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A MODEL FOR PRICING STOCKS AND BONDS WITH DEFAULT RISK ^{*}

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A MODEL FOR PRICING STOCKS AND BONDS WITH DEFAULT RISK

Abstract

This paper develops a tractable, dynamic, no-arbitrage model for the pricing of bonds and stocks that are subject to default risk. The model produces the bond pricing equations of the Duffie and Singleton (1999) framework. It is then shown that a particular choice of dividend process, characterized by affine dividend yields, along with the Duffie and Singleton (1999) default specification, produces stock prices that are exponential affine in the model's state variables. Importantly, the model allows for quite general interdependence between the prices of risky debt and equity. This, along with the model's tractability, makes it a natural platform for empirical investigations into the pricing of a firm's capital structure.

JEL Classification: G12, G13.

1 Introduction

This paper develops a dynamic, no-arbitrage model capable of pricing bonds and stocks that are subject to default risk. Importantly, the model produces bond and stock prices whose logs are affine functions of the economy's state variables. The existence of closed form solutions for the prices of stocks and bonds make the model easy to implement in practice. Furthermore, by allowing for a rich specification of state variable dynamics in a no-arbitrage setting, the paper provides an empirically flexible, theoretically coherent framework for future work on pricing corporate bonds and stocks.

It should be noted that deriving a model for defaultable debt and equity which produces closed form prices is nontrivial. Indeed, in the current framework, tractability relies crucially on three assumptions. First, the default convention in this model follows the default intensity with fractional recovery of market value assumption first introduced by Duffie and Singleton (1999). Second, the dividend process for equities follows the affine dividend yield specification proposed in Mamaysky (2002a).¹ Finally, factor dynamics, short-term interest rates, dividends yields, recovery rates, and default intensities are all restricted to be affine functions of the economy's state variables.² This paper shows that the above three assumptions, each one having been advantageously employed in other contexts, may be brought together to produce a convenient model for the valuation of corporate bonds and stocks.

To fix ideas, let us define what stocks, bonds, and default mean in the present setting:

1. Bonds are assumed to pay a fixed, one dollar dividend at some prespecified point in the future, assuming that default has not occurred before then.
2. Stocks are assumed to pay a continuous stochastic dividend stream, as well as a stochastic terminal dividend at some point (potentially infinitely far off) in the future. A stock stops making any dividend payments once default has occurred.
3. Default occurs at some random time, following which all future cashflows from the defaulting security cease. Furthermore, at the instant of default, the owner of the security in question is entitled to a default dividend (otherwise known as a recovery amount) representing some fraction of the security's value at the instant prior to the occurrence of default.

As will become clear, modeling default-free securities is straightforward in the present model. Indeed, a default-free security is simply a special case of a defaultable security where the parameters of the model are such that default never occurs.

¹Related papers on the modeling of stocks are Bakshi and Chen (1997a,b), Bekaert and Grenadier (2000), and Brennan, Wang, and Xia (2001). Mamaysky (2002a) contains a discussion of how these approaches differ from the one adopted in this paper.

²More precisely, the product of the loss rate, given default, and the default intensity must be affine. Also affine factor dynamics refers to the fact that instantaneous drifts and variances of the factors are affine functions of the factors themselves. A general theoretical treatment of bond pricing in such economies is in Duffie and Kan (1996).

In the context of bond pricing, Duffie and Singleton (1999) assume a very specific process for the occurrence of default and for the value of the default dividend conditional on such an occurrence: (i) Default is assumed to occur as a Poisson-type event, with a stochastic and time-varying default probability per unit time (or default intensity), and (ii) when default occurs, the owner of the risky bond (or stock) in question receives some fraction of the pre-default market value of that security as the default dividend. Duffie and Singleton (1999) then show that these assumptions, in effect, allow risky bonds to be priced by simply adding a default spread (equal to the default intensity multiplied by the loss rate conditional on default) to the short rate, and then discounting future promised cashflows at this adjusted interest rate under the risk-neutral measure. In this paper, we show that this insight applies as well to the pricing of stocks that are subject to default risk.

Since stocks, unlike bonds, represent a claim to a future stochastic dividend flow, valuation of stocks relies crucially on the way in which this future dividend stream is specified. Whereas it is straightforward to model the future dividend of a zero coupon bond, choosing an appropriate dividend process for equity turns out to be somewhat trickier. Following Mamaysky (2002a), we assume that the instantaneous dividend paid by a given stock can be written in the following form

$$\text{Dividend} = \text{Affine Dividend Yield} \times \text{Stock Price}. \quad (1)$$

The pricing of multiple stocks can be accommodated by endowing each stock with its own dividend process. It should be emphasized that the decomposition of the dividend in equation (1) is a result, rather than an assumption: We first assume that the dividend has the form given by Dividend, and then solve for the no-arbitrage stock price in the economy. We then shown that the original choice of dividend process, and the endogenously determined stock price, are related by an affine dividend yield, as in the above formula. The fact that the dividend specification in the paper leads to exponential affine stock prices is a major benefit of the present modeling approach.

As in Mamaysky (2002a), the present economy is specified so as to allow for bonds and stocks to be priced in a unified framework. Bond prices turn out to be identical to those found in Duffie and Singleton (1999). It is shown that a certain set of state variables, which we will call the Y -type factors, can be common to stock and bond price processes. As is usually done in the literature, the model's bond yields are assumed to be stationary, implying that the Y -type factors must possess steady-state distributions. Furthermore, we assume that dividend yields, default intensities, and recovery rates must also be stationary, and must therefore be driven by the Y -type factors. Note that it is possible, in fact likely, that dividend yields and interest rates may share a common component.³ This feature of the data can be captured by allowing the short rate and the dividend yield of a given stock to load (in perhaps different ways) on the same Y -type factors. Similarly the short rate, dividend yields, default intensities, and recovery rates may all share common Y -type factors.

³For instance, the simple Gordon dividend growth model generates a dividend yield equal to the interest rate minus the dividend growth rate.

An important empirical regularity associated with stocks is that, in all likelihood, dividends contain a random walk component.⁴ To accommodate this feature of the data, the present economy is assumed to contain another set of state variables, which we will call the Z -type factors, which have independent increments. While, by assumption, bonds do not depend on the Z -type factors, stock dividends may do so, which in turn induces stock prices to contain random walk components (via the relationship in (1)). The fact that stocks are able to load on random-walk like factors gives the model a great deal of flexibility in matching actual stock prices. Furthermore, because stocks and bonds can both load on the same set of the Y -type stationary factors, the present paper allows for a rich interdependence to exist between these two asset classes. For these reasons, the present model provides a natural platform for conducting empirical investigations into the joint behavior of defaultable bonds and stocks.

With this goal (of empirical implementation) in mind, the paper shows that the total returns process for a given stock (this being the value of a portfolio initially holding one share of the stock, and then reinvesting all of that stock's dividends back into the stock itself, up to the time of default) is also an exponential affine function of the model's state variables, once a straightforward change of variables is performed on the model's Z -type factors. The convenience of working with total returns processes, rather than directly with stocks and their associated dividend processes, is that total returns data is easily obtained (from CRSP, for example), and the necessity of worrying about the exact timing and nature of dividend payments is therefore obviated for purposes of model estimation. Also, because bond prices, default intensities, and recovery rates do not depend on the Z -type factors, this change of variable conveniently leaves all of these other quantities unaffected.

While the pricing of securities is accomplished under the risk-neutral measure, estimation of the model needs to proceed under the physical measure. In order to move between these measures, we need to specify a price of risk process. This paper shows that the “essentially affine” price of risk process proposed in Duffee (2001) (see also Dai and Singleton (2001)) can also be used in the present setting. It is shown that this particular choice of price of risk process, characterized by a continuous pricing kernel, leaves default intensities identical under both measures. Also, it is shown that the ability to express asset prices as expectations, under the physical measure, of kernel weighted future cashflows applies to dividends which are paid at random times, such as default dividends in the present model. This result allows us to deduce a convenient representation for the expected returns on risky stocks and bonds under the physical measure. In doing so, we are able to make an important distinction between the expected returns on securities where the expectation is restricted to paths of the economy on which default does not occur, and the expected returns on total returns securities, which do take into account the possibility of default.

Finally, we provide an example economy where all Y -type factors are of the Cox, Ingersoll,

⁴This would follow from stationary dividend yields and stock prices which contain random walk components. See Campbell and Shiller (2001) for evidence regarding the stationarity of dividend yields.

Ross (1985) type, with cross-sectionally independent innovations, and where the Z -type factor follows a random walk with a stochastic drift. In this economy, we solve for bond prices, the stock price, and the value of the total returns process for the stock. The model therefore provides a reduced form representation for the capital structure of a given firm. Furthermore, all technical conditions introduced in the development of the model are shown to hold for this example economy.

The remainder of the paper proceeds as follows. Section 2 specifies the underlying economy used in this paper. Section 3 shows how bond and stock prices are computed. Section 4 computes the total returns process for equities. Section 5 introduces the price of risk process, and shows that kernel pricing under the physical measure applies to dividends paid at random times. Section 6 derives risk premia for bonds and stocks, as well as the risk premia for total returns processes. Section 7 develops the example economy. Section 8 concludes. Those proofs which are not contained in the text are given in the Appendix.

2 The Model

2.1 Specification of the Uncertainty

We assume the existence of a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, along with an augmented filtration $\{\mathcal{F}_t : t \geq 0\}$ satisfying the usual conditions given in Protter (1995). Furthermore, we assume the existence of a measure \mathcal{Q} , equivalent to \mathcal{P} , such that the discounted gains process (to be defined shortly) for all securities in this economy are \mathcal{Q} martingales. It is a well known result that, under some regularity condition, the existence of the martingale measure \mathcal{Q} is equivalent to the absence of arbitrage opportunities in the economy (see, for example, Harrison and Pliska (1981) and Dybvig and Huang (1989)). Of importance for this paper is the fact that if all discounted gains processes are martingales under \mathcal{Q} , then arbitrage opportunities are precluded in the economy. Hence the economy developed in this paper is arbitrage-free by construction.

We assume the existence of an $N + M$ dimensional vector of state variables $X(t)$ which is admissible in the sense of Duffie and Kan (1996). By virtue of this assumption, $X(t)$ takes values in some open subset of \mathbb{R}^{N+M} . It will be convenient to write $X(t)$ as follows

$$X(t) = \begin{bmatrix} Y(t) \\ Z(t) \end{bmatrix},$$

for an N dimensional vector $Y(t)$, and an M dimensional vector $Z(t)$. Letting $\tilde{W}(t)$ indicate a standard $N + M$ dimensional Brownian motion under \mathcal{Q} , the dynamics of $Y(t)$ and $Z(t)$ are given by

$$dY(t) = \tilde{K}_Y(\tilde{\Theta} - Y(t))dt + \Sigma_Y \sqrt{V(Y(t))}d\tilde{W}(t) \quad (2)$$

and

$$dZ(t) = \tilde{\mu}dt - \tilde{K}_Z Y(t)dt + \Sigma_Z \sqrt{V(Y(t))}d\tilde{W}(t) \quad (3)$$

respectively. Here $\tilde{K}_Y \in \mathbb{R}^{N \times N}$, $\tilde{\Theta} \in \mathbb{R}^N$, $\tilde{\mu} \in \mathbb{R}^M$, $\tilde{K}_Z \in \mathbb{R}^{M \times N}$, $\Sigma_Y \in \mathbb{R}^{N \times (N+M)}$, $\Sigma_Z \in \mathbb{R}^{M \times (N+M)}$, and $V(Y)$ is an $(N+M) \times (N+M)$ dimensional diagonal matrix, with elements along the diagonal given by

$$[V(Y)]_{nn} \equiv \alpha_n + \beta'_n Y$$

for $\alpha_n \in \mathbb{R}$ and $\beta_n \in \mathbb{R}^N$. Also let us define Σ_X as follows

$$\Sigma_X \equiv \begin{bmatrix} \Sigma_Y \\ \Sigma_Z \end{bmatrix}.$$

Hence Σ_X is an $(N+M) \times (N+M)$ matrix. Duffie and Kan (1996) discuss the restrictions which need to be placed on the parameters of the above processes to insure admissibility. For future reference, let us define \mathcal{D}_X as the Ito operator associated with X under \mathcal{Q} . We then will have that

$$\begin{aligned} \mathcal{D}_X f(Y, Z) &= f'_Y \tilde{K}_Y (\tilde{\Theta} - Y) + f'_Z (\tilde{\mu} - \tilde{K}_Z Y) \\ &+ \frac{1}{2} \text{tr} \left(f_{YY'} \Sigma_Y V(Y) \Sigma'_Y \right) + \frac{1}{2} \text{tr} \left(f_{ZZ'} \Sigma_Z V(Y) \Sigma'_Z \right) + \text{tr} \left(f_{ZY'} \Sigma_Y V(Y) \Sigma'_Z \right). \end{aligned} \quad (4)$$

Each security in the economy (to be introduced shortly) is subject to default risk as follows. At some stopping time \mathcal{T} , a default event is said to occur. When default occurs, the defaulting security is assumed to immediately pay a default dividend, and to make no further payments to the security holder. This will be discussed in more detail below.

Let us write $\mathcal{N}(t)$ as the indicator function for this default event, or

$$\mathcal{N}(t) \equiv \begin{cases} 0 & \text{if } t < \mathcal{T}, \\ 1 & \text{if } t \geq \mathcal{T}. \end{cases}$$

We assume that $\mathcal{N}(t)$ has a decomposition as follows

$$\mathcal{N}(t) = \int_0^t (1 - \mathcal{N}(u)) h(Y(u), u) du + \mathcal{M}(t), \quad (5)$$

where $\mathcal{M}(t)$ is a martingale under \mathcal{Q} , and where $(1 - \mathcal{N}(t)) h(X, t) \geq 0$ is the default intensity process under \mathcal{Q} (see Lipster and Shirayev (2001b) for a discussion of random point processes such as \mathcal{N}). Our use of the term “default intensity” for $h(t)$ comes from the following, well-known result. Under \mathcal{Q} , the probability of no-default from time t through time T , conditional on no-default as of time t , is given by

$$\mathbb{E}_t^{\mathcal{Q}} \left[\exp \left(- \int_t^T h(u) du \right) \right]. \quad (6)$$

This corresponds to $h(t)dt$ representing the instantaneous probability of default under \mathcal{Q} .

For now, we will assume that a single default process $\mathcal{N}(t)$ drives all defaults in the economy. This corresponds to the case where all securities are associated with the same

issuer. An extension of this analysis to the case of security classes with different default processes is straightforward, and is discussed later in the paper.

We assume the existence of a short-rate process $r(t)$. Investments in a money market account are not subject to default risk, and therefore grow risklessly at the rate $r(t)$. Furthermore, we assume that markets are complete with respect to the evolution of uncertainty as defined in this section.

Bonds

We assume the existence of a continuum of bonds. Each bond is characterized by a time T at which it matures, as well as by a loss process $L_s(t)$ which determines the portion of the bond's value which will be lost if default occurs (in a sense which will be made precise momentarily). We will assume that $L_s(t) \leq 1$, which we will soon see implies limited liability for bond holders in the case of default. Once the bond has defaulted, it entitles its owner to no future cashflows. In particular, a defaulted zero-coupon bond will not pay anything upon maturity. If the time t price of such a zero-coupon bond is given by $\hat{P}_{s,T}(t)$, then the above assumptions imply the boundary condition that

$$\hat{P}_{s,T}(T) = 1 - \mathcal{N}(T). \quad (7)$$

Hence, if by its maturity, the bond has not defaulted, it pays \$1 to its holder; if default has occurred, implying that $\mathcal{N}(T) = 1$, then the bond pays nothing upon maturity.

We assume that the economy is characterized by $s = 1, \dots, S$ loss processes, all associated with the same default event $\mathcal{N}(t)$. The interpretation of these multiple loss processes is that bonds of a given issuer may have different seniority levels, and hence may face different recovery rates in the event of a default.

For reasons which will become clear later, in order to maintain the affine structure of the model, we assume that the product of the default intensity process and the loss rate process must be affine. Therefore, let us write

$$\phi_s(t) \equiv h(t) \times L_s(t) = \phi_{s,0} + \phi'_{s,Y} Y(t). \quad (8)$$

Here $\phi_{s,0} \in \mathbb{R}$ and $\phi_{s,Y} \in \mathbb{R}^N$. In our implementation of the model, one of $h(t)$ and $L_s(t)$ will be assumed to be affine, with the other assumed to be constant. Which assumption is the more reasonable one depends on the nature of the problem at hand.

If default occurs at some time $\mathcal{T} \leq T$, a bond which matures at time T and which has a loss process given by $L_s(t)$ pays a default dividend of $\tilde{D}_{s,T}(\mathcal{T})$, where

$$\tilde{D}_{s,T}(t) \equiv (1 - L_s(t)) \exp\left(A_{s,T}(t) - B_{s,T}(t)' Y(t)\right) \quad (9)$$

where $A_{s,T}(t)$ and $B_{s,T}(t)$ satisfy the following sets of ordinary differential equations

$$0 = -r_0 - \phi_{s,0} + \partial A_{s,T}(t)/\partial t - \tilde{\Theta}' \tilde{K}'_Y B_{s,T}(t) + \frac{1}{2} \sum_{n=1}^{N+M} [\Sigma'_Y B_{s,T}(t)]_n^2 \alpha_n, \quad (10)$$

$$0 = -r_Y - \phi_{s,Y} - \partial B_{s,T}(t)/\partial t + \tilde{K}'_Y B_{s,T}(t) + \frac{1}{2} \sum_{n=1}^{N+M} [\Sigma'_Y B_{s,T}(t)]_n^2 \beta_n, \quad (11)$$

subject to the constraints that $A_{s,T}(T) = 0$ and $B_{s,T}(T) = 0$. This is admittedly a strange looking default dividend. To understand why this is an appropriate choice, let us write $P_{s,T}(t)$ as the price of the bond in question, conditional on no-default. We will show in Section 3.1, that $P_{s,T}(t)$ is given by

$$P_{s,T}(t) = \exp\left(A_{s,T}(t) - B_{s,T}(t)' Y(t)\right). \quad (12)$$

This equation allows us to interpret the default dividend in (9) as representing a recovery of a fraction $(1 - L_s(\mathcal{T}))$ of the bond's price at the instant \mathcal{T}^- immediately before default. However, at this point, equation (9) should be interpreted simply as an assumption about the behavior of the bond at the time of default.⁵ The fact that the price turns out to have the form in (12) is a result of our analysis, one which just happens to provide us with a convenient interpretation for the exogenous default dividend specified in (9).

Let us emphasize at this point the distinction between $\hat{P}_{s,T}(t)$ and $P_{s,T}(t)$. The latter is the stock price, assuming that default has not yet occurred. The former, given by $\hat{P}_{s,T}(t) = (1 - \mathcal{N}(t))P_{s,T}(t)$, is the actual stock price. The crucial point, to which we will return in Section 6, is that the drift of the no-default stock price does not reflect the per unit probability of default, whereas the drift of the stock price $\hat{P}_{s,T}(t)$ does reflect this default probability.

Because of our assumption about the default dividend, it will be convenient to assume that, at the default time \mathcal{T} and thereafter, the value of the defaulted bond is zero, or

$$\hat{P}_{s,T}(t) = 0 \quad \text{for } t \geq \mathcal{T}. \quad (13)$$

Note that this assumption is simply a modeling tool, and could be changed to require that, at default, the value of the defaulted bond is equal to the default dividend. The difference between these assumptions on the behavior of the security's price at the instant of default show up in the way in which cashflows are accounted for in the discounted gains process of the in question. The assumption that the value of the defaulted security, at the instant of default, is zero allows for a convenient representation of the discounted gains process, and so will be maintained throughout the paper.

⁵This assumption about the behavior for the default dividend of the bond is slightly different from the modeling choice made in Duffie and Singleton (1999). We make this assumption because it facilitates modeling of the cashflows associated with stocks. In any case, bond prices in this model are identical to the ones in Duffie and Singleton (1999), despite the slightly different modeling formalisms.

Our assumed decomposition of the default process into a default dividend and a bond price worth zero allows us to write down the following discounted, cumulative-gains process for bond expiring at time T , with a loss process given by $L_s(t)$:

$$g_{s,T}(t) \equiv g_{s,T}(0) + e^{-\int_0^t r(s)ds} \hat{P}_{s,T}(t) + \int_{0+}^t e^{-\int_0^u r(s)ds} \tilde{D}_{s,T}(u) d\mathcal{N}(u). \quad (14)$$

Note that the above integral with respect to $\mathcal{N}(t)$ begins at 0^+ , indicating that if default has occurred at time 0, the value of the default dividend is in the $g_{s,T}(0)$ term. Also, the bond price must satisfy the boundary condition that $\hat{P}_{s,T}(T) = 1 - \mathcal{N}(T)$.

Under the risk-neutral measure \mathcal{Q} , it must be the case that this discounted gains process is a martingale. We will show in Section 3.1 how this observation allows us to conclude that no-default bond prices, $P_{s,T}(T)$ are indeed given by (12).

Stocks

In addition to bonds, the present economy is assumed to contain a set of I stocks, which are referenced by $i = 1, \dots, I$. Stocks differ from bonds in two ways:

1. Assuming default has not occurred, stocks pay a random terminal dividend $\bar{D}_i(T)$ at some point T in the future, whereas bonds only pay \$1 at time T .
2. In addition to the terminal dividend, stock i entitles its holder to a cumulative dividend of

$$\int_0^t D_i(u) du$$

through time t as long as default has not occurred. We will refer to $D_i(t)$ as the instantaneous dividend of stock i .

Like bonds, stocks pay a default dividend in the case of a default event. This default dividend for a stock is given by $\tilde{D}_i(t)$. We assume that when default occurs, the stock price at the time of default and thereafter is equal to zero. Writing $\hat{S}_{i,T}(t)$ for the ex-dividend price of the i^{th} stock, this implies that

$$\hat{S}_{i,T}(t) = 0 \quad \text{for } t \geq \mathcal{T}. \quad (15)$$

Recalling the discussion following equation (13), we see that this decomposition of the stock default event into a default dividend and a zero value stock price shows up in the specification of the discounted gains process for stocks in (26).

Once default has occurred, the stock entitles its holders to no future payments. With this, the instantaneous dividend of the stock may be written as

$$(1 - \mathcal{N}(t))D_i(t).$$

The terminal dividend, given by $(1 - \mathcal{N}(T))\bar{D}_i(T)$, implies the following boundary condition

$$\hat{S}_{i,T}(T) = (1 - \mathcal{N}(T))\bar{D}_i(T). \quad (16)$$

We make one further assumption about the terminal dividend process of a given stock: model parameters must be chosen so as to satisfy the following transversality condition

$$\lim_{T \rightarrow \infty} \mathbb{E}_t^{\mathcal{Q}} \left[(1 - \mathcal{N}(T))e^{-\int_t^T r(s)ds} \bar{D}_i(T) \right] = 0. \quad (17)$$

The role of this condition will be to insure the existence of a unique price for an infinitely lived stock. This idea will be developed in Section 3.2.1.

Each stock has associated with it a loss process $L_i(t) \leq 1$, which has the same interpretation as the bond loss processes $L_{s,T}(t)$. Again, to maintain the affine structure of the model we assume that the following product is affine for all i

$$\phi_i(t) \equiv h(t) \times L_i(t) = \phi_{i,0} + \phi'_{i,Y} Y(t). \quad (18)$$

Here $\phi_{i,0} \in \mathbb{R}$ and $\phi_{i,Y} \in \mathbb{R}^N$. Just as for bonds, we will take one of $h(t)$ and $L_i(t)$ to be affine, and one to be constant. Note that the default intensity process $h(t)$ in equation (18) is the same $h(t)$ as in equation (8). This follows from the assumption that stocks and bonds are subject to the same default risk process. Note as well that stocks in this model do not literally need to be stocks of a corporation, but may refer to any security which pays a dividend stream which may be modeled in the above fashion.

We will assume that for the i^{th} stock, the instantaneous dividend $D_i(t)$ is given by

$$D_i(t) \equiv (\delta_{i,0} + \delta'_{i,Y} Y(t)) \exp\left(a_i \times t - B'_i Y(t) - C'_i Z(t)\right) \quad (19)$$

where $\delta_{i,0} \in \mathbb{R}$, $\delta_{i,Y} \in \mathbb{R}^N$, $C_i \in \mathbb{R}^N$, and $a_i : \{\delta_{i,0}, \delta_{i,Y}, C_i, \phi_{i,0}, \phi_{i,Y}\} \rightarrow \mathbb{R}$ and $B_i : \{\delta_{i,0}, \delta_{i,Y}, C_i, \phi_{i,0}, \phi_{i,Y}\} \rightarrow \mathbb{R}^N$ solve the following equations

$$a_i = r_0 + \phi_{i,0} - \delta_{0i} + \tilde{\Theta}' \tilde{K}'_Y B_i + \tilde{\mu}' C_i - \frac{1}{2} \sum_{n=1}^{N+M} \left([\Sigma'_Y B_i]_n + [\Sigma'_Z C_i]_n \right)^2 \alpha_n, \quad (20)$$

$$0 = -r_Y - \phi_{i,Y} + \delta_{Yi} + \tilde{K}'_Y B_i + \tilde{K}'_Z C_i + \frac{1}{2} \sum_{n=1}^{N+M} \left([\Sigma'_Y B_i]_n + [\Sigma'_Z C_i]_n \right)^2 \beta_n. \quad (21)$$

Also we assume that the time T terminal dividend $\bar{D}_i(T)$, assuming that no default has occurred, is given by

$$\bar{D}_i(T) = \exp\left(a_i \times T - B'_i Y(T) - C'_i Z(T)\right), \quad (22)$$

and that in the case of default at time \mathcal{T} , the stock pays a default dividend of $\tilde{D}_i(\mathcal{T})$, where

$$\tilde{D}_i(t) \equiv (1 - L_i(t))\bar{D}_i(t). \quad (23)$$

Note that so far we have exogenously specified a dividend process of the stocks in the economy. The dividend process for stock i can be fully described by the parameters

$$\{\delta_{i,0}, \delta_{i,Y}, C_i, \phi_{i,0}, \phi_{i,Y}\}.$$

Of these, the first three serve to define the instantaneous and terminal dividends of the stock, and the last two define the default dividend. The a_i and B_i functions, which also determine properties of the dividend process, may not be chosen arbitrarily, but must satisfy equations (20) and (21) above. These restrictions, as will become clear in the proof of Proposition 3, are crucial in order to maintain an exponential affine form for stock prices in the model. The fact that (21) may have multiple solutions is addressed in Section 3.2.2 of the paper.

A comment is in order at this point about the above dividend specification. The development of the present model has been guided by two goals: tractability and empirical flexibility. As will become clear momentarily, the somewhat convoluted dividend specification given above leads to a very simple expression for the stock price. Furthermore, this simple stock price expression allows us to easily interpret the above specified dividend process. This happy outcome comes about at the cost of giving up some amount of empirical flexibility. An example of this is the fact that the instantaneous dividend $D_i(t)$ in (19) is restricted to be of the form of an affine process multiplying an exponential affine process, where some of the latter's coefficients are restricted to satisfy equations (20) and (21) above. If these restrictions were removed from the specification of the instantaneous dividend, we would still be able to obtain stock prices in the model, however stock prices would no longer have the convenient feature of being exponential affine functions of the model's factors. It is the hope that, with the restrictions given above, the dividend specification remains sufficiently rich to provide a reasonable fit for observed dividend series of equities. To the extent that this is not the case, the above restrictions also restrict the model's ability to provide an adequate description for the data.

To make the above discussion more concrete, let us write $S_{i,T}(t)$ for the ex-dividend price of a stock with a dividend process of $\{\delta_{i,0}, \delta_{i,Y}, C_i, \phi_{i,0}, \phi_{i,Y}\}$, conditional on no-default. The dividend process specified in the above discussion is convenient because, as we will show in Section 3.2, $S_{i,T}(t)$ is given by

$$S_{i,T}(t) = \exp\left(a_i \times t - B'_i Y(t) - C'_i Z(t)\right). \quad (24)$$

Hence, as promised, no-default stock prices are indeed exponential affine functions of the model's factors. Furthermore, it should be emphasized that this fact makes the model very well suited for empirical implementation.

With a no-default stock price of the form in (24), we can provide the following interpretation of the stock's dividend process:

- The instantaneous dividend may be written as

$$D_i(t) = (\delta_{i,0} + \delta'_{i,Y} Y(t)) S_i(t).$$

Because of the above relationship, which holds if the stock price turns out to satisfy (24), we see that the instantaneous dividend yield on a stock $\delta_i(t)$ is given by

$$\delta_i(t) = \delta_{i,0} + \delta'_{i,Y} Y(t). \quad (25)$$

Hence the specification of the instantaneous dividend in (19) amounts to specifying an exogenous dividend yield process. Also we exogenously specify the dependence of the instantaneous dividend on the Z -type factors in the economy. With this, though, the dependence of the stock price on time and on the Y -type factors (through the a_i and B_i coefficients) is fully determined by equations (20) and (21) respectively.

- The terminal dividend of the stock $\bar{D}_i(T)$ is seen to satisfy

$$\bar{D}_i(t) = S_{i,T}(t)$$

for all t, T . Because of this, we see that the stock price does not depend on the time T at which the terminal dividend is paid. Going forward we will simply write $S_i(t)$ for the no-default, ex-dividend, time t price of a stock with a dividend process $\{\delta_{i,0}, \delta_{i,Y}, C_i, \phi_{i,0}, \phi_{i,Y}\}$, regardless of the actual date of the terminal dividend.

- We see as well that the default dividend $\tilde{D}_i(t)$ is actually equal to a fraction $1 - L_i(t)$ of the stock price at the instant \mathcal{T}^- immediately before default. This justifies our calling $L_i(t)$ the loss rate for stock i .

It should be emphasized that all of these observations are results, rather than assumptions. That is, we have exogenously assumed a dividend process for a stock (consisting of the processes $D_i(t)$, $\tilde{D}_i(t)$, and $\bar{D}_i(t)$), and then if we show that the no-default, ex-dividend price of a stock with such a dividend process is given by (24), the interpretations given above apply. As we have already pointed out, it is our hope that the restricted dividend process described thus far does a good job of describing the data because the restrictions that have been imposed are, unfortunately, necessary in order to insure that the no-default stock price has the convenient form given in (24).

With the above dividend specification, the discounted, cumulative-gains process for a stock with dividend process $\{\delta_{i,0}, \delta_{i,Y}, C_i, \phi_{i,0}, \phi_{i,Y}\}$ which pays a terminal dividend at time T is defined as

$$\begin{aligned} g_{i,T}(t) \equiv & g_{i,T}(0) + \int_0^t (1 - \mathcal{N}(u)) e^{-\int_0^u r(s) ds} D_i(u) du + e^{-\int_0^t r(s) ds} \hat{S}_i(t) \\ & + \int_{0^+}^t e^{-\int_0^u r(s) ds} \tilde{D}_i(u) d\mathcal{N}(u), \end{aligned} \quad (26)$$

subject to the boundary condition that $\hat{S}_i(T) = (1 - \mathcal{N}(T)) \bar{D}_i(T)$ for some time $T > t$. Notice that the integral with respect to $\mathcal{N}(t)$ begins at 0^+ , implying that if default has occurred at time 0, the value of the default dividend is reflected in the $g_{i,T}(0)$ term.

As will be shown in Section 3.2, the requirement that $g_i(t)$ be a \mathcal{Q} martingale will allow us to prove that, conditional on no default, stock prices in this model are indeed given by (24).

At this point, we have enough structure to show one case in which the transversality condition in (17) holds.

Proposition 1 *Let us assume that for a stock with dividends process $\{\delta_{i,0}, \delta_{i,Y}, C_i, \phi_{i,0}, \phi_{i,Y}\}$, we have*

$$\delta_i(t) + h(t)(1 - L_i(t)) \geq \epsilon > 0. \quad (27)$$

Then the transversality condition in (17) holds.

Since $h(t) \geq 0$ by definition, we note that one sufficient condition for this proposition is to have $\delta_i(t) > 0$ and $L_i(t) \leq 1$ – that is, dividends must be strictly positive, and default losses cannot exceed the market value of the defaulted securities at the instant before default.

2.2 The Issuing Firm

So far we have specified the cashflow characteristics of a set of securities called bonds and stocks (or more accurately, stock-like securities). The assumption that the bonds and stocks, as defined in the previous sections, are driven by the same default process $\mathcal{N}(t)$ may be taken to indicate that these bonds and stocks are issued by a single firm. If we further assume that all of the cashflows generated by the firm's operations have financial claims (in the form of bonds and stocks) issued against them, then the value of the firm is, by definition, equal to the sum of the prices of the bonds and stocks which exist in the present economy. Because we have assumed that markets are complete and that we are working under an equivalent martingale measure \mathcal{Q} , we know that the economy has no arbitrage opportunities. This naturally implies that the Modigliani-Miller Theorem holds in the present context.

To make this point more precisely, let us assume that the firm represents a claim on the cashflows of a set of bonds and stocks. These bonds come from $s = 1, \dots, S$ credit classes, with maturities of $T_1^s, \dots, T_{N_s}^s$ within each credit class. Furthermore, the principal amount due at each maturity, for each credit class, is given by C_n^s for $n = 1, \dots, N_s$. Here N_s represents the number of maturities of zero-coupon bonds which exist inside each credit class. Therefore there are $\sum_{s=1}^S \sum_{n=1}^{N_s} C_n^s$ zero-coupon bonds issued by the firm. Furthermore, assume that there are $i = 1, \dots, I$ stocks (or stock-like) securities, each with a maturity T_i . The firm is therefore, by definition, a claim to the following sets of cashflows:

CF[i]. C_n^s one dollar dividends set to occur at time T_n^s for $s = 1, \dots, S$ and $n = 1, \dots, N_s$.

CF[ii]. A default dividend $\tilde{D}_{s,T_n^s}(t)$ associated with each of the zero-coupon bonds listed above.

CF[iii]. A stochastic dividend stream $D_i(t)$ for each $i = 1, \dots, I$.

CF[iv]. A terminal dividend $\bar{D}_i(T_i)$ associated with each stock-like security.

CF[v]. A default dividend $\tilde{D}_i(t)$ associated with each of the stock-like securities.

We will take this list of cashflows as the *definition* of the output of the firm's production technology, regardless of how these cashflows are allocated to financial securities. The fact that these cashflows are allocated to the set of bonds and stocks listed above is simply a convenience. Since markets are complete and no arbitrage opportunities exist, any reallocation of these cashflows (adapted to the filtration $\{\mathcal{F}_t : t \geq 0\}$) to a set of different securities will leave the value of the firm unchanged, as these new securities can be perfectly replicated by the existing set of stocks and bonds. If this were not the case, then an arbitrage opportunity would arise where the cheaper allocation could be bought and the more expensive allocation could be sold, for an instantaneous riskless profit (riskless because, by definition, both allocations represent claims to the same set of cashflows). Therefore, by construction, the Modigliani-Miller Theorem holds in the present context. This argument is basically identical to Merton's (1974) proof as to why the Modigliani-Miller Theorem holds in the context of his model for risky debt.

Going forward, let us refer to the value of a firm with the above-listed cashflows as $V(t)$.

2.3 Bonds and Stocks from Multiple Issuers

As has already been pointed out, we have thus far assumed that all securities in the model are driven by the same default process. This amounts to the assumption that all bonds and the stock(s) are issued by a single corporation, and hence the default event affects all of the securities in the model.

However, there is an alternative interpretation of the present setup. Assume that, in fact, the bonds and stocks in the economy are subject to default processes with the same intensity $h(t)$, though the actual default processes for different bonds and different stocks may be different. In this case, the pricing of bonds and stocks proceeds in exactly the same way as in the single issuer case, with the caveat that there are multiple $\mathcal{N}(t)$ processes. Because the values of all securities in the model, conditional on no-default, depend only on the distribution of the future payoffs, the presence of multiple issuers does not affect the values of the securities in the model, as long as default has not yet occurred. Of course, the fact that the $\mathcal{N}(t)$ processes are different across different securities implies that the paths of prices may not be the same, though the no-default prices are. For example, two stocks, subject to the same distribution of future default, may today have the same price. However, it may be the case that one of these stocks defaults tomorrow, whereas the other one does not, implying different price paths.

Whether we are assuming a single or a multiple-issuer economy turns out not to matter for most of the ensuing analysis, and so this point will not be discussed further unless it is directly relevant for the topic at hand.

2.4 No Default Risk

It should be noted that (with very few exceptions having to do with the mechanics of default) all of the analysis in this paper applies equally to an economy with no default risk. We simply set $h(t) = 0$ for all t . A detailed analysis of the no-default version of the present model, which includes several results not presented in this paper, is in Mamaysky (2002a).

3 Bond and Stock Prices

In this section we discuss how to price the bonds and stocks in the economy set out in the previous section. Also, we solve for the value of the firm defined in Section 2.2 (a task which is trivial, once we know how to value the bonds and stocks in the model).

3.1 Bond Prices

In this section, we solve for bond prices in the present model.⁶ Under the risk-neutral measure \mathcal{Q} , discounted gains processes must be martingales. Given the gains process in (14) and the boundary condition in (7), it is easy to check that $g_{s,T}(t) = E_t^{\mathcal{Q}}[g_{s,T}(t')]$ (see proof of Proposition 2), for $t < t' < T$, is equivalent to

$$\hat{P}_{s,T}(t) = \mathbb{E}_t^{\mathcal{Q}} \left[e^{-\int_t^T r(s)ds} (1 - \mathcal{N}(T)) + \int_{t+}^T e^{-\int_t^u r(s)ds} \tilde{D}_{s,T}(u) d\mathcal{N}(u) \right]. \quad (28)$$

An immediate consequence of this is that bond prices are zero at the instant of default, and thereafter. This is in keeping with our stated assumption in equation (13). To solve for the bond price in (28), it is enough to find a bond price which satisfies the boundary condition in (7), such that the associated discounted gains process $g_{s,T}(t)$ is a \mathcal{Q} martingale. The following proposition makes this argument in a precise way.

Proposition 2 *Assume that a function $\hat{P}_{s,T}$ is given by*

$$\hat{P}_{s,T}(t) = (1 - \mathcal{N}(t))P_{s,T}(t) \quad (29)$$

for $P_{s,T}(t)$ given in (12). Furthermore, assume that the following integral is a \mathcal{Q} martingale

$$\begin{aligned} & \int_0^T e^{-\int_0^t r(u)du} (1 - \mathcal{N}(t)) \frac{\partial P_{s,T}}{\partial X^i} \Sigma_X \sqrt{V(t)} d\tilde{W}(t) \\ & + \int_{0+}^T e^{-\int_0^t r(u)du} \left(\tilde{D}_{s,T}(t) - P_{s,T}(t) \right) d\mathcal{M}(t). \end{aligned} \quad (30)$$

⁶Note that this derivation differs slightly from the one in Duffie and Singleton (1999), though the resulting bond prices are identical.

Then $\hat{P}_{s,T}(t)$ is equal to the expectation in (28). Furthermore, subject to a regularity condition given in the Appendix, the no-default bond price satisfies the following

$$P_{s,T}(t) = \mathbb{E}_t^{\mathcal{Q}} \left[\exp \left(- \int_t^T (r(u) + \phi_s(u)) du \right) \right]. \quad (31)$$

The proof is given in the Appendix.

We note that if there are multiple issues in the economy, all being driven by the same default intensity process, then the no-default bond prices $P_{s,T}(t)$ will all be the same (assuming identical recovery rates). However, if there are multiple $\mathcal{N}(t)$ processes, then the actual price paths (in equation (29)) may be different, as one bond in a given credit class may default, whereas others do not. This comment applies as well to Proposition 3, which gives no-default and actual stock prices in the economy.

3.2 Stock Prices

Let us now turn our attention to stocks. From the requirement that discounted gains processes are martingales under the risk-neutral measure \mathcal{Q} , we have that

$$g_{i,T}(t) = \mathbb{E}_t^{\mathcal{Q}} \left[g_{i,T}(t') \right] \quad (32)$$

for $t < t' < T$. It is easy to check (see proof of Proposition 3) that this martingale condition and the boundary condition for stocks in (16) are equivalent to the following pricing equation for stocks

$$\begin{aligned} \hat{S}_{i,T}(t) = \mathbb{E}_t^{\mathcal{Q}} & \left[\int_t^T (1 - \mathcal{N}(u)) e^{-\int_t^u r(s) ds} D_i(u) du + (1 - \mathcal{N}(T)) e^{-\int_t^T r(s) ds} \bar{D}_i(T) \right. \\ & \left. + \int_{t+}^T e^{-\int_t^u r(s) ds} \tilde{D}_i(u) d\mathcal{N}(u) \right]. \end{aligned} \quad (33)$$

An immediate consequence of (33) is that, at the time of default and after, ex-dividend stock prices are equal to zero, in keeping with our assumption in (15). To solve for the stock price in (33), it suffices to find a stock price which satisfies the boundary condition in (16), and whose discounted gains process $g_{i,T}(t)$ satisfies equation (32). The following proposition formalizes this argument.

Proposition 3 *Assume that a function $\hat{S}_{i,T}(t)$ is given by*

$$\hat{S}_{i,T}(t) = (1 - \mathcal{N}(t)) S_{i,T}(t) \quad (34)$$

for $S_{i,T}(t)$ from (24). Furthermore, assume that the following integral is a \mathcal{Q} martingale

$$\begin{aligned} & \int_0^T e^{-\int_0^t r(u) du} (1 - \mathcal{N}(t)) \frac{\partial S_{i,T}}{\partial X^i} \Sigma_X \sqrt{V(t)} d\tilde{W}(t) \\ & + \int_{0+}^T e^{-\int_0^t r(u) du} \left(\tilde{D}_i(t) - S_{i,T}(t) \right) d\mathcal{M}(t). \end{aligned} \quad (35)$$

Then $\hat{S}_{i,T}(t)$ is equal to the expectation in (33). Furthermore, subject to a regularity condition given in the Appendix, the no-default stock price satisfies the following

$$S_i(t) = \mathbb{E}_t^{\mathcal{Q}} \left[\int_t^T e^{-\int_t^u (r(h) + \phi_i(h)) dh} D_i(u) du + e^{-\int_t^T (r(h) + \phi_i(h)) dh} \bar{D}_i(T) \right]. \quad (36)$$

The proof is given in the Appendix.

From (34) we see that the stock price at time t is equal to the ex-dividend no-default stock price $S_i(t)$, as long as default has not taken place. If default has occurred, then at the instant of default and thereafter, the price of the stock is equal to zero. Going forward we will refer to $S_i(t)$ as the no-default stock price for a stock with a dividend process given by $\{\delta_{i,0}, \delta_{i,Y}, C_i, \phi_{i,0}, \phi_{i,Y}\}$.

3.2.1 Infinitely Lived Stocks

With the above results, we are now ready to determine the price of an infinitely lived stock. Let us assume that we have an infinitely lived stock which pays a dividend stream for ever, unless a default event occurs. In this case, the stock pays its default dividend, and will make no further payments in the future. As for a finitely lived stock, it will be convenient to assume that at the time of default, the value of the defaulted, infinitely lived stock will be zero. This implies that $S_i^\infty(t) = (1 - \mathcal{N}(t))S_i^\infty(t)$. This cashflow specification is almost the same as for the finitely lived stocks which we have discussed upto this point, the one difference being that the infinitely lived stock characterized by $\{\delta_{i,0}, \delta_{i,Y}, C_i, \phi_{i,0}, \phi_{i,Y}\}$, does not pay a terminal dividend $\bar{D}_i(t)$. Let us write $S_i^\infty(t)$ for the price of such a stock.

The requirement that the discounted gains process associated with this stock is a \mathcal{Q} martingale, implies that for all T , we must have that

$$\begin{aligned} \hat{S}_i^\infty(t) &= \mathbb{E}_t^{\mathcal{Q}} \left[\int_t^T (1 - \mathcal{N}(u)) e^{-\int_t^u r(s) ds} D_i(u) du + (1 - \mathcal{N}(T)) e^{-\int_t^T r(s) ds} \hat{S}_i^\infty(T) \right. \\ &\quad \left. + \int_{t+}^T e^{-\int_t^u r(s) ds} \tilde{D}_i(u) d\mathcal{N}(u) \right]. \end{aligned} \quad (37)$$

We also impose the following transversality condition which requires that

$$\lim_{T \rightarrow \infty} \mathbb{E}_t^{\mathcal{Q}} \left[(1 - \mathcal{N}(T)) e^{-\int_t^T r(s) ds} \hat{S}_i^\infty(T) \right] = 0. \quad (38)$$

We will say that an infinitely lived stock is *admissible*, if its price process $S_i^\infty(t)$ satisfies both conditions (37) and (38). Since equation (37) holds for all T , we know that the limit of the right-hand side of that equation must exist. Therefore we can write that

$$\begin{aligned} \hat{S}_i^\infty(t) &= \lim_{T \rightarrow \infty} \mathbb{E}_t^{\mathcal{Q}} \left[\int_t^T (1 - \mathcal{N}(u)) e^{-\int_t^u r(s) ds} D_i(u) du + (1 - \mathcal{N}(T)) e^{-\int_t^T r(s) ds} \hat{S}_i^\infty(T) \right. \\ &\quad \left. + \int_{t+}^T e^{-\int_t^u r(s) ds} \tilde{D}_i(u) d\mathcal{N}(u) \right]. \end{aligned}$$

Noting the transversality condition in (38), we observe that the limit of the sum of the $D_i(t)$ and the $\tilde{D}_i(t)$ terms must also exist, and must equal the infinitely lived stock's price. Hence we can write that an admissible stock price satisfies

$$\hat{S}_i^\infty(t) = \lim_{T \rightarrow \infty} \mathbb{E}_t^\mathcal{Q} \left[\int_t^T (1 - \mathcal{N}(u)) e^{-\int_t^u r(s) ds} D_i(u) du + \int_{t^+}^T e^{-\int_t^u r(s) ds} \tilde{D}_i(u) d\mathcal{N}(u) \right]. \quad (39)$$

Because the terms in the above integral are all exogenously specified (via equations (19) and (23)), and because existence of an admissible stock price also implies existence of the above limit, we see that if an admissible stock price exists, it is unique.

Now consider the price of a finitely lived stock $S_i(t)$. We know that $S_i(t) = \bar{D}_i(t)$ for all t . Therefore, the finitely lived stock price satisfies the pricing equation in (37) and the transversality condition in (38). Hence it is admissible, and we have proved the following:

Proposition 4 *An admissible price process for an infinitely-lived stock exists, and is given by*

$$S_i^\infty(t) = S_i(t). \quad (40)$$

For this reason, going forward, we will use $S_i(t)$ to refer to the price of any stock, finitely or infinitely lived, that has a dividend process given by $\{\delta_{i,0}, \delta_{i,Y}, C_i, \phi_{i,0}, \phi_{i,Y}\}$.

3.2.2 Choosing a Solution for B_i in Equation (21)

We first note that any B_i which satisfies equation (21), with $\delta_{i,0} = 0$, $\delta_{i,Y} = 0$, $C_i = 0$, and $\phi_i = \phi_s$, is a particular solution to equation (11). However, subject to the boundary condition that $B_{s,T}(T) = 0$, the ordinary differential equation in (11) has a unique solution given by $B_{s,T}(t)$. Therefore when the limit as $T \rightarrow \infty$ of $B_{s,T}(t)$ exists, it will be unique, will have no dependence on t , and will satisfy equation (21) with the above parameter restrictions imposed. To choose a unique solution to (21) in a way which insures that a stock, which pays no intermediate dividends and whose terminal dividend has no loadings on the Z type factors, has the same risk loadings as a sequence of bonds whose maturities go to infinity, we need to choose a solution B_i to (21) which converges to

$$B_{s,\infty} \equiv \lim_{T \rightarrow \infty} B_{s,T}(t)$$

as $\delta_{i,0} \rightarrow 0$, $\delta_{i,Y} \rightarrow 0$, $C_i \rightarrow 0$, and as $\phi_i \rightarrow \phi_s$.

With this, we now have a procedure for choosing a solution to (21) which is consistent with no-arbitrage in the economy. First, we need to solve the bond pricing equations. Then, using the limiting case of these solutions for infinitely lived bonds, we can identify the solutions for B_i which are equal to the steady-state bond solution $B_{s,\infty}$ for the base case of the stock's dividend specification

$$\{\delta_{i,0}, \delta_{i,Y}, C_i, \phi_{i,0}, \phi_{i,Y}\} = \{0, 0, 0, \phi_{s,0}, \phi_{s,Y}\}.$$

Then the B_i which applies for other dividend specification must come from this base-case solution.

Of course, in the case where $\beta_n = 0$ for $n = 1, \dots, N + M$, the solution to (21) is unique, and this procedure does not have to be followed (though it would still give the correct result).

3.3 The Value of the Firm

Since we have computed the prices for bonds and stocks, it becomes trivial to find the value $V(t)$ of the firm specified in Section 2.2. From the specification of the firm's future cashflows in $\text{CF}[i] - \text{CF}[v]$, we see that the price of the firm is given by

$$V(t) = \sum_{s=1}^S \sum_{n=1}^{N_s} C_n^s \hat{P}_{s, T_n^s}(t) + \sum_{i=1}^I \hat{S}_i(t). \quad (41)$$

One interesting empirical question raised by this analysis is to see what happens to the value of firms prior to the time that they default on their debt obligations. On the one hand, since all bond and stock prices depend on $h(t)$ (though in varying degrees having to do with the $L(t)$ process for a given security), an increase in the likelihood of default causes security prices, and firm value, to fall. However, at the actual time of default, the above formula for $V(t)$ predicts a discontinuous drop in the firm value, reflecting default on all of the firm's constituent financial liabilities. The open question with regard to this is to find the specification of the present model which most closely matches the empirically observed pre-default drop in firm prices, as well as the (potentially) discontinuous drop in firm value when the default event actually occurs.

4 Total Returns Processes

In empirically implementing the present model it is often useful to work not with the dividend paying stock directly, but instead with the total returns process associated with that stock. Let us define a total returns process for a stock as the value of a portfolio which results from holding one share of the stock at time zero, and then reinvesting all dividends back into the stock itself. In the case of default, we assume that the post-default portfolio is reinvested into the short rate, and hence grows at a rate of $r(t)$. Let us write $\hat{s}_i(t)$ for the value of the total returns process associated with the stock price process $\hat{S}_i(t)$. Recalling that

$$\hat{S}_i(t) = (1 - \mathcal{N}(t))S_i(t),$$

we can write the dynamics of the total returns process as follows

$$\begin{aligned} d\hat{s}_i(t) &= n_i(t^-) \left((1 - \mathcal{N}(t)) \left(dS_i(t) + D_i(t)dt \right) + (\tilde{D}_i(t) - S_i(t))d\mathcal{N}(t) \right) \\ &\quad + \left(s_i(t) - n_i(t)(1 - \mathcal{N}(t))S_i(t) \right) r(t)dt, \end{aligned} \quad (42)$$

where $n_i(t)$ is the number of shares of the stock held in the portfolio at time t . As long as the stock has not yet defaulted we will require that the portfolio is fully invested in the stock. At the time of default and thereafter, the number of shares held is irrelevant, and we can set $n_i(t)$ equal to zero for all $t \geq \mathcal{T}$. Hence $n_i(t)$ is given by

$$n_i(t) = (\hat{S}_i(t))^\Delta s_i(t). \quad (43)$$

Here the notation $(x)^\Delta$ refers to the generalized inverse of x , and is equal to $1/x$ if $x \neq 0$, and zero otherwise. With this, we can solve for the value of the total returns process as of time t . The following proposition gives the result.

Proposition 5 *The total returns process $\hat{s}_i(t)$ for stocks with dividend process*

$$\{\delta_{i,0}, \delta_{i,Y}, C_i, \phi_{i,0}, \phi_{i,Y}\}$$

is given by

$$\hat{s}_i(t) = (1 - \mathcal{N}(t))s_i(t) + \mathcal{N}(t)(1 - L_i(\mathcal{T}))s_i(\mathcal{T}) \exp\left(\int_{\mathcal{T}}^t r(s)ds\right), \quad (44)$$

where \mathcal{T} is the random default time, and $s_i(t)$ is the value of the total returns process at time t , assuming that default has not yet occurred, and is given by

$$s_i(T) = s_i(t) \exp\left(\int_t^T \frac{dS_i(t)}{S_i(t)} + \delta_i(t)dt - \frac{1}{2} \frac{d[S_i(t)]}{S_i(t)^2}\right). \quad (45)$$

The proof is in the Appendix. Note that a crucial assumption for the proof to go through is that $L_i(t) \leq 1$, indicating limited liability in the case of default. This prevents the total returns process from ever becoming negative, and permits us to represent the total returns process in exponential form.

We have seen, therefore, that the total returns process for a stock subject to default risk can be written as the combination of the total returns formula for a stock without default risk $s_i(t)$, and a second term which governs the behavior of the total returns process once default has actually taken place. The next section shows why this is useful.

4.1 Exponential Affine Form of $s_i(t)$

The usefulness of the representation given in (44) for total returns processes lies in the fact that $s_i(t)$, the no-default value of the total returns process, has an exponential affine form, just like no-default stock prices, once an appropriate change of variables has been performed on the Z -type factors. Note that because bond prices do not depend on the Z -type factors this change of variables leaves bond prices completely unaffected.

It was shown in Mamaysky (2002a) that $s_i(t)$ from equation (45) can be written as follows

$$s_i(T) = s_i(t) \exp\left(a_i(T-t) + \int_t^T \delta_i(u)du - B'_i(Y(T) - Y(t)) - C'_i(Z(T) - Z(t))\right). \quad (46)$$

The dynamics of the Z -type factors, reproduced here from equation (3), are

$$dZ(t) = \tilde{\mu}dt - \tilde{K}_Z Y(t)dt + \Sigma_Z \sqrt{V(Y(t))}d\tilde{W}(t).$$

We now would like to construct another set of M factors, called the z -type factors, such that the integral in equation (46) could be incorporated into the change in the values of these factors. It turns out that it is possible to do this construction for some set \mathcal{S} of M stocks. For these M stocks, we will show that the expression in (46) can be written as an exponential affine function of the Y -type variables and of the new z -type variables.

To construct this new set of factors, let us choose an arbitrary subset \mathcal{S} of M stocks. Let us reference the stocks in \mathcal{S} by $1, \dots, M$. Furthermore let us define an $M \times M$ dimensional matrix C as follows

$$C \equiv \begin{bmatrix} C_1 & C_2 & \dots & C_M \end{bmatrix}$$

and an $N \times M$ dimensional matrix d_Y as

$$\delta_Y \equiv \begin{bmatrix} \delta_{1,Y} & \delta_{2,Y} & \dots & \delta_{M,Y} \end{bmatrix}.$$

Finally, let us define an M dimensional set of state variables $z(t)$, such that $z(0) = Z(0)$, where the dynamics of the new set of state variables is given by

$$dz(t) = \tilde{\mu}dt - \tilde{k}_Z Y(t)dt + \Sigma_Z \sqrt{V(Y(t))}d\tilde{W}(t). \quad (47)$$

The $M \times N$ dimensional matrix \tilde{k}_Z is defined as

$$\tilde{k}_Z \equiv \tilde{K}_Z + (C')^{-1} \delta_Y'. \quad (48)$$

With this, we are now ready to state the main proposition of this section.

Proposition 6 *Given a set \mathcal{S} of stocks referenced by $i = 1, \dots, M$, the no-default total returns process for each of these stocks from equation (46) is equivalent to*

$$s_i(T) = s_i(t) \exp \left((a_i + \delta_{i,0})(T - t) - B_i' \left(Y(T) - Y(t) \right) - C_i' \left(z(T) - z(t) \right) \right). \quad (49)$$

The proof of this proposition follows immediately from the dynamics of $z(t)$ and from the form of the no-default total returns processes in (46).

With this, and with equation (44) for total returns processes for defaultable stocks, we have a convenient representation for total returns processes. In practice, when the model is estimated using only data on stocks which have not defaulted, the only formula which needs to actually be used is the one for $s_i(t)$ in equation (49). If total returns data is being used for the estimation, we are able to extract the z -type factors associated with some set \mathcal{S} of stocks. The one drawback of doing the estimation with only the total returns processes, and with no dividend data outside of that contained in the actual total returns series, is that only \tilde{k}_Z can be estimated. From equation (48) we see that this matrix actually depends on \tilde{K}_Z and on δ_Y , which are not identifiable separately if only total returns data is being used. Mamaysky (2002b) uses this observation to conduct a test of the empirical implementation of the no-default-risk version of the present model.

5 Price of Risk and Change of Measure

Thus far, the development of the model has proceeded under the risk-neutral measure \mathcal{Q} . Implementation and estimation of the model typically involves specifying the dynamics of the model's state variables under the true physical measure \mathcal{P} , and then performing a change of measure to go to the risk-neutral world in which pricing is actually done. In this section, we will establish some results about how this change of measure may be performed in the present setting.

Our development starts with the specification of a pricing kernel process. Let us write $m(t)$ for the value of this pricing kernel at time t . The following proposition establishes that the pricing kernel can be used to price cashflows, involving default dividends, under the physical measure.

Proposition 7 *Assume that the regularity conditions of Lemma E.1 hold. Furthermore, assume that the pricing kernel $m(t)$ is given by*

$$m(t) = \mathbb{E}_t^{\mathcal{P}} \left[\frac{d\mathcal{Q}}{d\mathcal{P}} \right] e^{-\int_0^t r(u)du}, \quad (50)$$

and that cumulative, discounted gains processes for bonds and stocks are \mathcal{Q} martingales (i.e. $g_{s,T}(t)$ from (14) and $g_{i,T}(t)$ from (26) respectively). Then bond prices in the present economy satisfy

$$m(t)\hat{P}_{s,T}(t) = \mathbb{E}_t^{\mathcal{P}} \left[\int_{t^+}^{t'} m(u)\tilde{D}_{s,T}(u)d\mathcal{N}(u) + m(t')\hat{P}_{s,T}(t') \right]. \quad (51)$$

for $t < t'$, and stock prices satisfy

$$m(t)\hat{S}_i(t) = \mathbb{E}_t^{\mathcal{P}} \left[\int_t^{t'} m(u)(1 - \mathcal{N}(u))D_i(u)du + \int_{t^+}^{t'} m(u)\tilde{D}_i(u)d\mathcal{N}(u) + m(t')\hat{S}_i(t') \right]. \quad (52)$$

for $t < t'$.

The proof is in Appendix E.

If not for the default dividend, the above would be a standard result (see Duffie (2001) for example). However, the presence of the default dividend, which is paid at a random time, forces us to do some more work in order to verify that (51) and (52) do indeed hold. The proof, the main part of which is given in Lemma E.1 in the Appendix, shows that, under certain regularity conditions, both (51) and (52) follow from the assumption that the pricing kernel is given by (50) and from the assumption that \mathcal{Q} is an equivalent martingale measure.

We note that the decision for how to go from the risk-neutral to the physical measure therefore depends on the specification of the expectation of $d\mathcal{Q}/d\mathcal{P}$ in (50). If we choose to model the above expectation as

$$\mathbb{E}_t^{\mathcal{P}} \left[\frac{d\mathcal{Q}}{d\mathcal{P}} \right] = \exp \left(-\frac{1}{2} \int_0^t \Lambda(X(u))' \Lambda(X(u)) du - \int_0^t \Lambda(X(u))' dW(u) \right). \quad (53)$$

for some priocess $\Lambda : \mathbb{R}^{N+M} \rightarrow \mathbb{R}^{N+M}$, which we will refer to as the *price of risk* process, then from the Girsanov Theorem (see Protter (1995)), we can decompose the \mathcal{Q} Brownian motion \tilde{W} as follows

$$\tilde{W}(t) = W(t) + \int_0^t \Lambda(u) du, \quad (54)$$

where $W(t)$ is a Brownian motion under \mathcal{P} . Hence, for this choice of $m(t)$, we see that a process \tilde{W} , which under \mathcal{Q} behaves as a Brownian motion, will behave as a Brownian motion with a drift adjustment (given by $\Lambda(u)du$) under the physical measure \mathcal{P} .

The same type of decomposition exists for the default indicator process $\mathcal{N}(t)$. We note that $\mathcal{N}(t)$ is the same regardless of which measure we are concerned with: either default has taken place as of time t , or it hasn't. What the choice of measure affects is the decomposition of this indicator process into a drift component and a martingale component. From equation (5), we see that thus far we have assumed that, under \mathcal{Q} , the following decomposition exists

$$d\mathcal{N}(t) = (1 - \mathcal{N}(t))h(t)dt + d\mathcal{M}(t), \quad (55)$$

where $\mathcal{M}(t)$ is a \mathcal{Q} martingale. Indeed, from Protter (1995) (Theorem 18 of Chapter 3), we know that, under \mathcal{Q} , this is the unique decomposition of $\mathcal{N}(t)$ into a continuous, finite-variation process $\int_0^t (1 - \mathcal{N}(u))h(u)du$ and a \mathcal{Q} martingale $\mathcal{M}(t)$.

Recall that we have interpreted $h(t)$ as the default intensity process, in the sense that $h(u)du$ gives the instantaneous probability of default under \mathcal{Q} , assuming that default has not yet occurred (see equation (6)). With this, we might be interested in knowing whether $\mathcal{N}(t)$ retains an identical decomposition under \mathcal{P} . That is, in what cases does $h(t)$ give the default intensity under both the physical and the risk-neutral measure? The following proposition establishes the conditions under which this holds.

Proposition 8 *The default indicator process $\mathcal{N}(t)$ has the decomposition given by (55) under both \mathcal{P} and \mathcal{Q} if the pricing kernel $m(t)$ is continuous. In particular, for the pricing kernel given by (50) and (53), $\mathcal{N}(t)$ has an identical decomposition under both measures.*

The proof is in the Appendix.

Note that, even with an identical decomposition of $\mathcal{N}(t)$, the survival probability under \mathcal{P} , which is given by

$$\mathbb{E}_t^{\mathcal{P}} \left[\exp \left(- \int_t^T h(u) du \right) \right], \quad (56)$$

will not be the same as the survival probability under \mathcal{Q} , given in equation (6). The reason is obvious: $h(t)$ does not have the same probabilistic behavior under both measures.

It should also be noted that a choice of $\mathbb{E}_t^{\mathcal{P}} [d\mathcal{Q}/d\mathcal{P}]$ exists which will cause the decomposition of $\mathcal{N}(t)$ to change as we change measures (see the discussion along these lines in Lipster and Shirayev (2001b)). The specification of a tractable pricing kernel (which will need to have jumps according to Proposition 8) which allows for this change in decomposition to occur is an interesting area for future work (see Appendix I in Duffie (2001)).

5.1 Price Of Risk Process

For now, we proceed under the assumption that the pricing kernel is given by (50) and (53). Hence the default intensity $h(t)$ is the same under both measures. Recall that the \mathcal{Q} dynamics of the Y and Z -type factors are given by (2) and (3). Let us now assume that the price of risk process is given by

$$\Lambda(Y) = V(Y)^{-\frac{1}{2}} \left(\lambda_0 + \lambda_Y Y \right), \quad (57)$$

where $\lambda_0 \in \mathbb{R}^{N+M}$ and $\lambda_Y \in R^{(N+M) \times M}$. We require that this price of risk process be chosen in a way that insures that the exponential in equation (53) is indeed a \mathcal{P} martingale. Furthermore, we would like the price of risk process to induce a change of measure which leaves the Y -type factors as the sole determinants of conditional means and volatilities under the physical measure. Let us refer to a price of risk process which satisfies both of these conditions is called as *admissible*. Admissibility of the pricing kernel must be enforced in order to have a well-defined pricing model.

We will see shortly that this second requirement is satisfied by a price of risk process of the form in (57). However, the martingale condition on $\Lambda(Y)$ must be checked on a case by case basis. To see this first point, we simply use the relationship in (54) to rewrite the dynamics of the Y and Z -type factors in terms of $W(t)$. With this we get that

$$dY(t) = K_Y(\Theta - Y(t))dt + \Sigma_Y \sqrt{V(Y(t))} dW(t) \quad (58)$$

and

$$dZ(t) = \mu dt - K_Z Y(t)dt + \Sigma_Z \sqrt{V(Y(t))} dW(t). \quad (59)$$

The above matrixes, without the tildes to indicate that they are under the physical measure, are given by

$$\begin{aligned} K_Y &= \tilde{K}_Y - \Sigma_Y \lambda_Y, \\ \Theta &= K_Y^{-1} \left(\tilde{K}_Y \tilde{\Theta} + \Sigma_Y \lambda_0 \right), \\ K_Z &= \tilde{K}_Z - \Sigma_Z \lambda_Y, \\ \mu &= \tilde{\mu} + \Sigma_Z \lambda_0. \end{aligned}$$

Given factor dynamics under both measures, it is straightforward to show that λ_0 and λ_Y satisfy the following:

$$\lambda_0 = \begin{bmatrix} \Sigma_Y \\ \Sigma_Z \end{bmatrix}^{-1} \begin{pmatrix} K_Y \Theta - \tilde{K}_Y \tilde{\Theta} \\ \mu - \tilde{\mu} \end{pmatrix}, \quad (60)$$

$$\lambda_Y = \begin{bmatrix} \Sigma_Y \\ \Sigma_Z \end{bmatrix}^{-1} \begin{pmatrix} \tilde{K}_Y - K_Y \\ \tilde{K}_Z - K_Z \end{pmatrix}. \quad (61)$$

Hence there exists a unique mapping between factor dynamics and the parameters of the price of risk process in (57).

Note that the price of risk process in (57) is a generalization of the more traditional price of risk process given by

$$\Lambda(Y) = V(Y)^{\frac{1}{2}}l$$

for some $l \in \mathbb{R}^{N+M}$ (see, for example, Dai and Singleton (2001)). The above price of risk process obtains in the present setting when

$$\lambda_0 = \begin{bmatrix} l_1 \alpha_1 \\ \vdots \\ l_{N+M} \alpha_{N+M} \end{bmatrix} \quad \lambda_Y = \begin{bmatrix} l_1 \beta'_1 \\ \vdots \\ l_{N+M} \beta'_{N+M} \end{bmatrix}.$$

This follows from the fact that $V(Y)$ is a diagonal matrix whose n^{th} element is $\alpha_n + \beta'_n Y$.

6 Risk Premia

One of the main goals of financial economics is to understand the behavior of the risk premia of financial securities. With this in mind, we should note that an important advantage of using a model-based approach for working with price data is that such approaches typically produce a risk premium as one of their outputs. Indeed, the present model, once estimated, can produce a time series of risk premia for bonds and stocks which are subject to default risk. The hope is that a disciplined approach for extracting such risk premia may lead to many interesting insights into fundamental economic processes. In this section, we will show how to derive risk premia in the present setting.

6.1 Stocks

We will focus first on computing risk premia for stocks. The computations of risk premia for bonds are almost identical, and will be addressed in section 6.2. From the stock pricing equation in (52), we see that the following quantity must be a \mathcal{P} martingale:

$$\gamma_i(t) = \gamma_i(0) + m(t)\hat{S}(t) + \int_0^t m(u)(1 - \mathcal{N}(u))D_i(u)du + \int_{0+}^t m(u)\tilde{D}_i(u)d\mathcal{N}(u). \quad (62)$$

From equations (50) and (53), an application of Ito's lemma reveals that the dynamics of the pricing kernel $m(t)$ are given by

$$\frac{dm(t)}{m(t)} = -r(t)dt - \Lambda(t)'dW(t). \quad (63)$$

Recall that, from Proposition 3, stock prices in the present model are given by

$$\hat{S}_i(t) = (1 - \mathcal{N}(t))S_i(t),$$

for $S_i(t)$ given by equation (24). The timing convention is that, at the instant of default, the stock is worth zero, and the owner of the stock is entitled to a one time default dividend given by $\tilde{D}_i(t) = (1 - L_i(t))S_i(t)$, and to no future payments. Let us write the dynamics of $S_i(t)$ as follows

$$\frac{dS_i(t)}{S_i(t)} = \mu_i(t)dt + \sigma_i(t)'dW(t). \quad (64)$$

The observation that $\gamma_i(t)$ is a \mathcal{P} -martingale allows us to compute the drift μ_i for the no-default stock price in the above equation. The following proposition states the relevant result:

Proposition 9 *If the $\gamma_i(t)$ of equation (62) satisfies $\gamma_i(t) = \mathbb{E}_t^{\mathcal{P}}[\gamma_i(T)]$ then the rate of return of the no-default stock price $S_i(t)$ satisfies*

$$\mu_i(t) + \delta_i(t) = r(t) + \phi_i(t) + \Lambda(t)'\sigma_i(t), \quad (65)$$

where $\sigma_i(t)$ is an $N + M$ -dimensional vector which is given by

$$\sigma_i(t)' = -\left(B_i'\Sigma_Y + C_i'\Sigma_Z\right)\sqrt{V(t)}. \quad (66)$$

The proof is given in the Appendix.

We see from equation (65) that the the risk premium on the no-default stock price $S_i(t)$ contains two components: the first component, $\Lambda(t)'\sigma_i(t)$, is the price of risk multiplied by the risk exposures of the stock, and the second component, $\phi_i(t)$, is an apparent compensation for the possibility that the stock defaults, with a default dividend given by $(1 - L_i(t))S_i(t)$. However, these two components of the risk premium both occur only in the event that there is no default. The owner of the stock realizes that there is an instantaneous default probability of $h(t)dt$ per unit time, with an accompanying loss of $L_i(t)S_i(t)$. This possibility of default makes the expected return (per unit time) from holding the stock lower than it would be on a path of the economy in which no default occurs.

One way in which the possibility of default may be incorporated into our assessment of expected returns is to look at the expected return of the total returns process for a stock. Total returns processes were discussed at length in Section 4. The dynamics of the total returns process $\hat{s}_i(t)$ for a stock with a dividend specification of $\{\delta_{i,0}, \delta_{i,Y}, C_i, \phi_{i,0}, \phi_{i,Y}\}$ are given in equation (90). If we write the no-default stock dynamics as in (64), we can rewrite the dynamics for the total returns process for stocks as

$$\begin{aligned} \frac{d\hat{s}_i(t)}{\hat{s}_i(t^-)} &= (1 - \mathcal{N}(t))\left(\mu_i(t) + \delta_i(t) - \phi_i(t)\right)dt + \mathcal{N}(t)r(t)dt \\ &\quad + (1 - \mathcal{N}(t))\sigma_i(t)'dW(t) - L_i(t)d\mathcal{M}(t). \end{aligned}$$

Using the results of Proposition 65, we can rewrite this as

$$\begin{aligned} \frac{d\hat{s}_i(t)}{\hat{s}_i(t^-)} &= \left(r(t) + (1 - \mathcal{N}(t))\Lambda(t)'\sigma_i(t)\right)dt \\ &\quad + (1 - \mathcal{N}(t))\sigma_i(t)'dW(t) - L_i(t)d\mathcal{M}(t). \end{aligned} \quad (67)$$

Keep in mind that, by definition, the total returns process for a stock is a portfolio which starts out by holding a single share of the stock, then reinvests all dividends back into the stock itself, and in the event of default invests the remaining portfolio value at the riskless interest rate $r(t)$.

Let us refer to the drift of the total returns process as $\mu_i^{TR}(t)$. From (67), we see that the instantaneous expected return on the total returns process is given by

$$\mu_i^{TR}(t) = r(t) + (1 - \mathcal{N}(t))\Lambda(t)'\sigma_i(t). \quad (68)$$

This indicates that the expected return of the total returns process is equal to $r(t) + \Lambda(t)'\sigma_i(t)$ up until the time of default, and is equal to $r(t)$ thereafter (this not being surprising because post-default the total returns portfolio is fully invested in the riskless rate). Also, using the form for the price of risk process in (57), the forms of λ_0 and λ_Y from (60) and (61) respectively, and the form for $\sigma_i(t)$ from (66), we can write the risk premium $\Lambda(t)'\sigma_i(t)$ for stocks as

$$\begin{aligned} \Lambda(t)'\sigma_i(t) &= -B'_i \left((K_Y\Theta - \tilde{K}_Y\tilde{\Theta}) + (\tilde{K}_Y - K_Y)Y(t) \right) \\ &\quad - C'_i \left((\mu - \tilde{\mu}) + (\tilde{K}_Z - K_Z)Y(t) \right). \end{aligned} \quad (69)$$

We note that the drift of the total return process is analogous to the drift of an ex-dividend stock added to that stock's dividend yield. This reflects the fact that the total returns process reinvests all stock dividends back into the stock. With this, the quantity which is analogous to the drift of the total returns portfolio is $\mu_i(t) + \delta_i(t)$, which from (65), is equal to

$$\mu_i(t) + \delta_i(t) = r(t) + \phi_i(t) + \Lambda(t)'\sigma_i(t).$$

Conditional on no-default as of time t (i.e. $\mathcal{N}(t) = 0$), this drift differs from the drift of the total returns process by the presence of the $\phi(t)$ term. The interpretation of this discrepancy is the following: The drift of the no-default stock price does not take into account the possibility that a default might occur. The drift of the total returns process does account for this possibility, and hence is penalized by $\phi_i(t) = L_i(t)h(t)$ relative to the $\mu_i(t) + \delta_i(t)$. The amount of this penalty is exactly enough to make the expected return for the total returns process equal to the riskless rate plus a risk compensation, which is determined by the price of risk process $\Lambda(t)$ and the risk loadings $\sigma_i(t)$ of the total returns process on the diffusion source of uncertainty.

This last point is important. From (67), we see that the total returns process for stock i loads on two mean zero sources of uncertainty: $W(t)$ and $\mathcal{M}(t)$. However, the expected return of the total returns process only provides compensation for the diffusion risk (via the $\Lambda(t)'\sigma_i(t)$ term in (68)). The loading on the $\mathcal{M}(t)$ term provides no additional direct compensation in the risk premium of the total returns process. We use of the word “direct” because expected returns can depend on the behavior of the $\phi_i(t)$ process through the

volatility $\sigma_i(t)$ of returns.⁷ Hence the behavior of the default process can be reflected in the risk premium, though not directly through a dependence of the loading on the $\mathcal{M}(t)$ term.

Another modeling alternative is to allow the pricing kernel $m(t)$ to have jumps. As has already been pointed out, this would induce a change in the default intensity $h(t)$ as we move from the risk-neutral to the physical measure, and would therefore open up an additional degree of freedom in the specification of the risk premium for total returns processes. A tractable implementation of this approach is an interesting area for future research (see Appendix I in Duffie (2001)).

6.2 Bonds

The above discussion for stocks applies almost without modification to the case of bonds. The main difference between the expected returns on bonds compared to stocks is the absence of the dividend yield term in the bond equations. For bonds, the following quantity should be a \mathcal{P} martingale.

$$\gamma_{s,T}(t) = \gamma_{s,T}(0) + m(t)\hat{P}_{s,T}(t) + \int_{0+}^t m(u)\tilde{D}_{s,T}(u)d\mathcal{N}(u). \quad (70)$$

Recall that, from Proposition 2, bond prices in the present model are given by

$$\hat{P}_{s,T}(t) = (1 - \mathcal{N}(t))P_{s,T}(t),$$

for $P_{s,T}(t)$ given by equation (12). The timing convention is that, at the instant of default, the bond is worth zero, and the owner of the bond is entitled to a one time default dividend given by $\tilde{D}_{s,T}(t) = (1 - L_s(t))P_{s,T}(t)$, and to no future payments. Let us write the dynamics of $P_{s,T}(t)$ as follows

$$\frac{dP_{s,T}(t)}{P_{s,T}(t)} = \mu_{s,T}(t)dt + \sigma_{s,T}(t)'dW(t). \quad (71)$$

The observation that $\gamma_{s,T}(t)$ is a \mathcal{P} -martingale allows us to compute the drift $\mu_{s,T}$ for the no-default bond price in the above equation. The following proposition states the relevant result:

Proposition 10 *If the $\gamma_{s,T}(t)$ of equation (70) satisfies $\gamma_{s,T}(t) = \mathbb{E}_t^{\mathcal{P}}[\gamma_{s,T}(T)]$ then the rate of return of the no-default bond price $P_{s,T}(t)$ satisfies*

$$\mu_{s,T}(t) = r(t) + \phi_s(t) + \Lambda(t)'\sigma_{s,T}(t), \quad (72)$$

where $\sigma_{s,T}(t)$ is an $N + M$ -dimensional vector which is given by

$$\sigma_{s,T}(t)' = -B_{s,T}(t)'\Sigma_Y\sqrt{V(t)}. \quad (73)$$

⁷Recall from (66) that $\sigma_i(t)$ depends on B_i and therefore on $\phi_{i,Y}$ because of the equation (21) which B_i must satisfy.

The proof is almost identical to the proof for Proposition 9, and is not repeated.

The same caveat that applied for stocks applies to the interpretation of the drift of the no-default bond price process. This will be the expected return only on paths of the economy in which default does not occur. To overcome this difficulty with interpretation let us consider a total returns process for bonds, which invests in a single zero-coupon bond, and holds it until maturity. In the event of default, the remaining proceeds from the portfolio are reinvested at the riskfree rate $r(t)$. Let us write $\hat{p}_{s,T}(t)$ for the time t value of this total returns process on bonds. From the dynamics of the total return process for stocks in (90), we see that we can write the dynamics for the total return process for bonds as

$$\frac{d\hat{p}_{s,T}(t)}{\hat{p}_{s,T}(t^-)} = (1 - \mathcal{N}(t)) \frac{dP_{s,T}(t)}{P_{s,T}(t)} - L_s(t)d\mathcal{N}(T) + \mathcal{N}(t)r(t)dt.$$

Using the form of dP/P in (71), we can write this process as

$$\begin{aligned} \frac{d\hat{p}_{s,T}(t)}{\hat{p}_{s,T}(t^-)} &= (1 - \mathcal{N}(t)) \left(\mu_{s,T}(t)dt + \sigma_{s,T}(t)'dW(t) \right) \\ &\quad - L_s(t) \left((1 - \mathcal{N}(t))h(t)dt + d\mathcal{M}(t) \right) + \mathcal{N}(t)r(t)dt \\ &= (1 - \mathcal{N}(t)) \left(\mu_{s,T}(t) - \phi_s(t) \right) dt + \mathcal{N}(t)r(t)dt \\ &\quad + (1 - \mathcal{N}(t))\sigma_{s,T}(t)'dW(t) - L_s(t)d\mathcal{M}(t), \end{aligned}$$

where we recall that $\phi_s(t) \equiv L_s(t)h(t)$. Then applying the results of Proposition 10 to the dynamics of $\hat{p}_{s,T}(t)$, we can write the drift of the bond's total returns process as

$$\mu_{s,T}^{TR}(t) = r(t) + (1 - \mathcal{N}(t))\Lambda(t)'\sigma_{s,T}(t). \quad (74)$$

The interpretation of this risk premium for the total returns process for defaultable bonds is exactly analogous to the stock case of Section 6.1, and so will not be repeated. Also, using the form of the price of risk from (57), the equations for λ_0 and λ_Y from (60) and (61) respectively, and $\sigma_{s,T}(t)$ from (73), we see that we can write the bond risk premium $\Lambda(t)'\sigma_{s,T}(t)$ as

$$\Lambda(t)'\sigma_{s,T}(t) = -B_{s,T}(t)' \left((K_Y\Theta - \tilde{K}_Y\tilde{\Theta}) + (\tilde{K}_Y - K_Y)Y(t) \right). \quad (75)$$

7 An Example Economy

In this section, we give an example of an economy in which risky bonds and a risky stock can be jointly priced. The maintained assumption is that the stock and the bonds are all driven by the same default process $\mathcal{N}(t)$ – in other words, these are all securities issued by a single company. Bonds differ from the stock via their respective dividend processes. Also, bonds may differ from one another because of their maturities, and also because different

levels of seniority imply potentially different recovery rates conditional on the occurrence of a default event. Also the stock's recovery rate, conditional on the occurrence of default, can be specified separately from the recovery rates of the bonds.

7.1 Conditions for Admissibility

Before we proceed with our example economy, it is useful to collect all of the conditions which together guarantee that the economy in question is well posed. These admissibility conditions are the following:

- C1. The factor dynamics in (2) and (3) must be admissible in the sense of Duffie and Kan (1996).
- C2. The solution to the stock B_i coefficient equation in (21) must be chosen in accordance with the discussion in Section 3.2.2.
- C3. The stock transversality condition in (17) must be satisfied.
- C4. The bond integral in (30) must be a \mathcal{Q} martingale.
- C5. The stock integral in (35) must be a \mathcal{Q} martingale.
- C6. The price of risk process in (57) must be admissible, and in particular must satisfy the martingale condition in (53).

As general results regarding whether or not these conditions hold are not available, these conditions must be checked on a case by case basis. Some of these conditions are quite easy to verify, whereas others, such as the martingale conditions in (30) and (35), are more difficult. Fortunately, for sample economy presented below, all of these conditions are rather straightforward.

7.2 Specification of the Economy

We make the following assumptions about the structure of the economy in question:

- A1. There is a single issuer, and hence just one default process $\mathcal{N}(t)$ in the economy.
- A2. There exist N Y -type factors, all of which obey the following Cox, Ingersoll, Ross (1985) (CIR-type) dynamics

$$dY_n(t) = \tilde{K}_{Y_n} \left(\tilde{\Theta}_n - Y_n(t) \right) dt + \sigma_{Y_n} \sqrt{Y_n(t)} d\tilde{W}_n(t). \quad (76)$$

Admissibility of this process requires that the following parameter constraint is satisfied

$$2\tilde{K}_{Y_n} \tilde{\Theta}_n > \sigma_{Y_n}^2. \quad (77)$$

This condition is discussed in Cox, Ingersoll, Ross (1985) and in Duffie and Kan (1996), and guarantees that the Y -type factors remain strictly positive, or $Y_n(t) > 0$. Furthermore, note that $\sigma_{Y_n} \in \mathbb{R}$, implying that innovations to all the Y -type factors are independent.

A3. There exists one Z -type factor, whose dynamics are given by

$$dZ(t) = \tilde{\mu}dt - \tilde{K}_Z Y(t)dt + \sigma_Z d\tilde{W}_m(t), \quad (78)$$

where \tilde{K}_Z is a $1 \times N$ vector. Furthermore, this Brownian motion is assumed to be independent from the Brownian motions driving innovations in the Y -type factors. According to Duffie and Kan (1996), such a process is admissible. Note that we have assumed constant volatility for the random walk component of the stock price. This assumption proves useful in checking the martingale condition in (35).⁸

A4. The shortrate is given by

$$r(t) = r_0 + r'_Y Y(t),$$

where $r_0 \geq 0$ and $r_Y \geq 0$, with at least one strict inequality. Since $Y_n(t) > 0$, this implies that $r(t) > 0$.

A5. The stock's dividend yield is given by

$$\delta_i(t) = \delta_{i,0} + \delta'_{i,Y} Y(t),$$

where $\delta_{i,0} > 0$ and $\delta_{i,Y} \geq 0$. This implies that the dividend yield is strictly positive, and allows us to apply the result of Proposition 1 to insure that the transversality condition in (17) on the terminal dividend is satisfied.

A6. The default intensity process is $(1 - \mathcal{N}(t))h(t)$, where $h(t)$ is given by

$$h(t) = h_0 + h'_Y Y(t),$$

where $h_0 > 0$ and $h_Y \geq 0$. This implies that $(1 - \mathcal{N}(t))h(t) \geq 0$, which must be the case by the assumption that $h(t)$ is a default intensity.

A7. The loss rates for different bond classes are constant, and are given by L_s for $s = 1, \dots, S$. This implies that the $\phi_s(t) = h(t) \times L_s$ processes for bonds are indeed affine, and are given by $\phi_s(t) = \phi_{s,0} + \phi'_{s,Y} Y(t)$, where

$$\phi_{s,0} = h_0 \times L_s,$$

$$\phi_{s,Y} = h_Y \times L_s.$$

Furthermore, we assume that $L_s \leq 1$, implying that default is a limited liability event.

⁸It would obviously be preferable to relax this constant volatility assumption. And in principle there is no problem with doing this. The difficulty arises from the fact that checking the martingale condition in (35), or in (30) if we allow for a richer volatility specification for bonds, is non-trivial. See Duffie, Filipovic, and Schachermayer (2002) for a discussion of these issues.

A8. The stock has a constant loss rate given by L_i . This implies that the $\phi_i(t) = h(t) \times L_i$ process for the stock is affine, and is given by $\phi_i(t) = \phi_{i,0} + \phi'_{i,Y}Y(t)$, where

$$\begin{aligned}\phi_{i,0} &= h_0 \times L_i, \\ \phi_{i,Y} &= h_Y \times L_i.\end{aligned}$$

As for bonds, we assume that $L_i \leq 1$, implying limited liability for the stock in the case of default.

A9. We assume that the price of risk process $\Lambda(t)$ is given by

$$\Lambda(t) = \sqrt{V(t)}l \quad (79)$$

for some constant vector $l \in \mathbb{R}^{N+1}$ (this is the price of risk process used in Dai and Singleton (2001)).

A10. The stock has a dividend process given by $\{\delta_{i,0}, \delta_{i,Y}, C_i, \phi_{i,0}, \phi_{i,Y}\}$, with $C_i \in \mathbb{R}$, since there is only a single Z -type factor.

A11. Default-free bonds exist. These bonds have a default intensity process given by $h(t) = 0$. We will refer to the prices of the default-free bonds as $P_{0,T}(t)$.

7.3 Bond and Stock Prices

With the economy specified, we now turn to checking the admissibility conditions set out in Section 7.1. We first note that condition C1 is satisfied by the parameter restriction in (77). Condition C2 implies that given a dividend specification $\{\delta_{i,0}, \delta_{i,Y}, C_i, \phi_{i,0}, \phi_{i,Y}\}$, the B_i coefficient for the stock must be given by

$$[B_i]_n = \frac{1}{\sigma_{Yn}^2} \left(-\tilde{K}_{Yn} + \sqrt{\tilde{K}_{Yn}^2 + 2\sigma_{Yn}^2 \left(r_{Yn} + [\phi_{i,Y}]_n - [\delta_{i,Y}]_n - [\tilde{K}'_Z C_i]_n \right)} \right), \quad (80)$$

To see that this holds, notice that for $B_{s,T}(t)$ in (81) we have the following

$$\lim_{T \rightarrow \infty} [B_{s,T}(t)]_n = \frac{1}{\sigma_{Yn}^2} \left(-\tilde{K}_{Yn} + \sqrt{\tilde{K}_{Yn}^2 + 2\sigma_{Yn}^2 \left(r_{Yn} + [\phi_{s,Y}]_n \right)} \right).$$

Then (80) follows from the discussion in Section 3.2.2. Also we require that element by element

$$r_Y + \phi_{i,Y} - \delta_{i,Y} - \tilde{K}'_Z C_i > 0,$$

which will insure that $B_i > 0$. The transversality requirement in condition C3 is satisfied by Proposition 1 because the dividend yield $\delta_i(t)$ for the stock is positive and greater than zero.

To see that condition C4 is satisfied, let us first solve for the bond prices in the economy. By Proposition 2, bond prices are given by

$$\hat{P}_{s,T}(t) = (1 - \mathcal{N}(t))P_{s,T}(t),$$

where from equation (12) $P_{s,T}$ is given by $P_{s,T}(t) = \exp(A_{s,T}(t) - B_{s,T}(t)'Y(t))$, where $A_{s,T}(t)$ and $B_{s,T}(t)$ solve (10) and (11) respectively. The solutions to these are given by

$$A_{s,T}(t) = -(r_0 + \phi_{s,0})(T - t) + \sum_{n=1}^N \frac{2\tilde{K}_{Yn}\tilde{\Theta}_n}{\sigma_{Yn}^2} \log \left(\frac{2c e^{\frac{1}{2}(T-t)(c+\tilde{K}_{Yn})}}{(c + \tilde{K}_{Yn})(e^{c(T-t)} - 1) + 2c} \right).$$

and

$$[B_{s,T}(t)]_n = \frac{2(e^{c(T-t)} - 1)(r_{Yn} + [\phi_{s,Y}]_n)}{(e^{c(T-t)} - 1)(\tilde{K}_{Yn} + c) + 2c}, \quad (81)$$

where

$$c = \sqrt{\tilde{K}_{Yn}^2 + 2\sigma_{Yn}^2(r_{Yn} + [\phi_{s,Y}]_n)}.$$

Of course we need to assume that $r_Y + \phi_{i,Y} \geq 0$ element by element. One consequence of this assumption is that bond prices are always bounded between zero and one, i.e. $P_{s,T}(t) \in (0, 1]$. The following proposition verifies that condition C4 holds in the present economy, with no further parameter restrictions.

Proposition 11 *In the economy presented in this section, the integral in (30) is a \mathcal{Q} martingale.*

The proof of this is given in the Appendix.

Default-free bond prices are obtained in exactly the same way as above, except that $\phi_0(t) = 0$ for these bonds.

Turning our attention to the pricing of stocks, we note that the solution B_i to equation (21) is given by (80) above. Furthermore, as has already been pointed out, we assume that parameter values are such that $B_i > 0$. Therefore, according to Proposition 3, in the present setting the price for a stock with a dividend process given by $\{\delta_{i,0}, \delta_{i,Y}, C_i, \phi_{i,0}, \phi_{i,Y}\}$ is

$$\hat{S}_i(t) = (1 - \mathcal{N}(t))S_i(t),$$

where $S_i(t) = \exp(a_i \times t - B_i'Y(t) - C_i'Z(t))$, and where a_i satisfies equation (20) and is given by

$$a_i = r_0 + \phi_{i,0} - \delta_{i,0} + \tilde{\Theta}'\tilde{K}_Y' B_i + \tilde{\mu} C_i - \frac{1}{2}\sigma_Z^2 C_i^2. \quad (82)$$

With this, and with the fact that $Y_n(t) > 0$, we see that the stock price may be bounded as follows

$$0 < S_i(t) < \exp(a_i \times t - C_i'Z(t)).$$

We note from (78) that $Z(t)$ is equal to

$$Z(t) = Z(0) + \tilde{\mu} t - \tilde{K}_Z \int_0^t Y_n(s) ds + \sigma_Z(\tilde{W}_m(t) - \tilde{W}_m(0)).$$

Therefore we see that the stock price may be further bounded as follows

$$\begin{aligned} 0 < S_i(t) &< \exp(a_i \times t - C_i Z(t)) \\ &= \exp\left(a_i \times t - C_i \left[Z(0) + \tilde{\mu} t - \tilde{K}_Z \int_0^t Y(s) ds + \sigma_Z(\tilde{W}_m(t) - \tilde{W}_m(0))\right]\right) \\ &< \exp\left(a_i \times t - C_i \left[Z(0) + \tilde{\mu} t + \sigma_Z(\tilde{W}_m(t) - \tilde{W}_m(0))\right]\right), \end{aligned} \quad (83)$$

where the last inequality follows from the following assumption

$$C_i \tilde{K}_Z < 0$$

element by element. This, with $Y_n(t) > 0$, implies that $C_i \tilde{K}_Z \int_0^t Y(s) ds < 0$, from which the above bound on the stock price follows. With this bound on the stock price, we are now in a position to show that condition C5 holds, i.e. that the stock martingale condition in (35) is indeed satisfied in the present economy, with no further parameter restrictions. The following proposition states the result.

Proposition 12 *In the economy presented in this section, the integral in (35) is a \mathcal{Q} martingale for any finitely lived stock which pays a terminal dividend given by $\bar{D}_i(T)$ at some future time $T > 0$.*

The proof of this is in the Appendix. We note that the requirement that the stock pays a terminal dividend at time T is made only for technical convenience inside the proof, and has no economic content because T can be arbitrarily large, and because the stock price remains the same for any value of T . Indeed, it may be shown that the requirement that we are dealing with a finitely lived stock may be dropped, though at the expense of a sufficient condition which places additional restrictions on the parameters of the model. To avoid this, we simply assume that the stock is finitely lived until some (very large) future time T .

Finally, we need to check that the price of risk process $\Lambda(t)$ given in (79) satisfies condition C6. Given the factor dynamics specified in (76) and (78), this implies that the price of risk process may be written as

$$\lambda_0 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ l_{N+1} \end{bmatrix}, \quad \lambda_Y = \begin{bmatrix} l_1 \beta'_1 \\ \vdots \\ l_N \beta'_N \\ 0 \end{bmatrix}, \text{ and } \Lambda(Y) = \begin{bmatrix} l_1 \sqrt{Y_1} \\ \vdots \\ l_N \sqrt{Y_2} \\ l_{N+1} \end{bmatrix}.$$

In a setting with $l_{N+1} = 0$, it was shown in Cox, Ingersoll, and Ross (1985) that this price of risk process is consistent with no-arbitrage, which implies that it satisfies the martingale

condition in (53). However, if the right-hand side of (53) is a \mathcal{P} martingale for $l_{N+1} = 0$, then independence of the Brownian motions clearly implies that it is also a \mathcal{P} martingale for $l_{N+1} \neq 0$. Hence the price of risk process in equation (79) is consistent with no-arbitrage in the economy. Also the discussion in Section 5.1 implies that for this choice of price of risk process, we have that $K_Y \Theta = \tilde{K}_Y \tilde{\Theta}$, implying that condition (77) is satisfied under both measures.

7.4 Total Returns Processes

Empirical implementation of the above model will often require working with total returns processes rather than directly with the dividend paying stocks. From Proposition 6, we see that the no-default value of the total returns process $s_i(t)$ for the stock is given by

$$s_i(t) = s_i(0) \exp\left((a_i + \delta_{i,0})t - B'_i(Y(t) - Y(0)) - C'_i(z(T) - z(0))\right).$$

Recall that in this case, the z dynamics are given by

$$dZ(t) = \tilde{\mu}dt - \tilde{k}_Z Y(t)dt + \sigma_Z d\tilde{W}_m(t),$$

where $\tilde{k}_Z = \tilde{K}_Z + \frac{1}{C_i} \delta'_{i,Y}$. Recall that a_i and B_i must satisfy (82) and (80) respectively. Let us define a_i^{TR} as follows

$$a_i^{TR} \equiv a_i + \delta_{i,0} = r_0 + \phi_{i,0} + \tilde{\Theta}' \tilde{K}'_Y B_i + \tilde{\mu} C_i - \frac{1}{2} \sigma_Z^2 C_i^2.$$

Also we note that B_i can be rewritten as

$$\begin{aligned} [B_i]_n &= \frac{1}{\sigma_{Y_n}^2} \left(-\tilde{K}_{Y_n} + \sqrt{\tilde{K}_{Y_n}^2 + 2\sigma_{Y_n}^2 \left(r_{Y_n} + [\phi_{i,Y}]_n - [\delta_{i,Y}]_n - [\tilde{K}'_Z C_i]_n \right)} \right) \\ &= \frac{1}{\sigma_{Y_n}^2} \left(-\tilde{K}_{Y_n} + \sqrt{\tilde{K}_{Y_n}^2 + 2\sigma_{Y_n}^2 \left(r_{Y_n} + [\phi_{i,Y}]_n - [\tilde{k}'_Z C_i]_n \right)} \right). \end{aligned}$$

With this, we can write the total returns, conditional on no-default, as follows

$$s_i(t) = s_i(0) \exp\left(a_i^{TR} t - B'_i(Y(t) - Y(0)) - C'_i(z(T) - z(0))\right). \quad (84)$$

This is the form of the total returns process which is useful for estimation. Notice that a_i^{TR} does not have $\delta_{i,0}$ in it. Furthermore, B_i only depends on \tilde{k}_Z , and not on \tilde{K}_Z or on $\delta_{i,Y}$ separately. Hence only the behavior of $z(t)$, and not that of the dividend yield or of $Z(t)$, is needed to estimate the model using total returns data.

8 Conclusion

This paper has developed an affine pricing model for stocks and bonds which are subject to default risk. The principal advantages of the model are its tractability and the richness of the allowable factor dynamics. The key modeling assumptions which have led to the convenient pricing equations in the paper are the adoption of the Duffie and Singleton (1999) default mechanism, and the adoption of the Mamaysky (2002a) dividend process for equities.

The paper suggests many important and interesting directions for future research. An obvious next step would be to perform an empirical study of the pricing of risky debt and equity using the framework of the example in Section 7 (see Mamaysky (2002b) for an empirical implementation of a joint stock-bond pricing model with no default risk). Also, it is important to extend this example to allow for more realistic factor dynamics (such as stochastic volatilities, or time-varying long run factor means). The principal difficulty with doing so is not in deriving the pricing equations, which are already given in this paper, but in checking the regularity conditions C1–C6. In particular, checking the martingale properties in C4 and C5 is nontrivial for richer factors dynamics than the ones used in this paper (see Duffie, Filipovic, and Schachermayer (2002) for example).

A very promising direction for future work is to combine the present model for the valuation of a firm's capital structure with the structural approach of Merton (1974). In particular, how might we reconcile Merton's view of equity as a call option on the firm's assets with the firm value process given in (41)? Which of the models provides a better empirical description of the actual behavior of firms' capital structures?

Another interesting avenue for future work is to solve for prices of derivatives based on elements of the firm's capital structure, such as convertible bonds. The fact that the present model is set in continuous-time, as well as the empirical flexibility of the proposed factor dynamics, make the framework of this paper a very natural setting for the pricing of such derivative instruments.

Extending the pricing kernel in this paper to allow for default intensities to change as we move from the risk-neutral to the physical measure seems important. Also, extending the present model to allow for recovery rates and default intensities to be jointly affine, rather than requiring their product to be affine, would allow for a much richer specification of the default mechanism. Along these lines, the quadratic term structure technology proposed in Ahn, Dittmar, and Gallant (2002) may be advantageously applied.

Appendix

A Proof of Proposition 1 (Transversality Condition)

Assume a dividend process of the form $\{\delta_{i,0}, \delta_{i,Y}, C_i, \phi_{i,0}, \phi_{i,Y}\}$. Let us define $\zeta(t)$ as follows (note that the i subscript is suppressed)

$$\zeta(t) \equiv e^{\int_0^t (\delta(u) + h(u)(1-L(u)))du - \int_0^t r(u)du} (1 - \mathcal{N}(t)) \bar{D}(t).$$

It is easy to check that given the definitions of a and B in (20) and (21) we have that

$$\mathcal{D}_X \bar{D}(t) + \partial \bar{D}(t) / \partial t = \bar{D}(t)(r(t) + \phi(t) - \delta(t)).$$

Using this condition, an application of Ito's lemma for semimartingales reveals that

$$\zeta(T) = \zeta(t) + \mathcal{I}_t(T). \tag{85}$$

where

$$\begin{aligned} \mathcal{I}_t(T) &\equiv \int_t^T e^{\int_0^h (\delta(u) + h(u)(1-L(u)) - r(u))du} (1 - \mathcal{N}(h)) \frac{\partial \bar{D}(X, h)}{\partial X'} \Sigma_X \sqrt{V(u)} d\tilde{W}(u) \\ &\quad - \int_t^T e^{\int_0^h (\delta(u) + h(u)(1-L(u)) - r(u))du} \bar{D}(h) d\mathcal{M}(h). \end{aligned}$$

We have also used the fact that $d\mathcal{N}(t) = (1 - \mathcal{N}(t))h(t)dt + d\mathcal{M}(t)$. We know that $\mathcal{I}_t(T)$ is a local martingale (see Protter (1995)). However, $\zeta(t) \geq 0$ by construction. Hence for all $T \geq t$ we have that $\mathcal{I}_t(T)$ is bounded below by $-\zeta(t)$, known at time t . Therefore, conditional on \mathcal{F}_t , $\mathcal{I}_t(T)$ is a supermartingale (this follows from the fact that a local martingale which is bounded from below is a supermartingale (see Dybvig and Huang (1989))). From the definition of $\zeta(t)$ and (85) we can write that

$$\begin{aligned} e^{-\int_0^T r(u)du + \int_0^T (\delta(u) + h(u)(1-L(u)))du} (1 - \mathcal{N}(T)) \bar{D}(T) &= \zeta(t) + \mathcal{I}_t(T), \\ e^{-\int_0^T r(u)du + \epsilon \times T} (1 - \mathcal{N}(T)) \bar{D}(T) &\leq \zeta(t) + \mathcal{I}_t(T), \end{aligned}$$

where the second line follows from the assumption that $\delta(u) + h(u)(1-L(u)) \geq \epsilon$. Taking expectations of both sides we find that

$$\begin{aligned} e^{\epsilon \times T} \mathbb{E}_t^{\mathcal{Q}} \left[e^{-\int_0^T r(u)du} (1 - \mathcal{N}(T)) \bar{D}(T) \right] &\leq \zeta(t) + \mathbb{E}_t^{\mathcal{Q}}[\mathcal{I}_t(T)], \\ e^{\epsilon \times T} \mathbb{E}_t^{\mathcal{Q}} \left[e^{-\int_0^T r(u)du} (1 - \mathcal{N}(T)) \bar{D}(T) \right] &\leq \zeta(t) + \mathcal{I}_t(t), \end{aligned}$$

where the second line follows from the fact that $\mathcal{I}_t(s)$ with $s \geq t$ is a supermartingale. We can then write that

$$0 < \mathbb{E}_t^{\mathcal{Q}} \left[e^{-\int_0^T r(u)du} (1 - \mathcal{N}(T)) \bar{D}(T) \right] \leq e^{-\epsilon \times T} (\zeta(t) + \mathcal{I}_t(t)).$$

The result follows as $T \rightarrow \infty$.

Q.E.D.

B Proof of Proposition 2 (Bond Pricing)

Part 1.

We first note that the $P_{s,T}$ from (12) will satisfy the following partial differential equation

$$\mathcal{D}_X P_{s,T}(t) + \partial P_{s,T}(t)/\partial t = (r(t) + \phi_s(t))P_{s,T}(t), \quad (86)$$

subject to the boundary condition that $P_{s,T}(T) = 1$. Then, applying Ito's lemma for semi-martingales (see Protter (1995)) to $g_{s,T}(t)$ from (14), we find that

$$\begin{aligned} dg_{s,T}(t) &= e^{-\int_0^t r(u)du} (1 - \mathcal{N}(t)) \left(-r(t)P_{s,T}(t) + \mathcal{D}_X P_{s,T} + \partial P_{s,T}/\partial t \right) dt \\ &\quad + e^{-\int_0^t r(u)du} (1 - \mathcal{N}(t)) \frac{\partial P_{s,T}}{\partial X'} \Sigma_X \sqrt{V(t)} d\tilde{W}(t) \\ &\quad + e^{-\int_0^t r(u)du} \left(\tilde{D}_{s,T}(t) - P_{s,T}(t) \right) d\mathcal{N}(t). \end{aligned}$$

Since we have assume that

$$d\mathcal{N}(T) = (1 - \mathcal{N}(t))h(Y(t), t)dt + d\mathcal{M}(t),$$

we can write

$$\begin{aligned} dg_{s,T}(t) &= e^{-\int_0^t r(u)du} (1 - \mathcal{N}(t)) \times \\ &\quad \left(-r(t)P_{s,T}(t) + \mathcal{D}_X P_{s,T}(t) + \partial P_{s,T}(t)/\partial t + (\tilde{D}_{s,T}(t) - P_{s,T}(t))h(t) \right) dt \\ &\quad + e^{-\int_0^t r(u)du} (1 - \mathcal{N}(t)) \frac{\partial P_{s,T}}{\partial X'} \Sigma_X \sqrt{V(t)} d\tilde{W}(t) \\ &\quad + e^{-\int_0^t r(u)du} \left(\tilde{D}_{s,T}(t) - P_{s,T}(t) \right) d\mathcal{M}(t). \end{aligned} \quad (87)$$

Given the assumed forms of $\tilde{D}_{s,T}(t)$, (86) implies that the drift term in (87) is zero. Hence $g_{s,T}(t)$ is a \mathcal{Q} martingale by virtue of condition (30). We note that

$$g_{s,T}(t) = \mathbb{E}_t^{\mathcal{Q}}[g_{s,T}(t')]$$

implies that

$$e^{-\int_0^t r(s)ds} \hat{P}_{s,T}(t) = \mathbb{E}_t^{\mathcal{Q}} \left[e^{-\int_0^{t'} r(s)ds} \hat{P}_{s,T}(t') + \int_{t^+}^{t'} e^{-\int_0^u r(s)ds} \tilde{D}_{s,T}(u) d\mathcal{N}(u) \right]$$

for $t \leq t'$. Dividing by the exponential on the left-hand side, we find that

$$\hat{P}_{s,T}(t) = \mathbb{E}_t^{\mathcal{Q}} \left[e^{-\int_t^{t'} r(u)du} \hat{P}_{s,T}(t') + \int_{t^+}^{t'} e^{-\int_t^s r(u)du} \tilde{D}_{s,T}(s) d\mathcal{N}(s) \right].$$

This, together with the boundary condition that $\hat{P}_{s,T}(T) = 1 - \mathcal{N}(T)$ imply that $\hat{P}_{s,T}(t)$ does indeed equal the expectation in (28).

Part 2.

To show that the relationship in (31) holds, let us define the following

$$h_{s,T}(t) \equiv \exp \left(- \int_0^t (r(h) + \phi_s(h)) dh \right) P_{s,T}(t).$$

Applying Ito's lemma to $h_{s,T}(t)$ we see that

$$\begin{aligned} dh_{s,T}(t) &= e^{-\int_0^t (r(h) + \phi_s(h)) dh} \left(-(r(t) + \phi_s(t)) P_{s,T} + \partial P_{s,T}(t) / \partial t + \mathcal{D}_X P_{s,T}(t) \right) dt \\ &+ e^{-\int_0^t (r(h) + \phi_s(h)) dh} \frac{\partial P_{s,T}}{\partial X'} \Sigma_X \sqrt{V(t)} d\tilde{W}(t). \end{aligned}$$

Since $P_{s,T}(t)$ satisfies the partial differential equation in (86), we see that the drift of $h_{s,T}(t)$ is zero. Furthermore, if we assume that the following is a \mathcal{Q} martingale

$$\int_0^t e^{-\int_0^u (r(h) + \phi_s(h)) dh} \frac{\partial P_{s,T}(u)}{\partial X'} \Sigma_X \sqrt{V(u)} d\tilde{W}(u),$$

then we see that $h_{s,T}(t) = \mathbb{E}_t^{\mathcal{Q}}[h_{s,T}(t')]$ for $t' > t$. From this and from the definition of $h_{s,T}(t)$ we see that

$$P_{s,T}(t) = \mathbb{E}_t^{\mathcal{Q}} \left[\exp \left(- \int_t^{t'} (r(h) + \phi_s(h)) dh \right) P_{s,T}(t') \right].$$

The result in (31) follows from imposing the boundary condition that $P_{s,T}(T) = 1$.

Q.E.D.

C Proof of Proposition 3 (Stock Pricing)

Part 1.

We first note that the $S_{i,T}(t)$ from equation (24) satisfies the following partial differential equation

$$\mathcal{D}_X S_{i,T}(t) + \partial S_{i,T}(t) / \partial t = (r(t) + \phi_i(t)) S_{i,T}(t) - \delta_i(t) S_{i,T}(t), \quad (88)$$

with the boundary condition that $S_{i,T}(T) = \bar{D}_i(T)$. Then, applying Ito's lemma for semi-martingales (see Protter (1995)) to $g_{i,T}(t)$ from (26), we find that

$$\begin{aligned} dg_{i,T}(t) &= e^{-\int_0^t r(u) du} (1 - \mathcal{N}(t)) \left(-r(t) S_{i,T}(t) + D_i(t) + \mathcal{D}_X S_{i,T} + \partial S_{i,T} / \partial t \right) dt \\ &+ e^{-\int_0^t r(u) du} (1 - \mathcal{N}(t)) \frac{\partial S_{i,T}}{\partial X'} \Sigma_X \sqrt{V(t)} d\tilde{W}(t) \\ &+ e^{-\int_0^t r(u) du} \left(\tilde{D}_i(t) - S_{i,T}(t) \right) d\mathcal{N}(t). \end{aligned}$$

Since we have assume that

$$d\mathcal{N}(T) = (1 - \mathcal{N}(t)) h(Y(t), t) dt + d\mathcal{M}(t),$$

we can write

$$\begin{aligned}
dg_{i,T}(t) &= e^{-\int_0^t r(u)du} (1 - \mathcal{N}(t)) \times \\
&\quad \left(-r(t)S_{i,T}(t) + D_i(t) + \mathcal{D}_X S_{i,T} + \partial S_{i,T}/\partial t + (\tilde{D}_i(t) - S_{i,T}(t))h(t) \right) dt \\
&\quad + e^{-\int_0^t r(u)du} (1 - \mathcal{N}(t)) \frac{\partial S_{i,T}}{\partial X'} \Sigma_X \sqrt{V(t)} d\tilde{W}(t) \\
&\quad + e^{-\int_0^t r(u)du} \left(\tilde{D}_i(t) - S_{i,T}(t) \right) d\mathcal{M}(t).
\end{aligned} \tag{89}$$

Given that we have assumed that $D_i(t) = \delta_i(t)\bar{D}_i(t)$ and that $\tilde{D}_i(t) = (1 - L_i(t))\bar{D}_i(t)$, (88) implies that the drift term in (89) is zero. Hence $g_{i,T}(t)$ is a \mathcal{Q} martingale by virtue of condition (35). It is easy to check that

$$g_{i,T}(t) = \mathbb{E}_t^{\mathcal{Q}}[g_{i,T}(t')]$$

implies that

$$\begin{aligned}
e^{-\int_0^t r(s)ds} \hat{S}_{i,T}(t) &= \mathbb{E}_t^{\mathcal{Q}} \left[\int_t^{t'} (1 - \mathcal{N}(u)) e^{-\int_0^u r(s)ds} D_i(u) du + e^{-\int_0^{t'} r(s)ds} \hat{S}_{i,T}(t') \right. \\
&\quad \left. + \int_{t^+}^{t'} e^{-\int_0^u r(s)ds} \tilde{D}_i(u) d\mathcal{N}(u) \right].
\end{aligned}$$

for all $t < t'$. Dividing by the exponential on the left-hand side, we find that

$$\begin{aligned}
\hat{S}_{i,T}(t) &= \mathbb{E}_t^{\mathcal{Q}} \left[\int_t^{t'} (1 - \mathcal{N}(u)) e^{-\int_t^u r(s)ds} D_i(u) du + e^{-\int_t^{t'} r(s)ds} \hat{S}_{i,T}(t') \right. \\
&\quad \left. + \int_{t^+}^{t'} e^{-\int_t^u r(s)ds} \tilde{D}_i(u) d\mathcal{N}(u) \right],
\end{aligned}$$

This, together with the boundary condition that $\hat{S}_{i,T}(T) = (1 - \mathcal{N}(T))\bar{D}_i(T)$ imply that $\hat{S}_{i,T}(t)$ does indeed equal the expectation in (33).

Part 2.

To show that the relationship in (36), let us define the following

$$h_i(t) = \int_0^t e^{-\int_0^u (r(h) + \phi_i(h))dh} D_i(u) du + e^{-\int_0^t (r(h) + \phi_i(h))dh} S_i(t).$$

An application of Ito's lemma to $h_i(t)$ reveals that

$$\begin{aligned}
dh_i(t) &= e^{-\int_0^t (r(h) + \phi_i(h))dh} \left(-(r(t) + \phi_i(t))S_i(t) + D_i(t) + \frac{\partial S_i(t)}{\partial t} + \mathcal{D}_X S_i(t) \right) dt \\
&\quad + e^{-\int_0^t (r(h) + \phi_i(h))dh} \frac{\partial S_i(t)}{\partial X'} \Sigma_X \sqrt{V(t)} d\tilde{W}(t).
\end{aligned}$$

We note that the no-default stock price $S_i(t)$ satisfies equation (88). This, together with the fact that $D_i(t) = \delta_i(t)S_i(t)$, implies that the drift of $h_i(t)$ is zero. If we also have that the following integral is a \mathcal{Q} martingale

$$\int_0^t e^{-\int_0^u (r(h) + \phi_i(h))dh} \frac{\partial S_i(u)}{\partial X'} \Sigma_X \sqrt{V(u)} d\tilde{W}(u),$$

then we see that $h_i(t) = \mathbb{E}_t^{\mathcal{Q}}[h_i(t')]$. With this, we see that

$$e^{-\int_0^t (r(h) + \phi_i(h))dh} S_i(t) = \mathbb{E}_t^{\mathcal{Q}} \left[\int_t^{t'} e^{-\int_0^u (r(h) + \phi_i(h))dh} D_i(u) du + e^{-\int_0^{t'} (r(h) + \phi_i(h))dh} S_i(t') \right],$$

which implies that

$$S_i(t) = \mathbb{E}_t^{\mathcal{Q}} \left[\int_t^{t'} e^{-\int_t^u (r(h) + \phi_i(h))dh} D_i(u) du + e^{-\int_t^{t'} (r(h) + \phi_i(h))dh} S_i(t') \right],$$

from which (36) follows once we impose the boundary condition that $S_i(T) = \bar{D}_i(T)$.

Q.E.D.

D Proof of Proposition 5 (Total Returns)

Let us divide both sides of (42) by $\hat{s}_i(t^-)$ to find that

$$\begin{aligned} \frac{d\hat{s}_i(t)}{\hat{s}_i(t^-)} &= (\hat{S}_i(t^-))^{\Delta} \left((1 - \mathcal{N}(t)) \left(dS_i(t) + D_i(t) \right) dt + (\tilde{D}_i(t) - S_i(t)) d\mathcal{N}(t) \right) \\ &\quad + \left(1 - (\hat{S}_i(t))^{\Delta} \hat{S}_i(t) (1 - \mathcal{N}(t)) \right) r(t) dt \\ &= (\hat{S}_i(t^-))^{\Delta} S_i(t) \left[(1 - \mathcal{N}(t)) \left(\frac{dS_i(t)}{S_i(t)} + \delta_i(t) \right) dt + \left(\frac{\tilde{D}_i(t)}{S_i(t)} - 1 \right) d\mathcal{N}(t) \right] \\ &\quad + \left(1 - (\hat{S}_i(t))^{\Delta} \hat{S}_i(t) \right) r(t) dt \\ &= \left(1 - \mathcal{N}(t) \right) \left(\frac{dS_i(t)}{S_i(t)} + \delta_i(t) dt \right) - L_i(t) d\mathcal{N}(t) + \mathcal{N}(t) r(t) dt, \end{aligned} \tag{90}$$

where we have used the facts that $D_i(t) = \delta_i(t) S_i(t)$, that $\tilde{D}_i(t) = (1 - L_i(t)) S_i(t)$, and that $(\hat{S}_i(t))^{\Delta} S_i(t) = (\hat{S}_i(t))^{\Delta} \hat{S}_i(t) = (1 - \mathcal{N}(t))$.

With this, let us apply Ito's lemma for semimartingales to $\log(\hat{s}_i(t))$ to obtain that

$$\begin{aligned} d \log(\hat{s}_i(t)) &= \frac{1}{\hat{s}_i(t^-)} d\hat{s}_i(t) - \frac{1}{2} \frac{1}{\hat{s}_i(t^-)^2} d[\hat{s}_i(t)]^c \\ &\quad + \left(\log \hat{s}_i(t) - \log \hat{s}_i(t^-) - \frac{1}{\hat{s}_i(t^-)} (\hat{s}_i(t) - \hat{s}_i(t^-)) \right). \end{aligned}$$

Here $[\cdot]^c$ refers to the continuous part of the quadratic variation process for a semimartingale (see Protter (1995)). Notice that two of the above terms may be written as

$$\begin{aligned}\hat{s}_i(t) - \hat{s}_i(t^-) &= -L_i(t)\hat{s}_i(t^-)d\mathcal{N}(t), \\ \log \hat{s}_i(t) - \log \hat{s}_i(t^-) &= \log \left(\frac{\hat{s}_i(t)}{\hat{s}_i(t^-)} \right) d\mathcal{N}(t) = \log(1 - L_i(t))d\mathcal{N}(t).\end{aligned}$$

With these we can write the dynamics of the log portfolio value as

$$\begin{aligned}d \log(\hat{s}_i(t)) &= (1 - \mathcal{N}(t)) \left(\frac{dS_i(t)}{S_i(t)} + \delta_i(t)dt - \frac{1}{2} \frac{d[S_i(t)]}{S_i(t)^2} \right) + \mathcal{N}(t)r(t)dt \\ &\quad + \log(1 - L_i(t))d\mathcal{N}(t).\end{aligned}\tag{91}$$

Integrating we find that

$$\begin{aligned}\log(\hat{s}_i(T)) &= \log(\hat{s}_i(t)) + \int_t^T (1 - \mathcal{N}(s)) \left(\frac{dS_i(s)}{S_i(s)} + \delta_i(s)ds - \frac{1}{2} \frac{d[S_i(s)]}{S_i(s)^2} \right) \\ &\quad + \int_t^T \mathcal{N}(s)r(s)ds + \int_{t^+}^T \log(1 - L_i(s))d\mathcal{N}(s).\end{aligned}\tag{92}$$

From this we see that the total returns portfolio value can be written as

$$\begin{aligned}\hat{s}_i(T) &= \hat{s}_i(t) \exp \left(\int_t^T (1 - \mathcal{N}(s)) \left(\frac{dS_i(s)}{S_i(s)} + \delta_i(s)ds - \frac{1}{2} \frac{d[S_i(s)]}{S_i(s)^2} \right) + \int_t^T \mathcal{N}(s)r(s)ds \right) \\ &\quad \times \exp \left(\int_{t^+}^T \log(1 - L_i(s))d\mathcal{N}(s) \right).\end{aligned}\tag{93}$$

Note that because the integration starts at t^+ , the occurrence of a default event at time t is reflected in the value of $\hat{s}_i(t)$. The exponential containing the jump component is equal to the following

$$\begin{aligned}\exp \left(\int_{t^+}^T \log(1 - L_i(s))d\mathcal{N}(s) \right) &= \exp \left((\mathcal{N}(T) - \mathcal{N}(t)) \log(1 - L_i(\mathcal{T})) \right) \\ &= 1 - (\mathcal{N}(T) - \mathcal{N}(t))L_i(\mathcal{T}).\end{aligned}$$

where \mathcal{T} is the stopping time which indicates default. Let us write $s_i(t)$ for the value of the total returns process, assuming that default has not occurred as of time t . Then we have that

$$s_i(T) = s_i(t) \exp \left(\int_t^T \frac{dS_i(t)}{S_i(t)} + \delta_i(t)dt - \frac{1}{2} \frac{d[S_i(t)]}{S_i(t)^2} \right).$$

Using (93) we can rewrite $\hat{s}_i(T)$ as follows

$$\hat{s}_i(t) = (1 - \mathcal{N}(t))s_i(t) + \mathcal{N}(t)(1 - L_i(\mathcal{T}))s_i(\mathcal{T}) \exp \left(\int_{\mathcal{T}}^t r(s)ds \right).\tag{94}$$

Q.E.D.

E Proof of Proposition 7 (The Pricing Kernel)

It is well known that, with dividends paid at known times or with a continuous dividend stream, pricing under the risk-neutral measure implies the existence of a pricing kernel which allows for pricing to be done under the physical measure (see Duffie (2001) for example). For example, the time t price of a no-default zero coupon bond maturing at time $T > t$ is given by the following expectation under \mathcal{Q}

$$\mathbb{E}_t^{\mathcal{Q}} \left[\exp \left(- \int_t^T r(u) du \right) \right].$$

Under \mathcal{P} , for the pricing kernel $m(t)$ given in (50), we have that the zero's price is given by

$$\frac{1}{m(t)} \mathbb{E}_t^{\mathcal{P}} [m(T)].$$

Here we show that the same result applies to dividends which are paid at random times. Let us define

$$\zeta(t) \equiv \mathbb{E}_t^{\mathcal{P}} \left[\frac{d\mathcal{Q}}{d\mathcal{P}} \right].$$

It is then easy to show that (see Duffie (2001), for example) for some $x(T)$, measurable with respect to \mathcal{F}_T , the following holds

$$\mathbb{E}_t^{\mathcal{P}} [x(T)\zeta(T)] = \zeta(t) \mathbb{E}_t^{\mathcal{Q}} [x(T)]. \quad (95)$$

Before proceeding with the proof of Proposition 7, we first need to show the following result about the default dividend paid by a security at its (random) default time:

Lemma E.1 *For the default dividend $\tilde{D}(t)$, paid at the default time \mathcal{T} , we have that*

$$\mathbb{E}_t^{\mathcal{Q}} \left[\int_{t^+}^T e^{-\int_0^s r(u) du} \tilde{D}(s) d\mathcal{N}(s) \right] = \frac{1}{\zeta(t)} \mathbb{E}_t^{\mathcal{P}} \left[\int_{t^+}^T m(s) \tilde{D}(s) d\mathcal{N}(s) \right], \quad (96)$$

where $m(t)$ is the pricing kernel, and is given by

$$m(t) = \mathbb{E}_t^{\mathcal{P}} \left[\frac{d\mathcal{Q}}{d\mathcal{P}} \right] e^{-\int_0^t r(u) du},$$

as long as the following conditions are satisfied

$$\mathbb{E}_t^{\mathcal{Q}} \left[\sup_{s \in [t, T]} \left(e^{-\int_0^s r(u) du} |\tilde{D}(s)| \right) \right] < \infty, \quad (97)$$

$$\mathbb{E}_t^{\mathcal{P}} \left[\sup_{s \in [t, T]} \left(m(s) |\tilde{D}(s)| \right) \right] < \infty. \quad (98)$$

Before proceeding with the proof, let us note that it is somewhat surprising that this Lemma holds at all. The key observation is that stochastic integration with respect to $\mathcal{N}(t)$ is identical to the Stieltjes integral with respect to $\mathcal{N}(t)$, computed path by path (see Theorem 17 in Chapter II of Protter (1995)). This observation allows us to treat the integrals in (96) as limits of sums. The proof proceeds from this observation.

Proof. An application of (95) allows to start the calculations:

$$\begin{aligned}
& \mathbb{E}_t^{\mathcal{Q}} \left[\int_{t^+}^T e^{-\int_0^s r(u)du} \tilde{D}(s) d\mathcal{N}(s) \right] \\
&= \frac{1}{\zeta(t)} \mathbb{E}_t^{\mathcal{P}} \left[\zeta(T) \left(\int_{t^+}^T e^{-\int_0^s r(u)du} \tilde{D}(s) d\mathcal{N}(s) \right) \right] \\
&= \frac{1}{\zeta(t)} \mathbb{E}_t^{\mathcal{P}} \left[\int_{t^+}^T \zeta(T) e^{-\int_0^s r(u)du} \tilde{D}(s) d\mathcal{N}(s) \right] \\
&= \frac{1}{\zeta(t)} \mathbb{E}_t^{\mathcal{P}} \left[\lim_{i \rightarrow \infty} \sum_{\{\tau_{n-1}, \tau_n\} \in \Pi_i} \zeta(T) e^{-\int_0^{\tau_n} r(u)du} \tilde{D}(\tau_n) (\mathcal{N}(\tau_n) - \mathcal{N}(\tau_{n-1})) \right] \\
&= \frac{1}{\zeta(t)} \lim_{i \rightarrow \infty} \mathbb{E}_t^{\mathcal{P}} \left[\sum_{\{\tau_{n-1}, \tau_n\} \in \Pi_i} \zeta(T) e^{-\int_0^{\tau_n} r(u)du} \tilde{D}(\tau_n) (\mathcal{N}(\tau_n) - \mathcal{N}(\tau_{n-1})) \right] \\
&= \frac{1}{\zeta(t)} \lim_{i \rightarrow \infty} \sum_{\{\tau_{n-1}, \tau_n\} \in \Pi_i} \mathbb{E}_t^{\mathcal{P}} \left[\zeta(T) e^{-\int_0^{\tau_n} r(u)du} \tilde{D}(\tau_n) (\mathcal{N}(\tau_n) - \mathcal{N}(\tau_{n-1})) \right].
\end{aligned}$$

Here Π_i is the i^{th} partition of $[t, T]$, containing N_i points given by $t = \tau_1 < \tau_2 < \dots < \tau_{N_i} = T$. The third equality above follows from the definition of a Stieltjes integral with a continuous integrand. The fourth equality requires justification. First note that we have the following for all i

$$\sum_{\{\tau_{n-1}, \tau_n\} \in \Pi_i} \zeta(T) e^{-\int_0^{\tau_n} r(u)du} |\tilde{D}(\tau_n)| (\mathcal{N}(\tau_n) - \mathcal{N}(\tau_{n-1})) \leq \zeta(T) \sup_{s \in [t, T]} \left(e^{-\int_0^s r(u)du} |\tilde{D}(s)| \right).$$

If we can show that

$$\mathbb{E}_t^{\mathcal{P}} \left[\zeta(T) \sup_{s \in [t, T]} \left(e^{-\int_0^s r(u)du} |\tilde{D}(s)| \right) \right] < \infty$$

then we can apply Lebesgue's Dominated Convergence Theorem to justify the move of the expectation through the limit. But notice that by definition of $\zeta(t)$ we have

$$\mathbb{E}_t^{\mathcal{P}} \left[\zeta(T) \sup_{s \in [t, T]} \left(e^{-\int_0^s r(u)du} |\tilde{D}(s)| \right) \right] = \zeta(t) \mathbb{E}_t^{\mathcal{Q}} \left[\sup_{s \in [t, T]} \left(e^{-\int_0^s r(u)du} |\tilde{D}(s)| \right) \right].$$

This is finite by assumption (97). Hence the fourth step is justified. The final equality follows from linearity of the expectations operator. We now note the following holds by repeated

application of iterated expectations

$$\begin{aligned}
& \mathbb{E}_t^{\mathcal{P}} \left[\zeta(T) e^{-\int_0^{\tau_n} r(u) du} \tilde{D}(\tau_n) \left(\mathcal{N}(\tau_n) - \mathcal{N}(\tau_{n-1}) \right) \right] \\
&= \mathbb{E}_t^{\mathcal{P}} \left[\mathbb{E}_{\tau_n}^{\mathcal{P}} \left[\zeta(T) e^{-\int_0^{\tau_n} r(u) du} \tilde{D}(\tau_n) \left(\mathcal{N}(\tau_n) - \mathcal{N}(\tau_{n-1}) \right) \right] \right] \\
&= \mathbb{E}_t^{\mathcal{P}} \left[\mathbb{E}_{\tau_n}^{\mathcal{P}} [\zeta(T)] e^{-\int_0^{\tau_n} r(u) du} \tilde{D}(\tau_n) \left(\mathcal{N}(\tau_n) - \mathcal{N}(\tau_{n-1}) \right) \right] \\
&= \mathbb{E}_t^{\mathcal{P}} \left[\mathbb{E}_{\tau_n}^{\mathcal{P}} \left[\frac{d\mathcal{Q}}{d\mathcal{P}} \right] e^{-\int_0^{\tau_n} r(u) du} \tilde{D}(\tau_n) \left(\mathcal{N}(\tau_n) - \mathcal{N}(\tau_{n-1}) \right) \right] \\
&= \mathbb{E}_t^{\mathcal{P}} \left[m(\tau_n) \tilde{D}(\tau_n) \left(\mathcal{N}(\tau_n) - \mathcal{N}(\tau_{n-1}) \right) \right], \tag{99}
\end{aligned}$$

where the last step follows by definition of $m(t)$. Now returning to the summations we have that

$$\begin{aligned}
& \frac{1}{\zeta(t)} \lim_{i \rightarrow \infty} \sum_{\{\tau_{n-1}, \tau_n\} \in \Pi_i} \mathbb{E}_t^{\mathcal{P}} \left[\zeta(T) e^{-\int_0^{\tau_n} r(u) du} \tilde{D}(\tau_n) \left(\mathcal{N}(\tau_n) - \mathcal{N}(\tau_{n-1}) \right) \right] \\
&= \frac{1}{\zeta(t)} \lim_{i \rightarrow \infty} \sum_{\{\tau_{n-1}, \tau_n\} \in \Pi_i} \mathbb{E}_t^{\mathcal{P}} \left[m(\tau_n) \tilde{D}(\tau_n) \left(\mathcal{N}(\tau_n) - \mathcal{N}(\tau_{n-1}) \right) \right] \\
&= \frac{1}{\zeta(t)} \lim_{i \rightarrow \infty} \mathbb{E}_t^{\mathcal{P}} \left[\sum_{\{\tau_{n-1}, \tau_n\} \in \Pi_i} m(\tau_n) \tilde{D}(\tau_n) \left(\mathcal{N}(\tau_n) - \mathcal{N}(\tau_{n-1}) \right) \right] \\
&= \frac{1}{\zeta(t)} \mathbb{E}_t^{\mathcal{P}} \left[\lim_{i \rightarrow \infty} \sum_{\{\tau_{n-1}, \tau_n\} \in \Pi_i} m(\tau_n) \tilde{D}(\tau_n) \left(\mathcal{N}(\tau_n) - \mathcal{N}(\tau_{n-1}) \right) \right] \\
&= \frac{1}{\zeta(t)} \mathbb{E}_t^{\mathcal{P}} \left[\int_{t^+}^T m(s) \tilde{D}(s) d\mathcal{N}(s) \right],
\end{aligned}$$

where the first equality follows from (99) and the second step follows from linearity of expectations. The third step follows from the Lebesgue Dominated Convergence Theorem, which can be applied once we notice that for all i

$$\sum_{\{\tau_{n-1}, \tau_n\} \in \Pi_i} m(\tau_n) |\tilde{D}(\tau_n)| \left(\mathcal{N}(\tau_n) - \mathcal{N}(\tau_{n-1}) \right) \leq \sup_{s \in [t, T]} (m(s) |\tilde{D}(s)|),$$

which is integrable by virtue of assumption (98). The final step follows from the definition of a Stieltjes integral with continuous integrands.

Q.E.D.

It should be pointed out that checking that the regularity conditions (97) and (98) do indeed hold may be non-trivial. It seems that condition (97) is fairly straightforward in many cases. For example, it is easy to check that it holds in the example economy given in

Section 7. However, condition (98) is more difficult as it relies on pathwise properties of the price of risk process. Showing that this condition holds for price of risk processes typically used in the literature is an important technical question for future research.

We now are ready to prove Proposition 7. We will only show the result in (52) for stocks. The result for bonds in (51) follows from almost identical arguments.

Proof of Proposition 7. Since we have assumed that discounted gains processes for stocks are \mathcal{Q} martingales, we have the following representation for stock prices (from equation (33) in the text):

$$\begin{aligned}\hat{S}_{i,T}(t) &= \mathbb{E}_t^{\mathcal{Q}} \left[\int_t^T (1 - \mathcal{N}(u)) e^{-\int_t^u r(s)ds} D_i(u) du + (1 - \mathcal{N}(T)) e^{-\int_t^T r(s)ds} \bar{D}_i(T) \right. \\ &\quad \left. + \int_{t+}^T e^{-\int_t^u r(s)ds} \tilde{D}_i(u) d\mathcal{N}(u) \right].\end{aligned}$$

We first multiply both sides by $\exp(-\int_0^t r(s)ds)$ and then apply (96) from Lemma E.1 to find

$$\begin{aligned}e^{-\int_0^t r(s)ds} \hat{S}_{i,T}(t) &= \mathbb{E}_t^{\mathcal{Q}} \left[\int_0^T (1 - \mathcal{N}(u)) e^{-\int_0^u r(s)ds} D_i(u) du + (1 - \mathcal{N}(T)) e^{-\int_0^T r(s)ds} \bar{D}_i(T) \right] \\ &\quad + \frac{1}{\zeta(t)} \mathbb{E}_t^{\mathcal{P}} \left[\int_{t+}^T m(u) \tilde{D}_i(u) d\mathcal{N}(u) \right].\end{aligned}$$

We now apply (95) to the $\mathbb{E}^{\mathcal{Q}}$ term above to find that

$$\begin{aligned}e^{-\int_0^t r(s)ds} \hat{S}_{i,T}(t) &= \frac{1}{\zeta(t)} \mathbb{E}_t^{\mathcal{P}} \left[\zeta(T) \int_0^T (1 - \mathcal{N}(u)) e^{-\int_t^u r(s)ds} D_i(u) du \right] \\ &\quad + \frac{1}{\zeta(t)} \mathbb{E}_t^{\mathcal{P}} \left[\zeta(T) (1 - \mathcal{N}(T)) e^{-\int_0^T r(s)ds} \bar{D}_i(T) \right] \\ &\quad + \frac{1}{\zeta(t)} \mathbb{E}_t^{\mathcal{P}} \left[\int_{t+}^T m(u) \tilde{D}_i(u) d\mathcal{N}(u) \right].\end{aligned}$$

We now multiply both sides by $\zeta(t)$, and then from the definition of $m(t)$ in (50), and from standard results (as in Duffie (2001)), we write

$$\begin{aligned}m(t) \hat{S}_{i,T}(t) &= \mathbb{E}_t^{\mathcal{P}} \left[\int_t^T (1 - \mathcal{N}(u)) m(u) D_i(u) du + (1 - \mathcal{N}(T)) m(T) \bar{D}_i(T) \right. \\ &\quad \left. + \int_{t+}^T m(u) \tilde{D}_i(u) d\mathcal{N}(u) \right].\end{aligned}$$

This is identical to (52) because $\hat{S}_i(T) = (1 - \mathcal{N}(T)) S_i(T)$ and $S_i(t) = \bar{D}_i(t)$ for all t .

Q.E.D.

F Proof of Proposition 8 (Default Intensity)

Let us define $\zeta(t) \equiv \mathbb{E}_t^{\mathcal{P}}[d\mathcal{Q}/d\mathcal{P}]$. Then from (50), we have that $m(t) = \zeta(t) \exp(-\int_0^t r(u)du)$. Hence $m(t)$ is continuous if and only if $\zeta(t)$ is continuous. Also let $\mathcal{N}(t)$ be the default indicator process given in (5). Let us assume that $\mathcal{N}(t)$ has a decomposition given by $\mathcal{N}(t) = A(t) + \mathcal{M}(t)$, where $\mathcal{M}(t)$ is a \mathcal{P} martingale and $A(t)$ is a finite variation process with $A(0) = 0$. Then using Girsanov's Theorem for semimartingales (see Protter (1995), Theorem 20 of Chapter III) we can write $\mathcal{N}(t)$ as follows

$$\mathcal{N}(t) = C(t) + L(t),$$

where $L(t)$ is a \mathcal{Q} local martingale given by

$$L(t) = \mathcal{M}(t) - \int_0^t \frac{1}{\zeta(s)} d[\zeta(s), \mathcal{M}(s)],$$

where $[\cdot, \cdot]$ indicates the quadratic covariation process, and where

$$C(t) = \mathcal{N}(t) - L(t)$$

is a finite variation process. Because $\mathcal{M}(t)$ only has a drift and a jump component, if $\zeta(s)$ is continuous, then $[\zeta(s), \mathcal{M}(s)] = \zeta(0)\mathcal{M}(0)$ (see Protter (1995) Theorem 28 in Chapter II). Hence $d[\zeta(s), \mathcal{M}(s)] = 0$, and we will have that $L(t) = \mathcal{M}(t)$ and $C(t) = A(t)$ for all t .

Q.E.D.

G Proof of Proposition 9 (Expected Returns)

Recall that $d\mathcal{N}(t) = (1 - \mathcal{N}(t))h(t)dt + d\mathcal{M}(t)$. Applying Ito's lemma for semimartingales to $\gamma_i(t)$ from (62) we have that

$$\begin{aligned} d\gamma_i(t) &= (1 - \mathcal{N}(t)) \left(S_i(t)dm(t) + m(t)dS_i(t) + d[m(t), S_i(t)] + m(t)D_i(t) \right) \\ &\quad + m(t) \left(\tilde{D}_i(t) - S_i(t) \right) d\mathcal{N}(t). \end{aligned}$$

By virtue of Proposition 3, we have that

$$\begin{aligned} D_i(t) &= \delta_i(t)S_i(t), \\ \tilde{D}_i(t) &= (1 - L_i(t))S_i(t). \end{aligned}$$

Using these and the assumed dynamics of $m(t)$ and $S_i(t)$ from (63) and (64), we find that following

$$\begin{aligned} d\gamma_i(t) &= \frac{1 - \mathcal{N}(t)}{m(t)S_i(t)} \left(-r(t) + \mu_i(t) - \sigma_i(t)' \Lambda(t) + \delta_i(t) - \phi_i(t) \right) \\ &\quad + \frac{1 - \mathcal{N}(t)}{m(t)S_i(t)} \left(\sigma_i(t) - \Lambda(t) \right)' dW(t) - m(t)S_i(t)L_i(t)d\mathcal{M}(t). \end{aligned}$$

Since martingales must have zero drifts, and since $\gamma_i(t)$ is a \mathcal{P} martingale by assumption, the condition in (65) follows by setting the drift above to zero. Equation (66) follows by application of Ito's lemma to $S_i(t)$ given in (24).

Q.E.D.

H Proof of Proposition 11 (Check Condition C4)

In order to verify condition C4, we will use the following result. For a standard $N + M$ -dimensional Brownian motion $\tilde{W}(t)$ under \mathcal{Q} , and a $N + M$ -dimensional adapted process $x(t)$, the following integral

$$\int_0^t x(s)' d\tilde{W}(s)$$

is a \mathcal{Q} martingale if the following condition is satisfied

$$\mathbb{E}_0^{\mathcal{Q}} \left[\int_0^t x(s)' x(s) ds \right] < \infty \quad (100)$$

for all t (see, for example, Lipster and Shirayev (2001a)). This condition implies that the Brownian integral in (30) is a \mathcal{Q} martingale because the zero-coupon bond price is between zero and one, because $Y_n(t)$ has finite expectations, and because $r(t) > 0$. We still need to check that the $\mathcal{M}(t)$ integral in (30) is a \mathcal{Q} martingale. Lipster and Shirayev (2001b) give sufficient conditions for an integral of the following type

$$\int_{0+}^t x(u) d\mathcal{M}(u)$$

to be a \mathcal{Q} martingale. In particular, it is sufficient that for $d\mathcal{M}(t) = d\mathcal{N}(t) - (1 - \mathcal{N}(t))h(t)dt$ we have that the following integrability condition holds (see Theorem 18.7 of Lipster and Shirayev (2001b)):

$$\mathbb{E}_0^{\mathcal{Q}} \left[\int_0^\infty |x(s)|(1 - \mathcal{N}(s))h(s)ds \right] < \infty. \quad (101)$$

In our case, we see from (30) that

$$x(t) = -e^{-\int_0^t r(s)ds} L_s P_{s,T}(t)$$

because $\tilde{D}_{s,T}(t) = (1 - L_s)P_{s,T}(t)$ by definition (notice that L_s is a constant). Furthermore, we know that $r(t) > 0$ and that $P_{s,T}(t) \in (0, 1]$. So $|x(t)| < |L_s|$. Hence the integrability condition satisfies the following

$$\begin{aligned} \mathbb{E}_0^{\mathcal{Q}} \left[\int_0^\infty |x(s)|(1 - \mathcal{N}(s))h(s)ds \right] &< |L_s| \mathbb{E}_0^{\mathcal{Q}} \left[\int_0^\infty (1 - \mathcal{N}(s))h(s)ds \right] \\ &= |L_s| \int_0^\infty \mathbb{E}_0^{\mathcal{Q}} [(1 - \mathcal{N}(s))h(s)] ds \\ &< |L_s| \int_0^\infty \left(\mathbb{E}_0^{\mathcal{Q}} [1 - \mathcal{N}(s)] \mathbb{E}_0^{\mathcal{Q}} [h(s)^2] \right)^{\frac{1}{2}} ds. \end{aligned}$$

where we have first used Tonelli's Theorem to switch the order of integration, and then used Holder's inequality. First note that $h(s)$ has a steady-state distribution, and hence a second moment bounded by some constant K . Furthermore, it is easy to show that

$$\mathbb{E}_0^{\mathcal{Q}}[1 - \mathcal{N}(s)] = \mathbb{E}_0^{\mathcal{Q}} \left[e^{-\int_0^s h(u) du} \right] < e^{-h_0 \times s},$$

because $h(t) \geq h_0$ by assumption. Putting these together, we find that

$$|L_s| \int_0^\infty \left(\mathbb{E}_0^{\mathcal{Q}}[1 - \mathcal{N}(s)] \mathbb{E}_0^{\mathcal{Q}}[h(s)^2] \right)^{\frac{1}{2}} ds < |L_s| K^{\frac{1}{2}} \int_0^\infty e^{-\frac{1}{2} h_0 \times s} ds < \infty.$$

Q.E.D.

I Proof of Proposition 12 (Check Condition C5)

We first make note of the bound on the stock price given in (83). Because $\tilde{W}_m(t)$ is Normally distributed, we see that the Brownian integral in (35) is indeed a \mathcal{Q} martingale by the sufficiency of condition (100), the fact that $r(t) > 0$, and the fact that $Y_n(t)$ has finite expectations for all t .

Let us make precise the statement that the stock pays a terminal dividend at some time T . This implies that the \mathcal{M} integral in (35) should actually be written as

$$\int_{0+}^{t'} e^{-\int_0^t r(u) du} \left(\tilde{D}_i(t) - S_{i,T}(t) \right) \mathbf{1}[t \leq T] d\mathcal{M}(t).$$

This indicates that the default dividend does not get paid if the stock has already paid its terminal dividend $\tilde{D}_i(T)$. We now need to check that the above integral is indeed a \mathcal{Q} martingale. We will use an approach similar to the one in the proof of Proposition 11. In particular, for $x(t)$ defined by

$$\begin{aligned} x(t) &= e^{-\int_0^t r(u) du} (\tilde{D}_i(t) - S_{i,T}(t)) \mathbf{1}[t \leq T] \\ &= -e^{-\int_0^t r(u) du} L_i S_{i,T}(t) \mathbf{1}[t \leq T], \end{aligned}$$

the $\mathcal{M}(t)$ integral in (35) will be a \mathcal{Q} martingale if the condition in (101) is satisfied. Keep in mind that the above L_i is constant by assumption. Using the bound on the stock price in (83), as well as the fact that $r(t) \geq r_0$, we see that

$$\begin{aligned} |x(t)| &< |L_i| \exp \left((a_i - r_0 - C_i \tilde{\mu}) \times t - C_i \left[Z(0) + \sigma_Z (\tilde{W}_m(t) - \tilde{W}_m(0)) \right] \right) \mathbf{1}[t \leq T] \\ &= K_1 \exp \left((\phi_{i,0} - \delta_{i,0} + \tilde{\Theta}' \tilde{K}_Y' B_i - \frac{1}{2} \sigma_Z^2 C_i^2) \times t - C_i \sigma_Z (\tilde{W}_m(t) - \tilde{W}_m(0)) \right) \mathbf{1}[t \leq T], \end{aligned}$$

where we have used a_i from (82), and where K_1 is some constant. We note that $\phi_{i,0} = L_i h_0$ and that $L_i \leq 1$. We now need to show that (101) holds, or that:

$$\begin{aligned} \mathbb{E}_0^{\mathcal{Q}} \left[\int_0^\infty \exp \left((\phi_{i,0} - \delta_{i,0} + \tilde{\Theta}' \tilde{K}'_Y B_i - \tfrac{1}{2} \sigma_Z^2 C_i^2) \times s - C_i \sigma_Z (\tilde{W}_m(s) - \tilde{W}_m(0)) \right) \times \mathbf{1}[t \leq T] (1 - \mathcal{N}(s)) h(s) ds \right] = \\ \int_0^T \mathbb{E}_0^{\mathcal{Q}} \left[\exp \left((\phi_{i,0} - \delta_{i,0} + \tilde{\Theta}' \tilde{K}'_Y B_i - \tfrac{1}{2} \sigma_Z^2 C_i^2) \times s - C_i \sigma_Z (\tilde{W}_m(s) - \tilde{W}_m(0)) \right) \times (1 - \mathcal{N}(s)) h(s) \right] ds \\ < \infty. \end{aligned}$$

The change in the order of integration is justified because of Tonelli's Theorem. We now note that the Brownian motion $\tilde{W}_m(s)$ is independent of everything else in the economy, and also that $\mathbb{E}_0^{\mathcal{Q}}[e^{k(\tilde{W}_m(t) - \tilde{W}_m(0))}] = \exp(\frac{1}{2}k^2t)$. By virtue of this we can rewrite and bound the above integral as

$$\begin{aligned} \int_0^T \mathbb{E}_0^{\mathcal{Q}} \left[\exp \left((\phi_{i,0} - \delta_{i,0} + \tilde{\Theta}' \tilde{K}'_Y B_i) s \right) \times (1 - \mathcal{N}(s)) h(s) \right] ds < \\ \sup_{t \in \{0, T\}} \exp \left((\phi_{i,0} - \delta_{i,0} + \tilde{\Theta}' \tilde{K}'_Y B_i) t \right) \int_0^T \mathbb{E}_0^{\mathcal{Q}} [(1 - \mathcal{N}(s)) h(s)] ds < \infty, \end{aligned}$$

where finiteness follows because the integrand is non-negative and, as has already been shown in the proof of Proposition 11, the above integral is bounded even for $T = \infty$.

Q.E.D.

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