On the term structure of lending interest rates when a fraction of collateral is recovered upon default *

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First Version: May 29, 2002

Abstract

This article provides an arbitrage-free model to evaluate the term structure of lending interest rates when a fraction of collateral is recovered upon default of the borrower. Unlike the previous literature, we assume that the value of the collateral asset fluctuates over time with certain correlation to the risk-free interest rate as well as the default hazard rate of the borrower. It is shown that the bank loan is a sum of holding a coupon-bearing bond and selling a put option, both being callable upon default. A Gaussian model is considered, as a special case, to derive an analytic expression of the appropriate lending interest rates, and some numerical example is given to demonstrate that bank loans exhibit different properties from corporate bonds.

Keywords: risk-neutral valuation, Cox process, stochastic recovery rate, Gaussian model, collateral, prepayment risk.

JEL classification: G13, G33

*The authors are grateful to Katsumasa Nishide for careful reading of the manuscript.

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1 Introduction

Suppose that a commercial bank finances a project of a firm with lending interest rate α and maturity T. Assuming that the loan is F dollars, this means that the borrowing firm repays αF dollars to the bank every year and the principal F at the maturity, if the firm does not default before the maturity. However, it is common that the bank requires the firm to put some property as a collateral, which is used to recover a loss when the firm defaults. The collateral asset is usually financial securities or lands.

To be more specific, let $L(t)$ denote the time t value of the collateral asset, and let τ be the default epoch of the borrowing firm. At time 0, the bank lends F dollars to the firm. In compensation of the loan, the borrowing firm repays αF dollars to the bank every year as the interest, until either the maturity T of the contract or termination by the firm's default, whichever happens first. The firm repays the principal F at the maturity, only if it does not default. On the other hand, if the firm defaults before the maturity, the bank uses the collateral asset to recover the principal F. However, the value of the collateral may not be enough to cover the principal. Hence, the recovery upon default is given by the minimum of $L(\tau)$ and F. The problem is how to determine the lending interest rate α.

The lending interest rate α should reflect the credit quality of the borrowing firm. If the firm's credit quality is high (low, respectively), the credit spread compared to the default-free interest rate should be small (large). The lending rate α also depends on the value of the collateral asset. That is, if the value of the collateral upon default is sufficiently high, the lending bank can always recover the principal and actually has no risk on the loan.[1] On the other hand, if it is insufficient to cover the principal, the bank cannot recover it using the collateral asset.[2] Since the lending rate depends on the length of the loan, it is of great interest to study the term structure of lending interest rates that reflect credit quality of the borrower and the future value of a collateral asset.

Loans and bonds are both debt contracts and the same in the cashflow structure (see

[1] As we shall see, however, there exists a prepayment risk for the bank even in this case.

[2] Traditionally, Japanese banks have used lands as collateral assets. This makes sense before the economic recession, because the values of the lands keep increasing and banks had no risk. However, even after the collapse of economic bubbles, they kept the same scheme without evaluating credit risk; the values of lands in turn started decreasing. This is one of the reasons that Japanese banks still suffer the problem of non-performing loans.

Figure 1 below). However, while bonds are held by dispersed creditors, bank loans are typically held by a single creditor. This distinctive feature makes possible and desirable for the lending bank to monitor and renegotiate with a borrower (see, e.g., Blackwell and Winters (1997) and Gorton and Kahn (2000) for details). In fact, empirical work suggests that bank loans exhibit different properties from corporate bonds.[3] In order to explain what makes bank loans different from corporate bonds, many theories have been proposed by focusing on various kinds of screening and monitoring of borrowers.

The lending bank has an option to liquidate the loan. It also has first claim on the asset of the borrower in the event of default, since the loan is usually secured by a collateral asset. These features will also result in different properties of bank loans from corporate bonds. As explained above, the recovery of the loan depends on the value of the collateral asset. In order to manage the credit as well as market risks of the loan, it is essential to determine the fair value of the loan. In this article, we argue an arbitrage-free model for the term structure of lending interest rates from the standpoint of financial engineering.

The model of the present paper is based on the reduced-form approach for the term structure of credit spreads.[4] However, the existing models assume that the recovery rate is exogenously given and related to no firm-specific variables. In contrast, our model assumes that the recovery rate depends on the value of a collateral asset, whence it is also a stochastic process. Note that the value of the collateral asset fluctuates over time, and so is the recovery rate, with certain correlation to the default-free interest rates as well as the default hazard rate of the borrower.[5] Hence, we need to extend the existing models

[3]See, e.g., James (1987), Lummer and McConnell (1989) and Slovin, Sushka and Polonchek (1993) for details.

[4]The pricing of corporate debt subject to credit risk has been extensively studied in the literature. We refer to Duffie and Singleton (1999), Jarrow and Turnbull (2000) and Madan and Unal (2000) for the survey of such term structure models. In particular, in the reduced-form approach, researchers focus on the formulation of default intensity processes. Such a default model for the single asset case was first developed in the work of Madan and Unal (1998). Recently, the default model was extended to the multivariate asset case by Duffie (1998), Kijima (2000) and Kijima and Muromachi (2000a).

[5]There is a strong evidence that the default intensities of corporate bonds vary with the business cycle. Since asset values and the default-free interest rates are correlated to the business cycle, it is important to consider the correlation between the three variables. For example, when the likelihood of default is high, the value of the collateral asset may decrease. In this case, the recovery rate becomes very low, making a loss in the loan quite big. In practice, however, many banks assume that the recovery rate upon default is constant, e.g. a fraction of the *current* value of the collateral asset.

so as to obtain the term structure of lending interest rates.

In order to analyze the effect of the underlying variables on the lending interest rates, we construct a Gaussian model, as a special case, to derive an analytic expression of the lending rates. In particular, it is shown that the lending rate is increasing in the correlation, ρ say, between the collateral asset and the default-free spot rate, if the hazard rate is a deterministic function of time t. Also, if $\rho \geq 0$, the lending rate is increasing in the volatility of the collateral asset. However, in the case that $\rho < 0$, it can be decreasing with respect to the collateral volatility. That is, a larger volatility does not always result in a higher interest in bank loans. Some numerical example is also given to demonstrate that bank loans exhibit different properties from corporate bonds.

This paper is organized as follows. In the next section, we construct an arbitrage-free model for bank loans in the reduced-form approach. Our model is similar to the one for credit swaps considered in Duffie (1999) and Kijima (2000). The appropriate lending interest rate can then be obtained, at least, by using Monte Carlo simulation. Section 3 proposes a Gaussian term structure model to derive an analytic expression for the lending rate. Some special case is considered in Section 4 to infer the effect of the underlying variables on the lending interest rate. A numerical example is given to demonstrate the usefulness of our results. Section 5 concludes this paper, and proofs are given in Appendix.

2 The Model Framework

Throughout this paper, we fix the probability space $(\Omega, \mathcal{F}, P)$ and denote the expectation operator by E. The probability measure P is the *risk-neutral measure*, and we assume that such a P exists and is unique, since we are interested in the pricing of financial instruments. The canonical filtration generated by the underlying stochastic structure is denoted by $\{\mathcal{F}_t\}$, where $\mathcal{F}_t$ defines the information available at time t.

Suppose that the current time is 0 and the bank lends F dollars to a corporate firm with maturity $T > 0$. In compensation of the loan, the firm repays αF dollars to the bank at time epochs t_j, where $0 = t_0 < t_1 < t_2 < \cdots < t_m = T$, and the principal F at maturity T in the case of no default before the maturity. If default occurs during $(t_k, t_{k+1}]$, then the payment terminates at time t_k.

Let τ denote the default epoch of the firm, and let $L(t)$ be the time t value of the

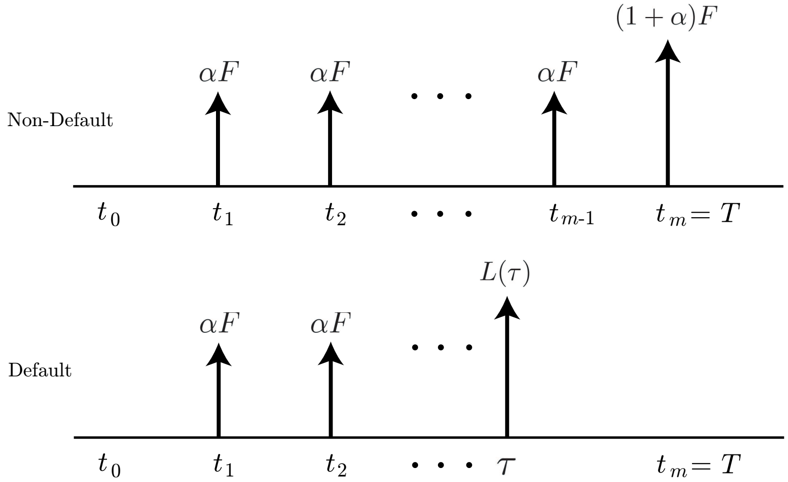

Figure 1: The cashflow of the payments of the firm

collateral asset. If the firm defaults before the maturity T, the recovery of the bank is given by $\min\{\beta L(\tau), F\}$ for some β, where $0 \leq \beta \leq 1$. The cashflow of the payments of the firm is depicted in Figure 1.

Let $r(t)$ denote the default-free spot rate at time t, and denote the *cumulative* spot rate by $R(t) = \int_0^t r(u)\mathrm{d}u$. The time t money-market account is then given by $B(t) = \mathrm{e}^{R(t)}$, $t \geq 0$. According to the risk-neutral method, the present value of the total payments of the firm is obtained as

$$T_R = E\left[\left(\sum_{j=1}^{m} \frac{\alpha F}{B(t_j)} + \frac{F}{B(T)}\right) 1_{\{\tau > T\}} + \sum_{k=0}^{m-1}\left(\sum_{j=1}^{k} \frac{\alpha F}{B(t_j)} + \frac{\min\{\beta L(\tau), F\}}{B(\tau)}\right) 1_{\{t_k < \tau \leq t_{k+1}\}}\right],$$

where 1_A denotes the indicator function meaning that $1_A = 1$ if event A is true and $1_A = 0$ otherwise. Note that the first term inside the above expectation corresponds to the payments for the case of no default, while the second term corresponds to the case of default at time τ before the maturity T. Since

$$\sum_{k=1}^{m-1}\sum_{j=1}^{k} \frac{1}{B(t_j)} 1_{\{t_k < \tau \leq t_{k+1}\}} = \sum_{j=1}^{m-1} \frac{1}{B(t_j)} 1_{\{t_j < \tau \leq T\}}$$

and

$$\sum_{k=0}^{m-1} \frac{\min\{\beta L(\tau), F\}}{B(\tau)} 1_{\{t_k < \tau \leq t_{k+1}\}} = \frac{\min\{\beta L(\tau), F\}}{B(\tau)} 1_{\{\tau \leq T\}},$$

it follows that

$$T_R = E\left[\alpha \sum_{j=1}^{m} \frac{F}{B(t_j)} 1_{\{\tau > t_j\}} + \frac{F}{B(T)} 1_{\{\tau > T\}} + \frac{\min\{\beta L(\tau), F\}}{B(\tau)} 1_{\{\tau \leq T\}}\right]. \tag{1}$$

Note that, if $\beta = 0$, then the bank has no collateral assets.

The present value T_R can be interpreted as follows. Since

$$\min\{x, y\} = y + \min\{x - y, 0\} = y - \max\{y - x, 0\},$$

we obtain from (1) that

$$T_R = E\left[\alpha \sum_{j=1}^{m} \frac{F}{B(t_j)} 1_{\{\tau > t_j\}} + \frac{F}{B(T \wedge \tau)}\right] - E\left[\frac{\max\{F - \beta L(\tau), 0\}}{B(\tau)} 1_{\{\tau \le T\}}\right], \qquad (2)$$

where $x \wedge y = \min\{x, y\}$. The first term in the right-hand side of (2) is the price of a coupon-bearing bond with coupon rate α and (guaranteed) face value F, while the second term is the price of a put option written on the collateral asset $L(t)$ with strike price being equal to the principal F. Both the bond and the put option are callable at default epoch. Note that the payoff in the first term is the same as the one when the loan is prepaid at time τ. Hence, even when a collateral is taken for a bank loan, the lending bank is exposed to the prepayment risk.[6] We thus conclude the following.

Proposition 1 *A bank loan with a collateral asset is equal to a sum of holding a coupon-bearing bond and selling a put option written on the collateral asset, both being callable at default epoch. The lending bank is always exposed to prepayment risk.*

Now, since the lending rate $\alpha = \alpha(T)$ should be determined such that the present values of the both sides (i.e. the bank and the firm) are equal,[7] we set $T_R = F$ in (1) to obtain

$$\alpha(T) = \frac{1 - E\left[B^{-1}(T) 1_{\{\tau > T\}}\right] - E\left[B^{-1}(\tau) \min\{\beta L(\tau)/F, 1\} 1_{\{\tau \le T\}}\right]}{\sum_{j=1}^{m} E\left[B^{-1}(t_j) 1_{\{\tau > t_j\}}\right]}. \qquad (3)$$

In this article, in order to evaluate the lending rate $\alpha(T)$ given in (3), we take the reduced-form approach in the finance literature.

Let τ be a default time of the corporate firm under consideration. The stochastic process $h(t)$ is called an *intensity process* for τ if

$$P\{\tau \le t + \Delta t | \tau > t, \mathcal{F}_t\} = h(t)\Delta t \qquad (4)$$

[6]Because of the prepayment risk and the fact that the bank is selling a put option, the bank never liquidates the loan as far as the lending interest rate is higher than the default-free interest rate.

[7]For simplicity, we assume that the bank is default-free.

for sufficiently small $\Delta t > 0$. That is, the intensity process $h(t)$ is the conditional rate of default just after time t given all the information available up to that time.[8] The *cumulative* default process is denoted by $H(t) = \int_0^t h(s)\mathrm{d}s$, $t \geq 0$. Also, throughout this paper, we shall use the notation

$$M(t) = R(t) + H(t) = \int_0^t \{r(s) + h(s)\}\mathrm{d}s, \quad t \geq 0, \tag{5}$$

as the *risk-adjusted* discount process.

As in Lando (1998) and Kijima (2000), suppose that the default process is a *Cox process*. That is, let $\mathcal{G}_t$ denote the σ-field generated from $\{(L(u), r(u), h(u)); u \leq t\}$.[9] Given the information $\mathcal{G}_T$ up to the maturity, i.e. for each realization of $(L(t), r(t), h(t))$, $0 \leq t \leq T$, the conditional survival probability of τ is given by

$$P\{\tau > t | \mathcal{G}_T\} = \mathrm{e}^{-H(t)}, \quad 0 \leq t \leq T, \tag{6}$$

where $\tau > 0$. It follows from the law of total probability that

$$P\{\tau > t\} = E\left[\mathrm{e}^{-H(t)}\right], \quad t \geq 0. \tag{7}$$

Also, from (4) and (6), we have

$$P\{t < \tau \leq t + \Delta t | \mathcal{G}_T\} = h(t)\mathrm{e}^{-H(t)}\Delta t, \quad 0 \leq t \leq T, \tag{8}$$

for sufficiently small $\Delta t > 0$. Equation (8) provides the *density function* of τ given the information $\mathcal{G}_T$.

Proposition 2 *Suppose that, under the risk-neutral probability measure P, the default process is a Cox process with intensity $h(t)$. Then, the lending interest rate is given by*

$$\alpha(T) = \frac{1 - E\left[\mathrm{e}^{-M(T)}\right] - \int_0^T E\left[h(u)\min\{\beta L(u)/F, 1\}\mathrm{e}^{-M(u)}\right]\mathrm{d}u}{\sum_{j=1}^m E\left[\mathrm{e}^{-M(t_j)}\right]}. \tag{9}$$

It remains to specify the stochastic structure of the value $L(t)$ of the collateral asset, the default-free spot rate $r(t)$, and the hazard rate $h(t)$. Here, following the ordinary models in the finance literature, we assume that they are modeled by stochastic differential equations (SDE's) under the risk-neutral probability measure P. That is,

$$\frac{\mathrm{d}L}{L} = r(t)\mathrm{d}t + \sigma_L(t)\mathrm{d}z_L, \quad 0 \leq t \leq T, \tag{10}$$

[8] See, e.g., Kijima and Muromachi (2000a) for detailed discussion of the intensity processes.

[9] Mathematically, the information, $\mathcal{F}_t$, available at time t in our model is given by $\mathcal{F}_t = \mathcal{G}_t \vee \sigma(1_{\{\tau \leq t\}})$.

for the collateral value $L(t)$,

$$\mathrm{d}r = \mu_r(t)\mathrm{d}t + \sigma_r(t)\mathrm{d}z_r, \quad 0 \le t \le T, \tag{11}$$

for the default-free spot rate $r(t)$, and

$$\mathrm{d}h = \mu_h(t)\mathrm{d}t + \sigma_h(t)\mathrm{d}z_h, \quad 0 \le t \le T, \tag{12}$$

for the hazard rate $h(t)$. Here, the correlations between the three variables are driven from those of the standard Brownian motions $z_L(t)$, $z_r(t)$ and $z_h(t)$. That is, we assume that $\mathrm{d}z_L(t)\mathrm{d}z_r(t) = \rho_{Lr}(t)\mathrm{d}t$, $\mathrm{d}z_L(t)\mathrm{d}z_h(t) = \rho_{Lh}(t)\mathrm{d}t$, and $\mathrm{d}z_r(t)\mathrm{d}z_h(t) = \rho_{rh}(t)\mathrm{d}t$.

We have completed the setup of our model. The lending interest rate $\alpha(T)$ given in (9) can be at least evaluated by Monte Carlo simulation in an obvious manner. That is, suppose that we are given a discrete-time model for the three processes $(L(k\Delta t), r(k\Delta t), h(k\Delta t))$, where $k = 0, 1, 2, \dots$ and Δt is fixed. Because we approximate the continuous-time model by the discrete-time counterpart, we will choose Δt sufficiently small. The processes are generated until maturity T to evaluate (9), and the procedure is repeated to obtain enough scenarios to calculate the lending rate $\alpha(T)$ based on the strong law of large numbers.

Note that, in our model, we have implicitly assumed that the collateral $L(t)$ is a traded asset in the market with no carrying cost. Hence, the mean rate of return in (10) is equal to the default-free spot rate $r(t)$ under the risk-neutral probability measure P. The drift terms in (11) and (12) are assumed to be adjusted by appropriate risk premia.[10] If there is no enough liquidity for the collateral asset $L(t)$, we of course need to impute the risk premium for the collateral by some means.

3 A Gaussian Model

In order to understand the impact of each underlying variable on the term structure of lending interest rates, it is convenient to derive an analytic solution of the lending rate $\alpha(T)$. To this end, we consider the following Gaussian model. That is, under the

[10] The risk premium for the spot rate $r(t)$ can be imputed from the market prices of default-free discount bonds based on, e.g., the Hull–White model (1990). Similarly, the risk premium for the hazard rate $h(t)$ can be imputed from the market prices of defaultable discount bonds based on, e.g., Kijima–Muromachi model (2000b).

risk-neutral probability measure P, suppose that

$$\frac{\mathrm{d}L}{L} = r(t)\mathrm{d}t + \sigma_L \mathrm{d}z_L, \quad 0 \le t \le T, \tag{13}$$

$$\mathrm{d}r = a_r(m_r - r)\mathrm{d}t + \sigma_r \mathrm{d}z_r, \quad 0 \le t \le T, \tag{14}$$

$$\mathrm{d}h = a_h(m_h - h)\mathrm{d}t + \sigma_h \mathrm{d}z_h, \quad 0 \le t \le T, \tag{15}$$

$\mathrm{d}z_L(t)\mathrm{d}z_r(t) = \rho_{Lr}\mathrm{d}t$, $\mathrm{d}z_L(t)\mathrm{d}z_h(t) = \rho_{Lh}\mathrm{d}t$, and $\mathrm{d}z_r(t)\mathrm{d}z_h(t) = \rho_{rh}\mathrm{d}t$, where all the parameters are constants. Here, m_r and m_h are positive constants,[11] representing the mean-reverting levels to which the default-free spot rate $r(t)$ and the hazard rate $h(t)$, respectively, tend to revert. The parameters a_r and a_h are also positive constants that represent the speed of mean reversion. While the volatilities σ_L, σ_r and σ_h are positive constants, the parameters ρ_{Lr}, ρ_{Lh} and ρ_{rh} represent the coefficients of correlation between the variables, which can be negative. In what follows, we assume for simplicity that the principal F is set to be unity (i.e. $F = 1$).

Since the default-free spot rate is the Vasicek model (1977), the SDE (14) can be solved in closed form as

$$r(t) = m_r + (r(0) - m_r)\mathrm{e}^{-a_r t} + \sigma_r \int_0^t \mathrm{e}^{-a_r(t-s)}\mathrm{d}z_r(s). \tag{16}$$

The process $\{r(t)\}$ is known as the *Ornstein–Uhlenbeck* process and $r(t)$ is normally distributed, whence the spot rates in the Vasicek model become negative with positive probability. However, the probability is often negligible and the merit from its analytical tractability makes the model have a practical interest.

From (16), we obtain the cumulative spot rate as

$$R(t) = m_r t + (r(0) - m_r)\frac{1 - \mathrm{e}^{-a_r t}}{a_r} + \sigma_r \int_0^t \frac{1 - \mathrm{e}^{-a_r(t-u)}}{a_r}\mathrm{d}z_r(u). \tag{17}$$

Note that $R(t)$ is normally distributed with mean

$$\mu_R(t) \equiv m_r t + (r(0) - m_r)\frac{1 - \mathrm{e}^{-a_r t}}{a_r}$$

[11] A simple extension of this model is to assume that the mean-reverting levels are deterministic functions of time t, $m_r(t)$ and $m_h(t)$ say. The functions can be determined so that they are consistent with the current term structures of the default-free discount bonds and the corporate discount bonds simultaneously. See Kijima (2001) for details.

and variance

$$S_R^2(t) \equiv \frac{\sigma_r^2}{a_r^2}\left[t - 2\frac{1-\mathrm{e}^{-a_r t}}{a_r} + \frac{1-\mathrm{e}^{-2a_r t}}{2a_r}\right].$$

The hazard-rate process $\{h(t)\}$ is also of the Vasicek type, whence the hazard rate becomes negative with positive probability. the SDE (15) is solved in closed form as

$$h(t) = m_h + (h(0) - m_h)\mathrm{e}^{-a_h t} + \sigma_h \int_0^t \mathrm{e}^{-a_h(t-s)}\mathrm{d}z_h(s). \tag{18}$$

Hence, the cumulative hazard rate $H(t)$ is normally distributed with mean

$$\mu_H(t) \equiv m_h t + (h(0) - m_h)\frac{1-\mathrm{e}^{-a_h t}}{a_h}$$

and variance

$$S_H^2(t) \equiv \frac{\sigma_h^2}{a_h^2}\left[t - 2\frac{1-\mathrm{e}^{-a_h t}}{a_h} + \frac{1-\mathrm{e}^{-2a_h t}}{2a_h}\right].$$

In particular, the risk-adjusted discount process $M(t) = R(t) + H(t)$ is also normally distributed with mean $\mu_R(t) + \mu_H(t)$ and variance $S_R^2(t) + S_H^2(t) + 2C_{RH}(t)$, where

$$C_{RH}(t) \equiv \rho_{rh}\frac{\sigma_r\sigma_h}{a_r a_h}\left[t - \frac{1-\mathrm{e}^{-a_r t}}{a_r} - \frac{1-\mathrm{e}^{-a_h t}}{a_h} + \frac{1-\mathrm{e}^{-(a_r+a_h)t}}{(a_r+a_h)}\right].$$

It is well known that the price of the default-free discount bond with maturity t is given by $v(t) = E\left[\mathrm{e}^{-R(t)}\right]$, $t \geq 0$. Note that, if random variable X is normally distributed with mean μ and variance σ^2, then its *moment generating function* is obtained as

$$E\left[\mathrm{e}^{\theta X}\right] = \exp\left\{\theta\mu + \frac{\theta^2\sigma^2}{2}\right\}. \tag{19}$$

It follows that

$$v(t) = \exp\left\{-\mu_R(t) + \frac{1}{2}S_R^2(t)\right\}, \tag{20}$$

which is another form of the well-known Vasicek formula (1977).

Similarly, we obtain

$$E\left[\mathrm{e}^{-M(t)}\right] = \exp\left\{-\mu_R(t) - \mu_H(t) + \frac{S_R^2(t) + S_H^2(t) + 2C_{RH}(t)}{2}\right\}.$$

It follows from (20) that

$$E\left[\mathrm{e}^{-M(t)}\right] = v(t)\exp\left\{-\mu_H(t) + \frac{1}{2}S_H^2(t) + C_{RH}(t)\right\}. \tag{21}$$

It remains to evaluate the term $E\left[h(t)\min\{\beta L(t), 1\}\mathrm{e}^{-M(t)}\right]$ in (9). For this purpose, the next result is useful.

Lemma 1 *Suppose that (X, Y, Z) follows a trivariate normal distribution. Then,*

$$E\Big[e^{-Z} f(X,Y)\Big] = E\Big[e^{-Z}\Big] E\Big[f\Big(X - C[X,Z],\, Y - C[Y,Z]\Big)\Big]$$

for which the expectations exist, where C denotes the covariance operator.

From (13) and Ito's formula, we obtain

$$\log \frac{L(t)}{L(0)} = R(t) + X(t); \quad X(t) = \sigma_L z_L(t) - \frac{\sigma_L^2}{2} t. \tag{22}$$

In order to evaluate the term $E\left[h(t)\min\{\beta L(t), 1\}e^{-M(t)}\right]$, we employ Lemma 1 as

$$E\left[e^{-M(t)} f(h(t), Y(t))\right] = E\Big[e^{-M(t)}\Big] E\Big[f\Big(h(t) - C[h(t), M(t)],\, Y(t) - C[Y(t), M(t)]\Big)\Big],$$

where $Y(t) = R(t) + X(t)$ and $f(h, y) = h\min\{\beta L(0)e^{y}, 1\}$. To this end, we need to calculate $C_{hM}(t) \equiv C[h(t), M(t)]$ and $C_{YM}(t) \equiv C[Y(t), M(t)]$.

From (17) and (18), we obtain

$$C_{hR}(t) \equiv C\left[h(t), R(t)\right] = \rho_{rh}\frac{\sigma_r \sigma_h}{a_r}\left[\frac{1 - e^{-a_h t}}{a_h} - \frac{1 - e^{-(a_r + a_h)t}}{a_r + a_h}\right].$$

Similarly, we have

$$C_{hH}(t) \equiv C\left[h(t), H(t)\right] = \frac{\sigma_h^2}{a_h}\left[\frac{1 - e^{-a_h t}}{a_h} - \frac{1 - e^{-2a_h t}}{2a_h}\right].$$

It follows that $C_{hM}(t) = C_{hR}(t) + C_{hH}(t)$. Also, from (17) and (22), we obtain

$$C_{LR}(t) \equiv C\left[X(t), R(t)\right] = \rho_{Lr}\frac{\sigma_L \sigma_r}{a_r}\left[t - \frac{1 - e^{-a_r t}}{a_r}\right],$$

since

$$X(t) = -\frac{\sigma_L^2}{2}t + \sigma_L \int_0^t dz_L(u).$$

Similarly, we have

$$C_{LH}(t) \equiv C\left[X(t), H(t)\right] = \rho_{Lh}\frac{\sigma_L \sigma_h}{a_h}\left[t - \frac{1 - e^{-a_h t}}{a_h}\right].$$

Therefore,

$$C_{YM}(t) = S_R^2(t) + C_{RH}(t) + C_{LR}(t) + C_{LH}(t),$$

where $S_R^2(t)$ and $C_{RH}(t)$ have been calculated above.

Proposition 3 *In the Gaussian model described above, the lending interest rate is given by*

$$\alpha(T) = \frac{1 - v(T)\pi_H(T) - \int_0^T \pi_H(t)\left\{\tilde{\mu}_h(t)(v(t) - p(t)) - C_{hY}(t)Q(t)/S_Y(t)\right\}\mathrm{d}t}{\sum_{j=1}^m v(t_j)\pi_H(t_j)},$$

where $\tilde{\mu}_h(t) = E[h(t)] - C_{hM}(t)$, $S_Y^2(t) = S_R^2(t) + \sigma_L^2 t + 2C_{LR}(t)$,

$$\pi_H(t) \equiv \exp\left\{-\mu_H(t) + \frac{1}{2}S_H^2(t) + C_{RH}(t)\right\},$$

$$C_{hY}(t) \equiv C_{hR}(t) + \rho_{Lh}\frac{\sigma_L\sigma_h}{a_h}\left(1 - \mathrm{e}^{-a_h t}\right),$$

$$d(t) = \frac{\log[L/v(t)]}{S_Y(t)} + \frac{1}{2}S_Y(t); \quad L = \beta L(0)\mathrm{e}^{-C_{RH}(t) - C_{LH}(t)},$$

$$Q(t) = L\phi(-d(t)) - v(t)\phi(S_Y(t) - d(t)) - LS_Y(t)\Phi(-d(t)) \tag{23}$$

and

$$p(t) = v(t)\Phi(S_Y(t) - d(t)) - L\Phi(-d(t)). \tag{24}$$

Here, $v(t)$ *denotes the default-free discount bond price with maturity* t*, and* $\phi(d)$ *and* $\Phi(d)$ *represent, respectively, the standard normal density and distribution functions.*

We note that the function $p(t)$ in (24) is the put option premium with initial underlying price L and volatility $S_Y(t)$. On the other hand, the function $Q(t)$ defined in (23) corresponds to the term

$$Q(t) \equiv v(t)E\left[Z\max\left\{1 - L\mathrm{e}^{S_Y(t)Z - S_Y^2(t)/2 - \log v(t)}, 0\right\}\right], \tag{25}$$

where $Z = (Y(t) - E[Y(t)])/S_Y(t)$.

4 Some Special Case

In this section, we assume that the hazard rate $h(t)$ is a deterministic function of time in order to study the impact of the correlation $\rho = \rho_{Lr}$ between the value of the collateral asset and the default-free spot rate on the term structure of lending rates.[12] In the following, we assume for the sake of simplicity that $\beta = 1$ and the coupons are paid continuously.

[12] Other cases can be studied similarly. Some numerical results are available upon request from authors.

Under these specifications, we have from Proposition 3 that

$$\alpha(T) = \frac{1 - v(T)\mathrm{e}^{-H(T)} - \int_0^T h(t)\mathrm{e}^{-H(t)}(v(t) - p(t))\mathrm{d}t}{\int_0^T v(t)\mathrm{e}^{-H(t)}\mathrm{d}t}, \tag{26}$$

since $\pi_H(t) = \mathrm{e}^{-H(t)}$, $\widetilde{\mu}_h(t) = h(t)$ and $C_{hY}(t) = 0$. Also, since $L = L(0)$, the put option premium (24) is given by

$$p(t) = v(t)\Phi(S_Y(t) - d(t)) - L(0)\Phi(-d(t)), \tag{27}$$

where

$$S_Y^2(t) = S_R^2(t) + \sigma_L^2 t + 2\rho\frac{\sigma_L\sigma_r}{a_r}\left[t - \frac{1 - \mathrm{e}^{-a_r t}}{a_r}\right] \tag{28}$$

and

$$d(t) = \frac{\log[L(0)/v(t)]}{S_Y(t)} + \frac{1}{2}S_Y(t).$$

It is well known that the put option premium decreases in the initial price $L(0)$ and increases in the volatility $S_Y(t)$. Also, the correlation coefficient ρ appears only in the volatility $S_Y(t)$ given in (28). Since $S_Y(t)$ is increasing in ρ, the next result is immediate.

Proposition 4 *In the Gaussian model described above, suppose that the hazard rate $h(t)$ is a deterministic function of time t. Then, the lending rate is decreasing in the initial value $L(0)$ of the collateral asset and increasing in the correlation coefficient ρ between the collateral asset and the default-free spot rate.*

From (28), we obtain

$$\frac{\partial S_Y^2(t)}{\partial \sigma_L} = 2\sigma_L t + 2\rho\frac{\sigma_r}{a_r}\left[t - \frac{1 - \mathrm{e}^{-a_r t}}{a_r}\right],$$

where σ_L is the volatility of the collateral asset. We then have the following.

Proposition 5 *In the Gaussian model described above, suppose that the hazard rate is a deterministic function of time t. If $\rho \geq 0$, then the lending rate is increasing in the volatility σ_L of the collateral asset. If $\rho < 0$, on the other hand, the lending rate is decreasing in σ_L if and only if*

$$\sigma_L \leq \Sigma_r \equiv \frac{-\rho\sigma_r}{a_r t}\left[t - \frac{1 - \mathrm{e}^{-a_r t}}{a_r}\right].$$

Table 1: Estimated parameter values for each credit rating

	Aaa	Aa	A	Baa	Ba	B
λ	5.116E-05	2.336E-04	2.889E-04	1.539E-03	1.249E-02	2.164
γ	2.0142	1.5656	1.6963	1.4221	1.1998	0.1725
η	0	0	0	0	0	9.721

Taken from Kijima and Muromachi (2000b)

From Proposition 5, when $\rho < 0$, the lending interest rate is decreasing in the collateral volatility σ_L as far as it is less than Σ_r. This is interesting, because it indicates that a larger volatility does not always result in a higher interest.

Next, we consider the effect of the hazard rate. For this purpose, we assume that the hazard function is given by

$$h(t) = \lambda\gamma(t+\eta)^{\gamma-1}, \quad t \geq 0.$$

That is, the default time epoch is independent and follows a *generalized Weibull distribution* with shape parameter γ, scale parameter λ and shift parameter η. Weibull distributions are one of the well-studied distributions in survival analyses, and their advantage is that they can express various shapes of the term structure of hazard rates. They are increasing if $\gamma > 1$, decreasing if $\gamma < 1$, and constant if $\gamma = 1$.

Kijima and Muromachi (2000b) estimated the parameters for each credit rating using the default data published by Moody's Investors Service. The estimation result is listed in Table 1. For other parameter values, we set $L(0) = 1$, $\sigma_L = 50\%$ for the collateral asset, $m_r = 8\%$, $\sigma_r = 10\%$ for the default-free spot rate, and $\rho = 0.5$ for the correlation between them. The par-yield of the government (default-free) bond $v(t)$ is given by

$$y(T) = \frac{1 - v(T)}{\int_0^T v(t)\mathrm{d}t}, \quad T \geq 0.$$

The *yield spread* is defined as $\alpha(T) - y(T)$.

The term structure of lending interest rates for each credit rating is calculated using Equation (26), and the numerical results are summarized in Tables 2 and 3. Table 2 (Table 3, respectively) corresponds to the case that the par-yield of the government bonds is increasing (decreasing). The number in the parenthesis below the lending interest rate is the corresponding yield spread. It is observed that the yield spread is increasing in

Table 2: The term structure of lending rates for each credit rating ($r(0) = 0.02$)

T	1	5	10	15	20
Aaa	3.1573%	4.8128%	5.3119%	5.4971%	5.5934%
	(0.0008%)	(0.0084%)	(0.0220%)	(0.0367%)	(0.0511%)
Aa	3.1599%	4.8215%	5.3224%	5.5059%	5.5987%
	(0.0034%)	(0.0171%)	(0.0325%)	(0.0455%)	(0.0564%)
A	3.1608%	4.8309%	5.3448%	5.5409%	5.6451%
	(0.0043%)	(0.0265%)	(0.0549%)	(0.0805%)	(0.1028%)
Baa	3.1784%	4.8919%	5.4406%	5.6602%	5.7804%
	(0.0219%)	(0.0875%)	(0.1507%)	(0.1998%)	(0.2381%)
Ba	3.3253%	5.2766%	5.9857%	6.3026%	6.4838%
	(0.1688%)	(0.4722%)	(0.6958%)	(0.8422%)	(0.9415%)
B	3.8456%	5.9958%	6.6357%	6.8492%	6.9396%
	(0.6891%)	(1.1914%)	(1.3458%)	(1.3888%)	(1.3973%)
$y(T)$	3.1565%	4.8044%	5.2899%	5.4604%	5.5423%

$m_r = 0.08,\ \sigma_r = 0.1,\ L(0) = 1,\ \sigma_L = 0.5,\ \rho = 0.5$

Table 3: The term structure of lending rates for each credit rating ($r(0) = 0.08$)

T	1	5	10	15	20
Aaa	7.8864%	7.1347%	6.7271%	6.5766%	6.5100%
	(0.0007%)	(0.0077%)	(0.0207%)	(0.0349%)	(0.0490%)
Aa	7.8888%	7.1435%	6.7393%	6.5888%	6.5203%
	(0.0032%)	(0.0166%)	(0.0329%)	(0.0472%)	(0.0593%)
A	7.8896%	7.1526%	6.7617%	6.6247%	6.5685%
	(0.0040%)	(0.0256%)	(0.0553%)	(0.0831%)	(0.1076%)
Baa	7.9958%	7.2119%	6.8596%	6.7496%	6.7125%
	(0.0202%)	(0.0849%)	(0.1532%)	(0.2080%)	(0.2515%)
Ba	8.0422%	7.5882%	7.4199%	7.4269%	7.4663%
	(0.1566%)	(0.4612%)	(0.7135%)	(0.8853%)	(1.0054%)
B	8.5279%	8.3057%	8.1122%	8.0335%	7.9885%
	(0.6422%)	(1.1787%)	(1.4058%)	(1.4919%)	(1.5275%)
$y(T)$	7.8856%	7.1270%	6.7064%	6.5416%	6.4610%

$m_r = 0.08,\ \sigma_r = 0.1,\ L(0) = 1,\ \sigma_L = 0.5,\ \rho = 0.5$

maturity for all the cases, which differs from the property of corporate bonds. That is, it is well known that the credit risk spread of a corporate bond with low credit class can be decreasing in maturity (see, e.g., Fons (1994) and Kijima (1998)). This is so, because the yield spread of the bank loan includes default risk and market risk of the collateral asset, while the credit spread of a corporate bond is mostly due to the likelihood of default. Also, the fact that the yield spread in the reversed par-yield case is increasing in maturity means that the term structure of lending interest rates can be inverse-humped.

Finally, we study the effect of the spot-rate volatility σ_r on the lending interest rates; however, the effect is not obvious from (26) since σ_r influences both the default-free discount bond price $v(t)$ and the put option premium $p(t)$. Hence, for this purpose, we perform numerical experiments to understand the impact of σ_r on the lending interest rates. In Tables 4 and 5 below, we set the basic parameter values to be $L(0) = 1$, $\sigma_L = 50\%$, $r(0) = 2\%$, $c_r = 50\%$, $m_r = 8\%$, $\rho = 0.5$ and $T = 1$. By these specifications, we observe that the par-yield of the government bonds is decreasing in the spot-rate volatility σ_r.

First, in Table 4, we consider the impact of correlation ρ between the value of the collateral asset and the default-free spot rate. In order to clarify the impact, we set the hazard function $h(t)$ to be constant ($h(t) = 0.05$). Table 4 lists the lending interest rates as well as the corresponding yield spreads in the parentheses below.

Recall from Proposition 5 that the lending interest rate is increasing in volatility σ_L of the collateral asset if $\rho \geq 0$, while it is decreasing first until some value of σ_L if $\rho < 0$. Table 4 reveals that the lending interest rate is *not* increasing in σ_r even if $\rho \geq 0$. This is so, because the government par-yield is a dominant factor and is decreasing in the spot-rate volatility σ_r. However, it is interesting to note that, when $\rho \geq 0$, the yield spread (not the lending rate itself) is increasing in σ_r, while it is decreasing first until some value of σ_r if $\rho < 0$. For example, when $\rho = -0.25$ ($\rho = -0.5$, respectively), the yield spread is decreasing until $\sigma_r = 0.15$ ($\sigma_r = 0.25$). Hence, we may conclude that the spot-rate volatility has the same impact as the collateral volatility on the yield spreads (not the lending rates itself).

On the other hand, Table 5 lists the yield spreads in each credit rating for various values of the spot-rate volatility σ_r. Since we set $\rho = 0.5$, the yield spread is increasing in σ_r. Of significance here is the fact that the yield spread becomes rapidly bigger as the

Table 4: The lending interest rates and yield spreads for various values of σ_r ($h(t) = 0.05$)

$\rho \backslash \sigma_r$	0.05	0.10	0.15	0.20	0.25	0.30
1.00	3.8771%	3.8095%	3.6852%	3.5041%	3.2660%	2.9708%
	(0.6340%)	(0.6529%)	(0.6730%)	(0.6941%)	(0.7164%)	(0.7398%)
0.75	3.8729%	3.8013%	3.6733%	3.4886%	3.2472%	2.9488%
	(0.6298%)	(0.6448%)	(0.6611%)	(0.6787%)	(0.6976%)	(0.7178%)
0.50	3.8687%	3.7931%	3.6611%	3.4727%	3.2278%	2.9261%
	(0.6256%)	(0.6366%)	(0.6489%)	(0.6628%)	(0.6782%)	(0.6951%)
0.25	3.8644%	3.7847%	3.6487%	3.4564%	3.2078%	2.9024%
	(0.6213%)	(0.6281%)	(0.6365%)	(0.6465%)	(0.6581%)	(0.6713%)
0.00	3.8601%	3.7761%	3.6360%	3.4396%	3.1869%	2.8777%
	(0.6170%)	(0.6196%)	(0.6238%)	(0.6297%)	(0.6373%)	(0.6466%)
−0.25	3.8558%	3.7674%	3.6230%	3.4223%	3.1653%	2.8518%
	(0.6127%)	(0.6109%)	(0.6108%)	(0.6123%)	(0.6157%)	(0.6208%)
−0.50	3.8514%	3.7586%	3.6096%	3.4043%	3.1427%	2.8246%
	(0.6083%)	(0.6021%)	(0.5974%)	(0.5944%)	(0.5931%)	(0.5935%)
−0.75	3.8470%	3.7496%	3.5959%	3.3857%	3.1191%	2.7957%
	(0.6039%)	(0.5931%)	(0.5837%)	(0.5758%)	(0.5694%)	(0.5647%)
−1.00	3.8426%	3.7405%	3.5818%	3.3663%	3.0941%	2.7649%
	(0.5995%)	(0.5840%)	(0.5696%)	(0.5564%)	(0.5445%)	(0.5339%)
$y(1)$	3.2431%	3.1565%	3.0122%	2.8099%	2.5496%	2.2310%

$L(0) = 1,\ \sigma_L = 0.5,\ r(0) = 0.02,\ c_r = 0.5,\ m_r = 0.08,\ h = 0.05,\ T = 1$

Table 5: The yield spreads in each credit ratings for various values of σ_r

σ_r	0.05	0.10	0.15	0.20	0.25	0.30
Aaa	0.0008%	0.0008%	0.0008%	0.0008%	0.0009%	0.0009%
Aa	0.0033%	0.0033%	0.0035%	0.0036%	0.0037%	0.0038%
A	0.0042%	0.0043%	0.0044%	0.0044%	0.0045%	0.0048%
Baa	0.0214%	0.0218%	0.0223%	0.0228%	0.0234%	0.0240%
Ba	0.1425%	0.1687%	0.1722%	0.1761%	0.1804%	0.1850%
B	0.6773%	0.6890%	0.7024%	0.7173%	0.7339%	0.7519%
$y(1)$	3.2431%	3.1565%	3.0122%	2.8099%	2.5496%	2.2310%

$r(0) = 0.02,\ m_r = 0.08,\ c_r = 0.5,\ L(0) = 1,\ \sigma_L = 0.5,\ \rho = 0.5,\ T = 1$

credit quality deteriorates.

5 Conclusion

In this article, we consider an arbitrage-free model to evaluate the term structure of lending interest rates when a fraction of collateral is recovered upon default of the borrower. Unlike the previous literature, we assume that the value of the collateral asset fluctuates over time with certain correlation to the default-free interest rate as well as the default hazard rate of the borrower. It is shown that the bank loan with a collateral asset is equal to a sum of holding a coupon-bearing bond and selling a put option written on the collateral asset, both being callable at default epoch, and the lending bank is always exposed to prepayment risk.

A Gaussian model is considered, as a special case, to derive an analytic expression of the appropriate lending interest rates and to investigate the impact of the underlying variables on the lending rates. In particular, when the hazard rate is a deterministic function of time, the lending interest rate is decreasing in the initial value of the collateral asset and increasing in the correlation coefficient ρ between the collateral asset and the default-free spot rate. Also, it is increasing in the volatility of the collateral asset if $\rho \geq 0$, while it can be decreasing in the collateral volatility if $\rho < 0$. That is, a larger volatility does not always result in a higher interest.

We demonstrate that bank loans exhibit different properties from corporate bonds, some being mathematically and the others through numerical experiments. In particular, it is observed that the term structure of lending interest rates can be inverse-humped. Also, the yield spreads of bank loans are always increasing in maturity, while credit risk spreads of corporate bonds with low credit class can be decreasing in maturity. This is so, because the yield spreads of bank loans include default risk and market risk of the collateral asset, while the credit risk spreads of corporate bonds are mostly due to default risk.

A Proofs

A.1 Proof of Proposition 2

From (6), we have

$$E\left[\frac{1_{\{\tau>t\}}}{B(t)}\right] = E\left[E\left[\frac{1_{\{\tau>t\}}}{B(t)}\Big|\mathcal{F}_T\right]\right] = E\left[\frac{1}{B(t)}P\{\tau>t|\mathcal{F}_T\}\right] = E\left[\frac{1}{B(t)}\mathrm{e}^{-H(t)}\right].$$

Similarly, from (8), we obtain

$$E\left[\frac{f(\tau)1_{\{\tau\le T\}}}{B(\tau)}\right] = E\left[\int_0^T \frac{f(u)}{B(u)}h(u)\mathrm{e}^{-H(u)}\mathrm{d}u\right],$$

where $f(u) = \min\{\beta L(u), 1\}$. Equation (9) can be obtained at once.

A.2 Proof of Lemma 1

Let $\xi(x, y, z)$ be the joint density function of (X, Y, Z) and define

$$\xi_Z(x, y) \equiv \int_{-\infty}^{\infty} \mathrm{e}^{-z}\xi(x, y, z)\mathrm{d}z.$$

Then,

$$\begin{aligned} E\Big[\mathrm{e}^{-Z}f(X,Y)\Big] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{-z}f(x,y)\xi(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\xi_Z(x,y)\mathrm{d}x\mathrm{d}y. \end{aligned}$$

Denoting the moment generating function of (X, Y, Z) by $\eta(s, t, u) = E\left[\mathrm{e}^{sX+tY+uZ}\right]$, we obtain

$$\eta(s, t, -1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{sx+ty}\xi_Z(x, y)\mathrm{d}x\mathrm{d}y.$$

Since $sX + tY - Z$ is normally distributed, it follows that

$$\begin{aligned}\eta(s,t,-1) &= \exp\Big\{sE[X] + tE[Y] - E[Z] + \frac{s^2}{2}V[X] + \frac{t^2}{2}V[Y] \\ &\quad + \frac{1}{2}V[Z] + stC[X,Y] - tC[Y,Z] - sC[X,Z]\Big\} \\ &= \mathrm{e}^{-E[Z]+V[Z]/2}\exp\Big\{s(E[X] - C[X,Z]) + \frac{s^2}{2}V[X] \\ &\quad + t(E[Y] - C[Y,Z]) + \frac{t^2}{2}V[Y] + stC[X,Y]\Big\} \\ &= E\Big[\mathrm{e}^{-Z}\Big]E\Big[\mathrm{e}^{s(X-C[X,Z])+t(Y-C[Y,Z])}\Big].\end{aligned}$$

Recall that the moment generating function determines the distribution uniquely, if it exists. Hence, we conclude that $\xi_Z(x,y)/E[\mathrm{e}^{-Z}]$ is a density function of $(X - C[X,Z],\, Y - C[Y,Z])$. It follows that

$$\begin{aligned}E\Big[\mathrm{e}^{-Z}f(X,Y)\Big] &= E\Big[\mathrm{e}^{-Z}\Big]\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\frac{\xi_Z(x,y)}{E\left[\mathrm{e}^{-Z}\right]}\mathrm{d}x\mathrm{d}y \\ &= E\Big[\mathrm{e}^{-Z}\Big]E\Big[f\Big(X - C[X,Z]\,,\, Y - C[Y,Z]\Big)\Big],\end{aligned}$$

completing the proof.

A.3 Proof of Proposition 3

Let $\widetilde{h}(t) = h(t) - C_{hM}(t)$ and $\widetilde{Y}(t) = Y(t) - C_{YM}(t)$. Then, $(\widetilde{h}(t), \widetilde{Y}(t))$ follows the bivariate normal distribution with means $\widetilde{\mu}_h(t) \equiv E[\widetilde{h}(t)]$ and $\widetilde{\mu}_Y(t) \equiv E[\widetilde{Y}(t)]$, variances $V[\widetilde{h}(t)] = V[h(t)]$ and $V[\widetilde{Y}(t)] = V[Y(t)]$ and covariance $C[\widetilde{h}(t), \widetilde{Y}(t)] = C[h(t), Y(t)]$. Note that $S_Y^2(t) = V[Y(t)]$. Hence, from (20), we have

$$\widetilde{\mu}_Y(t) = -\log v(t) - \frac{1}{2}S_Y^2(t) - C_{RH}(t) - C_{LH}(t).$$

We need to evaluate

$$D(t) \equiv E\left[\widetilde{h}(t)\min\left\{\beta L(0)\mathrm{e}^{\widetilde{Y}(t)},\, 1\right\}\right].$$

Since $(\widetilde{h}(t), \widetilde{Y}(t))$ follows the bivariate normal distribution, it is well known that the conditional expectation is given by

$$E\left[\widetilde{h}(t)\middle|\widetilde{Y}(t)\right] = \widetilde{\mu}_h(t) + \frac{C_{hY}(t)}{S_Y^2(t)}\left(\widetilde{Y}(t) - \widetilde{\mu}_Y(t)\right) = \widetilde{\mu}_h(t) + \frac{C_{hY}(t)}{S_Y(t)}Z,$$

where $Z = (Y(t) - E[Y(t)])/S_Y(t)$. It follows that

$$
\begin{aligned}
D(t) &= E\left[E\left[\widetilde{h}(t)\middle|\widetilde{Y}(t)\right]\min\left\{\beta L(0)\mathrm{e}^{\widetilde{Y}(t)},\,1\right\}\right] \\
&= E\left[\left(\widetilde{\mu}_h(t) + \frac{C_{hY}(t)}{S_Y(t)}Z\right)\min\left\{\beta L(0)\mathrm{e}^{\widetilde{\mu}_Y(t)+S_Y(t)Z},\,1\right\}\right] \\
&= E\left[\left(\widetilde{\mu}_h(t) + \frac{C_{hY}(t)}{S_Y(t)}Z\right)\min\left\{L\mathrm{e}^{S_Y(t)Z-\log v(t)-S_Y^2(t)/2},\,1\right\}\right],
\end{aligned}
$$

since $L = \beta L(0)\mathrm{e}^{-C_{RH}(t)-C_{LH}(t)}$.

Now, since $\min\{x,y\} = y - \max\{y-x,0\}$, we have

$$
D(t) = E\left[\left(\widetilde{\mu}_h(t) + \frac{C_{hY}(t)}{S_Y(t)}Z\right)\left(1 - \max\left\{1 - L\mathrm{e}^{S_Y(t)Z-\log v(t)-S_Y^2(t)/2},\,0\right\}\right)\right].
$$

Let $K = S_Y(t) - d(t)$ and let

$$
\begin{aligned}
p(t) &\equiv v(t)E\left[\max\left\{1 - L\mathrm{e}^{S_Y(t)Z-\log v(t)-S_Y^2(t)/2},\,0\right\}\right] \\
&= E\left[\max\left\{v(t) - L\mathrm{e}^{S_Y(t)Z-S_Y^2(t)/2},\,0\right\}\right].
\end{aligned}
$$

It is easily seen that

$$
p(t) = \int_{-\infty}^{K}\left\{v(t) - L\mathrm{e}^{S_Y(t)z-S_Y^2(t)/2}\right\}\phi(z)\mathrm{d}z = v(t)\Phi(K) - L\Phi(K - S_Y(t)),
$$

which is (24). Next, define $Q(t)$ as in (25). Since $\phi'(z) = -z\phi(z)$, we obtain

$$
\begin{aligned}
Q(t) &= \int_{-\infty}^{K} z\left\{v(t) - L\mathrm{e}^{S_Y(t)z-S_Y^2(t)/2}\right\}\phi(z)\mathrm{d}z \\
&= -v(t)\int_{-\infty}^{K}\phi'(z)\mathrm{d}z - L\int_{-\infty}^{K} z\phi(z - S_Y(t))\mathrm{d}z \\
&= -v(t)\phi(K) - L\int_{-\infty}^{K-S_Y(t)}(z + S_Y(t))\phi(z)\mathrm{d}z \\
&= -v(t)\phi(K) + L\int_{-\infty}^{K-S_Y(t)}\phi'(z)\mathrm{d}z - LS_Y(t)\Phi(K - S_Y(t)).
\end{aligned}
$$

Combining all the terms, we obtain the proposition.

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