A Two-Factor Hazard-Rate Model for Pricing Risky Debt and the Term Structure of Credit Spreads

Dilip Madan and Haluk Unal[1]
Robert H. Smith School of Business
University of Maryland, College Park, 20742
dmadan@rhsmith.umd.edu, hunal@rhsmith.umd.edu

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Abstract

This paper proposes a two-factor hazard-rate model, in closed-form, to price risky debt. The likelihood of default is captured by the firm's non-interest sensitive assets and default-free interest rates. The distinguishing features of the model are threefold. First, impact of capital structure changes on credit spreads can be analyzed. Second, the model allows stochastic interest rates to impact current asset values as well as their evolution. Finally, the proposed model is in closed form enabling us to undertake comparative statics analysis, compute parameter deltas of the model, calibrate empirical credit spreads and determine hedge positions. Credit spreads generated by our model are consistent with empirical observations.

1 Introduction

This paper proposes a simple closed form credit spread model with realistic short maturity spreads that are related to the structural characteristics of the firm's economic environment and accommodates stochastic interest rates. The distinguishing feature of the model is that it incorporates the attractive features of the diffusion based models (Merton (1974) and Longstaff and Schwartz (1995)) with the hazard rate approach (Jarrow and Turnbull (1995), Madan and Unal (1998), Duffie and Singleton (1999)).

Diffusion-based models of pricing risky debt define default as occurring either at maturity (Merton (1974)) or when the firm's asset value diffuses to a prespecified default boundary for the first time (Longstaff and Schwartz (1995)). An attractive feature of these models is that they express the default time in terms of firm specific structural variables.[1] These models can then answer questions about the implications for debt pricing of changes in firm specific variables such as capital structure reorganizations. However, this important feature is compromised by their inability to generate realistic credit spreads in the shorter term although Longstaff and Schwartz (1995) did succeed in obtaining such spreads in the medium term. In these models, time needs to pass to allow assets to diffuse for the default probability to materialize. Equivalently the probability of a positive-equity firm defaulting in the near term is negligible leading to near zero spreads for short maturities.[2]

The recent hazard rate approach to pricing risky debt of Jarrow and Turnbull (1995), Madan and Unal (1998), and Duffie and Singleton (1999), develops a class of models that allow for the possibility of default in the immediate future.[3] This literature proposes an exogenous model for the hazard rate, which is the *likelihood* of the firm defaulting over the next period.

[1] A non-exhaustive list includes Cooper and Mello (1992), Kim, Ramaswamy and Sunderesan (1993), Hull and White (1995), Leland (1994), Longstaff and Schwartz (1995), Leland and Toft (1996) and Briys and de Varenne (1997).

[2] This deficiency has long been recognized and diffusion based models are commonly criticized for this characteristic. For example, the theoretical yield curves reported in Kim, Ramaswamy and Sundaresan (1993) and Longstaff and Schwartz (1995) report zero credit spreads for maturities up to 4 years and remain below 50 basis points up to ten years for high rated firms. The classical reference for low spreads in this literature is Jones, Mason, and Rosenfeld (1984).

[3] In recent years an extensive literature developed incorporating the hazard-rate approach, including Artzner and Delbaen (1995), Lando (1997), Nielsen and Ronn (1995), Das and Tufano (1996), Jarrow, Lando, and Turnbull (1997), and Duffie and Lando (1997).

A major advantage of this approach is that they generate realistic short maturity credit spreads. However, these models lack a structural definition of the default event. As a consequence, the resulting hazard rate model is a reduced form with parameters that lack a structural interpretation and hence offers no guidance in the presence of a structural change in firm specific variables.

This paper seeks to propose a structural hazard rate model in closed form. The key assumption of the paper is that default is a consequence of a single jump loss event that drives the equity value to zero and requires cash outlays that can not be externally financed. A case in point is the near default of Long Term Capital Management, resulting from an adverse movement in interest rates. Another case is that of Barings where a large trading loss forced bankruptcy. Both these examples illustrate the phenomena of default arising from the arrival of an unforeseen loss. Such a sudden fatal loss can be caused by numerous surprise events including the outcome of lawsuits, unexpected devaluations, sudden default of a creditor, supplier or a customer, and catastrophes in production lines.

The model has a number of attractive features. First, consistent with the hazard-rate literature, the probability of such sudden loss arriving unexpectedly is captured in the pricing equations by "discounting" the promised payments by the hazard rate.[4] We endogenously derive the stochastic hazard-rate as a function of the firm's non-interest sensitive asset values and stochastic default-free interest rates. To achieve this we use a simple equity valuation framework and show that the hazard rate can be expressed as a first order approximation to the probability that a sudden loss exceeds the level of equity. Hence, we differ from Madan and Unal (1998) in that the parameters of the hazard process are interpretable and can be linked to firm specific information such as the firm's equity value, duration gap, the loss distribution, and its arrival rate. We also differ from Zhou (1997) and Duffie and Lando (1997) by not modeling default as occuring on first passage of value to a default boundary, but rather as a cash shortage caused by a loss arrival.

Second, our treatment of the interest-rate risk differs from the previous literature that explictly allows for the relationship between credit spreads and default-free interest rates (Longstaff and Schwartz (1995) and Kim, Ra-

[4]Alternatively, one may obtain short maturity credit spreads by modeling the asset value process as a jump diffusion (see for example Zhou (1997)) but this comes at the cost of tractability, as one must allow for multiple jump events in the determination of asset values.

maswamy and Sundaresan, 1993). In this literature because current asset values are assumed not to be interest sensitive (or they have zero duration) the firm's duration gap is implicitly assumed to be negative. Hence, these models would predict that an increase in interest rates benefits the firm's equity and reduces credit spreads. This is an overly simplified assumption and to the extent that a firm's assets have durations exceeding that of liabilities then such predictions will not be correct.[5] We allow for this possibility by letting the firm hold interest sensitive assets (such as marketable securities and growth opportunities).The resulting model can capture the impact of changes in default free interest rates on negative as well as positive duration gap balance sheets.

Finally, the proposed model, by virtue of its closed form, enables the researcher to undertake comparative statics analysis and enhances the empirical applicability of the model. Because the parameters have economic meanings, impact of policy changes and structural shifts can also be readily identified with a closed form model. In addition, hedge ratios and parameter sensitivity of credit spreads are easily computable.

The flexibility of the model is demonstrated by calibrating the model to data on the term structure of credit spreads. We observe that sufficiently rich term-structure of credit spread shapes can be fitted to data and large term premiums of up to 200 basis points can be obtained. We also undertake factor and parameter sensitivity analysis. They show that the proposed model is quite promising in its applications to explain short term credit spreads. We demonstrate the application of the model in constructing hedge ratios to manage credit risk.

The paper is organized as follows: Section 2 provides the framework we adopt to pricing risky-debt. Section 3 defines the default event in a hazard rate context. Section 4 develops the risky-debt equation and presents an analysis of the resulting credit spreads. Strategies for hedging the model's factor risks are demonstrated in section 5. Section 6 concludes.

[5] The empirical evidence on the relationship between credit spreads and interest rates is mixed. Duffee (1998) reports a negative relationship, Morris, Neal and Rolph (1998) present evidence of a positive long run relationship. However, these empirical tests do not control for equity duration gaps.

2 The Framework

Consider a frictionless economy with the time horizon $[0, \Upsilon]$. Traded in this economy are two classes of zero-coupon bonds: default-free and risky. Default-free bonds pay a sure dollar at time T , for $0 \leq T \leq \Upsilon$, with time t prices for maturity $(\tau = T - t)$ given by $p(t, \tau)$. A money-market account also exists and its time t unit account value is given by $B(t) = \exp(\int_0^t r(u)du)$, where $r(t)$ is the default-free instantaneous spot interest rate. Let $v(t, \tau)$ be the time t price of a zero-coupon bond subject to default risk, promising to pay a dollar at time T. Default occurs at a random time z, $z < T$, and a percentage write-down ω is applied to the security. The write-down is measured in terms of equivalent time T dollars lost per unit of promised face, F. In other words, bondholders receive reduced face of ωF at T. This is the formulation referred to in Duffie and Singleton (1999) as "Recovery of Face Value." The alternative definition "Recovery of Market Value" leads to a model in which recovery rates can not be identified from data on the prices of pure discount bonds alone as shown in Duffie and Singleton (1999).

We suppose that asset prices discounted by the money market account are martingales under a probability measure Q. It follows that the value of risky debt, $v(t, \tau)$, is defined by the following martingale representation:

$$\frac{v(t,\tau)}{B(t)} = E_t^Q \left[\frac{1}{B(T)} (1 - \omega \mathbf{1}_{z \leq T}) \right], \tag{1}$$

where, $\mathbf{1}_{z \leq T}$ is the default indicator operator. This representation has the intuitive structure that the value of risky debt can be viewed as the difference between the value of the default-free promise and the discount for the default risk of the bond. Jarrow and Turnbull (1995) and Madan and Unal (1998) evaluate this expectation under Q assuming independence between the default-free interest-rate process, $r(t)$, and the default process as represented by z and ω. Note that this crucial assumption eliminates the need to evaluate the covariance term in the expectation of equation (1).

To allow for dependence between these two processes we follow Jamshidian (1989) and Geman, El Karoui and Rochet (1995) and use the price of the risk-free bond, $p(t, \tau)$, as the numeriare (rather than the money market account $B(t)$) to develop an expression for the value of risky debt. Hence, we let Q^T be the unique equivalent martingale measure (called the *forward measure*), under which traded asset prices discounted by the price of the default-free bond, $p(t,\tau)$, are martingales. It follows that,

$$\frac{v(t,\tau)}{p(t,\tau)} = E_t^{Q^T}\left[\frac{1}{p(T,0)}(1-\omega \mathbf{1}_{z\leq T})\right]. \tag{2}$$

Given $p(T,0)=1$, and denoting $G(t,\tau)$ to be the survival probability under Q^T conditional on no default prior to T,

$$G(t,\tau) = Q^T(z > T) = E_t^{Q^T}[\mathbf{1}_{z>T}], \tag{3}$$

and y is the constant recovery rate defined as the proportion of the promised unit face paid to bondholders ($\omega = 1-y$), the risky debt valuation equation can be simply expressed as:

$$\frac{v(t,\tau)}{p(t,\tau)} = G(t,\tau) + (1-G(t,\tau))y, \tag{4}$$

Note that the write-down amount and the time at which default occurs are assumed to be independent in moving from equation (2) to (4).[6] Within this framework, it is clear that the spot price of the risky bond has two distinct components that need to be evaluated: the survival probability of the firm over the life of the risky debt, $G(t,\tau)$, and the price of the default-free bond, $p(t,\tau)$.

The hazard-rate models of Jarrow and Turnbull (1995), Duffie and Singleton (1999) and Madan and Unal (1998) view $G(t,\tau)$ as the likelihood of survival over the life of the risky debt. The equation relating the survival probability, $G(t,\tau)$, to the process for the hazard rate of default, $\phi(t)$, is established in Madan and Unal (1998). They show that:[7]

$$G(t,\tau) = E_t^{Q^T}\left[\exp\left(-\int_t^T \phi(u)du\right)\right]. \tag{5}$$

The intuition behind this equation is straightforward and $\phi(t)$ can be compared with the default-free interest-rate risk process, $r(t)$. The spot-price of

[6]This crucial assumption is commonly invoked in the default literature. One exception is Das and Tufano (1996) where ω and τ are both expressed as functions of the same state variables.

[7]Madan and Unal (1998) establish the relation between the hazard-rate and the survival probability under the measure Q. However, the hazard-rate of the default-time process under Q^T is the same as that under Q by virtue of the continuity of $dQ^T/dQ = p(t,T)/B(t)$ (see, Bjork, 1997)

a default-free discount bond discounts the promised payoff by the default-free interest-rate risk process, $r(t)$, for reasons related to the time value of money. In contrast, equation (5) "discounts" the promised payoff on a risky debt by the process for the hazard-rate of default, $\phi(t)$, for reasons related to the hazard of default. Hence, $(1 - G(t, \tau))$ can be viewed as the "price-of-default," for a given loss level $(1 - y)$. This value is then discounted by $r(t)$, to obtain the price of the risky debt given in equation (4).

In this framework the default event is not defined and one employs a reduced form for the evolution of the hazard rate, $\phi(t)$. For example, Duffie and Singleton (1997 and 1999) specify equation (1) in terms of the risky-discount rate, $R(t) = r + (1-\phi)(1-y)$ and apply the Cox, Ingersoll and Ross (1985) solution and Heath, Jarrow, and Morton (1992) framework to equation (5), respectively. Jarrow and Turnbull (1995), use the Heath-Jarrow-Morton for $v(t, \tau)$ and $p(t, \tau)$ and then derive the relation in an example based on $\phi(t)$ being a constant. Though plausible, such reduced form specifications do not offer any guidance on the implications for credit spreads of changes in economic conditions, for example, the impact of the firm's financing decisions on its credit spreads.

These considerations suggest that one should consider the possibility of both introducing jumps in the default occurrence and defining the default event. The next section achieves this by deriving $\phi(t)$ from an explicit default condition as a first order approximation to the probability that a sudden loss level exceeds equity.

3 Modeling The Hazard Rate of Default

Utilizing the framework described in the previous section we model the occurrence of default in the context of a firm with a simple capital structure that has a single zero coupon risky debt instrument outstanding. In this section the default event is first defined. This leads to a closed form expression for a two-factor hazard rate of default. The model is completed by specifying the risk neutral dynamics of the two factors.

3.1 Defining the Default Event

We suppose that at a random time z the firm faces the payment of a random loss amount L. Default occurs if this loss is larger than the equity (E) in place. The equity value of the firm equals value of its assets less its liabilities. On the asset side, the firm holds cash assets with a current market value, V, that is insensitive to the current level of interest rates, and other assets with an interest-sensitive current market value of, $g(t,r)$. The literature has by and large assumed that all of the firm's assets are cash assets, ignoring $g(t,r)$. Examples of interest sensitive assets are marketable securities where the current value of such assets vary with the interest rate. More importantly, the firm's growth opportunities are real options with current values functionally dependent on the current interest rate. Liabilities, $\overline{v}(t,r)$, are the present value of promised payments, discounted at risk free Treasury rates, and hence are interest sensitive by definition.

Hence, the equity value of a firm can be viewed as a function of V and the level of default free interest rates r as follows:

$$E = V + g(t,r) - \overline{v}(t,r), \tag{6}$$

Note that duration gap arises when the firm's interest sensitive assets $g(t,r)$ and $\overline{v}(t,r)$ have different sensitivities to r. For example, firms with large amounts of short term receivables outstanding that are financed by long term debt (negative duration gap) benefit from interest rate increases and vice versa for firms holding the opposite position (a positive duration gap). Hence, sensitivity of equity to interest rates can be positive or negative.[8]

Default arises when

$$L \geq E = V + g(t,r) - \overline{v}(t,r) \tag{7}$$

and a payment in the amount of L is due. Cash asset values and interest rates therefore directly affect the probability of default. Note that this direct impact of interest rates on default probabilities has so far not been recognized in the literature that defines the default event (for example Longstaff and Schwartz (1995) and Leland and Toft (1996)). In this literature the firm's asset value is viewed as consisting only of cash assets V. Interest rates have a secondary impact only, arising from correlations in the evolution of V. This ignores the impact of the presence of interest-sensitive assets $(g(t,r))$ on default probabilities. Hence, these models predict that an increase in interest rates lowers credit spreads because liability values decline.

[8] The impact of other variables on the firm's equity value, such as firm's earnings process, foreign exchange rates and macro-economic variables can also be incorporated by adopting a more general equity valuation model.

Additionally, the size of these secondary impacts can be small. Indeed, Leland and Toft (1996) abstract their analysis from the complexities of stochastic interest rates upon observing that Kim, Ramaswamy and Sundaresan (1993) and Longstaff and Schwartz (1995) predict credit spreads to be about 5 to 7 basis points less than when the default-free interest rate is nonstochastic. However, as shown below when we incorporate the existence of interest sensitive assets in modeling the price of risky debt we can obtain opposite predictions and results.

3.2 The Hazard Rate of Default

Formally, the hazard rate $\phi(t)dt$ is the probability of default in the interval $(t, t+dt)$:

$$\phi(t)dt = Prob\ [z \in (t, t+dt)\ and\ L \geq E] = h(V, r)dt, \tag{8}$$

which is the probability of the arrival of loss times the conditional probability that this loss is large enough to drive the firm into default. The intuition behind equation (8) is as follows. There exists only one sudden loss size arrival that can cause instant default. However, as the equity level increases a larger-size sudden loss is needed to cause the firm to default and because the likelihood of such a larger-size jump is small the default probability correspondingly becomes small.

The specific determination of $h(V, r)$ is based on the distribution of loss sizes contingent on loss arrival and the process for the arrival rate of the loss event. Suppose, that the loss event has, prior to arrival, a constant Poisson arrival rate of λ per unit time. On the arrival of the loss event, further suppose that the distribution of the size of the loss has density $m(L)$ with cumulative distribution function $M(L)$. In this case, the function h is given by

$$h(V, r) = \lambda \left(1 - M(V + g(t, r) - \overline{v}(t, r))\right). \tag{9}$$

A first order approximation of this function around reference levels of the logarithm of cash assets, $log(V_0)$, and interest rates, r_0, yields

$$h(V(t), r(t)) \simeq h(V_0, r_0) - \lambda m\left(E_0\right) V_0(\Delta \ln V) - \lambda m\left(E_0\right)(g_r - \overline{v}_r)(\Delta r). \tag{10}$$

where $\Delta \ln V = \ln V(t) - \ln V_0$ and $\Delta r = r(t) - r_0$. We may then write on combining equations (8) and (10) that

$$\phi(t) = a - blnV(t) + cr(t) \tag{11}$$

where,

$$\begin{aligned} a &= h(V_0, r_0) + b \ln(V_0) - c r_0 && (12) \\ b &= \lambda m\,(E_0)\,E_V^0 V_0 && (13) \\ c &= -\frac{b}{E_V^0 V_0}\,(g_r - \overline{v}_r) && (14) \end{aligned}$$

and E_V^0 is the partial of equity with respect to cash assets evaluated at the reference point.

Equation (11) provides us with a simple two factor structural hazard rate model. The two factors are the value of the firms' cash assets and the level of default free interest rates. Furthermore, its parameters may be interpreted in terms of firm specific information such as the equity level, its sensitivity to cash assets and the duration gap. The model is completed by choosing an appropriate loss distribution.

The linear model for the hazard rate of equation (11) has the potential of yielding negative credit spreads as hazard rates can go negative. However, this problem can be mitigated in practice by calibrating the resulting model to positive credit spread data over a finite horizon of debt maturities.

3.3 An example of a fully specified hazard rate model

We consider the simple case of a one parameter exponentially distributed loss level with a mean loss level of μ_L. This distribution has the feature that larger losses have a lower probability. The cumulative distribution function is given by

$$F(L) = 1 - \exp(-L/\mu_L). \tag{15}$$

For this case the reference point loss probability is

$$h(V_0, r_0) = \lambda \left(1 - F\left(V_0 + g(0, r_0) - \overline{v}(0, r_0)\right)\right). \tag{16}$$

The hazard rate for an exponentially distributed loss level is given by

$$h(V(t), r(t)) \simeq \lambda(1-F(E_0)) - \frac{\lambda}{\mu_L}\exp(-\frac{E_0}{\mu_L})V_0(\Delta \ln V) + \frac{\lambda}{\mu_L}\exp(-\frac{E_0}{\mu_L})D(\Delta r). \tag{17}$$

The coefficients a, b, and c of equation (11) are

$$a = \lambda(1 - F(E_0)) + b\ln(V_0) - cr_0 \quad (18)$$

$$b = \frac{\lambda}{\mu_L}\exp(-\frac{E_0}{\mu_L})E_V^0 V_0 \quad (19)$$

$$c = \frac{b}{E_V^0 V_0}D \quad (20)$$

For given levels of the firm's cash asset holdings V_0, the market value of equity E_0, the sensitivity of equity to cash assets at the reference point E_V^0 and the level of interest rates r_0,the hazard rate model of equation (11) is completely specified from estimates of the mean loss level μ_L, its arrival rate λ, and the duration gap D. Alternatively, one may also invert these relations to ascertain the levels of μ_L, λ, and D implied by any estimates of a, b, and c. Other choices of loss distributions would lead to other similar formulations.

3.4 The Hazard Rate Factors

Consistent with the literature, we suppose that the value of cash assets is risk neutrally a geometric Brownian motion:

$$\frac{dV}{V} = rdt + \sigma dW_v, \quad (21)$$

where, σ is the return volatility and, W_v is a standard Brownian motion. [9]

The risk-neutral process for the default-free interest-rates draws from the term structure model of Vasicek (1977):

$$dr = (\theta - \kappa r)dt + \eta dW_r, \quad (22)$$

where $\frac{\theta}{\kappa}$ is the long-term mean rate of interest, κ is the speed at which the interest-rate r approaches to its long-term mean, η is the volatility of changes in the instantaneous default free interest rate and W_r is a standard Brownian motion. The correlation between dW_v and dW_r is ρdt. Equation (22) also establishes the basis to derive the pricing equation for the default-free bond,

[9] The jump nature of default can also be modeled by specifying a jump-diffusion process for asset values. In such a case default can either occur on asset values diffusing to a threshold or a firm can face more than one negative jump in equity cumulating to default. For reasons of tractability we focus on the single jump case.

$p(t,\tau)$ needed in equation (4). Vasicek (1977) provides the default-free bond-price expression and it is given in the Appendix for completeness.

Note that the asset value and interest-rate processes given in equations (21) and (22) are risk-neutral processes under the martingale measure Q. Under the forward measure Q^T the dynamics of $V(t)$ and $r(t)$ are shown in the Appendix to be given by

$$d\ln V = (r - \sigma^2/2 - \rho\sigma\eta N(\tau))dt + \sigma d\widetilde{W}_v, \tag{23}$$

$$dr = (\theta - \kappa r - \eta^2 N(\tau))dt + \eta d\widetilde{W}_r, \tag{24}$$

where $\widetilde{W}_v$ and $\widetilde{W}_r$ are Q^T Brownian motions with correlation ρ per unit time and $N(\tau) = (1 - \exp(-\kappa\tau))/\kappa$.

4 Risky Debt Price and the Analysis of Credit Spreads

The closed form expression for the price of risky debt, $v(t,\tau)$, is derived by first obtaining a solution for the survival probability, $G(t,\tau)$, as given in equation (5) then substituting this solution into equation (4).[10] The following proposition achieves this.

Proposition 1 *The value of a risky discount bond, $v(t,\tau)$, for maturity $\tau = T - t$, with constant recovery y, is given by*

$$v(\tau) = yp(t,\tau) + p(t,\tau)(1-y)G(t,\tau), \tag{25}$$

where

$$G(t,\tau) = \exp\left(H(\tau) - a\tau + b\tau\ln V(t) - \left((b+c)N(\tau) - \frac{b}{\kappa}\tau\right)r(t)\right), \tag{26}$$

and the default-free debt price is given by the Vasicek formula

$$p(t,\tau) = \exp\left(A(\tau) - N(\tau)r\right). \tag{27}$$

Proof in the Appendix.

[10] Note that the present value of fully paid liabilities is $\overline{v}(t,r)$ while the market value of a particular defaultable liability of maturity τ that we price is $v(t,\tau)$.

The functions $H(\tau)$, and $A(\tau)$ are defined in the Appendix. Equation (26) gives the explicit solution for the survival probability. This closed form expression for $v(\tau)$ has eight parameters: (a, b, and c or their equivalents in the case of an exponential loss distribution, μ_L, λ, and D as given by equations (18), (19), and 20)) that come from the hazard-rate process; σ the cash asset volatility; (θ, κ, and η) the parameters of the Vasicek interest rate process and ρ the cash asset interest rate correlation.

The price of risky debt in equation (25) synthesizes many influences impacting this price. There are the effects of the diffusions in the value of cash assets and interest rates, as well as the probabilities of large losses dominating the equity at hand. To better appreciate these impacts, we conduct an analysis, separately, of short maturity spreads, the term structure of credit spreads and the factor and parameter sensitivities of these spreads.

4.1 The Short Term Credit Spread

To evaluate the impact of cash asset values and default-free interest rates on credit spreads (defined as the difference between the yields of a risky and a default-free bond with identical maturity), we consider the limiting expression for short maturity spreads. It is shown in the appendix that equation (25) yields the following instantaneous credit spread:

$$\begin{aligned} CS_0 \cong\ & a - b\ln V + cr \\ & + \frac{1}{\kappa^2}(b\theta + \eta b\rho c(1-\sigma) + \eta^2 c) \\ & + \frac{1}{k^3}(\eta b^2\rho - \eta^2 b^2 - \eta^2 b - \eta^2 bc(1-\rho)) \\ & + \frac{1}{k^4}(\eta^2 b^2) \end{aligned} \tag{28}$$

In contrast to the extant literature, equation (28) shows that default-free interest rates have a direct impact on the short-term credit spreads. For example, short term credit spreads are insensitive to interest rates in Longstaff and Schwartz (1995) and independent of them in Duffie and Singleton (1999). However there exist empirical evidence that short-term credit spreads are indeed sensitive to changes in interest rates. For example, Duffee (1998) documents a negative relation between interest rates and short maturity credit spreads. Hence, equation (28) is the first model consistent with the empirical literature on sensitivity of short-term credit spreads to movements in the interest rates. The direction of this relation is, however, as we have noted earlier, dependent on the duration gap between the firms' interest-sensitive assets and liabilities. Empirical studies of this relation that do not control for the duration gap are misspecified. Furthermore, in empirical studies one also needs to differentiate between short term responses of credit spreads to interest rate changes and the long term equilibrium adjustment that is the content of equation (28) as well. In addition, we observe that long-term interest rates (as proxied by θ) also can impact positively short term credit spreads. However, this impact is mitigated when mean reversion (κ) is high.

4.2 The Term Structure of Credit Spreads

We next consider the shapes of the credit spread term structures that are consistent with the model. For this exercise we assume a typical upward sloping yield curve with Vasicek parameters of $\theta = .10$, $\kappa = 1$, and $\eta = .0333$ and an initial interest rate of 4 percent. The other inputs of the model are the value of cash assets V, the level of equity E, the mean loss level μ_L, the loss arrival rate λ, the duration gap D, the asset volatility σ_V, its correlation with interest rates ρ, and the recovery level y. To determine reasonable values for these parameters we calibrated the model to data on the term structure of credit spreads obtained from Bloomberg for April 29, 1999.

The term structure data include credit spreads for 14 bond ratings at 11 maturities. We consider the hypothetical case of a firm with a unit face value of debt, an equity level of .5, the level of cash assets at 2, and equity sensitivity to cash assets at the expansion point E_V^0, to be unity. Placing this capital structure in each rating class we estimated the determinants of the hazard rate structure (μ_L, λ, D) the cash asset volatility σ_V and the level of recovery w. We assume that ρ is equal to zero. This calibration exercise is conducted for two rating categories, $AA1-AA2$, and $B3$.[11] The purpose of the calibration exercise is to infer reasonable parameter estimates and demonstrate the flexibility of the proposed model.

We observe from figures 1 and 2 that the model can fit a double humped credit spread curve reasonably well, and it is also capable of generating a

[11] Bloomberg reports yields for 14 rating categories of Industrial Firms. The category AA1-AA2 is the second rating category while B3 is the fourteenth category. We thank Sanjiv Das for suggesting the use of this data.

sizable term premium of up to 200 basis points. The estimated parameter values are very intuitive as well. For the high rated bond the loss distribution has a mean (μ_L) of .2455. Such a loss is expected to arrive (λ) at a rate of .0315 or once in 32 years. The expected recovery at time of default (y) is 40.66 percent. The duration gap (D) is negative at 3.1 and the volatility of cash asset value (σ_V) is about 90 percent.

For the low rated bond, as expected, the loss distribution has a much higher mean $\mu_L = 10.1449$, an arrival rate, $\lambda = .0419$ (or once in 24 years) and lower conditional recovery of $y = 30.14$ percent. Also, the estimate of the duration gap is a large positive of 19.71 which is consistent with the option value that such low rated firms may be holding. Cash asset volatility is also much higher than the high rated bond at $\sigma_V = 1.5463$.

4.3 Factor Deltas and Parameter Sensitivity of Credit Spreads

For the analysis of the sensitivity of credit spreads to changes in cash asset value and interest rates we compute numerical deltas using the valuation expression in equation (25). The delta computations are conducted for the fitted curve of the AA1-AA2 rating category.

Figure 3 displays the results for cash asset values. We observe that an increase (decrease) in cash assets yields a decrease (increase) in credit spreads. In addition, the sensitivity of the credit spreads is of a comparable order for the entire term structure. Equivalently, the exposure to firm value risk must be hedged for all maturities.

Figure 4 shows the sensitivity of credit spreads to interest rate movements. An increase in interest rates reduces the credit spreads. This is a consequence of the negative duration gap implied by the term-structure. As interest rates rise the hazard rate decreases, decreasing the likelihood of default. Note that this effect is diminished and even reversed if the firm has more interest sensitive assets and the duration gap is positive, as in the case of the B3 rating curve.

We next assess the impact of the model parameter choices on the level and shape of the credit spread curve. Figure 5 presents the derivative of the credit spreads of the curve in figure 1, to changes in the arrival rate λ, the mean loss level μ_L,the duration gap D, the cash asset volatility σ, and the recovery rate y.[12] We observe that the effects are quite varied. The ability of the model to explain a wide variety of credit spread term structures is a consequence of the diversity of term structure sensitivities reflected in Figure 5. The impact of changes in the parameters λ and μ_L affect short term credit spreads more than the long term spreads. On the contrary, the changes in duration gap and cash asset volatility affect the longer maturity spreads. Such an impact is expected because these are diffusion parameters and σ and D require passage of time before they have an impact. The effect of σ can also be explained by noting that the drift in $\ln V$ is $(r - \frac{\sigma^2}{2})$ and hence higher values of σ impact credit spreads positively at longer maturities.

5 Hedging Credit Risk

An advantage of closed form expressions for the price of risky debt, as given in (25), is that they may be easily employed to hedge the factor risks embedded in credit sensitive instruments. For the holders of risky debt, the exposure to the firms' cash asset value risk and the risk of movements in interest rates can be hedged by positions in the firms' equity and default-free instruments.

Consider the perspective of a holder of the firms' risky debt within the context of Proposition 1. The value of this debt is then sensitive to the two factors V and the default-free interest rate r and has a value given by

$$v(t, T) = \Phi(V, r, \tau). \tag{29}$$

The exposure to V may be hedged by holding the firms' equity with value E, while the interest rate uncertainty may be hedged by holding default free bonds with a price $p(t, T) = \Psi(r, \tau)$. With a position of α shares and β Treasury bills, the hedged portfolio has a value, Y, given by

$$Y(V, r, \tau) = \Phi(V, r, \tau) + \alpha E + \beta\Psi(r, \tau). \tag{30}$$

From equation (6) we have that

$$E = V + g(r) - \Phi(V, r, \tau) \tag{31}$$

and hence

$$Y(V, r, \tau) = (1 - \alpha)\,\Phi(V, r, \tau) + \alpha V + \alpha g(r) + \beta\Psi(r, \tau) \tag{32}$$

[12] The derivative is numerically obtained using perturbation of the initial value. Furthermore, the curves graphed have been relativized to a unit value at the five year point.

For delta neutrality with respect to cash asset value and interest rate risk, one may determine equity and Treasury bill positions as

$$\alpha = \frac{-\Phi_V}{1-\Phi_V} \tag{33}$$

$$\beta = \frac{-\Phi_r}{\Psi_r} - \frac{-\Phi_V}{1-\Phi_V}\frac{g_r - \Phi_r}{\Psi_r}. \tag{34}$$

The position in equity (α) is negative and one shorts the equity, essentially creating a synthetic put to protect against declining cash asset values. One also shorts Treasury bonds such that the dollar Treasury interest rate exposure ($-\beta\Psi_r$) offsets the interest rate sensitivity of the defaultable bond adjusted for the duration gap inherited by the short equity position ($\Phi_r + \alpha(g_r - \Phi_r)$).

6 Conclusion

In this paper we show how hazard rate models can be structurally enriched by defining the default event in the short term as arising from the occurrence of a sudden loss event. The hazard rates are then derived endogenously as the probability of losses exceeding the firm's equity. Equity itself is modeled as composed of both cash (non-interest sensitive) assets and interest sensitive assets (such as growth opportunities) less liabilities. Hence, the two factor risks driving credit spreads are the value of cash assets and the level of stochastic default free interest rates. In addition to the parameters of the default free interest rate process, the structural determinants of default are seen to be the rate of loss arrival, its expected magnitude, the duration gap of equity, the volatility of cash asset values and the rate of recovery in default. It is shown that the factor risks can be hedged by appropriate positions in the firm's equity and Treasury bonds.

The resulting model for credit spreads is tractable and can be readily implemented. This is demonstrated by calibrating the model to data on credit spreads. We observe that a wide variety of realistic credit spread shapes can be generated by the model. However, going beyond the mere generation of term structure shapes, the model provides a deeper and richer understanding of the differences in these shapes in terms of the structural determinants of default.

A full scale empirical study of corporate bond yields using the proposed model of this paper requires attention to four major points. First, as most of the data are for coupon bonds, the model must be extended to pricing these securities. Second, as it is desirable to work with a panel of data for bond yields across time for a number of firms one needs to account of variations in duration. Third, one must differentiate between partial and full adjustment of credit spreads to movements in default free interest rates. Finally, one needs to model the effects of differences in the liquidity of traded bonds on credit spreads.

7 Appendix

Vasicek (1977) solution for $p(t,\tau)$: The price of default-free bonds, $p(t,\tau)$, is:

$$p(t,\tau) = \exp\left(A(\tau) - N(\tau)r\right) \tag{35}$$

where

$$\begin{aligned} A(\tau) &= \left(\frac{\eta^2}{2\kappa^2} - \frac{\theta}{\kappa}\right)\tau + \left(\frac{\eta^2}{\kappa^3} - \frac{\theta}{\kappa^2}\right)(\exp(-\kappa\tau) - 1) \\ &\quad - \left(\frac{\eta^2}{4\kappa^3}\right)(\exp(-2\kappa\tau) - 1), \end{aligned}$$

$$N(\tau) = \frac{1 - \exp(-\kappa\tau)}{\kappa},$$

and $\tau = T - t$.

Dynamics of $\mathbf{V(t)}$ and $\mathbf{r(t)}$ under $\mathbf{Q^T}$

The change of measure density process from Q to Q^T is given by the Q martingale

$$\frac{dQ^T}{dQ} = \frac{p(t,\tau)}{B(t)}. \tag{36}$$

This follows on noting that for traded asset prices on non-dividend paying claims $S(t)$, one must have that $S(t)/p(t,\tau)$ is a Q^T martingale by definition of Q^T. We also have that X is a Q^T martingale if and only if $X\frac{dQ^T}{dQ}$ is a Q martingale (see Elliott). The above choice for the measure change suffices as $S(t)/B(t)$ is a Q martingale by construction of Q (for further details see Baxter and Rennie page 191).

By Girsanov's theorem the original Brownian motions under Q are transformed into Brownian motions with drift under Q^T, whereby under Q^T

$$dW_v(t) = d\widetilde{W}_v(t) + \frac{dp(t,\tau)}{p(t,\tau)}dW_v(t) \tag{37}$$

$$dW_r(t) = d\widetilde{W}_r(t) + \frac{dp(t,\tau)}{p(t,\tau)}dW_r(t) \tag{38}$$

for new Q^T Brownian motions $\widetilde{W}_v(t), \widetilde{W}_r(t)$. Noting the definition of $p(t,\tau)$ in equation (35) and the dynamics of r in equation (22) we have that

$$\frac{dp(t,\tau)}{p(t,\tau)}dW_v(t) = -\eta N(\tau)\rho dt \quad (39)$$

$$\frac{dp(t,\tau)}{p(t,\tau)}dW_r(t) = -\eta N(\tau)dt \quad (40)$$

The Q dynamics for $\ln V(t)$ and $r(t)$ are given by

$$d\ln V(t) = (r - \frac{\sigma^2}{2})dt + \sigma dW_v(t) \quad (41)$$

$$dr(r) = (\theta - \kappa r)dt + \eta dW_r(t) \quad (42)$$

Substituting (39) into (37) and the resulting expressions for dW_v, dW_r into (41) yields

$$d\ln V = (r - \sigma^2/2 - \rho\sigma\eta N(\tau))dt + \sigma d\widetilde{W}_v, \quad (43)$$

$$dr = (\theta - \kappa r - \eta^2 N(\tau))dt + \eta d\widetilde{W}_r. \quad (44)$$

Proof of Proposition 1: Note that equation (4) can be written as

$$v(t,\tau) = p(t,\tau)y + (1-y)p(t,\tau)G(t,\tau), \quad (45)$$

Substituting equation (35) in equation (45) we obtain,

$$v(t,\tau) = y\exp\left[A(\tau) - N(\tau)r\right] + (1-y)\exp\left[A(\tau) - N(\tau)r\right]G(t,\tau) \quad (46)$$

The solution for G(t,T) is obtained as follows. Recall that the survival probability $G(t,\tau)$ is

$$G(t,\tau) = E_t^{QT}\left[\exp\left(-\int_t^T \phi(u)du\right)\right], \quad (47)$$

An immediate consequence of the assumptions that asset value process is log-normal (as in equation (23)) and the interest-rate process is normal (as in 24) is that the integral of the hazard-rates in equation (47), $\phi(t,\tau)$, is normally distributed, $N(\mu, s^2)$. Hence, it follows that,

$$G(V,r,t,T;\mu,s) = \left[e^{-\mu+\frac{1}{2}s^2}\right]. \quad (48)$$

The solution for the survival probability, $G(t,\tau)$, follows by integrating the dynamics for $\ln V(t,\tau)$ and $r(t)$ under Q^T, substituting them in equation (11) and evaluating equation (48).

Integrating the dynamics for r from time zero to time t gives the explicit solution for $r(t)$ as

$$\begin{aligned} r(t) &= r(0)\exp(-\kappa t) + \left(\frac{\theta}{\kappa} - \frac{\eta^2}{\kappa^2}\right)(1-\exp(-\kappa t)) \\ &+ \frac{\eta^2}{2\kappa^2}\exp(-\kappa T)(\exp(\kappa t) - \exp(-\kappa t)) \\ &+ \eta\exp(-\kappa t)\int_0^t \exp(\kappa s)\, d\widetilde{W}_r(s). \end{aligned} \tag{49}$$

The explicit solution for $V(t)$ is obtained by observing first that

$$\ln V(t) = \ln V(0) + \int_0^t \left(r(u) - \frac{\sigma^2}{2} - \rho\sigma\eta\frac{(1-e^{-\kappa(T-u)})}{\kappa}\right) du + \sigma\bar{W}_v(t). \tag{50}$$

Substituting equation (49) in equation (50) yields

$$\begin{aligned} \ln V(t) &= \ln V(0) + r(0)\left(\frac{1-e^{-\kappa t}}{\kappa}\right) \\ &+ \left(\frac{\theta}{\kappa} - \frac{\eta^2}{\kappa^2}\right)\left(t - \frac{1-e^{-\kappa t}}{\kappa}\right) \\ &+ \frac{\eta^2}{2\kappa^2}e^{-\kappa T}\left(\frac{e^{\kappa t} + e^{-\kappa t} - 2}{\kappa}\right) + \eta\int_0^t \frac{(1-e^{-\kappa(t-s)})}{\kappa} d\widetilde{W}_r(s) \\ &- \frac{\sigma^2 t}{2} - \rho\sigma\eta\left(\frac{t}{\kappa} - \frac{e^{-\kappa T}(e^{\kappa t} - 1)}{\kappa^2}\right) + \sigma\widetilde{W}_v(t). \end{aligned} \tag{51}$$

Now we can substitute equation (51) and equation (49) in equation (11) and collecting terms we obtain

$$\begin{aligned} \phi(t) &= a - b\ln V(0) + \left(ce^{-\kappa t} - \frac{b(1-e^{-\kappa t})}{\kappa}\right) r(0) \\ &+ \left(\frac{\theta}{\kappa} - \frac{\eta^2}{\kappa^2}\right)\left(c(1-e^{-\kappa t}) - b\left(t - \frac{(1-e^{-\kappa t})}{\kappa}\right)\right) \end{aligned}$$

$$
\begin{aligned}
&+\frac{\eta^2}{2\kappa^2}\left(ce^{-\kappa t}(e^{\kappa t}-e^{-\kappa t})-be^{-\kappa t}\left(\frac{e^{\kappa t}+e^{-\kappa t}-2}{\kappa}\right)\right)\\
&+\int_0^t\left(c\eta e^{-\kappa t}-\frac{b\eta}{\kappa}(e^{-\kappa s}-e^{-\kappa s})\right)e^{\kappa s}d\widetilde{W}_r(s)\\
&+\frac{b\sigma^2 t}{2}+b\rho\sigma\eta\left(\frac{t}{\kappa}-\frac{e^{-\kappa t}(e^{\kappa t}-1)}{\kappa^2}\right)-b\sigma\widetilde{W}_v(t). \qquad (52)
\end{aligned}
$$

Evaluating $\int_t^T \phi(u)du$ for $t < u$ we obtain the expressions for mean μ and variance s^2 as:

$$
\begin{aligned}
\mu \;=\;& a\tau - b\tau\ln V(t)\\
&+\left(c\left(\frac{(1-e^{-\kappa\tau})}{\kappa}\right)-b\left(\frac{\tau}{\kappa}-\frac{(1-e^{-\kappa\tau})}{\kappa}\right)\right)r(t)\\
&+\left(\frac{\theta}{\kappa}-\frac{\eta^2}{\kappa^2}\right)\left(c\left(\tau-\frac{1-e^{-\kappa\tau}}{\kappa}\right)-\frac{b\tau^2}{2}+\frac{b}{\kappa}\left(\tau-\frac{1-e^{-\kappa\tau}}{\kappa}\right)\right)\\
&+\frac{\eta^2}{2\kappa^2}\left[ce^{-\kappa\tau}\left(\frac{e^{\kappa\tau}+e^{-\kappa\tau}-2}{\kappa}\right)-be^{-\kappa\tau}\left(\frac{e^{\kappa\tau}-e^{-\kappa\tau}}{\kappa^2}-\frac{2\tau}{\kappa}\right)\right]\\
&+\frac{b\sigma^2\tau^2}{4}+\frac{b\rho\sigma\eta\tau^2}{2\kappa}-\frac{b\rho\sigma\eta e^{-\kappa\tau}}{\kappa^2}\left(\frac{(e^{\kappa\tau}-1)}{\kappa}-\tau\right). \qquad (53)
\end{aligned}
$$

The variance is:

$$
\begin{aligned}
s^2 \;=\;& \int_t^T\left[c\eta\left(\frac{e^{-\kappa(s-t)}-e^{-\kappa\tau}}{\kappa}\right)-\frac{b\eta}{\kappa}(T-s)+\frac{b\eta}{\kappa^2}(1-e^{-\kappa(T-s)})\right]^2 ds\\
&+b^2\sigma^2\int_t^T(T-s)^2ds\\
&-2\rho b\sigma\int_t^T(T-s)\left[c\eta\left(\frac{e^{-\kappa(s-t)}-e^{-\kappa\tau}}{\kappa}\right)-\frac{b\eta}{\kappa}(T-s)\right.\\
&\left.+\frac{b\eta}{\kappa^2}(1-e^{-\kappa(T-s)})\right]ds \qquad (54)
\end{aligned}
$$

Integrating and collecting terms, we obtain

$$
\begin{aligned}
s^2 \;=\;& f_1\tau+f_2\tau^2+f_3\tau^3\\
&+g_1\tau e^{-\kappa\tau}+g_2\tau^2e^{-\kappa\tau}+g_3\tau e^{-2\kappa\tau}\\
&+h_1(1-e^{-\kappa\tau})+h_2(1-e^{-\kappa\tau})\\
&+h_3e^{-\kappa\tau}(1-e^{-\kappa\tau}) \qquad (55)
\end{aligned}
$$

where,

$$\begin{aligned}
f_1 &= \frac{b^2\eta^2}{\kappa^2} - \frac{2b\rho c\eta^2}{\kappa^3} - \frac{2b\rho c\sigma\eta}{\kappa^2} \\
f_2 &= -\frac{b^2\eta^2}{\kappa^3} - \frac{b^2\rho\sigma\eta}{\kappa^2} \\
f_3 &= \frac{1}{3}\left(\frac{b^2\eta^2}{\kappa^2} + \frac{2b^2\rho\sigma\eta}{\kappa} + b^2\sigma^2\right) \\
g_1 &= -\frac{4bc\eta^2}{\kappa^3} - \frac{2b^2\eta^2}{\kappa^4} - \frac{2b^2\rho\sigma\eta}{\kappa^3} \\
g_2 &= \frac{bc\eta^2}{\kappa^2} + \frac{b\rho\sigma c\eta}{\kappa} \\
g_3 &= \frac{c^2\eta^2}{\kappa^2} \\
h_1 &= \frac{2bc\eta^2}{\kappa^4} - \frac{2b^2\eta^2}{\kappa^3} + \frac{2bc\eta^2}{\kappa^4} + \frac{2b^2\eta^2}{\kappa^4} + \frac{2bc\rho\sigma\eta}{\kappa^3} + \frac{2b^2\rho\sigma\eta}{\kappa^3} \\
h_2 &= \frac{c^2\eta^2}{\kappa^3} + \frac{b^2\eta^2}{\kappa^3} \\
h_3 &= \frac{2bc\eta^2}{\kappa^4} - \frac{2c^2\eta^2}{\kappa^3}
\end{aligned}$$

Substituting equation (53) and equation (55) in equation (48) and collecting terms we obtain

$$G = \exp\left(-a\tau + H(\tau) + b\tau\ln V - \left(\frac{c(1-e^{-\kappa\tau})}{\kappa} - \frac{b\tau}{\kappa} + \frac{b(1-e^{-\kappa\tau})}{\kappa}\right) r\right) \tag{56}$$

where

$$\begin{aligned}
H(\tau) &= -\left(\frac{\theta}{\kappa} - \frac{\eta^2}{\kappa^2}\right)\left(c\tau - cN(\tau) - \frac{b\tau^2}{2} + \frac{b}{\kappa}(\tau - N(\tau))\right) \qquad (57) \\
&\quad -\frac{\eta^2}{2\kappa^2} c e^{-\kappa\tau}\left(\frac{e^{\kappa\tau} + e^{-\kappa\tau} - 2}{\kappa}\right) + -b e^{-\kappa\tau}\left(\frac{e^{\kappa\tau} - e^{-\kappa\tau}}{\kappa^2} - \frac{2\tau}{\kappa}\right) \\
&\quad -\frac{b\sigma^2\tau^2}{4} - \frac{b\rho\sigma\eta\tau^2}{2\kappa} + -\frac{b\rho\sigma\eta e^{-\kappa\tau}}{\kappa^2}\left(\frac{e^{\kappa\tau} - 1}{\kappa} - \tau\right) \\
&\quad +\frac{\tau}{2}\left(f_1 + f_2\tau + f_3\tau^2\right) + \frac{\tau e^{-\kappa\tau}}{2}\left(g_1 + g_2\tau + g_3 e^{-\kappa\tau}\right) \\
&\quad +\frac{\kappa N(\tau)}{2}(h_1 + h_3 e^{-\kappa\tau}) + \frac{h_2}{2}(1 - e^{-2\kappa\tau})
\end{aligned}$$

Hence, substituting equation (56) into equation (46) we obtain,

$$\begin{aligned} v(\tau) &= y\exp(A(\tau)-N(\tau)r) \\ &+(1-y)\exp\left(A(\tau)+H(\tau)+b\tau\ln V-\left((1+b+c)N(\tau)-\frac{b}{\kappa}\tau\right)r\right) \end{aligned} \quad (58)$$

Q.E.D.

Derivation of the short maturity spread.

We re-write proposition 1 as

$$\frac{v(t,\tau)}{p(t,\tau)} = y+(1-y)G \quad (59)$$

The credit spread, $CS(t,\tau)$, can be expressed as

$$CS(t,\tau) = \frac{-\ln\frac{v(t,\tau)}{p(t,\tau)}}{\tau} = \frac{-\ln(y+(1-y)G)}{\tau} \quad (60)$$

For low levels of y we may write

$$\begin{aligned} CS(t,\tau) &= \frac{-\ln G}{\tau} \\ &= \frac{\mu-\frac{s^2}{2}}{\tau} \end{aligned} \quad (61)$$

Hence, subtracting equation (55) from equation (53) gives the expression for credit spread for short maturities (deleting higher order terms in (τ)) as:

$$\begin{aligned} CS_0 &= a-b\ln V+cr \\ &+\frac{1}{\kappa^2}(b\theta+\eta b\rho c(1-\sigma)+\eta^2 c) \\ &+\frac{1}{k^3}(\eta b^2\rho-\eta^2 b^2-\eta^2 b-\eta^2 bc(1-\rho)) \\ &+\frac{1}{k^4}(\eta^2 b^2) \end{aligned} \quad (62)$$

8 References

Artzner, P. and F. Delbaen. "Default Risk Insurance and Incomplete Markets." *Mathematical Finance*, 5 (July 1995), 187-195.

Bjork, T. "Interest Rate Theory." *Lecture Notes in Mathematics,* Springer-Verlag, Berlin (1997).

Briys, E. and F. de Varenne. "Valuing Risky Fixed Debt: An Extension." *Journal of Financial and Quantitative Analysis*, 32 (June 1997), 239-248.

Cox, J.C., J.E. Ingersoll, Jr., and S.A. Ross. "A Theory of TheTerm Structure of Interest Rates." *Econometrica*, 53 (March 1985), 385-407.

Cooper, I. A. and A. S. Mello. "The Default Risk of Swaps." *Journal of Finance*, 47 (June 1992), 597-620.

Das, S. R. and P. Tufano. "Pricing Credit-sensitive Debt when Interest Rates, Credit Ratings and Credit Spreads are Stochastic." *Journal of Financial Engineering 5 (June 1996),* 161-198.

Duffee, G.R. "Treasury Yields and Corporate Bond Yield Spreads: An Empirical Analysis." *Journal of Finance*, 53 (December 1998), 2225-2242.

Duffie, D. and D. Lando. "Term Structure of Credit Spreads with Incomplete Accounting Information." Working Paper, Stanford University, (1997).

Duffie, D. and K. J. Singleton. "An Econometric Model of The Term Structure of Interest Rate Swap Yields." *Journal of Finance*, 52 (September 1997), 1287-1323.

Duffie, D. and K. J. Singleton."Modelling Term Structures of Defaultable Bonds." *Review of Financial Studies, 12 (1999), 687-720.*

Geman, H., N. El Karoui, and J.C. Rochet. "Change of Numeriare, Changes of Probability Measures and Pricing of Options." *Journal of Applied Probability*, 32 (1995), 443-458.

Heath, D., R. Jarrow, and A. Morton. "Bond Pricing and The Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation." *Econometrica*, 60 (January 1992), 77-106.

Hull J. and A. White. "The Impact of Default Risk on The Prices of Options and Other Derivative Securities." *Journal of Banking and Finance*, 19 (May 1995), 299-322.

Jamshidian, F. "An Exact Bond Option Formula." *Journal of Finance*, 44, 205-209.

Jarrow, R. A. and S. M. Turnbull. "Pricing Derivatives on Financial Securities Subject to Credit Risk." *Journal of Finance*, 50 (March 1995), 53-85.

Jarrow, R. A , D. Lando and S. M. Turnbull. "A Markov Model for The Term Structure of Credit Risk Spreads." *Review of Financial Studies*, 10 (1997), 481-523.

Jones, E., S. Mason, and E. Rosenfeld. "Contingent Claims Analysis of Corporate Capital Structures: An Empirical Investigation." *Journal of Finance*, 39 (July 1984), 611-627.

Kim, J., K. Ramaswamy, and S. Sundaresan. "Does Default Risk in Coupons Affect The Valuation of Corporate Bonds?: A Contingent Claims Model." *Financial Management*, 22 (Autumn 1993), 117-131.

Lando, D. "Modelling Bonds and Derivatives with Default Risk." in *Mathematics of Derivative Securities*, M. Dempster and S. Pliska (eds.) pp. 369-393, Cambridge University Press (1997).

Leland, H. "Corporate Debt Value, Bond Covenants, and Optimal Capital Structure." *Journal of Finance*, 49 (September 1994), 1213-1252.

Leland, H. E. and K. B. Toft. "Optimal Capital Structure, Endogenous Bankruptcy, and The Term Structure of Credit Spreads." *Journal of Finance*, 51 (July 1996), 987-1019.

Longstaff, F. and E. Schwartz. "A Simple Approach to Valuing Risky Fixed and Floating Rate Debt." *Journal of Finance*, 50 (September 1995), 789-819.

Madan, D. and H. Unal. "Pricing The Risks of Default." *Review of Derivatives Research*, 2 (1998), 121-160.

Merton, R. C. "On The Pricing of Corporate Debt: The Risk Structure of Interest Rates." *Journal of Finance*, 29 (May 1974), 449-470.

Morris, C., R. Neal and D. Rolph. "Credit Spreads in Interest Rates: A Cointegration Approach." Working Paper, Indiana University (1998).

Nielsen, S. and E. Ronn. "The Valuation of Default Risk in Corporate Bonds and Interest Rate Swaps." Working Paper, University of Texas at Austin (1995).

Vasicek, O. "An Equilibrium Characterization of The Term Structure." *Journal of Financial Economics*, 5 (1997), 177-188.

Zhou, Chunsheng. "A Jump-Diffusion Approach to Modeling Credit Risk and Valuing Defaultable Securities." Working Paper, Federal Reserve Board, (1997).

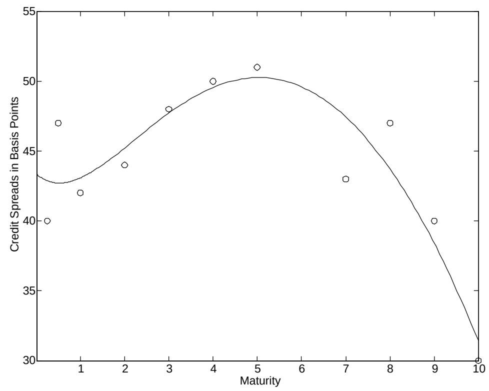

Figure 1: **Model Calibration on Rating Category AA1 and AA2 for April 29, 1999.** The data points are represented by circles and the solid line is the model fit. The estimated parameters are $\mu_L = .2455$, $\lambda = .0315$, $D = -3.1061$, $\sigma_V = .8907$ and $w = .4066$.

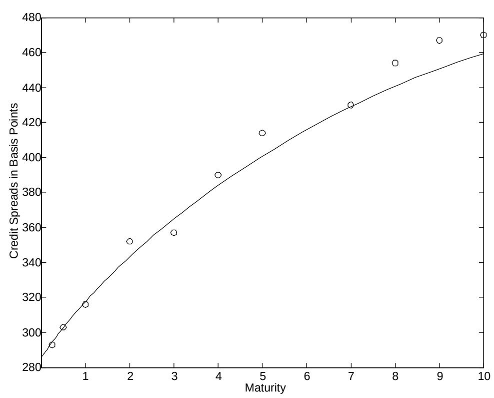

Figure 2: **Model Calibration on Rating Category B3 for April 29, 1999.** The data points are represented by circles and the solid line is the model fit. The estimated parameters are $\mu_L = 10.1449$, $\lambda = .0419$, $D = 19.7089$, $\sigma_V = 1.5463$ and $w = .3014$.

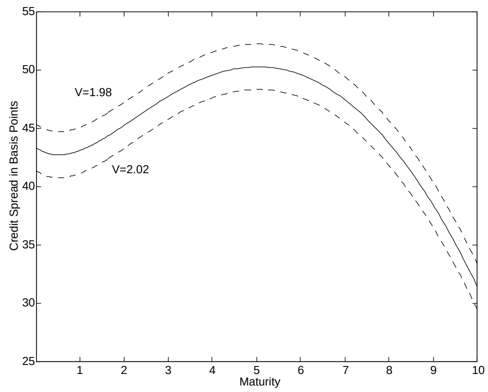

Figure 3: **Credit Spread Sensitivity to Cash Asset Value** The solid line represents the base case depicted in Figure 1. The impact of changes in the cash asset value are given by the dashed curves.

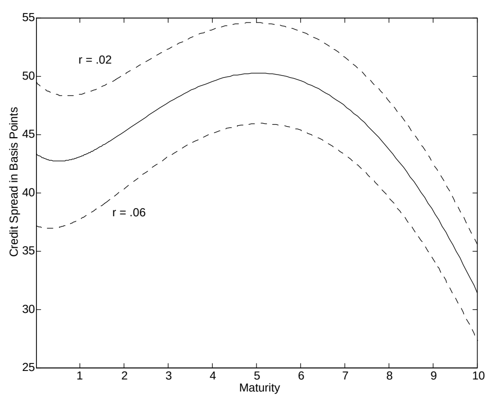

Figure 4: **Credit Spread Sensitivity to Interest Rates** The solid line represents the base case depicted in Figure 1. The impact of changes in the interest rate are given by the dashed curves.

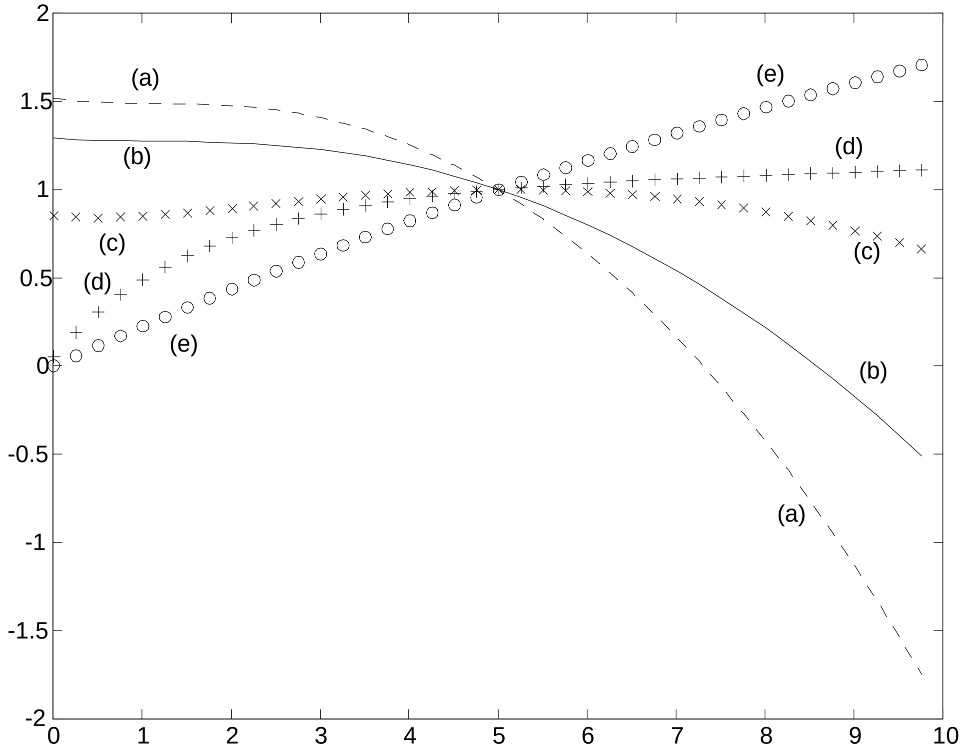

Figure 5: **Parameter Sensitivity of Credit Spreads.** The graphs show the derivative of credit spreads with respect to the parameters normalized to a value of unity at the five year point. Curve (a) through Curve (e) represent the sensitivity with respect to the arrival rate, the mean loss level, the recovery rate, duration and cash asset volatility, respectively.