new equity value is \$100, a clear contradiction. This means that the new equity owner pays \$100 in cash but in return receives a portion of \$75. Therefore, any rational investor would not invest equity in this firm. Since the firm cannot raise capital to continue its operation, it has to declare bankruptcy. There is a (default) point where the new equity owner is indifferent and this is the default point of the company. Suppose the asset value at one year is \$186.01. Assuming new equity being raised, the second debt is \$86. Consequently, the new equity owner has \$100 and the old equity has \$0.01. And we know that the default point is \$86. This value is precisely the “implied strike price”, K_{ij}, in the Geske-Johnson model.

However, in reality, default is usually triggered when the company fails to make an interest rate payment. As a result, a company can keep operating while it should be under bankruptcy. Under this situation, the equity owner will take huge risk (projects of large volatility) in an attempt to bring the company back to survival. The large volatility projects hurt the value of the debt. As a result, the debt owner instead of receiving the asset value if the firm is forced bankrupt takes a lower value due to excess risk the equity owner is undertaking.

3. THE CONSISTENCY BETWEEN EXTENDED GESKE-JOHNSON AND REDUCED FORM MODELS

The discrete barrier model and the Geske model are consistent with the default/no-default binomial structure that usually is used to explain the reduced form models because the binomial structure that is represented by the Bernulli event coincides with a Poisson process. The binomial structure can be depicted as the following diagram:



Hence, their solution to the risky fixed rate coupon bond can be written as:[12]

[12] The solution to the risky floating rate bond is:

$$\begin{aligned}V^{\mathrm{FLT}}(0,T_n) &= E_0^{\mathbb{Q}}\left[\sum_{i=1}^{n} \exp\left(-\int_0^{T_i} r(u)du\right)\ell(T_{i-1},T_i)h1_{\{\tau>T_i\}} + \exp\left(-\int_0^{T_n} r(u)du\right)1_{\{\tau>T_n\}}\right.\\ &\qquad \left. + \exp\left(-\int_0^{\tau} r(u)du\right)w_n(\tau)1_{\{\tau<T_n\}}\right]\\ &= \sum_{i=1}^{n} P(0,T_i)Q(0,T_i)\left(\frac{P(0,T_{i-1})}{P(0,T_i)}-1\right) + h\,\mathrm{cov}[\ell(T_{i-1},T_i),1_{\{\tau>T_n\}}] + P(0,T_n)Q(0,T_n) + R_n(0)\end{aligned}$$

where $\ell(T_{i-1},T_i)$ is the floating risk free term rate (between T_{i-1} and T_i), h is the coupon interval. As we can see, the difference between a floater and a fixed rate bond is the coupon

$$V(0,T_n) = E_0^{\mathbb{Q}}\left[\sum_{i=1}^{n} \exp\left(-\int_0^{T_i} r(u)du\right)c_n h 1_{\{\tau > T_i\}} + \exp\left(-\int_0^{T_n} r(u)du\right)1_{\{\tau > T_n\}}\right.$$

$$(12) \qquad \left. + \exp\left(-\int_0^{\tau} r(u)du\right)w_n(\tau)1_{\{\tau < T_n\}}\right]$$

$$= \sum_{i=1}^{n} P(0,T_i)Q(0,T_i)c_n h + P(0,T_n)Q(0,T_n) + R_n(0)$$

where

$Q(0,T_i) \equiv E_0^{\mathbb{F}(T_i)}\left[1_{\{\tau > T_i\}}\right]$ is the forward survival probability under the T_i-forward measure, [13]

τ is default time,

$1_{\{\cdot\}}$ is the indicator function,

c_n is the fixed coupon of the T_n-maturity bond,

h is the coupon interval (measured in years),

$1 - w_n(t)$ is the recovery of the T_n-maturity bond, that is a function of state variables, and

$R_n(0) = E_0^{\mathbb{Q}}\left[\exp\left(-\int_0^{\tau} r(u)du\right)w_n(\tau)1_{\{\tau < T_n\}}\right]$ is the current value of expected recovery.

Note that the first two terms of the second line represent the value of no default (under which the bond pays all coupons and the face value). The last term represents the current recover value under default, which is an expected present value of the discounted recovery amount upon default.

This assumption validates the use of the Geske model and the discrete barrier model. Rewrite (5) and (9) as follows:

$$(13) \qquad \begin{aligned} V_{\mathrm{DB}}(0,T_n) &= A(0)[1 - \mathsf{N}_n^{+}] + \sum_{i=1}^{n} P(0,T_i)X_i\mathsf{N}_i^{-} \\ &= R_n(0) + \sum_{i=1}^{n} P(0,T_i)Q(0,T_i)hc_n + P(0,T_n)Q(0,T_n) \end{aligned}$$

and

payment series, where the former is taken from the forward curve and the latter is a fixed constant. Hence, for the rest of the paper, we simply concentrate on fixed rate bond.

[13] The separation theorem states that:

$$E_0^{\mathbb{Q}}\left[\exp\left(-\int_0^{T_i} r(u)du\right)1_{\{\tau > T_i\}}\right] = E_0^{\mathbb{Q}}\left[\exp\left(-\int_0^{T_i} r(u)du\right)\right]E_0^{\mathbb{F}(T_i)}\left[1_{\{\tau > T_i\}}\right]$$
$$= P(0,T_i)Q(0,T_i)$$

For the discussions of the forward measure, see, for example, Jamshidian (1987) and Hull (2000).

$$
\begin{aligned}
V_{\mathrm{GJ}}(0,T_n) &= A(0)[1-\Pi_n^+(K_{1n},K_{2n},\cdots,K_{nn})] + \sum\nolimits_{i=1}^{n} P(0,T_i)X_i\Pi_i^-(K_{1n},\cdots,K_{in}) \\
&= R_n(0) + \sum\nolimits_{i=1}^{n} P(0,T_i)Q(0,T_i)hc_n + P(0,T_n)Q(0,T_n)
\end{aligned} \tag{14}
$$

where the forward survival probability notation $Q(t,T_i)$ replaces multi-variate normal probabilities N_i^- and Π_i^- in the discrete barrier model and the Geske-Johnson model respectively Equation (13) and (14) resembles (12) remarkably. The only difference is the recovery assumption. Note that the first term $1-\mathsf{N}_i^+$ and $1-\Pi_i^+$ represent the total unconditional default probability (under a different measure). While (5) and (9) are much simpler than (12), they requires the knowledge the underlying asset. It is interesting to see that both the discrete barrier and the Geske-Johnson models satisfy the condition $Q(0,T_i) > Q(0,T_{i+1})$.

The most widely cited reduced form models are the Jarrow-Turnbull (1995) and Duffie-Singleton (1999) models. The Jarrow-Turnbull model assumes:[14]

$$
\begin{aligned}
Q(0,T_i) &= \exp\left(-\int_0^{T_i} \lambda(t)dt\right) \\
R_n(0) &= E_0^{\mathbb{Q}}\left[\exp\left(-\int_0^{T_n} r(u)du\right)\overline{w}1_{\{\tau<T_n\}}\right] = \overline{w}P(0,T_n)[1-Q(0,T_n)]
\end{aligned} \tag{15}
$$

where $\overline{w}$ is a constant recovery rate. Note that in the Jarrow-Turnbull model, the recovery is paid at maturity, T_n, as opposed to the default time, τ. A more practical Jarrow-Turnbull model is to allow recovery to be paid upon default which happens discretely (at coupon payment times.) Hence, we can write the *modified* Jarrow-Turnbull model as:

$$
\begin{aligned}
V_{\mathrm{JT}}(0,T_n) = &\sum\nolimits_{i=1}^{n} P(0,T_i)Q(0,T_i)c_nh + P(0,T_n)Q(0,T_n) \\
&+\sum\nolimits_{i=1}^{n} P(0,T_i)[Q(0,T_{i-1})-Q(0,T_i)]\overline{w}
\end{aligned} \tag{16}
$$

The Duffie-Singleton model assume recovery to be paid upon default and to equal a fraction of the value of the bond if it were not defaulted. Formally:

$$
w(t) = \delta V(t,T_n) \tag{17}
$$

[14] When the intensity parameter is stochastic, then $Q(0,T_i) = E_0^{\mathbb{Q}}\left[\exp\left(-\int_0^{T_i} \lambda(t)dt\right)\right]$, see Lando (1998).

where $0 < \delta < 1$ is constant. Substituting (17) into (12) and rearrange, the result of the Duffie-Singleton model is,

$$(18) \qquad V_{\mathrm{DS}}(0,T_n) = \sum\nolimits_{i=1}^{n} P(0,T_i)Q^*(0,T_i)c_n h + P(0,T_n)Q^*(0,T_n)$$

where c_n and h are the fixed coupon and coupon interval respectively, and $Q^*(0,T_i) = Q(0,T_i)^{1-\delta}$. Note that since the recovery is incorporated inside Q^*, there is no need to have a separate term for recovery.

Equation (12) represents a general formula for all risky bond pricing models, reduced form and structural models included. It is seen that the difference only lies in the recovery assumption: recovery of the face value yields the Jarrow-Turnbull model; recovery of the market value yields the Duffie-Singleton model, and recovery of the asset value yields the Geske-Johnson model.

4. CALIBRATION AND MODEL COMPARISON

In this section, we demonstrate how different models can calibrate to the same market data and imply different parameter values. This is possible due to (12), (13), (14), and (18). We shall first use a two-period model as an example to demonstrate the computational details. Then we provide analysis in a multi-period setting. The Vasicek model (1977) for the risk free term structure of interest rates is considered.

A. A Two-Period Example

We need three pieces of information to complete the calibration: risk free zero yield curve, a set of risky bond prices, and a recovery assumption. In the case of reduced form models, the recovery rate is given exogenously. In the case of structural models, the recovery rate is computed endogenously (using equity price, equity volatility, debt value, and the term structure of interest rates).

To demonstrate the calibration procedures for both reduced form and structural models, we use a two-period example. The following table describes the base case:

time	1	2
coupon	10	10
face	100	100
bond price	100	100
yield curve	5% flat	5% flat

The Jarrow-Turnbull model can be calibrated as follows. Note that the Jarrow-Turnbull model assumes a constant recovery amount, i.e. $\overline{w}$. From equation (16), we obtain the one year bond formula:

$$V_{\mathrm{JT}}(0,1) = P(0,1)[Q(0,1)(1+c_1) + (1-Q(0,1))\overline{w}] \tag{19}$$

We assume a fixed recovery rate of 0.4 of the principal and accrued interest. In this one-year example, the recovery amount is \$44 if default occurs. Given coupon $c_1 = \$10/\$100 = 10\%$ and one-year discount $P(0,1) = e^{-5\%} = 0.9512$, we can solve for the survival probability to be $Q(0,1) = 0.9262$. Again, using equation (16) for the two-year bond, we obtain:

$$\begin{aligned} V_{\mathrm{JT}}(0,2) &= P(0,1)Q(0,1)c_2 + P(0,2)Q(0,2)(1+c_2) \\ &\quad + P(0,1)[1-Q(0,1)]\overline{w} + P(0,2)[Q(0,1)-Q(0,2)]\overline{w} \end{aligned} \tag{20}$$

With the knowledge of $Q(0,1)$, together with $P(0,2) = e^{-10\%} = 0.9048$, we can then solve for the second period survival probability to be $Q(0,2) = 0.8578$.

To calibrate the Duffie-Singleton model, we learn from (18) that,

$$V_{\mathrm{DS}}(0,1) = P(0,1)Q^*(0,1)(1+c_1) \tag{21}$$

for the one-year bond and

$$V_{\mathrm{DS}}(0,2) = P(0,1)Q^*(0,1)c_2 + P(0,2)Q^*(0,2)(1+c_2) \tag{22}$$

for the two-year bond. Hence, we solve for $Q^*(0,1) = 0.9557$ and $Q^*(0,2) = 0.9134$. For $\delta < 1$, we know that $Q^*(0,t) > Q(0,t)$ always due to non-negative recovery. By assuming a recovery rate of 0.4, we get $Q(0,1) = 0.8929$ and $Q(0,2) = 0.7973$, which are both lower than the survival probabilities calculated by the Jarrow-Turnbull model. Note that in both models, the recovery present values for one and two-year bonds are 3.09 and 5.81 for Jarrow-Turnbull and 6.57 and 12.15 for Duffie-Singleton, respectively. It is seen that the recovery amounts for Duffie-Singleton are higher, and hence to maintain the same the bond values, the survival (default) probabilities of Duffie-Singleton have to be lower (higher) to balance it out.

In both the Jarrow-Turnbull model and the Duffie-Singleton model, given that there is no other random factor other than the interest rate, there is no need to identify a specific term structure model, given that survival and default probabilities are computed under the forward measure. However, in the Geske-Johnson model, in addition to random interest rates, there is a random “asset price” state variable. As a result, a specific term structure model needs to be specified in order to carry out survival and default probabilities.

To simplify the calculation and without any loss of generality, we assume a deterministic yield curve in this section. The Vasicek term structure model is assumed in the next sub-section. In the Geske-Johnson model, the recovery amount is random and endogenous. Under deterministic interest rates, the one-period Geske-Johnson model is a Black-Scholes-Merton solution:

$$V_{\mathrm{GJ}}(0,1) = D(0,1) = P(0,1)X_1N(h_1^-(X_1)) + A(0)[1 - N(h_1^+(X_1))] \tag{23}$$

where $h_1^\pm$ is defined in (4). Since interest rates are non-stochastic, $P(0,1) = \exp(-\int_0^1 r(u)du)$. Equation (23) is identical to (4) when $\sigma_P = 0$ and $T_n = T_1 = 1$. For the two-year bond, i.e. $T_n = T_2 = 2$ and $T_1 = 1$, we have two zero bond components:

$$D(0,1) = P(0,1)X_1N(h_1^-(X_1)) + A(0)[1 - N(h_1^+(X_1))] \tag{24}$$

and

$$\begin{aligned} D(0,2) &= P(0,1)X_1[N_1(h_1^-(K_{12})) - N_1(h_1^-(X_1))] + P(0,2)X_2N_2(h_1^-(K_{12}), h_2^-(X_2); \sqrt{1/2}) \\ &\quad + A(0)[N_1(h_1^+(X_1)) - N_2(h_1^+(K_{12}), h_2^+(X_2); \sqrt{1/2})] \end{aligned} \tag{25}$$

where $h_i^\pm$ for $i = 1, 2$ is defined in (4), $X_1 = c_2$, $X_2 = 1 + c_2$. Recall that K_{12} is the solution to $A(1) = D(1,2) + X_1$ where $D(1,2)$ is a bond price at $t = 1$. Finally, the two-year coupon bond value is the sum of

(24) and (25):

$$\begin{aligned} V_{\mathrm{GJ}}(0,2) &= D(0,1) + D(0,2) \\ &= P(0,1)X_1N(h_1^-(X_1)) + P(0,2)X_2N_2(h_1^-(K_{12}), h_2^-(X_2)) \\ &\quad + A_0[1 - N_2(h_1^+(K_{12}), h_2^+(X_2); \sqrt{1/2})] \end{aligned} \tag{26}$$

In the equation, we need to evaluate three probabilities. The first one is $N(h_1^-)$ which has already been evaluated. The second one is a bivariate normal probability with two separate strikes, K_{12} that has to be internally solved and X_2 that is the face and coupon value at maturity.

Note that X_1 is the first cash flow of the company, hence $X_1 = 10 + 110 = 120$. The value of $D(0,1)$ hence contains the cash flow of the first bond and the coupon amount of the second bond. Here, we assume that the split of $D(0,1)$ is proportional. That is the value of the one-year bond is $\frac{110}{120} D(0,1)$, and the value of the two-year bond is $\frac{10}{120} D(0,1) + D(0,2)$.

In the original Geske-Johnson model, the volatility is assumed flat (as in Black-Scholes (1973)). Unfortunately under this condition, the calibration of the second bond becomes impossible. Hence, we extend the model to include a volatility curve, i.e. $v(0,2)^2 = v(0,1)^2 + v(1,2)^2$. This flexibility allows us to calibrate the model to the two-year bond price. The results are $A(0) = 2351.65$, $v(0,1) = 1.5$, and $v(1,2) = 0.69$. Under such results, the survival probabilities are $Q(0,1) = N_1^-(K_{12}) = 0.9426$ and $Q(0,2) = N_2^-(K_{12}, X_2) = 0.8592$ respectively. And the recovery values for the two bonds are computed by subtracting the coupon value from the corresponding discount debt value. For example, $D(0,1) = 109.09$ which is split into two parts – the one-year bond of \$100 and the coupon of the two-year bond of \$9.09. The one-year bond has a coupon portion of $110 \times Q(0,1) \times P(0,1) = 110 \times 0.9426 \times 0.9512 = 98.63$, and hence has a recovery value of $100 - 98.63 = 1.37$. A similar calculation gives the recovery value of the two-year bond as 5.52.[15]

For easy comparison, we put together all the numbers in the following table. We observe that the Duffie-Singleton (DS) model has the lowest survival probabilities and therefore should have the highest recovery values, due to the fact that they both contribute positively to the bond price. The Geske-Johnson (GJ) model has the highest survival probabilities and lowest recovery values. The Jarrow-Turnbull (JT) model is in between. Since the bond price is traded at par, higher survival probabilities need to be balanced by lower recovery values.

	JT	DS	GJ
$Q(0,1)$	0.9262	0.8929	0.9426
$Q(0,2)$	0.8578	0.7923	0.8592
total recovery value (i.e. $R(0)$)	8.90	18.68	6.89
recovery of first bond	3.09	6.53	1.37
recovery of second bond	5.81	12.15	5.52

B. Multi-period Analysis

In this sub-section, we examine the multi-period behavior of reduced form models, namely Jarrow-Turnbull and Duffie-Singleton and the structural model of Geske-Johnson. We examine the default and recovery implication under an n-period setting. We also incorporate stochastic

[15] Note that the total recovery value needs to be 6.89 by (26).

interest rates in our analysis. We replace the interest rate process in (1) by the following Ornstein-Uhlenbeck process:

$$dr(t) = \alpha\left(\mu - \tfrac{\gamma q}{\alpha} - r(t)\right)dt + \gamma dW_r^{\mathbb{Q}}(t) \tag{27}$$

where the parameter values are given as follows:

Vasicek Model Parameters	
α (reverting speed)	0.40
μ (reverting level)	0.08
γ (volatility)	0.05
q (mkt. price of risk)	-0.10
$r(0)$ (initial rate)	0.05

These parameter values of the Vasicek model (1977) represent an upward sloping yield curve. We first examine the case of extremely low coupons.[16] This is the case where we can see the fundamental difference between the structural model of Geske-Johnson and the reduced form models of Jarrow-Turnbull and Duffie-Singleton. Note that the Jarrow-Turnbull model is (16) and the Duffie-Singleton model is (18), both with $c_n = 0$. The Geske-Johnson model is (9) with $X_i = 0$ for $i < n$. In all the models, the zero coupon bond price, $P(t,T)$, should follow the Vasicek model (formula given in an appendix.) The face value of debt is 110. We run the Geske-Johnson model with an asset value of $A(0) = 184$ and various volatility levels: 0.4, 0.6, 1.0, and 1.6. The results computed are summarized as follows.

GJ Model				
Volatility	0.4	0.6	1.0	1.6
equity value	169.25	177.92	183.49	183.94
debt value	14.74	6.08	0.51	0.06
Recovery	4.06	2.08	0.19	0.00
JT Model				
recovery rate	3.69%	1.89%	0.17%	0.00%
Intensity	1.79%	4.85%	13.19%	20.00%

[16] The Vasicek model is re-stated in Appendix F. See equation (F2).

As the volatility goes up, the equity value in the Geske-Johnson model goes up (i.e. call option value goes up.) Since the asset value is fixed at 184, the debt value goes down. The survival probability curves under various volatility scenarios of the Geske-Johnson model are plotted in Figure 1; and the default probability curves (unconditional, i.e. $Q(T_{i-1}) - Q(T_i)$) are plotted in Figure 1a. We observe several results. First, as the risk of default becomes eminent (i.e. high volatility and low debt value), the likelihood of default shifts from far term (peak at year 30 for volatility = 0.4) to near term (peak at year 5 for volatility = 1.6). Second, it is seen that the asset volatility has a huge impact on the shape of the survival probability curve. The Geske-Johnson model is able to generate humped default probability curve, often observed empirically. Third, these differently shaped probability curves are generated by one single debt, something not possible by reduced form models. Both the Jarrow-Turnbull and Duffie-Single models cannot generate such probability curves with one single bond, due to the lack of information of intermediate cash flows. Corresponding to the recovery amounts under the Geske-Johnson model, we set the fixed recovery rate of the Jarrow-Turnbull model as shown in the above table. Given that one bond can only imply one intensity parameter value, we set it (under each scenario) so that the zero coupon bond price generated by the Geske-Johnson model is matched. As shown in the above table, the intensity rate goes from 1.79% per annum to 20% per annum. Note that flat intensity value is equivalent to a flat *conditional* default probability curve. To visualize the difference this flat conditional default probability curve of the Jarrow-Turnbull model with the non-flat curve generated by the Geske-Johnson model, we plot in Figure 1c the case of volatility = 1.6. The conditional default probabilities are calculated as $\frac{Q(T_{i-1})-Q(T_i)}{Q(T_{i-1})}$.

Our next analysis is to keep the bond value fixed, so that we can examine the default probability curve with the risk of the bond controlled. We assume that the company issues only one coupon bond at 10%. At the volatility level of 0.4, the asset value is $123, at 0.6, it is $290, and at 1.0, it is $15,000. Figure 2 and Figure 2a demonstrate the Geske-Johnson model for various volatility levels but keep the bond at par. We can see that for the same par bond, the default and survival probability curves are drastically different as the asset/volatility combination changes. This is a feature not captured by either the Duffie-Singleton or the Jarrow-Turnbull model.

To compare to the Jarrow-Turnbull and Duffie-Turnbull models, we keep the case where the volatility level is 0.6 and asset value is 290. The recover rates of both Jarrow-Turnbull and Duffie-Singleton models are assumed to be 0.4. Figure 3 and Figure 3a show the survival and default probability curves of the three models. The flat conditional forward default probability for the Jarrow-Turnbull model is solved to be 7.60% (or equivalently the intensity rate is 7.74%.) The conditional forward default probabilities for the Duffie-Singleton model are certainly non-constant. The "recovery-adjusted" continuously compounded discount rate is 9.52%. From the survival probability curves (Figure 3), it is seen that the Geske-Johnson and Jarrow-Turnbull models can be close. But the default probability curves (Figure 3a) demonstrate that the default pattern can be quite different.

5. CREDIT DEFAULT SWAP PRICING

Credit default swaps are the most widely traded credit derivative contract today. A default swap contract offers protection against default of a pre-specified corporate issue. In the event of default, a default swap will pay the principal (with or without accrued interest) in exchange for the defaulted bond.[17]

Default swaps, like any other swap, have two legs. The premium leg contains a stream of payments, called spreads, paid by the buyer of the default swap to the seller till either default or maturity, whichever is earlier. The other leg, protection leg, contains a single payment from the seller to the buyer upon default if default occurs and 0 if default does not occur. Under some restrictive conditions, credit default swap spreads are substitutes for par floater spreads.[18] In many occasions, the traded spreads off credit default swaps are more representative than those of risky corporate bonds. The valuation of a credit default swap is straightforward. For the default protection leg:

$$
\begin{aligned}
W(0,T_n) &= \int_0^{T_n} (1-w(A(t),r(t)))P(0,t)[-dQ(0,t)] \\
&= \int_0^{T_n} P(0,t)[-dQ(0,t)] - \int_0^{T_n} w(A(t),r(t))P(0,t)[-dQ(0,t)] \\
&= \int_0^{T_n} P(0,t)[-dQ(0,t)] - R_n(0)
\end{aligned} \tag{28}
$$

In discrete time, we can write (28) as:

$$
W(0,T_n) = \sum_{i=1}^{n} P(0,T_i)[Q(0,T_{i-1}) - Q(0,T_i)] - R_n(0) \tag{28a}
$$

This is called the protection leg or the floating leg of the swap. For the premium leg, or the fixed leg:

---

[17] Default swaps can also be designed to protect a corporate name. These default swaps were used to be digital default swaps. Recently these default swaps have a collection of "representative" reference bonds issued by the corporate name. Any bond in the reference basket can be used for delivery.

[18] See, for example, Chen and Soprazetti (2002) for a discussion of the relationship between the credit default swap spread and the par floater spread.

$$W(0,T_n) = s_n \sum_{i=1}^{n} P(0,T_i)Q(0,T_i) \tag{29}$$

Combining (28) and (29), we can use market credit default swap spreads to back out default probability curve. As the default swap market grows, more and more investors seek arbitrage trading opportunities between corporate bonds and default swaps. This suggests that we should use the calibrated corporate bond curves (last section) to compute default swap spreads. We use the results of Figure 3 to compute credit default swap values for various tenors (1~30 years). The recovery rate in the swap contract is assumed to be 0.[19] Figure 4 shows how different models can imply different default swap values.[20] The Geske-Johnson model is close to the Duffie-Singleton model at near terms but close to the Jarrow-Turnbull model at far terms. As we have seen, same par bond implies very different default probabilities from the Jarrow-Turnbull, Duffie-Singleton, and Geske-Johnson models, which in turn imply very different credit default swap values.

The default swap market has grown very fast in the past few years[21] and many fixed income traders and fund managers try to arbitrage between corporate bonds and credit default swaps should they see discrepancies in spreads. Here, we demonstrate that such arbitrage trading strategies can be misleading. Arbitrage profits can be entirely due to model specification. To see that, we suppose that the Geske-Johnson model is the correct model. Hence the probabilities that calculate the par bond are shown in Figure 5 for the case where asset value is \$290 and volatility is 0.6 (so that the coupon debt is at par). The 30-year default swap spread implied by the Geske-Johnson model is 438 basis points. This is done by implementing (28a) to obtain the default swap value (\$32.87) and implementing (29) to solve for the spread. Note that the credit default swap value and spread computed using the Geske-Johnson model are consistent with the recovery assumption of the Geske-Johnson model. To use the Jarrow-Turnbull model, we need a fixed recovery rate. To get such value, we use that the probabilities generated by the Geske-Johnson model and a fixed recovery rate to compute the default swap value. This implied recovery rate is 0.5840. Now we let the Jarrow-Turnbull model to calibrate to the data (by changing the intensity paramter). The Jarrow-Turnbull bond price is it is 110.10 at an intensity level of 7.19%, 10% overvalued due to model error.

---

[19] As long as the recovery rate is fixed, it simply scales up/down the curves and does not change the shapes.

[20] We should note that the default swap price cannot be computed directly from the Duffie-Singleton model because there is no cash flow paid if there is no default on the protection leg. Hence, for the Duffie-Singleton model, it is possible to compute the default swap value once the underlying bond is available; but not possible to compute the bond price when the default swap spread is available.

[21] According to a Lehman Brothers credit research report (O'kane (2001)), the credit derivative markets are estimated to be \$1 trillion in notional at the time the report was written and near half of which is the market of default swaps.

The situation can be even more severe if we allow the Geske-Johnson model to generate more humped shaped default probability curve. Set volatility to 0.4 and asset value to 184, we price another 30-year par bond by the Geske-Johnson model. We also use the Vasicek model for the term structure. Again, assume the Geske-Johnson model to be correct. It implies the probability curves as shown in Figure 6. The 30-year default swap value is \$11.18 and the spread is 163 basis points. A flat recovery rate implied by such a spread is 0.3117. Using this recovery rate, we obtain the Jarrow-Turnbull price to be 83.74 at the intensity level of 4.99%, which is 16% undervalued due to model error. If we set volatility to 0.6 and asset value to 184, we price the 30-year bond by the Geske-Johnson model at 80. Again, assume the model to be correct. It implies the probability curves as shown in Figure 7. The 30-year default swap value is \$29.06 and the spread is 562 basis points. A flat recovery rate implied by such a spread is 0.5485. Using this recovery rate, we obtain the Jarrow-Turnbull price to be 81.10 at the intensity level of 9.84%, 1% overvalued due to model error.

6. CONCLUSION

In this paper, we extend the Geske-Johnson model (1979, 1984) to n periods and incorporate stochastic interest rates. This extension is necessary to price risky bonds and credit derivatives because they are extremely sensitive to interest rates. The resulting formula is analytical in the Geske-Johnson sense – it is represented as multi-variate integrals over Gaussian densities.

The comparison between the *extended* Geske-Johnson model with other structural models shows that only the Geske-Johnson model is free of arbitrage. Arbitrarily specified barrier could cause incorrect survival. While failure to fulfill interest payments is regarded as default in reality, barrier structural models that assume coupon barrier implies that there is a wealth transfer from bond holders to share holders.

Finally, we are able to show that the Geske-Johnson model is consistent with the reduced form models. The general binomial structure for the reduced form models can be fit well by the Geske-Johnson model (or any structural model). The consistency allows us to compare all default models. Equation (12) shows clearly that all models differ in their recovery assumptions. Numerical examples of bonds and default swaps are provided to demonstrate how the Geske-Johnson model can differ from the Jarrow-Turnbull and Duffie-Singleton models.

APPENDIX

A. Distribution of $\ln A(t)$ *under the Forward Measure*

Assume an interest rate process under the $\mathbb{Q}$ measure generally as:

(A1) $$dr(t) = \alpha(r,t)dt + \gamma(r,t)dW^{\mathbb{Q}}(t)$$

Define the Radon-Nykodym derivative as follows:

(A2) $$\eta(t,T) = \frac{\exp\left(-\int_t^T r(u)du\right)}{P(t,T)}$$

Then, Apply Ito's lemma on the bond price $P(t,T)$:

(A3)

$$\begin{aligned}
0 &= \ln P(T,T) \\
&= \ln P(t,T) + \int_t^T \frac{1}{P(u,T)}\left[\frac{\partial P(u,T)}{\partial u}du + \frac{\partial P(u,T)}{\partial r}dr + \frac{1}{2}\frac{\partial^2 P(u,T)}{\partial r^2}(dr)^2\right] - \int_t^T \frac{1}{2}\left[\frac{\gamma(r,u)}{P(u,T)}\frac{\partial P(u,T)}{\partial r}\right]^2 du \\
&= \ln P(t,T) + \int_t^T \frac{1}{P(u,T)}\left[\frac{\partial P(u,T)}{\partial u}du + \frac{\partial P(u,T)}{\partial r}\alpha(r,u) + \frac{1}{2}\frac{\partial^2 P(u,T)}{\partial r^2}\gamma(r,u)^2\right]du \\
&\quad + \int_t^T \frac{1}{P(u,T)}\frac{\partial P(u,T)}{\partial r}\gamma(r,u)dW^{\mathbb{Q}}(u) - \int_t^T \frac{1}{2}\left[\frac{\gamma(r,u)}{P(u,T)}\frac{\partial P(u,T)}{\partial r}\right]^2 du \\
&= \ln P(t,T) + \int_t^T r(u)du + \int_t^T \frac{\gamma(r,u)}{P(u,T)}\frac{\partial P(u,T)}{\partial r}dW^{\mathbb{Q}}(u) - \int_t^T \frac{1}{2}\left[\frac{\gamma(r,u)}{P(u,T)}\frac{\partial P(u,T)}{\partial r}\right]^2 du
\end{aligned}$$

Letting:

(A4) $$\theta(t,T) = -\frac{\gamma(r,t)}{P(t,T)}\frac{\partial P(u,T)}{\partial r}$$

and moving the first two terms to the left:

(A5)

$$\begin{aligned}
-\int_t^T r(u)du - \ln P(t,T) &= -\int_t^T \theta(u,T)dW^{\mathbb{Q}}(u) - \int_t^T \frac{1}{2}\theta(u,T)^2 du \\
\frac{\exp\left(-\int_t^T r(u)du\right)}{P(t,T)} &= \eta(t,T) = \exp\left(-\int_t^T \theta(u,T)dW^{\mathbb{Q}}(u) - \int_t^T \frac{1}{2}\theta(u,T)^2 du\right)
\end{aligned}$$

This implies the Girsonav transformation of the following:

$$\text{(A6)} \qquad \begin{aligned} W^{\mathbb{Q}}(t) &= W^{\mathbb{F}(T)}(t) + \int_t^T \theta(u,T)du \\ &= W^{\mathbb{F}(T)}(t) - \int_t^T \gamma(r,u)\frac{\partial P(u,T)/\partial r}{P(u,T)}du \end{aligned}$$

The interest rate process under the forward measure henceforth becomes:

$$\text{(A7)} \qquad dr(t) = \left[\alpha(r,t) - \gamma(r,t)^2 \frac{\partial P(t,T)/\partial r}{P(t,T)}\right]dt + \gamma(r,t)dW^{\mathbb{F}(T)}(t)$$

Note that the forward measure derived above is quite general. It does not depend on any specific assumption on the interest rate process. Incorporating the random interest rate process into equation (1), we have:

$$\text{(A8)} \qquad \begin{bmatrix} \frac{dA(t)}{A(t)} \\ dr(t) \end{bmatrix} = \begin{bmatrix} r(t) \\ \alpha(r,t) \end{bmatrix} dt + \begin{bmatrix} \sigma & 0 \\ 0 & \gamma(r,t) \end{bmatrix} \times \begin{bmatrix} \sqrt{1-\rho_{Ar}^2} & \rho_{Ar} \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} dW_A^{\mathbb{Q}}(t) \\ dW_r^{\mathbb{Q}}(t) \end{bmatrix}$$

To derive the analytical result for the extended Geske-Johnson model, we assume normally distributed interest rates, or there is no solution to option pricing. In such a case, $\gamma(r,t)$ must not be a function of r (i.e. $\gamma(r,t) = \gamma(t)$) then $A(T)$ is log normally distributed. Without loss of generality and for the ease of exposition, we let $\gamma(r,t) = \gamma$. Using the forward measure, we get:

$$\text{(A9)} \qquad \begin{bmatrix} \frac{dA(t)}{A(t)} \\ dr(t) \end{bmatrix} = \begin{bmatrix} r(t) - \rho\sigma\gamma\frac{\partial P(t,T)/\partial r}{P(t,T)} \\ \alpha(r,t) - \gamma^2 \frac{\partial P(t,T)/\partial r}{P(t,T)} \end{bmatrix} dt + \begin{bmatrix} \sigma & 0 \\ 0 & \gamma \end{bmatrix} \times \begin{bmatrix} \sqrt{1-\rho_{Ar}^2} & \rho_{Ar} \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} dW_A^{\mathbb{Q}}(t) \\ dW_r^{\mathbb{F}(T)}(t) \end{bmatrix}$$

The solution of the asset price to the above equation is:

$$\text{(A10)} \qquad A(T) = A(t)\exp\left(\int_t^T \left[r(u) - \rho\sigma\gamma\frac{\partial P(u,T)/\partial r}{P(u,T)}\right]du + \int_t^T \sigma dW^{\mathbb{F}(T)}(u) - \frac{1}{2}\int_t^T \sigma^2 du\right)$$

where

$$dW^{\mathbb{F}(T)}(t) = \sqrt{1-\rho_{Ar}^2}\, dW_A^{\mathbb{Q}}(t) + \rho_{Ar} dW_r^{\mathbb{F}(T)}(t)$$

From the result of Ito's lemma, we can then arrive at the following result:

$$\text{(A11)} \quad \begin{aligned} A(T) = \frac{A(t)}{P(t,T)} \exp\Bigg(\int_t^T & \left[-\frac{1}{2}\left(\gamma \frac{\partial P(u,T)/\partial r}{P(u,T)}\right)^2 + \rho_{Ar}\gamma \frac{\partial P(u,T)/\partial r}{P(u,T)} - \frac{1}{2}\sigma^2 \right] du \\ & - \int_t^T \gamma \frac{\partial P(u,T)/\partial r}{P(u,T)} dW_r^{\mathbb{F}(T)}(u) + \int_t^T \sigma dW^{\mathbb{F}(T)}(u) \Bigg) \end{aligned}$$

As a result, $A(T)$ is log normally distributed and the log mean and variance of the asset value under the forward measure can be computed as:

$$\text{(A12)} \quad \begin{aligned} & E_t^{\mathbb{F}(T)}[\ln A(T)] = \ln A(t) - \ln P(t,T) + \frac{1}{2} v(t,T)^2 \\ & \text{var}_t^{\mathbb{F}(T)}[\ln A(T)] = v(t,T)^2 \end{aligned}$$

where

$$v(t,T)^2 = \int_t^T \left[\left(\gamma \frac{\partial P(u,T)/\partial r}{P(u,T)}\right)^2 - 2\rho_{Ar}\sigma\gamma \frac{\partial P(u,T)/\partial r}{P(u,T)} + \sigma^2 \right] du$$

B. Derivation of (4)

With the forward measure result in Part A, we shall derive (4) by induction. A T_1-maturity zero coupon bond with a barrier X_1 is:

$$\text{(B1)} \quad \begin{aligned} D(0,T_1) &= E_0^{\mathbb{Q}}\left[\exp\left(-\int_0^{T_1} r(u)du \right) \min\{A(T_1), X_1\} \right] \\ &= P(0,T_1) E_0^{\mathbb{F}(T_1)}\left[\min\{A(T_1), X_1\} \right] \\ &= P(0,T_1) E_0^{\mathbb{F}(T_1)}\left[A(T_1) - \max\{A(T_1) - X_1, 0\} \right] \\ &= P(0,T_1) \frac{A(0)}{P(0,T_1)} - P(0,T_1) E_0^{\mathbb{F}(T_1)}\left[\max\{A(T_1) - X_1, 0\} \right] \\ &= A(0)[1 - N(h_1^+(X_1))] + P(0,T_1) X_1 N(h_1^-(X_1)) \end{aligned}$$

where $h_1^{\pm}(X_1)$ is defined in the text. The last step, especially the second term of the second to last line, is the standard Black-Scholes derivation. A T_2-maturity zero coupon bond with a barrier X_2 has the following value at T_1:

$$\text{(B2)} \qquad \begin{cases} E_{T_1}^{\mathbb{Q}}\left[\exp\left(-\int_{T_1}^{T_2} r(u)du\right)\min\{A(T_2), X_2\}\right] & A(T_1) > X_1 \\ 0 & A(T_1) \leq X_1 \end{cases}$$

Hence, it has the following value today:

$$\begin{aligned} D(0,T_2) &= E_0^{\mathbb{Q}}\left[\exp\left(-\int_0^{T_1} r(u)du\right)E_{T_1}^{\mathbb{Q}}\left[\exp\left(-\int_{T_1}^{T_2} r(u)du\right)\min\{A(T_2),X_2\}\right]1_{\{A(T_1)>X_1\}}\right] \\ &= E_0^{\mathbb{Q}}\left[E_{T_1}^{\mathbb{Q}}\left[\exp\left(-\int_0^{T_2} r(u)du\right)\left(A(T_2)1_{\{A(T_2)<X_2\}} + X_2 1_{\{A(T_2)>X_2\}}\right)\right]1_{\{A(T_1)>X_1\}}\right] \\ \text{(B3)} \qquad &= P(0,T_2)E_0^{\mathbb{F}(T_2)}\left[A(T_2)1_{\{A(T_2)<X_2 \wedge A(T_1)>X_1\}} + X_2 1_{\{A(T_2)>X_2 \wedge A(T_1)>X_1\}}\right] \\ &= P(0,T_2)E_0^{\mathbb{F}(T_2)}\left[A(T_2)\left(1_{\{A(T_1)>X_1\}} - 1_{\{A(T_2)>X_2 \wedge A(T_1)>X_1\}}\right) + X_2 1_{\{A(T_2)>X_2 \wedge A(T_1)>X_1\}}\right] \\ &= A(0)[\mathsf{N}_1^+ - \mathsf{N}_2^+] + P(0,T_2)X_2\mathsf{N}_2^- \end{aligned}$$

where the second to last line is obtained by appropriately dividing the integration region. Carrying out the expectation using the standard log normal results should yield the result desired. Similar procedure is applied to any arbitrary T_n:

$$\text{(B3)} \qquad D(0,T_n) = A(0)[\mathsf{N}_{n-1}^+ - \mathsf{N}_n^+] + P(0,T_n)X_n\mathsf{N}_n^-$$

which is identical to equation (4) in the text.

C. N_n^+ *and* N_n^-

In this appendix, we show that N_n^+ and N_n^- are both survival probabilities, but under different probability measures. For the ease of composition, we use $\mathsf{N}_1^{\pm} = N(h_1^{\pm}(X_1))$ where $N(\cdot)$ is a standard univariate normal probability function, as an example. The general case, while tedious algebraically, is straightforward.

To simplify notation, we drop subscript "1" from $h_1^{\pm}$, X_1, T_1, and measure $\mathbb{F}(T_1)$. Note that:

$$N(h^{+}) = \frac{P(0,T)}{A(0)} E_0^{\mathbb{F}(T)}\left[A(T)1_{\{A(T)>X\}}\right] \tag{C1}$$
$$N(h^{-}) = E_0^{\mathbb{F}(T)}\left[1_{\{A(T)>X\}}\right]$$

$N(h^{+})$ in equation (C1) can be written as:

$$\begin{aligned} E_0^{\mathbb{F}(T)}\left[A(T)1_{\{A(T)>X\}}\right] &= E_0^{\mathbb{F}(T)}[A(T)]E_0^{\mathbb{F}^*(T)}\left[1_{\{A(T)>X\}}\right] \\ &= \frac{A(0)}{P(0,T)} E_0^{\mathbb{F}^*(T)}\left[1_{\{A(T)>X\}}\right] \end{aligned} \tag{C2}$$

where $1_{\{\bullet\}}$ is an indicator function whose value is 1 if the statement in {} is true, and 0 otherwise. The change of measure from $\mathbb{F}$ to $\mathbb{F}^*$ is done as follows. The Radon-Nykodim derivative for the change of measure is defined as:

$$\eta^*(T) = \frac{d\mathbb{F}^*(T)}{d\mathbb{F}(T)} = \frac{A(T)}{E_0^{\mathbb{F}(T)}[A(T)]} \tag{C3}$$

Given that A is a log normal process under (A8), the denominator of (C3) can be easily computed using (A12) as:

$$E_0^{\mathbb{F}(T)}[A(T)] = \frac{A(0)}{P(0,T)} \tag{C4}$$

Then, using (A11), (C3) becomes:

$$\begin{aligned} \eta^*(T) = \exp\Bigg(&\int_t^T \left[-\frac{1}{2}\left(\gamma \frac{\partial P(u,T)/\partial r}{P(u,T)}\right)^2 + \rho_{Ar}\gamma \frac{\partial P(u,T)/\partial r}{P(u,T)} - \frac{1}{2}\sigma^2 \right] du \\ &- \int_t^T \gamma \frac{\partial P(u,T)/\partial r}{P(u,T)} dW_r^{\mathbb{F}(T)}(u) + \int_t^T \sigma dW^{\mathbb{F}(T)}(u) \Bigg) \\ = \exp\Bigg(&\int_t^T -\frac{1}{2} v(u,T)^2 - \int_t^T v(u,T) dZ^{\mathbb{F}(T)}(u) \Bigg) \end{aligned} \tag{C5}$$

where

$dZ^{\mathbb{F}(T)}(t) = a dW_r^{\mathbb{F}(T)}(t) + b dW^{\mathbb{F}(T)}(t)$

$$a = \frac{1}{v(u,T)} \gamma \frac{\partial P(u,T) / \partial r}{P(u,T)}$$
$$b = \frac{\sigma}{v(u,T)}$$

As a result, the Girsonav's Theorem can be written as:

$$dZ^{\mathbb{F}(T)}(t) = dZ^{\mathbb{F}^*(T)} + v(t,T)dt$$

We immediately obtain that the above derivative as a change of measure of $\sqrt{\mathrm{var}^{\mathbb{F}(T)}[\ln(A(T)]}$ under the $\mathbb{F}$ measure. As a result, $E^{\mathbb{F}(T)}[1_{\{\bullet\}}]$ and $E^{\mathbb{F}^*(T)}[1_{\{\bullet\}}]$ differ by $\frac{v^2}{2}$.

D. Derivation of (7)

The Geske-Johnson model is usually written as the multi-variate normal probability format. As a result, the final solution looks more complex than it really is. Once, we understand the recursive relationship in zero coupon bond formulas, the final result is very straightforward to recognize. Hence, in this appendix, we provide a simple three period example to demonstrate how we can easily derive and streamline the Geske-Johnson model. The rest can be obtained by induction.

For the one-period zero bond, it is identical to the Merton-Rabinovitch model:

$$\text{(D1)} \qquad \begin{aligned} D(0,T_1) = V_{\mathrm{GJ}}(0,T_1) &= P(0,T_1)K_{11}[N_1(h_1^-(K_{11})) - 0] + A(0)[1 - N_1(h_1^+(K_{11}))] \\ &= P(0,T_1)K_{11}\Pi_1^-(K_{11}) + A(0)[1 - \Pi_1^+(K_{11})] \end{aligned}$$

Note that $K_{11} = X_1$, the first coupon. The second line of the above equation is merely a notation change. The simpler notation allows to more easily label high dimension normal probabilities with different strikes.

For the two-year zero, we need to solve for an internal solution K_{12} that equates

$$\text{(D2)} \qquad A(T_1) = D(T_1,T_2) + X_1$$

where

$$\text{(D3)} \qquad \begin{aligned} D(T_1,T_2) &= E_{T_1}^{\mathbb{Q}}[\Lambda(T_1,T_2)\min\{A(T_2),X_2\}] \\ &= P(T_1,T_2)N(d_-) + A(T_1)[1 - N(d_+)] \end{aligned}$$

and

$$(D4)\qquad d_{\pm} = \frac{\ln A(T_1) - \ln P(T_1,T_2) - \ln X_2 \pm \frac{1}{2} v^2(T_1,T_2)}{v(T_1,T_2)}$$

is a Merton-Rabinovitch result again. Note that $v^2(\cdot,\cdot)$ is defined in (4). We should note that since $P(T_1,T_2)$ is a function of the interest rate at time T_1, i.e. $r(T_1)$. As a result, K_{12} is also a function of $r(T_1)$. For the convenience of later derivation, we rewrite (C3) in its original integral form as follows. Also for notational convenience, we shorten the following notation: $D(T_i,T_j) = D_{ij}$, $P(T_i,T_j) = P_{ij}$, $A(T_i) = A_i$ and $r(T_i) = r_i$.

$$(C3a)\qquad \begin{aligned} D_{12} &= \int_{-\infty}^{\infty}\int_{X_2}^{\infty} \exp\left(-\int_{T_1}^{T_2} r(u)du\right) X_2 \varphi(A_2,r_2)dA_2dr_2 \\ &+ \int_{-\infty}^{\infty}\int_{0}^{X_2} \exp\left(-\int_{T_1}^{T_2} r(u)du\right) A_2 \varphi(A_2,r_2)dA_2dr_2 \end{aligned}$$

This expression allows us to easily integrate with other integrals. The value of the two-period zero price at T_1 is:

$$(D5)\qquad \begin{cases} D_{12} & A_1 > K_{12} \\ A_1 - X_1 & X_1 < A_1 \le K_{12} \\ 0 & A_1 \le X_1 \end{cases}$$

To obtain the current value of the two-period zero bond, we simply integrate these payoffs at its corresponding region. Note that K_{12} is a function of the interest rate.

$$(D6)\qquad \begin{aligned} D(0,2) &= \underbrace{\int_{-\infty}^{\infty}\int_{K_{12}}^{\infty} e^{-\int_0^{T_1} r(t)dt} E[e^{-\int_{T_1}^{T_2} r(t)dt} \min\{A_2,F_2\}]\varphi(A_1,r_1)dA_1dr_1}_{\Delta_1} \\ &+ \underbrace{\int_{-\infty}^{\infty}\int_{X_1}^{K_{12}} e^{-\int_0^{T_1} r(t)dt} (A_1 - X_1)\varphi(A_1,r_1)dA_1dr_1}_{\Delta_2} \end{aligned}$$

The second term can be shown as:

$$
\text{(D7)}\quad \begin{aligned}
\Delta_2 &= \int_{-\infty}^{\infty}\int_{K_{11}}^{\infty} \exp\left(-\int_0^{T_1} r(u)du\right)(A_1 - X_1)\varphi^-(A_1, r_1)dA_1dr_1 \\
&\quad -\int_{-\infty}^{\infty}\int_{K_{12}}^{\infty} \exp\left(-\int_0^{T_1} r(u)du\right)(A_1 - X_1)\varphi^-(A_1, r_1)dA_1dr_1 \\
&= A_0[\Pi_1^+(K_{11}) - \Pi_1^+(K_{12})] - P_{01}X_1[\Pi_1^-(K_{11}) - \Pi_1^-(K_{12})]
\end{aligned}
$$

The first term can be valued as:

(D8)

$$
\begin{aligned}
\Delta_1 &= \int_{-\infty}^{\infty}\int_{K_{12}}^{\infty} \exp\left(-\int_0^{T_1} r(u)du\right)\left[\int_{-\infty}^{\infty}\int_0^{K22} \exp\left(-\int_{T_1}^{T_2} r(u)du\right)A_2\varphi^-(A_2, r_2 \mid A_1, r_1)dA_2dr_2\right]\varphi^-(A_1, r_1)dA_1dr_1 \\
&\quad +\int_{-\infty}^{\infty}\int_{K_{12}}^{\infty} \exp\left(-\int_0^{T_1} r(u)du\right)\left[\int_{-\infty}^{\infty}\int_{K_{22}}^{\infty} \exp\left(-\int_{T_1}^{T_2} r(u)du\right)X_2\varphi^-(A_2, r_2 \mid A_1, r_1)dA_2dr_2\right]\varphi^-(A_1, r_1)dA_1dr_1 \\
&= \int_{-\infty}^{\infty}\int_{K_{12}}^{\infty} \exp\left(-\int_0^{T_1} r(u)du\right)\left[\int_{-\infty}^{\infty}\int_0^{\infty} \exp\left(-\int_{T_1}^{T_2} r(u)du\right)A_2\varphi^-(A_2, r_2 \mid A_1, r_1)dA_2dr_2\right]\varphi^-(A_1, r_1)dA_1dr_1 \\
&\quad -\int_{-\infty}^{\infty}\int_{K_{12}}^{\infty} \exp\left(-\int_0^{T_1} r(u)du\right)\left[\int_{-\infty}^{\infty}\int_{K_{22}}^{\infty} \exp\left(-\int_{T_1}^{T_2} r(u)du\right)A_2\varphi^-(A_2, r_2 \mid A_1, r_1)dA_2dr_2\right]\varphi^-(A_1, r_1)dA_1dr_1 \\
&\quad +\int_{-\infty}^{\infty}\int_{K_{12}}^{\infty} \exp\left(-\int_0^{T_1} r(u)du\right)\left[\int_{-\infty}^{\infty}\int_{K_{22}}^{\infty} \exp\left(-\int_{T_1}^{T_2} r(u)du\right)X_2\varphi^-(A_2, r_2 \mid A_1, r_1)dA_2dr_2\right]\varphi^-(A_1, r_1)dA_1dr_1 \\
&= A_0\Pi_1^+(K_{12}) - A_0\Pi_2^+(K_{12}, K_{22}) + P_{02}X_2\Pi_2^-(K_{12}, K_{22})
\end{aligned}
$$

Hence, the two-year coupon bond, returning to the original notation, is:

$$
\text{(D9)}\quad \begin{aligned}
V_{\mathrm{GJ}}(0, T_2) &= D(0, T_1) + D(0, T_2) \\
&= D(0, T_1) + \Delta_1 + \Delta_2 \\
&= P(0, T_1)X_1\Pi_1^-(X_1) + P(0, T_2)X_2\Pi_2^-(K_{12}, X_2) + A_0[1 - \Pi_2^+(K_{12}, X_2)]
\end{aligned}
$$

where

$$
\text{(D10)}\quad \begin{aligned}
\Pi_1^{\pm}(K) &= \int_{-\infty}^{\infty}\int_K^{\infty} \varphi^{\pm}(A_1, r_1)dA_1dr_1 \\
\Pi_2^{\pm}(K_1, K_2) &= \int_{-\infty}^{\infty}\int_{K_1}^{\infty}\int_{K_2}^{\infty} \varphi^{\pm}(A_1, A_2, r_1)dA_2dA_1dr_1
\end{aligned}
$$

It is shown in Appendix B that the last two integrals in the above equation can be written as:

$$\text{(D11)} \quad \int_K^\infty A_1 \varphi^-(A_1 \mid r_1) dA_1 = A_0 \int_K^\infty \varphi^+(A_1 \mid r_1) dA_1$$

and

$$\text{(D12)} \quad \int_{K_1}^\infty \int_{K_2}^\infty A_2 \varphi^-(A_1, A_2 \mid r_1) dA_2 dA_1 = A_0 \int_{K_1}^\infty \int_{K_2}^\infty \varphi^+(A_1, A_2 \mid r_1) dA_2 dA_1$$

where the density is adjusted by the volatility.

Following the same procedure, although tedious, we can derive the three-year zero as:

$$\text{(D13)} \quad \begin{aligned} D(0,T_3) &= P(0,T_1)X_1[\Pi_1^-(K_{13}) - \Pi_1^-(K_{12})] + P(0,T_2)X_2[\Pi_2^-(K_{13},K_{23}) - \Pi_2^-(K_{12},K_{22})] \\ &\quad + P(0,T_3)X_3\Pi_3^-(K_{13},K_{23},K_{33}) + A(0)[\Pi_2^+(K_{12},K_{22}) - \Pi_3^+(K_{13},K_{23},K_{33})] \end{aligned}$$

Now, we should observe a pattern for the zero coupon bond prices, which gives (7).

E. Implementation of (7)

The closed form Gekse-Johnson model (with constant volatility and constant interest rates) can be computed efficiently only when $n \le 2$. When $n > 2$, then the multi-variate normal probability functions can not be implemented efficiently, particularly when n is large. We use the standard equity binomial model with various payoffs to obtain the survival probabilities, zero bond values, and the equity (compound option) value.[22] Note that in the binomial model, there is no need to solve for the implied strike price, K_{ij}. The (compound) option value is directly computed off the actual strikes, X_i.

In (7), we have both volatility and interest rate to be non-constant. We allow volatility to be a deterministic function of time (for calibration) and interest rates to be random (to capture interest rate risk).

First, the deterministic volatility function is handled by changing time interval, suggested by Amin (1991). In the binomial model, the up and down sizes are determined by:

$$\text{(E1)} \quad \begin{aligned} u &= e^{\sigma\sqrt{\Delta t}} \\ d &= e^{-\sigma\sqrt{\Delta t}} \end{aligned}$$

[22] We learned via private conversation that the implementation of the Geske-Johnson model in Eom, Hedweg, and Huang (2002) is computed using the binomial model.

If the volatility is changing over time, we simply adjust Δt so that $\sigma\sqrt{\Delta t}$ is constant. In this case, both u and d are constants over time and the tree recombines.

To incorporate stochastic interest rates, we first build a lattice for a "risk-neutral" bivariate Brownian motion process of (A8) by setting (i.e. the Vasicek model):

$$\begin{aligned} \alpha(r,t) &= \alpha(\mu - r) \\ \gamma(r,t) &= \gamma \end{aligned} \tag{E2}$$

We then set up a binomial lattice as follows:

$$\begin{array}{cc} & \sqrt{\Delta t} \\ 0 & \\ & -\sqrt{\Delta t} \end{array}$$

We label asset value and interest rate at various nodes as follows:

$$\begin{array}{ccc} & & < \ln A_{22}, r_{22} > \\ & < \ln A_{11}, r_{11} > & \\ < \ln A_0, r_0 > & & < \ln A_{21}, r_{21} > \\ & < \ln A_{10}, r_{10} > & \\ & & < \ln A_{20}, r_{20} > \end{array}$$

Then we first show that the asset values recombine. Given that:

$$\begin{aligned} \ln A_{11} &= \ln A_0 + (r_0 - \sigma^2/2)\Delta t + \sigma\sqrt{\Delta t} \\ \ln A_{10} &= \ln A_0 + (r_0 - \sigma^2/2)\Delta t - \sigma\sqrt{\Delta t} \end{aligned} \tag{E3}$$

It is straightforward that:

$$\begin{aligned} \ln A_{21} &= \ln A_{10} + (r_0 - \sigma^2/2)\Delta t + \sigma\sqrt{\Delta t} \\ &= \ln A_{11} + (r_0 - \sigma^2/2)\Delta t - \sigma\sqrt{\Delta t} \end{aligned} \tag{E4}$$

We now show that the interest rate lattice recombines approximately. Note that:

$$\begin{aligned} r_{11} &= r_0(1 - \alpha\Delta t) + \alpha\mu\Delta t + \gamma\sqrt{\Delta t} \\ r_{10} &= r_0(1 - \alpha\Delta t) + \alpha\mu\Delta t - \gamma\sqrt{\Delta t} \end{aligned} \tag{E5}$$

Hence,

$$\text{(E6)}\quad \begin{aligned} r_{21} &= r_{10}(1-\alpha\Delta t)+\alpha\mu\Delta t+\gamma\sqrt{\Delta t} \\ &= [r_0(1-\alpha\Delta t)+\alpha\mu\Delta t-\gamma\sqrt{\Delta t}](1-\alpha\Delta t)+\alpha\mu\Delta t+\gamma\sqrt{\Delta t} \\ &= r_0(1-2\alpha\Delta t)+2\alpha\mu\Delta t \\ &= r_{11}(1-\alpha\Delta t)+\alpha\mu\Delta t-\gamma\sqrt{\Delta t} \end{aligned}$$

This result is an approximated one because higher order terms are ignored.[23]

F. Closed Form Solution for $v^2(0,T_i)$ – *Rabinovitch* (1989)

Following the interest rate model defined in Appendix D and let:

$$\text{(F1)}\quad \begin{aligned} \alpha(r,t) &= \alpha(\mu-r) \\ \gamma(r,t) &= \gamma \end{aligned}$$

(i.e. the Vasicek model). Then, the zero coupon bond price satisfies the following closed form equation:

$$\text{(F2)}\quad P(0,t)=e^{-rF(0,t)-G(0,t)}$$

where

$$\begin{aligned} F(0,t) &= \frac{1-e^{-\alpha t}}{\alpha} \\ G(0,t) &= (t-F(0,t))\left(\mu-\tfrac{\gamma^2}{2\alpha}\right)+\frac{\gamma^2F(0,t)^2}{4\alpha} \end{aligned}$$

Hence, by Ito's lemma we have,

[23] If higher order terms are not ignored, then the lattice does not combine. In the case where the lattice does not recombine, we take the average of two non-recombining values at each node and proceed. We discover that this method produces extremely close results when we compare the zero coupon bond price against the Vasicek closed form model. We use the same technique with both deterministic volatility function and random interest rate are used. Note that in here, since Δt is different period by period, the interest rate dimension of the lattice does not recombine.

(F3) $$\sigma_P^2(t,T_i) = \left[\frac{\partial P(t,T_i)}{\partial r}\gamma\right]^2 = F(t,T_i)^2\gamma^2$$

and

(F4) $$\begin{aligned}\int_0^t \sigma_P^2(u,T_i)du &= \int_0^t F(u,T_i)^2\gamma^2 du = \left((t-2F(0,t)) + \tfrac{1-e^{-2\alpha t}}{2\alpha}\right)\left(\tfrac{\gamma}{\alpha}\right)^2 \\ \int_0^t \sigma_P(u,T_i)du &= \int_0^t F(u,T_i)\gamma du = (t-F(0,t))\left(\tfrac{\gamma}{\alpha}\right)\end{aligned}$$

Substituting (F4) back to the (4) we get:

(F5) $$v^2(0,T_i) = \sigma_A^2 T_i + \left((t-2F(0,T_i)) + \tfrac{1-e^{-2\alpha T_i}}{2\alpha}\right)\left(\tfrac{\gamma}{\alpha}\right)^2 - 2\rho_{Ar}\sigma_A(t-F(0,T_i))\tfrac{\gamma}{\alpha}$$

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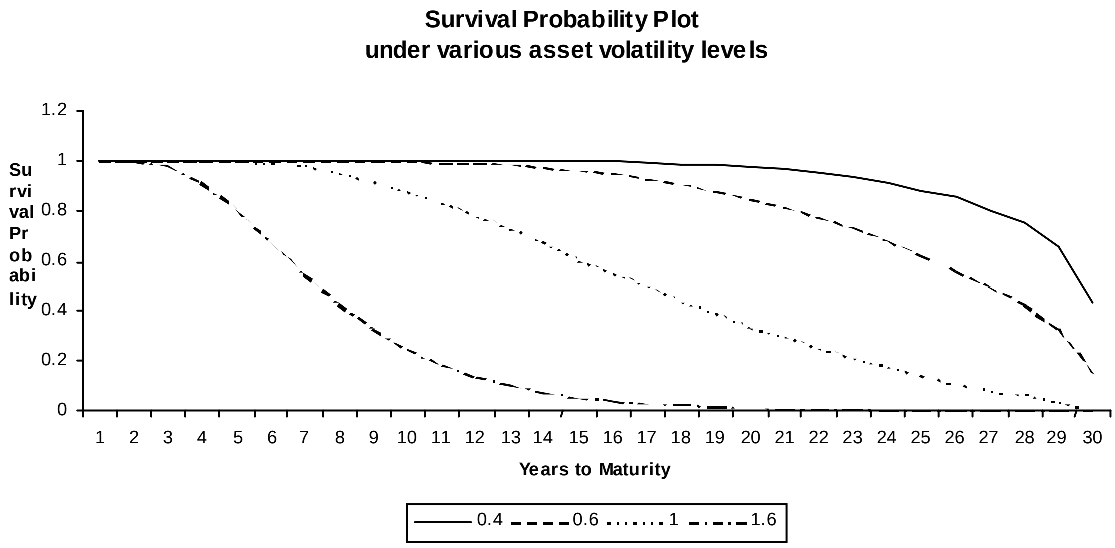
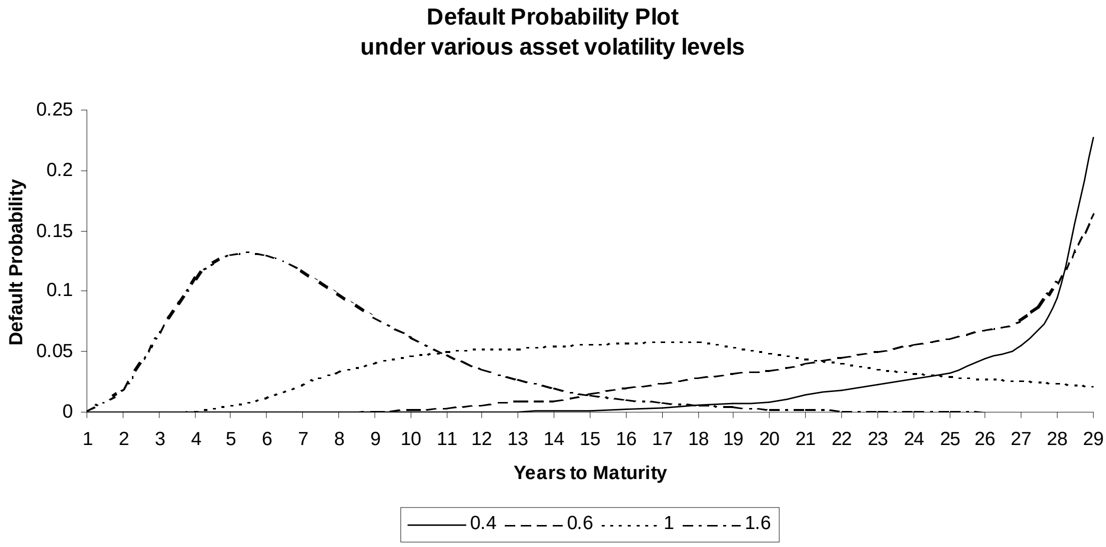
Figure 1: The GJ Model: Zero coupon bond



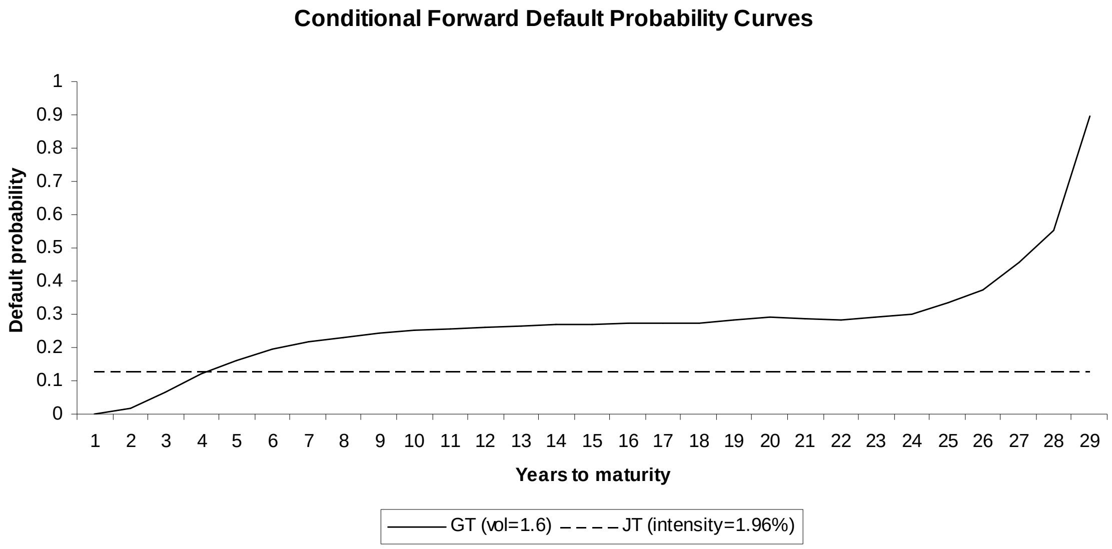
Note: Figure 1 illustrates the survival probability curves under various asset volatility scenarios of the 30-year zero coupon bond under the Geske-Johnson model. The Asset value is set to be $184. The bond has no coupon and a face value of $110. The yield curve is flat at 5%

Figure 1a: The GJ Model: Zero Coupon Bond



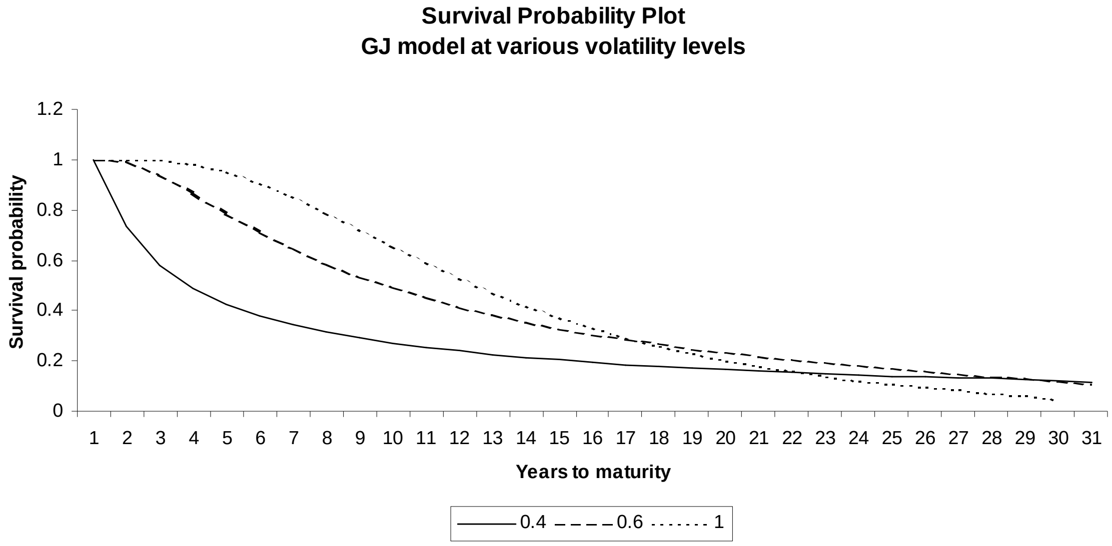
Note: Figure 1a illustrates the default probability curve. All parameters are identical to those in Figure 1.

Figure 1b: Comparison of GJ and JT Models: Zero Coupon Bond



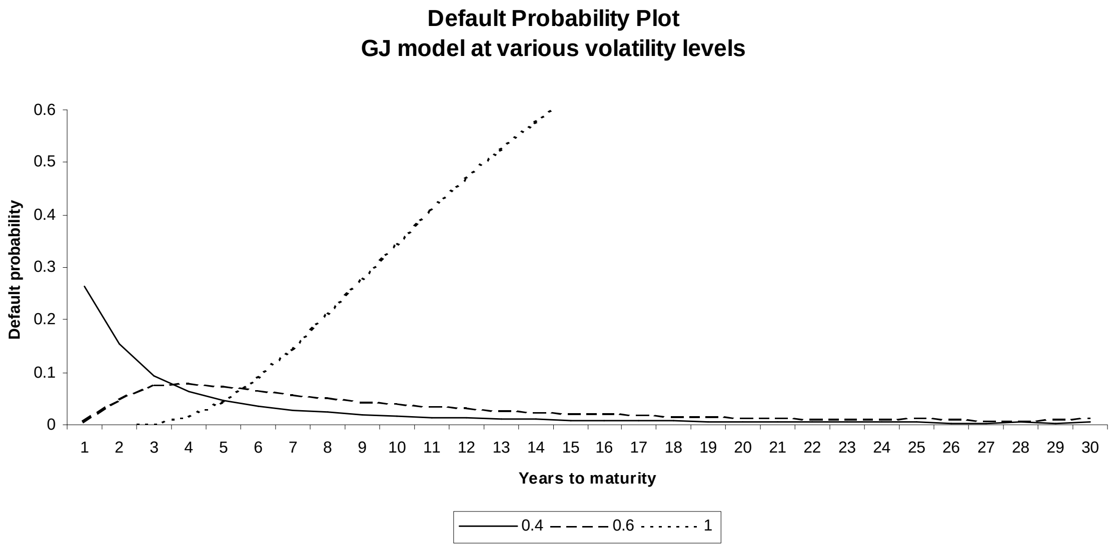
Note: Figure 1b illustrates the default probability curves under GJ and JT models for the volatility level of 1.6. All parameters are identical to those in Figure 1.

Figure 2: The GJ model: 10% Coupon Bond



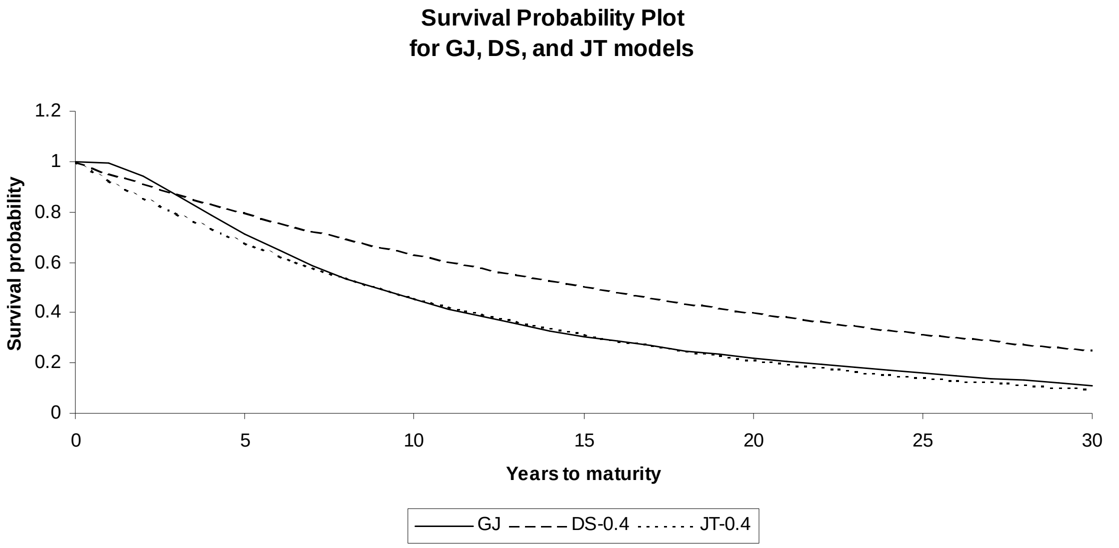
Note: Figure 2 illustrates the survival probability curves under various asset volatility scenarios of the 30-year Geske-Johnson model. The Asset values are 123, 290, 15,000 for volatility levels of 0.4, 0.6, and 1.0 repsectively. The bond has a coupon of $10 and price of par.

Figure 2a: The GJ Model: 10% Coupon Bond



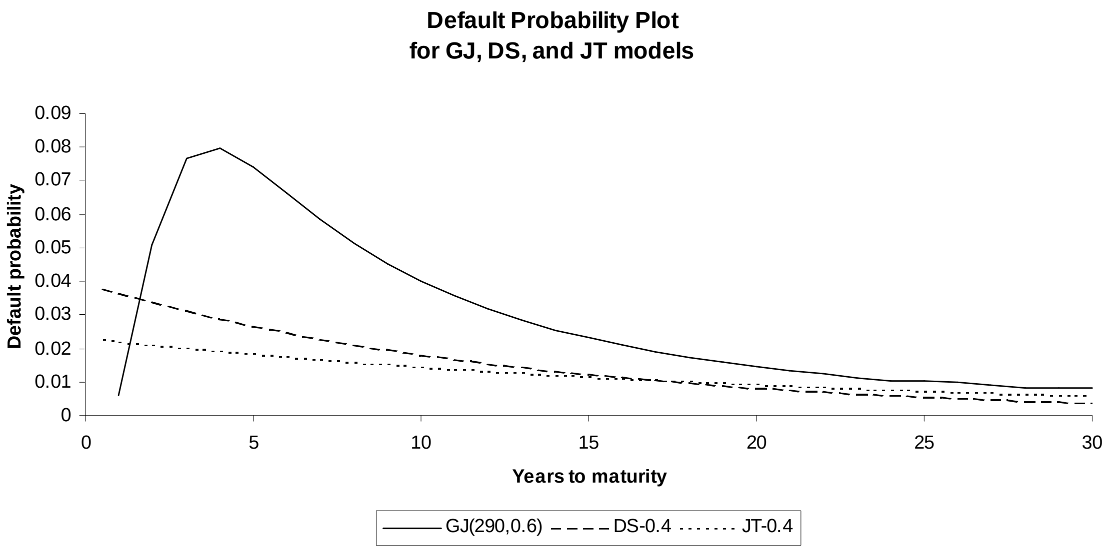
Note: Figure 2a illustrates the default probability curves under various asset volatility scenarios of the 30-year Geske-Johnson model. The Asset values are 123, 290, 15,000 for volatility levels of 0.4, 0.6, and 1.0 repsectively. The bond has a coupon of $10 and price of par. All parameters are identical to those in Figure 2.

Figure 3: Comparison of the GJ, DS and JT Models: Par Coupon Bond



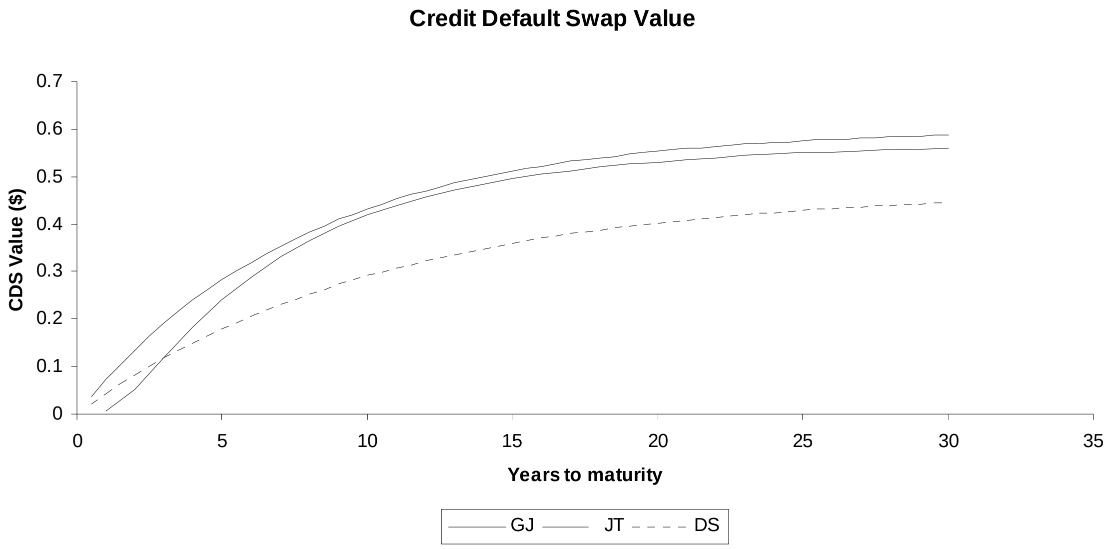
Note: Figure 3 illustrates the survival probability curves under the 30-year Geske-Johnson, Duffie-Singleton with 0.4 recovery ratio, and Jarrow-Turnbull with 0.4 recovery ratio models. The Asset value is set to be $290 and volatility 0.6 so that the bond is priced at par. The bond has $10 coupon and a face value of $110.

Figure 3a: (Comparison of the GJ, DS and JT Models: Par Coupon Bond



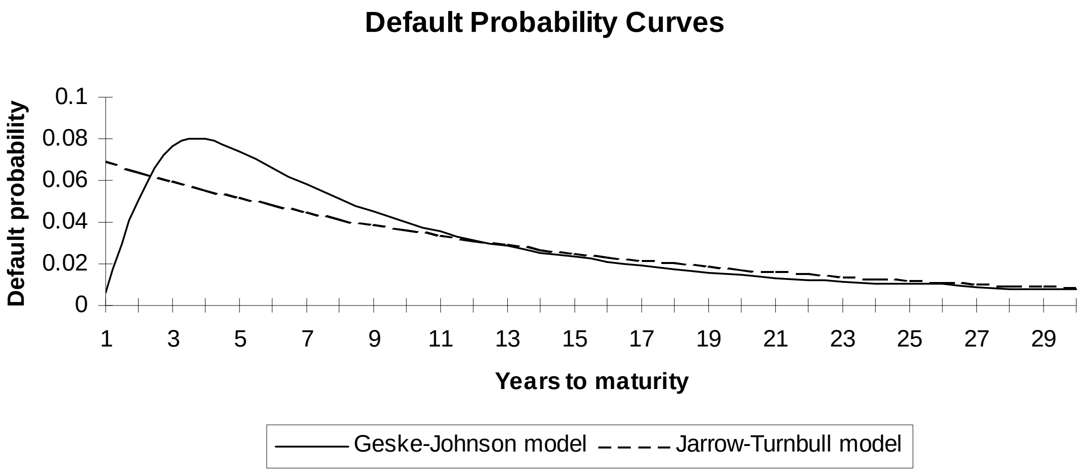
Note: Figure 3a illustrates that under a 10% coupon bond, the Geske-Johnson model can generate desired default probability curve. All parameters are identical to those in Figure 3.

Figure 4: Credit Default Swap Values for the GJ, JT, and DS models



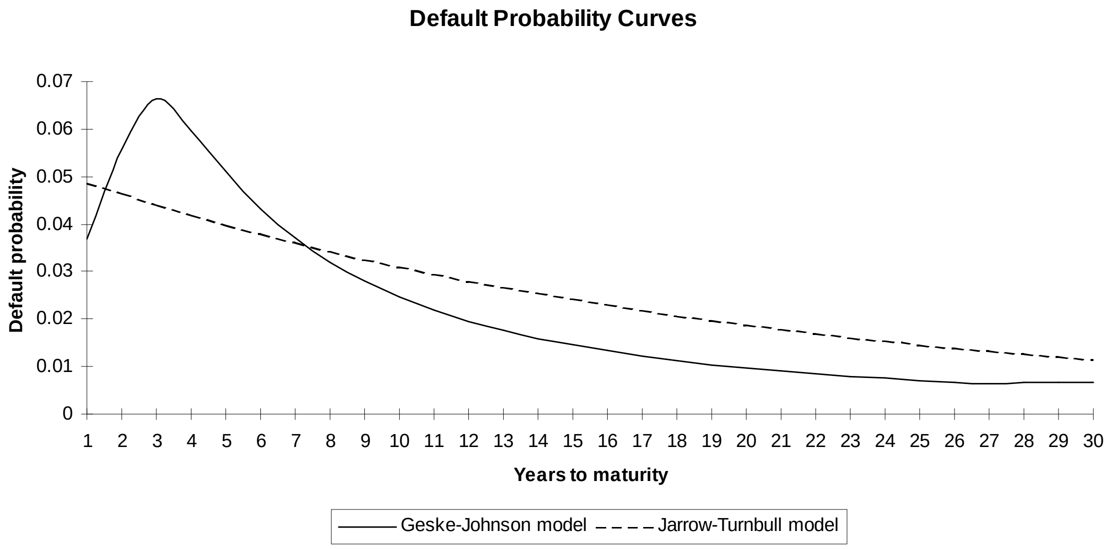
Note: Data that compute this figure are taken from those generate Figure 3a.

Figure 5: (Unconditional) Default Probability Curve for the GJ and JT models



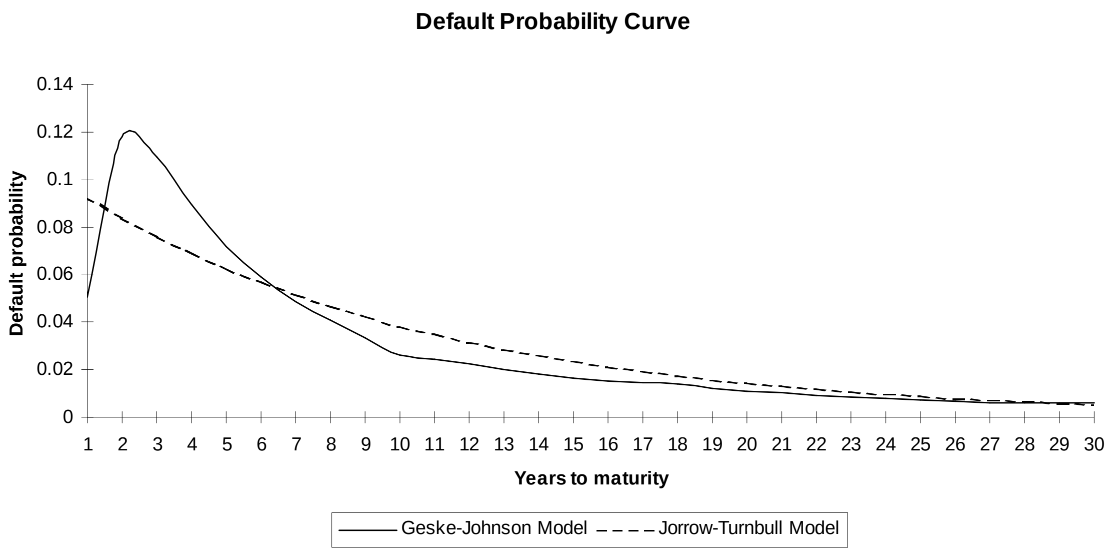
Note: In the Geske-Johnson model, asset=290, volatility=0.6 (so that debt=100), coupon=10, face=100, and yield curve is flat at 5%. The credit default swap value is $32.87. A recovery rate that satisfies a flat recovery is 0.5840. The intensity value is 7.19% and the bond price is 110.1.

Figure 6: (Unconditional) Default Probability Curve for the GJ and JT models



Note: In the Geske-Johnson model, asset=184, volatility=0.4 (so that debt=100), coupon=10, face=100, and yield curve is generated by the Vasicek model given in the text. The credit default swap value is $11.18. A recovery rate that satisfies a flat recovery is 0.3117. The intensity value is 4.99% and the bond price is 83.74.

Figure 7: (Unconditional) Default Probability Curve for the GJ and JT models



Note: In the Geske-Johnson model, asset=184, volatility=0.6 (debt=80), coupon=10, face=100, and yield curve is generated by the Vasicek model given in the text. The credit default swap value is $29.06. A recovery rate that satisfies a flat recovery is 0.5485. The intensity value is 9.84% and the bond price is 81.05.