Affine Model for Credit Risk Analysis

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Mars 2003
This version september 2003

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Abstract

This paper develops a coherent analysis of term structures of T-bonds and corporate bonds. The approach is based on a model in which general and firm specific factors can influence the stochastic discount factor and individual defaults. An appropriate choice of the specification of the sdf, of the survivor intensity and of the factor distribution leads to an affine term structure of the different types of bonds. A decomposition of the spread is obtained for corporate bonds and first-to-default baskets.

Keywords : Term Structure, Credit Risk, Affine Model, Stochastic Discount Factor, Affine Process, CAR Process.

JEL number C41, G21.

Modèles affines pour l'analyse du risque de crédit

Résumé

L'article propose une analyse cohérente des structures par terme des bons du Trésor et des obligations d'entreprise. La démarche s'appuie sur un modèle où des facteurs généraux et spécifiques des firmes peuvent influencer la défaillance et le facteur d'escompte stochastique. Les formulations des facteurs d'escompte, des intensités de survie et la loi des facteurs sont choisies de façon à obtenir des modèles affines de structure par terme. Ceci permet aussi d'introduire des décompositions des spreads pour les obligations d'entreprises et pour les "first-to-default basket".

Mots clés : structure par terme, risque de crédit, modèle affine, facteur d'escompte stochastique, processus affine, processus CAR.

1. Introduction

In the general strategy followed by the Basle Committee for monitoring the risk included in their financial investments, the banks will have to compute a CreditVaR. This Value at Risk defines the amount of reserve necessary to hedge the risk of their credit portfolios, including retail credits (consumer credit, mortgage, revolving credit, over-the-counter corporate loans) and corporate bonds. This paper will focus on portfolios of corporate loans which are traded on bond markets, and on the introduction of appropriate models for describing the associated risk. These models have to incorporate borrowers' heterogeneity with both industry-wide and firm-specific effects. This heterogeneity concerns the default intensity patterns at various maturities, but also the so-called default correlation, which is important since it accounts for what the regulators call *concentration risk*.

The first generation of credit risk simulators proposed by the industry is already in place[4]. Even though these models fulfil the basic need for simple credit risk measurement, they have some obvious limitations with yet unknown consequences. The following is a list of important empirical facts that are not accounted for in the current approaches:

- Riskfree interest rates evolve stochastically, but in most industry models they are assumed independent of the default phenomenon. This prohibits the use of such models for pricing assets other than plain vanilla debt.
- Default correlation is often taken into account in a very crude way, implicitely assuming similar default correlation for the different categories of firms, or for the short and long term credits. However the defaults are driven by general, sector and firm specific factors, whose influences vary in time.
- The models have to use the available information, including microeconomic data on firms (such as size, industrial sector, financial ratios, rating by the agencies) and market information concerning the equity or bond prices. It is well known that the KMV simulator is based on

[4]See Crouhy, Galai and Mark (2000) for a comparative analysis of the major industry models, proposed by KMV, J.P. Morgan (Credit Metrics) or Credit Swiss First Boston (Credit Risk +).

Merton's model and incorporates observed equity prices. But more information is included in the corporate bond prices and this information is often not taken into account.

- The returns on credit portfolios are well-known to be heavily skewed to the left. Indeed the introduction of a firm in a watchlist for downgrade by Moody's or Standard & Poor has an immediate impact on bond and equity prices, whereas the impact of an expected upgrade is slower.

Our objective in this paper is to develop a consistent valuation model for portfolio of corporate bonds which addresses these issues. This model can be used for pricing, but also to compute a CreditVaR. In this model, two types of factors generate *default correlation*. Firstly, defaults are instantaneously correlated because all firms are influenced by a common pool of economic factors. They can concern the so-called aggregate state of the economy or some sectorial effects, if we consider a restricted set of firms. Indeed the industry and the Basle Committee suggest to partition the firms by industrial sector and country. In a first step the CreditVaR will be computed separately for these different classes [5]. In the sequel this type of factor is called general factor. For example a source of default correlation could be information about the term structure (i.e. short term yields, slope, curvature, etc.) and/or equity markets. It can also be due to the natural common dependence of same industry companies on various norms related to that industry alone. For example profitability, intensity of competition, entrance of new players and introduction of new products, just to name a few, are potential sources of default correlation for firms in the same industry. Secondly it is also necessary to link the default occurrences of a given corporate at different ages. This can be done by introducing corporate specific factors. These idiosyncratic factors represent for example the effect of the strategy of the firm. Typically an investment decision has an impact on several future financial results.

Understanding and modelling such instantaneous and serial default correlations is important since it significantly impacts the overall credit risk in a portfolio. Whereas some factors are easily observable, other constitute a much more elusive, and thus a priori difficult to measure, set of factors . Fortunately, when such latent factors generate common default patterns,

[5] Even if it is known that the CreditVaR of a portfolio does not coincide with the sum of the CreditVaR of its sector components.

their values can still be recovered from the observable corporate and Treasury bond prices, something possible due to the *affine* character of our model.

Besides capturing the above empirical facts, the proposed model specifications will allow for the assessment of credit risk in a tractable manner. They will also allow for pricing of defaultable claims and their derivatives. For practical purposes, it is not necessary to arrive at closed-form expressions of the future default probabilities and risk derivative prices, but only to explain how to compute them by simple numerical methods, for instance by applying recursive equations. Moreover the computational burden should not become excessive when the number of credits included in the portfolio increases. In practice such portfolios can include thousands of different securities.

In section 2, we introduce the default arrival model. The survivor intensity rate depends on both systematic and firm specific factors. We compare the model with alternative specifications introduced in the literature to capture default correlation and with models written in continuous time. By assuming that individual defaults, conditional on state variables, are independent, and by employing affine dynamics for the factors, we recursively compute the conditional joint survivor function in section 3. The formulas are simplified when the effect of the state factors on the intensity rate is time independent . The aim of section 4 is to specify the pricing model. For this purpose we assume an exponential affine pricing kernel (stochastic discount factor), depending on general factors affecting default. Then the term structure of T-bonds, corporate bonds and first-to-default baskets can be derived recursively. In particular we discuss the decomposition and interpretation of the term structure of the spread. Monte-Carlo studies are presented in section 5. Section 6 concludes.

2. The default arrival model

2.1 Assumptions

Let us consider a cohort of n obligor firms with the same birth date fixed by convention at $t = 0$. The birth date can be defined in different ways according to the problem of interest. It can correspond to the date of creation of the firms if we focus on new industrial sectors, or to the date of the first rating by agencies such as Moody's or Standard and Poor if we consider the introduction on bond markets, or even an arbitrary initial date corresponding to the period of analysis. We denote by $\tau_i, i = 1, \ldots, n$ the failure date for corporation i. The aim of this section is to specify a joint distribution of the durations compatible with various patterns of the term structure of default correlation. The distribution of the set of durations is defined in two steps. We first assume that the durations are independent conditionally to the past, present and future values of a set of systematic and corporate specific factors. Then the future realization of the factors is integrated out to derive the joint distribution of durations conditional on information available at time t and to create the default dependence.

Since credit risk analysis is useful to satisfy the requirement of the Basle Committee concerning the reserves, and since the Committee demands a reporting of the CreditVaR (Value at Risk) at regular discrete dates, it is preferable to develop the model in discrete time. Therefore we will assume that $t = 1, 2, \ldots$

Assumption A.1 : There exist underlying systematic and firm specific factors[6], denoted by $(Z_t), (Z_t^i), i = 1, \ldots, n$, respectively. These factors are independent, Markovian and their transitions are such that :

$$E(\exp u' Z_{t+1} | Z_t) = \exp[a_g(u)' Z_t + b_g(u)],$$

$$E(\exp u' Z_{t+1}^i | Z_t^i) = \exp[a_c(u)' Z_t^i + b_c(u)], i = 1, \ldots, n,$$

where the above relations hold for all arguments u for which the expectation is well defined. Moreover we assume that the marginal distributions of $Z_o^i, i = 1, \ldots, n$ are identical.

Note that the factors can be multi-dimensional.

Thus the factors satisfy a compound autoregressive (CAR) process [see Darolles, Gourieroux, Jasiak (2002), Polimenis (2001)] (also called affine process in the continuous time framework by Duffie et al (2001)]. The conditional distributions are defined by means of the conditional Laplace transform, or moment generating function, restricted to real arguments u [7]. Since the functions a_g, b_g, a_c, b_c are not highly constrained a priori [8], the CAR dynamics

[6]The model can be easily extended to also include sector specific factors. These additional factors are not introduced for expository purpose.

[7]We assume in the sequel that the real Laplace transform characterizes the factor distribution. This condition is satisfied for nonnegative or bounded variables [see Feller (1971)]. In the general case it may require power conditional moments at any order and the possibility to get a series expansion of the Laplace transform in a neighborhood of zero.

[8]This is a consequence of the discrete time framework. In a continuous time framework

can be used to represent a large pattern of nonlinear serial dependence, including long memory or default clustering around an economic recession [see Jarrow, Yu (2001)]. By Assumption A.1. the population is assumed homogenous, that is the distributions of the corporate specific factor processes are independent of the firm. Thus the cohort is both homogenous with respect to the birth date and to the individual characteristics such as the industrial sector, or the initial rating. Finally note that the general factor Z, is defined for any t, whereas the firm specific factor Z^i can only exist until the default date τ_i of the i^{th} firm.

In this paper, instantaneous default correlation arises only because of the existence of common factors that drive individual firms' default intensities. That is to say that, given those common factors, default arrivals of different firms become independent:

Assumption A.2 : Conditionally on the realization path of the factors[9], $\underline{Z}, \underline{Z^i}, i = 1, \ldots, n$, default arrival times $\tau_i, i = 1, \ldots, n$ are independent. Moreover the conditional survivor intensities are such that :

$$\begin{aligned} & P[\tau_i > t+1 | \tau_i > t, \underline{Z}, \underline{Z^j}, j = 1, \ldots, n] \\ = \; & P[\tau_i > t+1 | \tau_i > t, Z_{t+1}, Z^i_{t+1}] \\ = \; & \exp[-(\alpha_{t+1} + \beta'_{t+1} Z_{t+1} + \gamma'_{t+1} Z^i_{t+1})] \\ = \; & \exp(-\lambda^i_{t+1}), \text{ say}, \forall t. \end{aligned}$$

Since the conditional survivor probability has to be smaller than unity, we get : $\lambda^i_t = \alpha_t + \beta'_t Z_t + \gamma'_t Z^i_t \geq 0, \forall t$. These restrictions are in particular satisfied with nonnegative factors and sensitivities.

The survivor intensity depends on time by means of the factors Z_{t+1}, Z^i_{t+1} and of the sensitivities $\alpha_{t+1}, \beta_{t+1}, \gamma_{t+1}$. This double time dependence can be interpreted in the following way. Let us assume for a moment time independent factors $Z_{t+1} = z, Z^i_{t+1} = z^i$, say [10]. For instance the general environment

the affine processes are essentially based on the geometric brownian motion, the CIR process and some specific branching process [see the discussion in Gourieroux, Monfort, Polimenis (2002)].

[9] $\underline{Z} = (Z_t, \forall t)$, $\underline{Z^i} = (Z^i_t, \forall t)$, and, $\underline{Z_h} = (Z_t, t \leq h)$, $\underline{Z^i_h} = (Z^i_t, t \leq h)$.

[10] When both factors and sensitivities are age independent, we get the so-called mul-

is stable, and the characteristics of the corporation such as its size, its financial ratios.. stay the same. Even with constant factors, the survivor rate is not the same for a young corporation and an old one. This age effect is captured by the age dependent sensitivities. At the opposite the factor Z_t for instance will capture the calendar time effect. Of course age and calendar time increase at the same rates.

The presence of common factors Z_t, which are time dependent, can create various term structures of default correlation. Assumption A.2 can easily be weakened to allow for corporate dependent sensitivities. In the extended framework, if Z_t includes current and lagged values of a factor $Z_t = (z_t, z_{t-1})$, say, and if the set of corporates is partitioned into two subsets with β sensitivities $(\beta_{1,t+1}, 0), (0, \beta_{2,t+1})$, respectively, the model allows to distinguish primary and secondary bonds [see Jarrow, Yu (2001) IV,B for another constrained specification for this problem].

2.2. The link to continuous time.

As mentioned before the survival model is defined in discrete time for practical purpose. However the main stream of the theoretical literature on credit risk considers continuous time specifications[11] and it is useful to link both approaches.

In continuous time the factors Z, Z^i will be continuous time affine processes [Duffie, Filipovic, Schachermayer (2001)], whereas durations become continuous positive variables. The distribution of a duration conditional on the factors path is defined through the (stochastic) infinitesimal default intensity :

$$\tilde{\lambda}_t = \lim_{dt \to 0} \frac{1}{dt} P[t < \tau_i < t + dt | \tau_i > t, \underline{Z}, \underline{Z^i}].$$

tivariate mixed proportional hazard (MMPH) model described in Van den Berg (1997), (2001). An exponential affine factor representation of the individual heterogeneity in the MMPH framework is usually assumed in the applications to labour [see e.g. Flinn, Heckman (1982), Heckman, Walker (1990), Bonnal, Fougere, Seradour (1997)]. In their labour framework the model of section 2.1 allows for a term structure of the individual heterogeneity and for a description of the latent dynamic effort variable underlying moral hazard.

[11] Note that the simulators for credit risk proposed by the industry follow discrete time approaches for the same practical reason as in our paper.

The associated survivor probability at horizon 1 is :

$$P[\tau_i > t+1 | \tau_i > t, \underline{Z}, \underline{Z^i}] = \exp - \int_t^{t+1} \tilde{\lambda}_u du.$$

If the infinitesimal default intensity is rather smooth, this survivor probability can be approximated by $\exp - \tilde{\lambda}_{t+1}$. Thus the discrete time specification introduced in the section above is the counterpart of a continuous time model with affine stochastic default intensity [see Lando (1994), (1998), Duffie, Singleton (1999)] :

$$\tilde{\lambda}_t = \tilde{\alpha}_t + \tilde{\beta}_t' Z_t + \tilde{\gamma}_t' Z_t^i.$$

However it is important to note that the model introduced in section 2.1 is not the discretization of the continuous time affine model for credit risk. First the continuous time factors generally satisfy stochastic differential equation, which implies non smooth trajectories. Then the approximation above cannot be applied in general; we get :

$$\lambda_{t+1} \sim \int_t^{t+1} (\tilde{\alpha}_u + \tilde{\beta}_u' Z_u + \tilde{\gamma}_u' Z_u^i) du,$$

and the survivor probability does not depend on the factors through Z_{t+1}, Z_{t+1}^i only. Moreover it is known that the time discretized affine processes form a small subset of CAR processes [see Darolles, Gourieroux, Jasiak (2002)]. As a consequence the discrete time specification will allow for a much larger set of patterns for the term structure of spreads and of default correlations.

2.3. Comparison with alternative approaches of default correlation

Default dependence has been introduced in the specification by means of stochastic factors with joint influence on the underlying corporate default intensities. However other modelling approaches have been considered in the literature.

i) A natural approach consists in specifying directly a joint distribution for the set of duration variables $\tau_1, \ldots, \tau_n$. This is usually done by introducing a copula in order to separate the treatment of marginal distribution from the dependence features [see e.g. Van den Berg (1997), Li (1999)]. Then the joint cdf is given by :

$$F(t_1, \dots, t_n) = C[G(t_1), G(t_2), \dots, G(t_n)],$$

where C denotes the copula and G the common marginal cdf. In practice the copula is selected in a parametric family, or depends on a functional parameter (see e.g. Gagliardini, Gourieroux (2002), Gagliardini (2003)]. For instance, it is possible to use an Archimedean copula, which admits a factor interpretation in a proportional hazard model framework (see e.g. Joe (1997)]. However this strategy has several drawbacks. First when a parametric family is selected, the number of parameters is generally small and the family is not compatible with various patterns of the term structure of default. Second the copula C provides the dependence between the duration at age 0. But we are also interested in the dependence between durations for still alive obligors at any age h. It is difficult to study how the copulas evolve with age and even if they will belong to the same family as the initial one.

As an illustration, let us consider an Archimedean copula :

$$C(u_1, \dots, u_n) = \psi^{-1}[\psi(u_1) + \dots + \psi(u_n)],$$

where ψ is the Laplace transform of the heterogeneity distribution [12]. The nonlinear dependence at age h when all individuals are still alive is also summarized by an Archimedean copula. This copula is given by :

$$C_h(u_1, \dots, u_n) = \psi_h^{-1}[\psi_h(u_1) + \dots + \psi_h(u_n)],$$

where $\psi_h(u) = \psi(u+nh)/\psi(nh)$ is the Laplace transform of the heterogeneity distribution at age h [Gourieroux, Monfort (2002)]. Typically the choice of the Archimedean copula implies restrictions between the dependencies at different ages.

ii) Another approach is based on the interpretation of the formula giving the survivor function. Indeed we have (in continuous time) :

[12] Recall that this copula corresponds to a MMPH specification with constant survivor intensity : $P[\tau_i > t+1 | \tau_i > t, Z] = \exp -Z$, where Z is a time invariant heterogeneity factor.

$$P[\tau_i > t] = \exp - \int_o^t \tilde{\lambda}_u^i du$$

$$= P[\int_o^t \tilde{\lambda}_u^i du < E_i],$$

where E_i is an exponential variate independent of the stochastic intensity. Thus we get the same model by defining the duration in terms of stopping time as [Bremaud (1981)] : $\tau_i = \inf\{t : \int_o^t \tilde{\lambda}_u^i du > E_i\}$.

It has been proposed in the literature to consider intensities $\tilde{\lambda}$, which are independent between individuals, and to extend the joint distribution of the thresholds $E_1, \ldots, E_n$ by allowing for dependence between the thresholds. Then a copula is introduced for defining the joint distribution of $E_1, \ldots, E_n$ [see e.g. Schonbucher and Schubert (2001)]. Moreover it has been advocated that this strategy allows for a larger range of admissible correlations between the durations than the stochastic intensity model [see e.g. the discussion in Van den Berg (1997)]. It is proved in Appendix 3 that this argument is not valid.

The main drawback of this approach is the introduction of default correlation by means of age independent factors $E_i, i = 1, \ldots, n$ which does not allow to manage the term structure of default correlation. Indeed, let us consider stochastic baseline intensities independent between individuals and independent of the thresholds. We get :

$$P[\tau_i > t+1 | \tau_i > t, E_1, \ldots, E_n]$$

$$= P[\int_o^{t+1} \tilde{\lambda}_u^i du < E_i] / P[\int_o^t \tilde{\lambda}_u^i du < E_i]$$

$$= \tilde{F}_i(t+1; E_i) / \tilde{F}_i(t; E_i) = A^i(t, E_i), \text{ say,}$$

where $\tilde{F}_i(t; .)$ denotes the cumulative distribution function of the cumulated baseline hazard. Thus we get a factor representation of the survivor intensity with time invariant factors. In some sense this approach is a slight modification of the basic MMPH Model. Typically with this strategy it is difficult to select a copula providing high default correlation in the short run and low default correlation in the long run [13] .

[13] At the beginning of the development of the internet sector, the default probabilities

3. Duration distribution

We first consider the case of general factor sensitivities and then the case of constant sensitivities.

3.1 General case

Under Assumptions A.1-A.2, it is possible to compute the joint conditional survivor function. This function depends on the information available at date t. When this information is complete, that is includes the current and lagged values of the factors together with the information on corporate default, we define the conditional survivor function for any subset S of the set I_t of firms which are still operational at time t :

$$\begin{aligned} & S_t^c(h_i, i \in S) \\ = \; & P[\tau_i > t + h_i, i \in S | I_t, S \subseteq I_t, (\tau_j | j \in \bar{I}_t), \underline{Z_t}, \underline{Z_t^j}, j = 1, \ldots, n], \end{aligned}$$

where $I_t = \{i : \tau_i > t\}$ and $\bar{I}_t$ denotes the complement of I_t.

The property below is proved in Appendix 2.

Proposition 1 :

$$\begin{aligned} & S_t^c(h_i, i \in S) \\ = \; & P[\tau_i > t + h_i, i \in S | I_t, S \subseteq I_t, (\tau_j | j \in \bar{I}_t), \underline{Z_t}, \underline{Z_t^j}, j = 1, \ldots n] \\ = \; & P[\tau_i > t + h_i, i \in S | I_t, S \subseteq I_t, Z_t, Z_t^i, i \in S]. \end{aligned}$$

Thus, given the evolution of factors up to now, the default history is not informative.

Proposition 2 : The conditional survivor function is given by :

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and default correlations were both high. After some years the remaining firms are more specialized and the default correlation has diminished.

$$
\begin{aligned}
& S_t^c(h_i, i \in S) \\
= \;& \exp[-\sum_{k=1}^{\bar{h}} n_{t+k}\alpha_{t+k} + B_g^{-[n][\beta]}(t, t+\bar{h}) + A_g^{-[n][\beta]}(t, t+\bar{h})' Z_t \\
+ \;& \sum_{i \in S} B_c^{-[\gamma]}(t, t+h_i) + \sum_{i \in S} A_c^{-[\gamma]}(t, t+h_i)' Z_t^i],
\end{aligned}
$$

where, for any deterministic sequence[14] $[u]$, operators $A^{[u]}, B^{[u]}$ are recursively defined by :

$$
\begin{aligned}
A^{[u]}(t, t+h) &= a[u_{t+1} + A^{[u]}(t+1, t+h)], \\
B^{[u]}(t, t+h) &= b[u_{t+1} + A^{[u]}(t+1, t+h)] + B^{[u]}(t+1, t+h),
\end{aligned}
$$

for $h > 0$, with terminal conditions :

$$A^{[u]}(t, t) = 0, B^{[u]}(t, t) = 0 \ , \ \forall t.$$

Furthermore $\bar{h} = \max_{i \in S} h_i$, the deterministic sequence $[n]$ is defined by $n_{t+k} = \text{ Card } \{h_i \geq k \ , \ i \in S\}$, and the product sequence $[n][\lambda]$ by $n_t \lambda_t$.

Proof : See Appendix 2.

Proposition 2 can be used to derive the distribution of a given failure time. It can also be used to derive the distribution of the time to the first failure in a basket of securities. These computations are especially important for credit derivatives currently traded on the market [see sections 4.4 and 4.5]. The credit default swaps (CDS) are options on default occurrence during a given period, whereas other options concern the occurrence of a default in a basket of securities (first-to- default basket security).

Let us consider a given obligor i. Then the conditional survivor function specific to this firm is $P[\tau_i > t+h | \tau_i > t, Z_t, Z_t^i]$ and corresponds to the joint survivor function with $S = \{i\}, h_i = h$, and $n_{t+k} = 1$ for $1 \leq k \leq h$.

[14] The [u] sequence may be infinite or not, provided it is defined at least for the interval $t+1$ to $t+h$, corresponding to the values included in the recursive computation.

Corollary 1 : We get :

$$
\begin{aligned}
& P(\tau_i > t+h | \tau_i > t, Z_t, Z_t^i) \\
= \; & \exp[-\sum_{k=1}^{h} \alpha_{t+k} + B_g^{-[\beta]}(t,t+h) + A_g^{-[\beta]}(t,t+h)' Z_t \\
+ \; & B_c^{-[\gamma]}(t,t+h) + A_c^{-[\gamma]}(t,t+h)' Z_t^i],
\end{aligned}
$$

where functions $A_g^{[u]}, A_c^{[u]}, B_g^{[u]}, B_c^{[u]}$ are recursively defined in Proposition 2.

Let us now consider the first to default time for a set S of still operating obligors, $\tau_S^* = \min_{\{i \in S \subseteq I_t\}} \tau_i$. The survivor function of τ_S^* is defined by :

$P[\tau_S^* > t+h | I_t, S \subseteq I_t, Z_t, Z_t^i, i \in S]$. It corresponds to the joint survivor function evaluated at $h_i = h, \forall i \in S$. Thus, for all $1 \leq k \leq h$, we have $n_{t+k} = n =$ Card (S).

Corollary 2 : We get :

$$
\begin{aligned}
& P[\tau_S^* > t+h | I_t, S \subseteq I_t, Z_t, Z_t^i, i \in S] \\
= \; & \exp[-n \sum_{k=1}^{h} \alpha_{t+k} + B_g^{-n[\beta]}(t,t+h) + A_g^{-n[\beta]}(t,t+h)' Z_t \\
+ \; & n B_c^{-[\gamma]}(t,t+h) + A_c^{-[\gamma]}(t,t+h)' \textstyle\sum_{i \in S} Z_t^i],
\end{aligned}
$$

where $n[\beta]$ denotes the sequence $n\beta_{t+k}$.

Even if the expressions of the survivor functions given in Proposition 1 and Corollaries 1 and 2 are rather cumbersome, they are easily computed from recursive equations. An interesting observation is that the distribution of the first to default time does not depend on the state of individual obligors since the average state $\frac{1}{n} \sum_{i \in S} Z_t^i$ is a sufficient statistic. That is the first to default probability of a security basket that contains some severely distressed corporations is identical with another basket where none of the corporations is distressed, but their average state is the same. This is a direct consequence of the homogeneity assumption.

3.2 Constant factor sensitivities

In the general framework of Assumption A.2, the conditional survivor probability for a given obligor i (for instance) varies with t because of the stochastic evolution of the factors Z_t, Z_t^i, but also because of the deterministic aging of coefficients $\alpha_t, \beta_t, \gamma_t$. An important special case arises when the sensitivity functions α, β, γ are constant. In this framework formulas of Corollaries 1 and 2 can be simplified.

Let us first consider the slope operator $A^{[u]}(t, t+h)$ when the deterministic sequence $[u]$ becomes a constant vector u. The recursive equation of Proposition 2 becomes :

$$\begin{aligned} A^u(t,t+h) &= a[u + A^u(t+1,t+h)] \\ &= a_u[A^u(t+1,t+h)], h > 0, \end{aligned}$$

where $a_u(s) = a(u+s)$ denotes a shifted version of function a and $A^u(t+h, t+h) = 0$. We deduce that :

$$A^u(t,t+h) = a_u^{oh}(0),$$

where a^{oh} denotes function $a(.)$ compounded h times. Similarly, for a constant sequence $u_t = u$, the intercept operator B^u is given by :

$$B^u(t,t+h) = \sum_{j=0}^{h-1} b_u(a_u^{oj}(0)).$$

Corollary 3 : When the factor sensitivities are constant :

$$\begin{aligned} & P[\tau_i > t+h | \tau_i > t, Z_t, Z_t^i] \\ = \; & \exp\{-h\alpha + \sum_{j=0}^{h-1} b_{g,-\beta}[(a_{g,-\beta}^{oj}(0)] + a_{g,-\beta}^{oh}(0)' Z_t + \sum_{j=0}^{h-1} b_{c,-\gamma}[a_{c,-\gamma}^{oj}(0)] + a_{c,-\gamma}^{oh}(0)' Z_t^i\}, \\ & P[\tau_S^* > t+h | I_t, S \subseteq I_t, Z_t, Z_t^i, i \in S] \\ = \; & \exp\{-nh\alpha + \sum_{j=0}^{h-1} b_{g,-n\beta}[a_{g,-n\beta}^{oj}(0)] + a_{g,-n\beta}^{oh}(0)' Z_t + n \sum_{j=0}^{h-1} b_{c,-\gamma}[a_{c,-\gamma}^{oj}(0)] + a_{c,-\gamma}^{oh}(0)' \sum_{i \in S} Z_t^i\}, \end{aligned}$$

where the shifted functions are :

$$a_{g,-n\beta}(u) = a_g(u - n\beta), a_{c,-\gamma}(u) = a_c(u-\gamma), \quad \text{etc.}$$

The later expression, concerning the first to default survivor probability, clearly shows the different effects of the general and corporate specific factors. The corresponding survivor probability involves the number n of firms in the basket, as well as their average state $\sum_{i \in S} Z_t^i$. An interesting point is the way the number of corporations n enters by scaling the shift in the a_g and b_g functions.

4. Affine term structure and credit risk

The joint historical distribution of default is an important element for comparing the term structures of T-bonds and corporate bonds. However the model has to be completed by specifying a stochastic discount factor (sdf). The sdf is used for pricing both future money and invididual defaults. Indeed it is not realistic to study independently the term structure of interest rates and default risk, which are both related to business cycles. For instance in a period of high activity, we can expect both an increase of the difference between the long and short term riskfree rates and an improvement of credit quality [see e.g. the study by Duffee (1998)]. For this reason we assume that some general factors Z appearing in the conditional survivor probability can also influence the sdf. Moreover we select a stochastic discount factor which is an exponential affine function of the general factors. As a consequence the benchmark term structure of interest rates will be affine [see Gourieroux, Monfort, Polimenis (2002)].

4.1 Specification of the stochastic discount factor

The pricing model is completed by specifying the stochastic discount factor $M_{t,t+1}$ for period $(t, t+1)$. The sdf is the basis for pricing any derivative written on the underlying factors and the default dates. A sdf exists as a consequence of the absence of arbitrage opportunity assumption, even in this incomplete market framework due to time discreteness. Typically the price at t of a European derivative paying g_{t+h} at date $t+h$ is :

$$\begin{aligned} C_t(g,h) &= E_t[M_{t,t+1} \dots M_{t+h-1,t+h} g_{t+h}] \\ &= E_t[M_{t,t+h} g_{t+h}], \text{ say}, \end{aligned} \tag{4.1}$$

where E_t denotes the historical expectation conditional on the information including the current and lagged values of the state variables, the knowledge of the set of corporates which are still alive at date t, and the default dates of the other firms. In particular pricing formula (4.1) can be used to determine the prices of zero-coupon bonds for both the T-bonds and corporate bonds. For the T-bonds, we get :

$$B(t, t+h) = E_t(M_{t,t+h}), \forall t, h. \tag{4.2}$$

For the h-year zero-coupon, zero-recovery corporate bond corresponding to corporate $i \in I_t$, we get :

$$C_i(t, t+h) = E_t[M_{t,t+h} \mathbb{1}_{\tau_i > t+h}], \forall t, h. \tag{4.3}$$

For a first-to-default basket which pays nothing in case of a single default and 1$ otherwise, we get :

$$C_S^*(t, t+h) = E_t[M_{t,t+h} \mathbb{1}_{\tau_S^* > t+h}], \forall t, h. \tag{4.4}$$

To restrict the set of admissible risk neutral distributions [15], we select a sdf which is exponential affine in the general factors.

Assumption A.3 :

$$M_{t,t+1} = \exp[\nu_o + \nu' Z_{t+1}]. \tag{4.5}$$

It would be possible to also introduce the corporate specific factors in the expression of the sdf. However the interpretation would become more complicated because the set of alive corporate specific factors depends on the date. When a corporation fails the factor associated with it ceases to exist. By introducing the effect of individual factors Z_i, we would also introduce risk corrections for the number and structure of corporations, which is beyond the scope of the present paper [16].

Also note that some components of ν [resp. β_t] can be zero. Therefore general factors can influence the sdf, the default intensity or both.

[15] In this incomplete market framework the set of admissible sdf is infinite. It is restricted in the model and the parameters ν_o, ν can in practice be estimated from yield data.

[16] The number of corporations can have an effect on default correlation. If we consider an industrial sector with two firms only, the default of a firm will increase the monopolistic power of the remaining one and likely diminish its default probability.

4.2 The price of the zero-coupon T-bond.

The prices of the zero-coupon T-bonds are given by :

$$\begin{aligned} B(t,t+h) &= E_t[M_{t,t+1}\dots M_{t+h-1,t+h}] \\ &= \exp(\nu_o h) E_t \exp[\nu' Z_{t+1} + \dots + \nu' Z_{t+h}]. \end{aligned}$$

The analytic expression of the price follows from Appendix 1 [see also Gourieroux, Monfort, Polimenis (2002)].

Property 3 : The price of a zero-coupon T-bond is :

$$B(t,t+h) = \exp[\nu_o h + \sum_{j=0}^{h-1} b_{g,\nu}(a_{g,\nu}^{oj}(0)) + a_{g,\nu}^{oh}(0)' Z_t].$$

In particular the geometric yields defined by :

$$\begin{aligned} r(t,t+h) &= -\frac{1}{h} \ln B(t,t+h) \\ &= -\nu_o - \frac{1}{h}\sum_{j=0}^{h-1} b_{g,\nu}(a_{g,\nu}^{oj}(0)) - \frac{1}{h} a_{g,\nu}^{oh}(0)' Z_t, \end{aligned} \qquad (4.6)$$

generate an affine space driven by the general factors. Thus we get an affine term structure of interest rates [Duffie, Kan (1996)].

4.3 The price of the zero-coupon corporate bond

This price is given by :

$$\begin{aligned} C_i(t,t+h) &= E_t[M_{t,t+1}\dots M_{t+h-1,t+h} \mathbf{1}_{\tau_i > t+h}] \\ &= E_t\left[\exp\left(\nu_o h + \nu' \textstyle\sum_{j=1}^h Z_{t+j}\right) \exp\left(-\textstyle\sum_{j=1}^h [\alpha_{t+j} + \beta'_{t+j} Z_{t+j} + \gamma'_{t+j} Z^i_{t+j}]\right)\right] \\ &= E_t\left[\exp\left(\nu_o h - \textstyle\sum_{j=1}^h \alpha_{t+j} + \sum_{j=1}^h [\nu - \beta_{t+j}]' Z_{t+j} - \sum_{j=1}^h \gamma'_{t+j} Z^i_{t+j}\right)\right]. \end{aligned}$$

Thus by applying Lemma 1 in Appendix 1, we derive the term structure of corporate bonds.

Property 4 : The price of the zero-coupon corporate bond is :

$$C_i(t,t+h) = \exp\left[\nu_o h - \sum_{j=1}^{h} \alpha_{t+j} + B_g^{\nu-[\beta]}(t,t+h) + A_g^{\nu-[\beta]}(t,t+h)' Z_t + B_c^{-[\gamma]}(t,t+h) + A_c^{-[\gamma]}(t,t+h)' Z_t^i\right].$$

The geometric yields $y_i(t,t+h) = -\frac{1}{h}\log C_i(t,t+h)$ for different h also generate an affine term structure, now driven by both the general and specific factors. The spread is given by :

$$s_i(t,t+h) = y_i(t,t+h) - r(t,t+h)$$

$$= \frac{1}{h}\sum_{j=1}^{h}\alpha_{t+j} - \frac{1}{h}B_g^{\nu-[\beta]}(t,t+h) - \frac{1}{h}[A_g^{\nu-[\beta]}(t,t+h) - a_{g,\nu}^{oh}(0)]' Z_t - \frac{1}{h}B_c^{-[\gamma]}(t,t+h)$$

$$+\frac{1}{h}\sum_{j=0}^{h-1} b_{g,\nu}(a_{g,\nu}^{oj}(0)) - \frac{1}{h}A_c^{-[\gamma]}(t,t+h)' Z_t^i].$$

From (4.2), (4.3), we know that :

$$s_i(t,t+h) = -\frac{1}{h}\log\frac{C_i(t,t+h)}{B(t,t+h)}$$

$$= -\frac{1}{h}\log E_t^f[\mathbb{1}_{\tau_i > t+h}],$$

where E_t^f denotes the forward risk adjusted measure for term h [see Merton (1973), Pedersen, Shiu (1994), Geman, El Karoui, Rochet (1995), for definition and use of the forward risk adjusted measure].

Property 4 shows that the forward risk adjusted measure is easy to use in the affine framework (The risk neutral measure is derived in section 4.6 below). Of course the spread is nonnegative due to its expression in terms of forward risk adjusted survivor function.

When the sensitivities $\alpha_t, \beta_t, \gamma_t$ are constant we get :

$$\begin{aligned} C_i(t,t+h) &= \exp\Bigg\{ \nu_o h - \alpha h + \sum_{j=1}^{h-1} b_{g,\nu-\beta}[a^{oj}_{g,\nu-\beta}(0)] + a^{oh}_{g,\nu-\beta}(0)' Z_t \\ &+ \sum_{j=1}^{h-1} b_{c,\gamma}[a^{oj}_{c,-\gamma}(0)] + a^{oh}_{c,-\gamma}(0)' Z^i_t \Bigg\}, \end{aligned}$$

$$\begin{aligned} y_i(t,t+h) &= -\nu_o + \alpha - \frac{1}{h}\sum_{j=o}^{h-1} b_{g,\nu-\beta}[a^{oj}_{g,\nu-\beta}(0)] - \frac{1}{h} a^{oh}_{g,\nu-\beta}(0)' Z_t \\ &- \frac{1}{h}\sum_{j=0}^{h-1} b_{c,-\gamma}\left[a^{oj}_{c,-\gamma}(0)\right] - \frac{1}{h} a^{oh}_{c,-\gamma}(0)' Z^i_t, \end{aligned}$$

$$\begin{aligned} s_i(t,t+h) &= \alpha + \frac{1}{h}\sum_{j=0}^{h-1} \left[b_{g,\nu}\left[a^{oj}_{g,\nu}(0)\right] - b_{g,\nu-\beta}[a^{oj}_{g,\nu-\beta}(0)] - b_{c,-\gamma}[a^{oj}_{c,-\gamma}(0)] \right] \\ &+ \frac{1}{h}[a^{oh}_{g,\nu}(0) - a^{oh}_{g,\nu-\beta}(0)]' Z_t - \frac{1}{h} a^{oh}_{c,-\gamma}(0)' Z^i_t. \end{aligned}$$

4.4 Decomposition of the spread for a zero coupon corporate bond

In the standard actuarial approach default is assumed independent of the riskfree rates and is priced according to the historical probability [Fons (1994)]. Thus the actuarial value of a corporate bond is :

$$C^a_i(t,t+h) = B(t,t+h) P[\tau_i > t+h | \tau_i > t, Z_t, Z^i_t]. \tag{4.7}$$

By considering the associated actuarial yields we get :

$$\begin{aligned} y^a_i(t,t+h) &= r(t,t+h) + \pi_i(t,t+h), \\ \text{where : } \pi_i(t,t+h) &= -\frac{1}{h}\log P[\tau_i > t+h | \tau_i > t, Z_t, Z^i_t] \\ &= -\frac{1}{h}\sum_{k=1}^{h} \log P[\tau_i > t+k | \tau_i > t+k-1, Z_t, Z^i_t] \\ &= \frac{1}{h}\sum_{h=1}^{h} \lambda^{i,f}_{t+k}, \text{ say,} \end{aligned} \tag{4.8}$$

can be interpreted as an averaged (forward) default intensity [17]. However the actuarial formula (4.7), which is frequently used by the markets to estimate the default probabilities from the spreads $s_i(t, t+h)$, is not valid in the general framework. The aim of this subsection is to derive a more accurate decomposition of the spread.

Let us first note that, from Corollary 1, the averaged default intensity is given by :

$$\begin{aligned}\pi_i(t,t+h) &= \frac{1}{h}\sum_{j=1}^{h}\alpha_{t+j} - \frac{1}{h}B_g^{-[\beta]}(t,t+h) - \frac{1}{h}A_g^{-[\beta]}(t,t+h)'Z_t \\ &- \frac{1}{h}B_c^{-[\gamma]}(t,t+h) - \frac{1}{h}A_c^{-[\gamma]}(t,t+h)'Z_t^i.\end{aligned}$$

We deduce the Proposition below.

Proposition 5 :

$$\begin{aligned}& s_i(t,t+h) - \pi_i(t,t+h) \\ = \; & -\frac{1}{h}[A_g^{\nu-[\beta]}(t,t+h) - A_g^{-[\beta]}(t,t+h) - a_{g,\nu}^{oh}(0)]'Z_t \\ & -\frac{1}{h}[B_g^{\nu-[\beta]}(t,t+h) - B_g^{-[\beta]}(t,t+h) - \sum_{j=0}^{h-1} b_{g,\nu}[a_{g,\nu}^{oj}(0)]].\end{aligned}$$

The averaged default intensity absorbs all the idiosyncratic variability in spreads. Even if both the spread term structure $s_i(t, t+h)$, as well as the term structure of averaged default intensity $\pi_i(t, t+h)$ depend on the stochastic state of the i^{th} corporation, their difference does not and is the same for all corporations of this industry.

The correcting term appearing in the decomposition of the spread given in Proposition 5 measures the price of the correlation between default and sdf, due to common factors. Thus Proposition 5 provides the following decomposition of the term structure of corporate bonds :

[17] Indeed the (forward) intensity $\lambda_{t+k}^{i,f} = -\log P(\tau_i > t+k | \tau_i > t+k-1, Z_t, Z_t^i)$ differs from the (spot) intensity $\lambda_{t+k}^{i} = -\log P[\tau_i > t+k | \tau_i > t+k-1, Z_{t+k-1}, Z_{t+k-1}^i]$.

term structure of corporate bonds
= term structure of T-bonds
+ term structure of averaged default intensity
+ term structure of correlation between sdf and default.

The correcting term, which can be of any sign, takes a simplified form when the sensitivities are time independent.

Corollary 4 : When the sensitivities $\alpha_t, \beta_t, \gamma_t$ are time independent, we get :

$$\begin{aligned} & s_i(t,t+h) - \pi_i(t,t+h) \\ = & -\frac{1}{h}[a^{oh}_{g,\nu-\beta}(0) - a^{oh}_{g,-\beta}(0) - a^{oh}_{g,\nu}(0)]' Z_t \\ - & \frac{1}{h}\sum_{j=0}^{h-1}\left[b_{g,\nu-\beta}[a^{oj}_{g,\nu-\beta}(0)] - b_{g,-\beta}[a^{oj}_{g,-\beta}(0)] - b_{g,\nu}[a^{oj}_{g,\nu}(0)]\right]. \end{aligned}$$

It is easily checked that this correcting factor is zero whenever Z_t is partitioned into two independent subvectors Z_{1t}, Z_{2t} and the conformable partitionings of ν and β are $(\nu_1', 0)'$ and $(0, \beta_2')'$, respectively. Thus the correcting term disappears when default and sdf are influenced by independent factors.

4.5 Term structure of yield and spread for a first-to-default basket

A first-to-default basket with residual maturity h, written on n firms provides at $t+h$ a cash-flow of 1\$, if no firms default before $t+h$. The price at t of this basket is :

$$C(t,t+h) = E_t[M_{t,t+h} \mathbb{1}_{\tau^* > t+h}], \forall t, h,$$

where $\tau^* = \min_{i=1,\dots,n} \tau_i$.

Computations similar to those presented in the previous sections give the following results.

Proposition 6 :

$$
\begin{aligned}
C(t,t+h) &= \exp[\nu_o h - n\sum_{j=1}^{h}\alpha_{t+j} + B_g^{\nu-n[\beta]}(t,t+h) + A_g^{\nu-n[\beta]}(t,t+h)'Z_t \\
&+ nB_c^{-[\gamma]}(t,t+h) + A_c^{[\gamma]}(t,t+h)'[Z_t^1+\ldots+Z_t^n]], \\
s(t,t+h) &= \frac{n}{h}\sum_{j=1}^{h}\alpha_{t+j} - \frac{1}{h}B_g^{\nu-n[\beta]}(t,t+h) - \frac{1}{h}\left[A_g^{\nu-n[\beta]}(t,t+h) - a_{g,\nu}^{oh}(0)\right]' Z_t \\
&- \frac{n}{h}B_c^{-[\gamma]}(t,t+h) + \frac{1}{h}\sum_{j=0}^{h-1} b_{g,\nu}(a_{g,\nu}^{oh}(0)) \\
&- \frac{1}{h}A_c^{-[\gamma]}(t,t+h)'(Z_t^1+\ldots+Z_t^n), \\
\pi(t,t+h) &= \frac{n}{h}\sum_{j=1}^{h}\alpha_{t+j} - \frac{1}{h}B_g^{-n[\beta]}(t,t+h) - \frac{1}{h}A_g^{-n[\beta]}(t,t+h)'Z_t \\
&- \frac{n}{h}B_c^{-[\gamma]}(t,t+h) - \frac{1}{h}A_c^{-[\gamma]}(t,t+h)(Z_t^1+\ldots+Z_t^n), \\
s(t,t+h) &- \pi(t,t+h) = -\frac{1}{h}[A_g^{\nu-n[\beta]}(t,t+h) - A_g^{-n[\beta]}(t,t+h) - a_{g,\nu}^{oh}(0)]'Z_t \\
&- \frac{1}{h}[B_g^{\nu-n[\beta]}(t,t+h) - B_g^{-\nu[\beta]}(t,t+h) - \sum_{j=1}^{h-1} b_{g,\nu}[a_{g,\nu}^{oj}(0)].
\end{aligned}
$$

Thus we still get affine term structures for first-to-default basket. The results are simplified if the sensitivities are constant.

Corollary 6 : If the sensitivities are constant, we get :

$$
\begin{aligned}
C(t,t+h) &= \exp[\nu_o h - n\alpha h + \sum_{j=0}^{h-1} b_{g,\nu-n\beta}(a^{oj}_{g,\nu-n\beta}(0)) + a^{oh}_{g,\nu-n\beta}(0)' Z_t \\
&+ n\sum_{j=1}^{h-1} b_{c,-\gamma}[a^{oj}_{c,-\gamma}(0)] + a^{oh}_{c,-\gamma}(0)'[Z^1_t + \ldots + Z^n_t], \\
s(t,t+h) &= n\alpha + \frac{1}{h}[\sum_{j=1}^{h-1} b_{g,\nu}(a^{oj}_{g,\nu}(0))] - b_{g,\nu-n\beta}[a^{oj}_{g,\nu-n\beta}(0)] \\
&- nb_{c,-\gamma}[a^{oj}_{c,-\gamma}(0)] + \frac{1}{h}[a^{oh}_{g,\nu}(0) - a^{oh}_{g,\nu-n\beta}(0)' Z_t \\
&- \frac{1}{h} a^{oh}_{c,-\gamma}(0)'[Z^1_t + \ldots + Z^n_t], \\
\pi(t,t+h) &= n\alpha - \frac{1}{h}\sum_{j=0}^{h-1}[b_{g,-n\beta}(a^{oj}_{g,-n\beta}(0)) + nb_{c,-\gamma}[a^{oj}_{c,-\gamma}(0)]] \\
&- \frac{1}{h} a^{oh}_{g,-n\beta}(0)' Z_t - \frac{1}{h} a^{oh}_{c,-\gamma}(0)'[Z^1_t + \ldots + Z^n_t], \\
s(t,t+h) &- \pi(t,t+h) = \frac{1}{h}\sum_{j=0}^{h-1}[b_{g,\nu}[a^{oj}_{g,\nu}(0)) + b_{g,-n\beta}(a^{oj}_{g,\nu-n\beta}(0)) \\
&- b_{g,\nu-2\beta}[a^{oj}_{g,\nu-2\beta}(0)]] \\
&+ \frac{1}{h}[a^{oh}_{g,\nu}(0) + a^{oh}_{g,-n\beta}(0) - a^{oh}_{g,\nu-n\beta}(0)]' Z_t.
\end{aligned}
$$

Moreover the term structure of survivor default intensity $\pi(t, t+h)$ can be decomposed into two parts. The first part corresponds to the marginal effect of individual default risks, obtained under the independence assumption [18]:

$$
\pi^*(t,t+h) = \sum_{i=1}^{n} \pi_i(t,t+h).
$$

[18] keeping the same marginal risks.

The second part corresponds to the residual $\pi(t,t+h)-\pi^*(t,t+h)$. Note that the sign of the residual term depends on the type of dependence between the durations. Let us for illustration consider two firms $n=2$. We get :

$$\begin{aligned}
& \pi(t,t+h)-\pi^*(t,t+h) \\
= & -\frac{1}{h}\log P[\tau_1>t+h,\tau_2>t+h|\tau_1>t,\tau_2>t] \\
+ & \frac{1}{h}\log P[\tau_1>t+h|\tau_1>t,\tau_2>t)+\frac{1}{h}\log P[\tau_2>t+h|\tau_1>t,\tau_2>t].
\end{aligned}$$

This quantity is nonnegative if and only if :

$$\begin{aligned}
& P[\tau_1>t+h,\tau_2>t+h|\tau_1>t,\tau_2>t]\leq P[\tau_1>t+h|\tau_1>t,\tau_2>t]P[\tau_2>t+h|\tau_1>t,\tau_2>t] \\
\Leftrightarrow & \text{ Cov }[\mathbb{1}_{\tau_1>t+h},\mathbb{1}_{\tau_2>t+h}|\tau_1>t,\tau_2>t]\leq 0.
\end{aligned}$$

Thus $\pi(t,t+h)$ is larger than $\pi^*(t,t+h)$, when τ_1 and τ_2 features negative dependence [19].

To summarize, the term structure of first-to-default basket yields can be decomposed as :

$$\begin{aligned}
y(t,t+h) &= r(t,t+h)+\pi^*(t,t+h)+[\pi(t,t+h)-\pi^*(t,t+h)] \\
&+ [s(t,t+h)-\pi(t,t+h)] \qquad (4.9)
\end{aligned}$$

corresponding, respectively, to :

(1) the term structure of T-bond yields,
(2) the term structure of marginal defaults effect,
(3) the term structure of default correlation,

[19] This result is a consequence of standard properties on positive quadrant dependence [see e.g. Joe (1997)]. More precisely if (τ_1,τ_2) and (τ_1^*,τ_2^*) are two pairs of variables such that τ_1 and τ_1^* [resp. τ_2 and τ_2^*] have the same marginal distribution, if the copula of (τ_1,τ_2) is larger than the copula of (τ_1^*,τ_2^*), then $\min(\tau_1^*,\tau_2^*)$ stochastically dominates $\min(\tau_1,\tau_2)$ at first order.

(4) the term structure of the effect of the correlation between stochastic discount factor and default.

4.6 Risk neutral dynamics

Let us introduce the state vector $Y_t = (Y_{i,t}, i = 1, \ldots, n)$, where $Y_{i,t}$ denotes the state of borrower i at date t :

$$
\begin{aligned}
Y_{i,t} &= 1, \text{ if firm } i \text{ is alive at t,} \\
&= 0, \text{ otherwise.}
\end{aligned}
$$

Under the historical probability the joint distribution of the processes is characterized by the successive conditional densities of $Y_{t+1}, Z_{t+1}, \tilde{Z}_{t+1}$ given $\underline{Y_t}, \underline{Z_t}, \underline{\tilde{Z}_t}$, where $\tilde{Z}_t = (Z_t^1, \ldots, Z_t^n)$ [20] :

$$
\begin{aligned}
& f(y_{t+1}, z_{t+1}, \tilde{z}_{t+1} | \underline{y_t}, \underline{z_t}, \underline{\tilde{z}_t}) \\
= \Pi_{i \in \bar{I}_t} \mathbb{1}_{\{0\}}(y_{i,t+1}) \quad & \Pi_{i \in I_t} [\exp -y_{i,t+1}(\alpha_{t+1} + \beta'_{t+1} z_{t+1} + \gamma'_{t+1} z^i_{t+1})] \\
& [1 - \exp -(\alpha_{t+1} + \beta'_{t+1} z_{t+1} + \gamma'_{t+1} z^i_{t+1})]^{1-y_{i,t+1}} f(z_{t+1}|z_t) \tilde{f}(\tilde{z}_{t+1}|\tilde{z}_t) \\
= \quad & f(y_{t+1}|y_t, z_{t+1}, \tilde{z}_{t+1}) f(z_{t+1}|z_t) \tilde{f}(\tilde{z}_{t+1}|\tilde{z}_t),
\end{aligned}
$$

with clear notations. The decomposition of the historical conditional density will imply a similar decomposition of the conditional risk neutral density.

Let us recall that the conditional risk neutral density is given by :

$$
\begin{aligned}
& f^*(y_{t+1}, z_{t+1}, \tilde{z}_{t+1} | \underline{y_t}, \underline{z_t}, \underline{\tilde{z}_t}) \\
= \quad & \frac{f(y_{t+1}, z_{t+1}, \tilde{z}_{t+1} | \underline{y_t}, \underline{z_t}, \underline{\tilde{z}_t}) M_{t,t+1}}{\int f(y_{t+1}, z_{t+1}, \tilde{z}_{t+1} | \underline{y_t}, \underline{z_t}, \underline{\tilde{z}_t}) M_{t,t+1} dy_{t+1} dz_{t+1} d\tilde{z}_{t+1}}.
\end{aligned}
$$

By using the expression of the SDF, we deduce that,

[20] For expositional purpose, we consider latent individual factors defined even after individual defaults.

$$
\begin{aligned}
& f^*(y_{t+1}, z_{t+1}, \tilde{z}_{t+1} | \underline{y_t}, \underline{z_t}, \underline{\tilde{z}_t}) \\
= \quad & f(y_{t+1}|y_t, z_t, \tilde{z}_t) f^*(z_{t+1}|z_t) f(\tilde{z}_{t+1}|\tilde{z}_t), \qquad (4.10) \\
\text{where : } \quad & f^*(z_{t+1}|z_t) = \frac{f(z_{t+1}|z_t)\exp(\nu_o + \nu' z_{t+1})}{\int f(z_{t+1}|z_t)\exp(\nu_o + \nu' z_{t+1}) dz_{t+1}}. \qquad (4.11)
\end{aligned}
$$

Proposition 7 : Under the risk neutral distribution the processes $(Y_t), (Z_t), (\tilde{Z}_t)$ satisfy Assumptions A.1, A.2, except that the CAR dynamic of the general factor is modified ; we get :

$$E^*(\exp u' Z_{t+1} | Z_t) = \exp[a_g^*(u)' Z_t + b_g^*(u)],$$

$$
\begin{aligned}
\text{where : } \quad a_g^*(u) &= a_g(\nu + u) - a_g(\nu), \\
b_g^*(u) &= b_g(\nu + u) - b_g(\nu).
\end{aligned}
$$

Proof : Proposition 7 is a direct consequence of decomposition (4.10). We have just to compute the conditional Laplace transform of process (Z_t). We get :

$$
\begin{aligned}
E^*(\exp u' Z_{t+1} | Z_t) &= \frac{E(\exp[\nu_o + (\nu + u)' Z_{t+1}] | Z_t)}{E(\exp[\nu_o + \nu' Z_{t+1}] | Z_t)} \\
&= \frac{\exp[a_g(\nu + u)' Z_t + b_g(\nu + u)]}{\exp[a_g(\nu)' Z_t + b_g(\nu)]}.
\end{aligned}
$$

The results follows.

QED

Thus the risk correction is only performed by means of the dynamics of the general factor. In particular the risk on the idiosyncratic factors is priced to zero. It has to be noted that this correction will affect the observable survivor functions, since the associated duration distribution is derived by integrating out the future values of Z (and $\tilde{Z}$). Typically :

$$f^*(y_{t+1}|y_t, z_t, \tilde{z}_t) = f(y_{t+1}|y_t, z_t, \tilde{z}_t)[\text{see (4.10)}],$$

whereas :

$$f^*(y_{t+h}|y_t, z_t, \tilde{z}_t) \neq f(y_{t+h}|y_t, z_t, \tilde{z}_t), \text{ for } h \geq 2.$$

4.7 Observability of the factors

Until now we have not deeply discussed the interpretation of the general and corporate specific factors and especially their observability. It is well-known that the general factors included in the sdf Z^* (say) can be recovered from the observed T-bond prices, due to the affine structure (see e.g. Duffie, Kan (1996)]. Equivalently the factors Z^* can be replaced by mimicking factors with yield interpretations (see equation 4.6).

A similar argument apply to the general factors which influence default only and to the corporate specific factors. They can be recovered from the corporate yields, taking into account the observability of Z^*. Thus the discussion of factor observability is equivalent to the discussion of observability of corporate term structure of yields, that is of the number and design of the liquid corporate bonds.

4.8 Determination of the CreditVaR.

The results above can be directly used to compute the CreditVaR of a corporate bond portfolio. Let us consider at date t a portfolio involving a set S of firms. For corporation i the portfolio includes a quantity $x_{i,t}(h)$ of zero-coupon bonds with residual maturity $h, h = 1, \ldots, H$. The current value of the credit portfolio is :

$$W_t = \sum_{i \in S} \sum_{h=1}^{H} x_{i,t}(h) C_i(t, t+h).$$

whereas its future value is :

$$W_{t+1} = \sum_{i \in S \cap I_{t+1}} \sum_{h=1}^{H} x_{i,t}(h) C_i(t+1, t+h),$$

where $S \cap I_{t+1}$ is the set of individuals of S, who are still alive at $t+1$. The future portfolio value is doubly stochastic : first we don't know the set of firms which will be still alive at $t+1$; second we have to evaluate the future term structure of corporate bonds.

Due to the affine structure of the model, this future value can be written in terms of the values of the factors corresponding to date $t+1$:

$$W_{t+1} = \sum_{i \in S \cap I_{t+1}} \sum_{h=1}^{H} x_{it}(h) \exp[\bar{a}(h-1) + \bar{b}'(h-1) Z_{t+1} + \bar{c}'(h-1) Z^i_{t+1}],$$

say, where the coefficients $\bar{a}, \bar{b}, \bar{c}$ are deduced from the pricing formulas. Then the CreditVaR is defined from the quantile of the conditional distribution of W_{t+1} given the information available at time t, that is $Z_t, Z^i_t, i \in S$ (which are deduced from the observed term structures, see section 4.7). This distribution can be approximated by Monte-Carlo, in the following way.

i) First draw the future value of the factor Z^s_{t+1}, [resp. $Z^{i,s}_{t+1}$] in the conditional distribution of Z_{t+1} given Z_t [resp. Z^i_{t+1} given Z^i_t].

ii) Second simulate independently the default occurrence between t and $t+1$ for the different firms in S and the drawn values of the factors. $S \cap I^s_{t+1}$ denotes the simulated set of surviving firms.

iii) Deduce the simulated future value of the portfolio by :

$$W^s_{t+1} = \sum_{i \in S \cap I^s_{t+1}} \sum_{h=1}^{H} x_{it}(h) \exp[\bar{a}(h-1) + \bar{b}'(h-1) Z^s_{t+1} + \bar{c}'(h-1) Z^{i,s}_{t+1}].$$

iv) Replicate the procedure for $s = 1, \ldots, \bar{s}$, where $\bar{s}$ is the total number of replications.

v) Approximate the CreditVaR by the associated empirical quantile from the sample distribution of $W^1_{t+1}, \ldots, W^{\bar{s}}_{t+1}$ [21].

5. Monte-Carlo study

The decompositions of the spreads are illustrated in this section by Monte-Carlo studies.

[21] The number of required computations depend on the risk level chosen to define the VaR. If the VaR is computed at 1%, the approach may require for instance $\bar{s} = 500$ replications, and simply consists in retaining the 5^{th} future simulated value of the portfolio ranked in increasing order. Of course it does not seem reasonable to select a much smaller threshold, such as 0.03%, say, even if a similar value can be suggested by the practice of the rating agencies. Indeed this critical level is retained for instance for a AAA rating.

5.1 Case of a corporate bond

Let us consider a model with one general factor and one specific factor, and let us suppose that both factors follow autoregressive gamma processes. Therefore the functions a_g, b_g, a_c, b_c have the following expressions :

$$\begin{aligned} a_g(u) &= \frac{\rho_g u}{1 - u d_g}, \rho_g > 0, d_g > 0, u < 1/d_g, \\ b_g(u) &= -\lambda_g \log(1 - u d_g), \lambda_g > 0, \\ a_c(u) &= \frac{\rho_c u}{1 - u d_c}, \rho_c > 0, d_c > 0, u < 1/d_c, \\ b_c(u) &= -\lambda_c \log(1 - u d_c), \lambda_c > 0. \end{aligned}$$

We assume constant sensitivities and take the following numerical values :

sensitivities : $\alpha = 0.1, \beta = 2, \gamma = .1$;
s.d.f.: $\nu_o = -.01, \nu = -.2$;
initial factor values : $Z_0 = .003, Z_0^i = .3$;
factor dynamics : $\rho_g = .9, d_g = .1, \lambda_g = .1, \rho_c = .9, d_c = .1, \lambda_c = .1$.

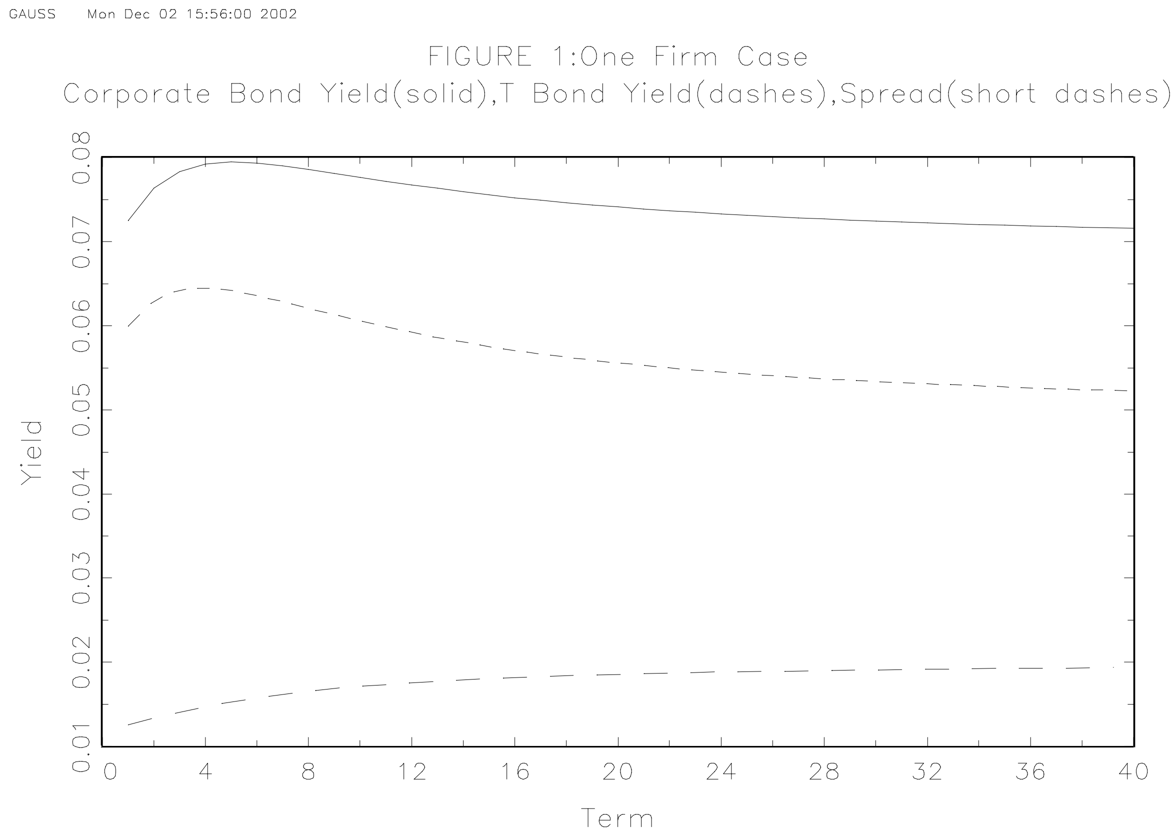

Figure 1 displays the term structures of the corporate yield of the T-bond yield and of the spread up to $h = 40$,. Bump shapes of the corporate yield and of the spread, often observed on bond markets, can easily be obtained even with fixed sensitivities.

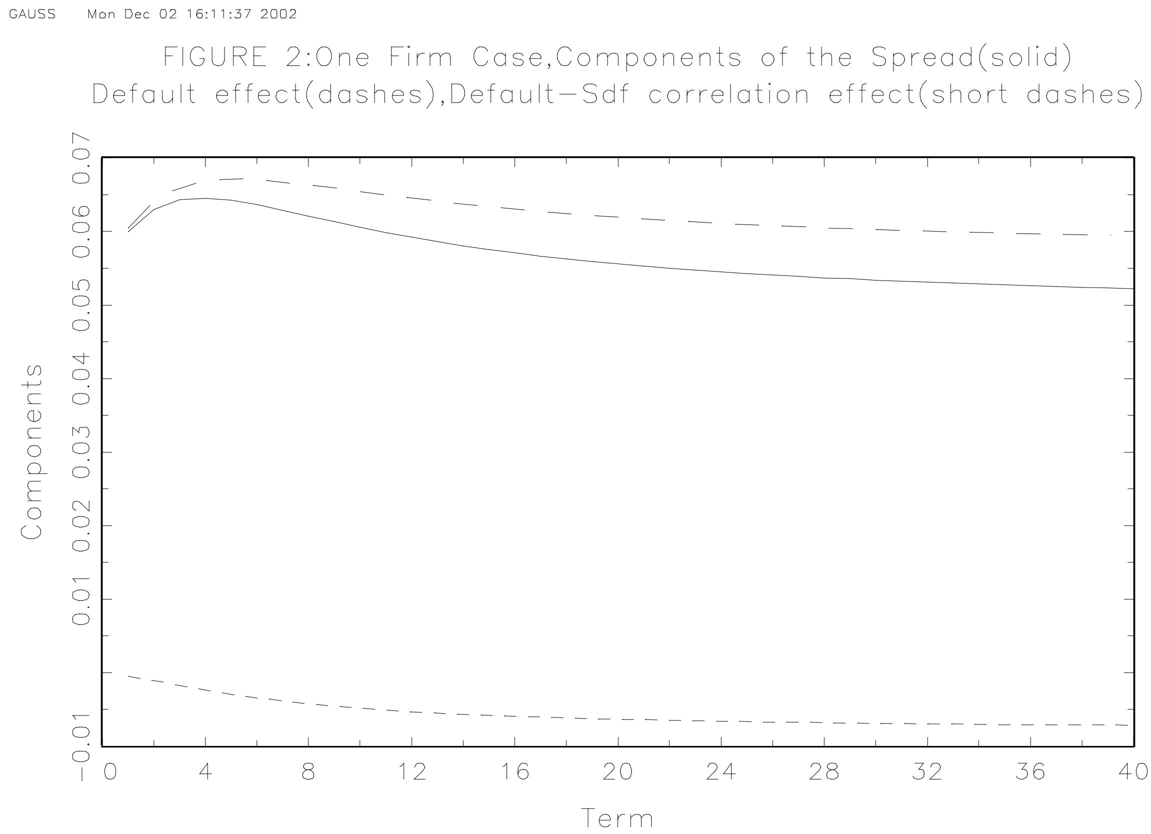

Figure 2 displays the decomposition of the spread into a default effect and a default-sdf correlation effect. In the simulation the latter effect is negative.

5.2 Case of a first-to-default basket.

Let us now consider a first-to-default basket on n firms. We still assume a general factor and univariate specific factors, following autoregressive gamma processes. We also assume fixed sensitivities. The numerical values are :

portfolio size : $n = 3$;
sensitivities : $\alpha = .01, \beta = .05, \gamma = .01$;
s.d.f. : $\nu_o = -.15, \nu = .05$;
initial values : $Z_0 = 1, Z_0^i = 1, i = 1, 2, 3$;
factor dynamics : $\rho_g = .9, d_g = .1, \lambda_g = 1; \rho_c = .9, d_c = .1, \lambda_c = 1$.

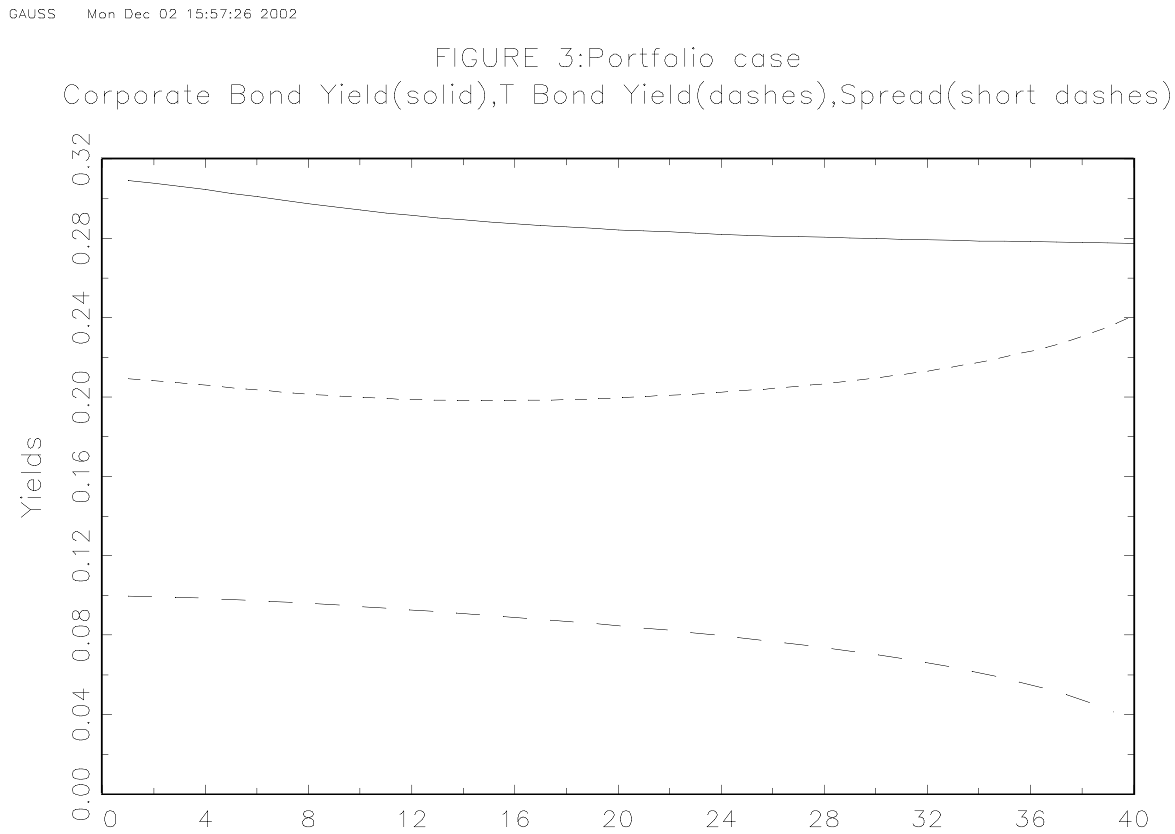

Figure 3 displays the term structures of the first-to-default yield, of the T-bond yield and of the spread. In the example both yields are decreasing, whereas the spread is first decreasing, and then increasing.

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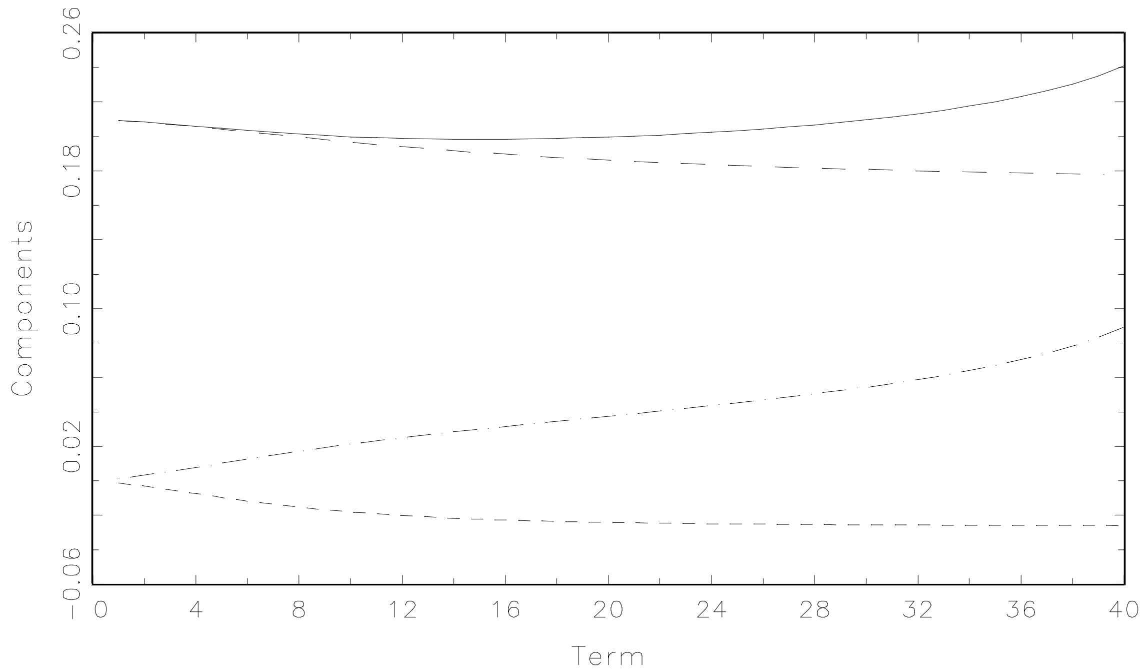
FIGURE4:Portfolio Case,Components of the Spread(solid)
Marginal Default Effect(dashes),Defaults correlation effect(short dashes)
Default−Sdf correlation effect(dots and dashes))



The decomposition of the spread is provided in Figure 4. The main part corresponds to the marginal default effect. The default correlation effect is negative (since the durations τ_1, τ_2, τ_3 feature positive quadrant dependence), whereas the default sdf correlation effect is positive.

6. Concluding remarks

We have described in this paper the affine framework for the analysis of credit risk. This framework assumes stochastic discount factor and survivor intensities, which are exponential affine functions of underlying affine factor processes. The affine framework offers a coherent description of the T-bond and corporate bond prices, and tractable methods for predicting the risk included in a portfolio of corporate bonds.

The affine framework can be used in a similar way to relax the assumption of zero recovery rate and incorporate the indeterminacy on magnitude and timing of recovery when default occurs. Similarly, it can also be used to analyse some events different from default, such as the up or down-grade by rating agencies. Indeed such an analysis has to be done before using the basic migration models.

Appendix 1

Conditional multivariate Laplace transform for a CAR process

Let us consider a multivariate CAR process, which satisfies :

$$E(\exp u' Z_{t+1} | Z_t) = \exp[a(u)' Z_t + b(u)].$$

Given a deterministic sequence [u] of vectors $\{u_s, s = 1, \ldots\}$, let us define the transformation:

$$E[\exp(u'_{t+1} Z_{t+1} + \ldots + u'_{t+h} Z_{t+h}) | Z_t],$$

which provides the conditional joint Laplace transform of $Z_{t+1}, \ldots, Z_{t+h}$.
The following lemma holds:
Lemma 1 : For any deterministic sequence [u] of vectors $\{u_s, s = 1, \ldots\}$, we have :

$$\begin{aligned} & E[\exp(u'_{t+1} Z_{t+1} + \ldots + u'_{t+h} Z_{t+h}) | Z_t] \\ = \; & \exp[A^{[u]}(t, t+h) Z_t + B^{[u]}(t, t+h)], \end{aligned}$$

where the operators $A^{[u]}$ and $B^{[u]}$, depend on functions $a(.)$, $b(.)$ as well as on sequence [u], and satisfy the backward recursion :

$$\begin{aligned} A^{[u]}(t, t+h) &= a[u_{t+1} + A^{[u]}(t+1, t+h)], \\ B^{[u]}(t, t+h) &= b[u_{t+1} + A^{[u]}(t+1, t+h)] + B^{[u]}(t+1, t+h), \end{aligned}$$

for $h > 0$, with terminal conditions :

$$A^{[u]}(t, t) = 0, B^{[u]}(t, t) = 0 \;, \forall \; t.$$

Proof : We get :

$$\begin{aligned}
& E[\exp(u'_{t+1}Z_{t+1}+\ldots+u'_{t+h}Z_{t+h})|Z_t] \\
= \; & E\left(E[\exp(u'_{t+1}Z_{t+1}+\ldots+u'_{t+h}Z_{t+h})|Z_{t+1}]|Z_t\right) \\
= \; & E(\exp[u'_{t+1}Z_{t+1}+A^{[u]}(t+1,t+h)'Z_{t+1}+B^{[u]}(t+1,t+h)]|Z_t) \\
= \; & E(\exp([u_{t+1}+A^{[u]}(t+1,t+h)]'Z_{t+1})|Z_t)\exp(B^{[u]}(t+1,t+h)) \\
= \; & \exp\{a[u_{t+1}+A^{[u]}(t+1,t+h)]'Z_t+B^{[u]}(t+1,t+h)+b[u_{t+1}+A^{[u]}(t+1,t+h)]\}.
\end{aligned}$$

The recursion follows by identification.

Finally the terminal conditions are satisfied, since :

$$A^{[u]}(t,t+1)=a(u_{t+1}), B^{[u]}(t,t+1)=b(u_{t+1}),$$

are deduced from the recursive equations applied with $A^{[u]}(t+1,t+1)=0, B^{[u]}(t+1,t+1)=0$.

QED

Appendix 2

Conditional survivor functions

i) Let us first compute the unconditional survivor function, given the stochastic factor path. By iterated expectation we deduce from Assumption A.2 that :

$$P[\tau_i > h|\underline{Z}, \underline{Z^i}] = \Pi_{t=1}^{h} \exp\{-(\alpha_t + \beta_t' Z_t + \gamma_t' Z_t^i)\},$$

where as before $\underline{Z} = (Z_t, \forall t \geq 0)$, $\underline{Z^i} = (Z_t^i, \forall t \geq 0)$.

In particular $P(\tau_i > h|\underline{Z}, \underline{Z^i}) = P(\tau_i > h|\underline{Z}_h, \underline{Z^i}_h)$,

where $\underline{Z}_h = (Z_t, t \leq h), \underline{Z_h^i} = (Z_t^i, t \leq h)$. We also deduce that :

$$P[\tau_i \leq h|\underline{Z}, \underline{Z^i}] = P[\tau_i \leq h|\underline{Z}_h, \underline{Z^i}_h],$$

$$P[\tau_i = h|\underline{Z}, \underline{Z^i}] = P[\tau_i = h_i|\underline{Z}_h, \underline{Z^i}_h].$$

ii) Then the joint survivor function given the entire realization of the stochastic path of the factor processes is :

$$\begin{aligned}
& P[\tau_i > h_i, i \in S \subseteq I_o|\underline{Z}, \underline{Z^j}, j = 1, \ldots, n] \\
= \;& P[\tau_i > h_i, i \in S \subseteq I_o|\underline{Z}, \underline{Z^j}, j \in S] \\
= \;& \Pi_{i \in S} P[\tau_i > h_i|\underline{Z}, \underline{Z^i}] \\
= \;& \Pi_{i \in S} \Pi_{j=1}^{h_i} \exp\{-(\alpha_j + \beta_j' Z_j + \gamma_j' Z_j^i)\}.
\end{aligned}$$

iii) We deduce that :

$$P[\tau_i > t + h_i, i \in S | I_t, S \subseteq I_t, \underline{Z}_t, \underline{Z^j}_t, j = 1, \ldots, n]$$

$$= \frac{P[\tau_i > t + h_i, i \in S, \tau_k > t, k \in I_t - S | \underline{Z}_t, \underline{Z^j}_t, j \in I_t]}{P[\tau_i > t, i \in I_t | \underline{Z}_t, \underline{Z^j}_t, j \in I_t]}$$

$$= \frac{E\left[\Pi_{i\in S}\Pi_{j=1}^{t+h_i} \exp\left(-(\alpha_j + \beta_j' Z_j + \gamma_j' Z_j^i\right) | \underline{Z_t}, \underline{Z_t^j}, j \in S\right]}{\Pi_{i\in S}\Pi_{j=1}^{t} \exp[-(\alpha_j + \beta_j' Z_j + \gamma_j' Z_j^i]}$$

$$= E\left[\Pi_{i\in S}\Pi_{j=1}^{h_i} \exp\left(-(\alpha_{t+j} + \beta_{t+j}' Z_{t+j} + \gamma_{t+j}' Z_{t+j}^i)\right) | \underline{Z_t}, \underline{Z_t^j}, j \in S\right]$$

$$= E[\Pi_{i\in S}\Pi_{k=1}^{h_i} \exp[-\beta_{t+k}' Z_{t+k} - \alpha_{t+k}] | Z_t]$$

$$\Pi_{i\in S} E[\Pi_{k=1}^{h_i} \exp(-\gamma_{t+k}' Z_{t+k}^i) | Z_t^i], \text{ from Assumption A.1,}$$

$$= E[\Pi_{k=1}^{\bar{h}} \exp(-n_{t+k}\beta_{t+k}' Z_{t+k}) | Z_t]$$

$$\Pi_{i\in S} E[\Pi_{k=1}^{h_i} \exp[-\gamma_{t+k}' Z_{t+k}^i] | Z_t^i]$$

$$\Pi_{k=1}^{\bar{h}} \exp[-n_{t+k}\alpha_{t+k}], \qquad \text{(A.1)}$$

where : $\bar{h} = \max_{i\in I} h_i$ and n_{t+k} denotes the number of firms in set I with $h_i > k$.

iv) By a similar argument this quantity is also equal to :

$$P[\tau_i > t + h_i, i \in S | I_t, S \subseteq I_t, \tau_j = y_j, j \in \bar{I}_t, \underline{Z}_t, \underline{Z}_{j,t}, j = 1, \ldots, n],$$

where the values $y_j \leq t, \forall j \in \bar{I}_t$, and Property 1 is shown.

v) Proof of Proposition 2

This is a direct consequence of Lemma 1 of Appendix 1 applied to formula (A.1).

Appendix 3

Default correlation in the stochastic intensity framework

The aim of this appendix is to compare the correlation values that can be reached in a framework where stochastic intensities are driven by a common factor Z with those that can be attained in a framework where the joint default distribution is directly specified. To simplify the exposition let us consider a unitary horizon H=1, and two companies with survivor times τ_1 and τ_2, respectively, with identical marginal distribution. The duration variables can only take value 0 and 1 and the joint survival distribution is completely characterized by the 2x2 contingency table:

$$\begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix},$$

subject to identical marginal distributions: $p_{10} + p_{11} = p_{01} + p_{11} = p$. The contingency table can also be written as:

$$\begin{pmatrix} 1 - 2p + p_{11} & p - p_{11} \\ p - p_{11} & p_{11} \end{pmatrix},$$

where p and p_{11} have to satisfy the inequalities below which ensure that the probabilities are well defined:

$$\max(0, 2p - 1) \leq p_{11} \leq p.$$

i) The correlation between τ_1 and τ_2 is equal to $\dfrac{p_{11} - p^2}{p(1-p)}$. The maximal [resp. minimal] value of the correlation is obtained for $p_{11} = p$ (resp. $p_{11} = \max(0, 2p-1)$]. Thus the range for the correlation is $[-\dfrac{p}{1-p}, 1]$, if $p < \dfrac{1}{2}$, $[-\dfrac{1-p}{p}, 1]$, if $p > \dfrac{1}{2}$.

This range always include $[0, 1]$, that is the maximal range for positive correlation.

ii) We have now to check that this range of positive correlation can also be reached with a factor model. Let us consider a general factor Z. Conditionally to Z, the durations are independent with identical distribution $\mathcal{B}(1, \exp(-\alpha - \beta Z))$. By denoting $W = \exp(-\alpha - \beta Z)$, we note that τ_1

and τ_2 are conditionally independent with identical Bernoulli distribution $\mathcal{B}(1, W)$. It is easily checked that Corr $(\tau_1, \tau_2) = \dfrac{VW}{VW + E[W(1-W)]}$. It is equal to zero if W is constant ; it is equal to one if the factor W only takes value 0 and 1.

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