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Counterparty Risk and the Pricing of Defaultable Securities

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ABSTRACT

Motivated by recent financial crises in East Asia and the United States where the downfall of a small number of firms had an economy-wide impact, this paper generalizes existing reduced-form models to include default intensities dependent on the default of a counterparty. In this model, firms have correlated defaults due not only to an exposure to common risk factors, but also to firm-specific risks that are termed "counterparty risks." Numerical examples illustrate the effect of counterparty risk on the pricing of defaultable bonds and credit derivatives such as default swaps.

IT HAS BEEN WELL DOCUMENTED in Moody's reports on historical default rates of corporate bond issuers that the number of defaults, the number of credit rating downgrades, and credit spreads are all strongly correlated with the business cycle. This has motivated reduced-form models such as Duffie and Singleton (1999) and Lando (1994, 1998), which assume that the intensity of default is a stochastic process that derives its randomness from a set of state variables such as the short-term interest rate. This approach has the convenient features that conditioning on the state variables, defaults become independent events, and default correlation arises due to the common influence of these state variables. On the other hand, a default intensity that depends linearly on a set of smoothly varying macroeconomic variables is unlikely to account for the clustering of defaults around an economic recession. This is evident from a casual inspection of the exhibits in the latest Moody's report (see Keenan (2000)). Within the usual affine framework, one can model the state variables as jump diffusions (see Duffie, Pan, and Single-

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ton (2000)). But presently there is no evidence whether the magnitude of jumps needed to justify the clustering of defaults is consistent with well-defined, observable macroeconomic variables.[1]

This paper complements the current literature on default risk modeling by introducing the concept of counterparty risk. In our model, each firm has a unique (firm-specific) counterparty structure that arises from its relation with other firms in the economy. Counterparty risk is the risk that the default of a firm's counterparty might affect its own default probability. This approach has two benefits. First, to the extent that the public is aware of the counterparty relations, the prices of marketed securities will reflect the market's assessment of the importance of counterparty risk. Relying on the notion of market efficiency, our model allows the extraction of this information, which is potentially important for the pricing of defaultable bonds and credit derivatives, as we demonstrate below. Second, the additional default correlation introduced by counterparty relations makes it straightforward to account for the observed clustering of defaults. For instance, a group of firms can be so highly interdependent that a single default can trigger a cascade of defaults. Because the likelihood of default is higher for all firms during a recession, this cascading effect is much more likely to be observed then. This has important implications for the management of credit risk portfolios, where default correlation needs to be explicitly modeled.

Our concept of counterparty risk has been motivated by a series of recent events in which firm-specific risks figure prominently. Several such examples are given below:

- The South Korean banking crisis was commonly attributed to nonperforming loans made to a handful of chaebols (industrial conglomerates).
- Long Term Capital Management's potential default had implications for the likelihood of default of other major investment banks. This was the given reason for the rescue effort led by the Federal Reserve Bank of New York in September 1998.
- Bankers Trust revealed its $350 million position in Russian assets on August 31, 1998. A month later, Standard and Poor's downgraded its senior debt.
- Goldman Sachs committed $2 billion in bridge loan to United Pan-Europe Communications on March 22, 2000. Two months later, UPC's stock price was down 69 percent and Goldman Sachs was unable to find a buyer for this loan.
- First Boston extended a $457 million bridge loan to the purchaser of Ohio Mattress in 1989. It ended up owning Ohio Mattress and could not recover its loan. This resulted in a takeover by Credit Suisse.

[1] As Duffee (1999) points out, the empirical test of these reduced-form models is still in its infancy. However, this body of literature has been growing rapidly. For example, see Duffee (1999), Hund (1999), Keswani (1999), Taurén (1999), and Zhang (1999). None of these studies address the issue of default correlation, however.

While there are numerous stories like these in the financial press, a common thread is that a firm will face significant counterparty risk when its portfolio is concentrated in just a few positions. As these positions change, often unexpectedly, the firm is likely to run into financial difficulties. The rational anticipation of these future events implies that there should be a response in the term structure of credit spreads today. It is this discernible shift that reveals the relevant information. In this context, traditional structural or reduced-form approaches are inadequate because they ignore this firm-specific source of risk.

The concept of counterparty risk, however, is not limited to the portfolio perspective. With the adoption of just-in-time manufacturing processes, firms are increasingly reliant on the smooth operation of other firms upstream, and a broken link is likely to have an impact on a firm's likelihood of bankruptcy. A case in point is the heavy loss suffered by GM when its Delphi plant went on strike in 1998. Our model can capture this sort of industrial organization interdependencies in default intensities. Similarly, in the study of mortgage prepayment behavior, one could specify an empirical prepayment function that incorporates not only macroeconomic factors such as interest rates, but also regional demographic and economic variables that could affect prepayment. For example, the default of a key local industry could increase the likelihood of prepayment in the related locality as unemployed workers relocate to find new opportunities. These are but a few of the potential applications of our framework.

The modeling of counterparty risk is achieved through an extension of the existing reduced-form models.[2] In our model, the intensity of default is influenced not only by a set of economy-wide state variables, but also by a collection of counterparty-specific jump terms capturing interfirm linkages. This setup allows us to maintain the simplicity of a reduced-form model while incorporating firm-specific information provided by the market.[3] In the limit as firms hold well-diversified credit risk portfolios, the counterparty risk part of their default intensities will disappear and our model reduces to a standard one.

We start Section I by assuming the existence of a collection of doubly stochastic Poisson processes representing default. The default intensities could potentially depend on the status of all the other firms in the economy. Within this general framework, we provide a pricing formula for defaultable bonds that generalizes a key result in Lando (1994, 1998). In Section II, we discuss the subtleties involved in the simultaneous construction of several default

[2] Examples of reduced-form models of default risk are Lando (1994, 1998), Jarrow and Turnbull (1995), Jarrow, Lando and Turnbull (1997), Madan and Unal (1998), and Duffie and Singleton (1999). A comprehensive survey of the literature can be found in Jeanblanc and Rutkowski (1999).

[3] A consideration for counterparty risk should also be possible within structural models such as Black and Scholes (1973), Merton (1974), and Longstaff and Schwartz (1995). One would simply model the counterparty relation as the holding of a compound option. Indeed, such an analysis can be implemented using the method of Delianedis and Geske (1998).

processes. Through a simple example, we show that the complexity of the model explodes whenever there exist "loops" in the counterparty structure. We then place a further restriction on the model to simplify subsequent exposition. In Sections III and IV, we explore the implication of counterparty risk on bond pricing through several examples. These include simple ones where default is independent of the default-free term structure, and more realistic ones where the default intensity depends linearly on a Gaussian spot rate. In Section V, we extend our model to the pricing of credit derivatives, focusing on default swaps and the importance of default correlation in pricing a default swap. We conclude with Section VI. All proofs and some detailed derivations are contained in the Appendices.

I. The Model

Our model generalizes Lando (1994, 1998) to include counterparty default risk. Lando's reduced-form model allows for dependency between credit and market risk through the use of a doubly stochastic Poisson process (also called a "Cox process"). In a Cox process, the intensity of default, which measures the likelihood of default per unit time, is itself a stochastic process that depends on a set of economy-wide state variables. Lando models these state variables as continuously varying diffusion processes. Our innovation is the inclusion of jump processes in this set of state variables, thereby capturing the interdependence among several default processes. Because of this feature, the construction of the default processes becomes recursive, since a complete default history of all other interdependent firms is needed to construct the default process for a given firm. This added complexity is fully analyzed below.

As the primary goal of this section is to extend Lando's defaultable bond pricing formula to include interdependent default risk, we initially assume the existence of these default processes and defer their construction to Section II. In the following, we outline the three ingredients of our model: the default process, the recovery mechanism, and the pricing formula.

A. Default Process

Let the uncertainty in the economy be described by the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t=0}^{T^*}, P)$, where $\mathcal{F} = \mathcal{F}_{T^*}$ and P is an equivalent martingale measure under which discounted bond prices are martingales. We assume the existence and uniqueness of P, so that bond markets are complete and priced by arbitrage,[4] as shown in discrete time by Harrison and Kreps (1979)

[4] For a set of conditions leading to the existence and uniqueness of the equivalent martingale measure when bond prices are driven by diffusions as well as a finite number of jump processes, see Jarrow and Madan (1995).

and in continuous time by Harrison and Pliska (1981). Subsequent specifications of the model are all under the equivalent martingale measure P.

On this probability space there is an R^d-valued process X_t, which represents d economy-wide state variables. There are also I point processes, N^i, $i = 1,\ldots,I$, initialized at 0. These represent the default processes of the I firms in the economy such that the default of the ith firm occurs when N^i jumps from 0 to 1. The properties satisfied by these point processes will be specified shortly.

The filtration is generated collectively by the information contained in the state variables and the default processes:[5]

$$\mathcal{F}_t = \mathcal{F}_t^X \vee \mathcal{F}_t^1 \vee \cdots \vee \mathcal{F}_t^I, \tag{1}$$

where

$$\mathcal{F}_t^X = \sigma(X_s, 0 \le s \le t) \text{ and } \mathcal{F}_t^i = \sigma(N_s^i, 0 \le s \le t) \tag{2}$$

are the filtrations generated by X_t and N_t^i, respectively.

Define a new filtration $\mathcal{G}_t^i$ as follows. First, we let the filtration generated by the default processes of all firms other than that of the ith be denoted by

$$\mathcal{F}_t^{-i} = \mathcal{F}_t^1 \vee \cdots \vee \mathcal{F}_t^{i-1} \vee \mathcal{F}_t^{i+1} \vee \cdots \vee \mathcal{F}_t^I. \tag{3}$$

Then, let

$$\mathcal{G}_t^i = \mathcal{F}_t^i \vee \mathcal{F}_{T^*}^X \vee \mathcal{F}_{T^*}^{-i}. \tag{4}$$

Since $\mathcal{F}_0^i$ is the trivial σ-field, $\mathcal{G}_0^i = \mathcal{F}_{T^*}^X \vee \mathcal{F}_{T^*}^{-i}$. We note that $\mathcal{G}_0^i$ contains complete information on the state variables and the default processes of all firms other than that of the ith, all the way up to time T^*.

Conditioning upon $\mathcal{G}_0^i$ is equivalent to conditioning upon the state of the macro economy and the default status of all the remaining firms. In a world that evolves according to the filtration $\mathcal{G}_t^i$, it is possible to select, at time 0, a complete history of a process λ_t^i, nonnegative, $\mathcal{G}_0^i$-measurable and satisfying $\int_0^t \lambda_s^i ds < \infty$, P-a.s. for all $t \in [0,T^*]$, so that an inhomogeneous Poisson process N^i can be defined, using the realized history of the process λ_t^i as its intensity function. This will be the point process governing default for firm i, since it has an intensity process that depends on state variables as well as the default status of all the other firms. This two-step randomization procedure, in fact, provides the intuition behind a formal definition of a doubly stochastic Poisson process in Brémaud (1981).

[5] The symbol $\mathcal{F} \vee \mathcal{G}$ represents the smallest σ-field containing both $\mathcal{F}$ and $\mathcal{G}$.

Let τ^i denote the first jump time of N^i. Then the conditional and unconditional distributions of τ^i are given by

$$P(\tau^i > t \mid \mathcal{G}_0^i) = \exp\left(-\int_0^t \lambda_s^i ds\right), \quad t \in [0, T^*] \tag{5}$$

and $$P(\tau^i > t) = E \exp\left(-\int_0^t \lambda_s^i ds\right), \quad t \in [0, T^*]. \tag{6}$$

This completes our specification of the default process.

We note that the computation of bond prices with counterparty risk generally requires the joint distribution of several first jump times. Due to the presence of counterparty risk, these first jump times may no longer be assumed independent conditional on the complete history of the state variables. We again defer the discussion to Section II.

B. Recovery Rate

The second ingredient in the modeling of defaultable bonds is the recovery rate. Our specification of the recovery rate is as follows.

Let $p(t,T)$ denote the time-t price of a default-free zero-coupon bond that pays one dollar at time T where $0 \leq t \leq T \leq T^*$. Let $v^i(t,T)$ denote the time-t price of a zero-coupon bond maturing at time T, issued by firm i where $i = 1, .., I$. These corporate bonds are subject to default. When firm i defaults, a unit of its bond will pay an exogenously specified constant fraction $\delta^i \in [0,1)$ of a dollar at maturity, so that the value of the bond after default is given by δ^i times the price of a default-free bond.[6] This is the "recovery of Treasury" assumption in Jarrow and Turnbull (1995) and Jarrow, Lando, and Turnbull (1997).

An alternative to the above is the "recovery of market value" where a fraction of predefault market value is recovered immediately upon default. However, with this alternative assumption, only the loss rate $\lambda(1 - \delta)$ can be recovered from zero-coupon bond prices.[7] To estimate the default intensity and the recovery rate simultaneously, one would have to resort to either the prices of credit derivatives, as suggested by Duffie and Singleton (1999), or the prices of equity or equity options, as suggested by Jarrow (1999).

[6] The recovery rate δ could in general be an $\mathcal{F}_t$-adapted stochastic process. This allows recovery to vary with the business cycle and the status of counterparties. The assumption of a constant recovery rate facilitates the derivation of a simple pricing formula.

[7] Using this recovery assumption, Duffie and Singleton (1999) show that the defaultable bond price can be expressed as the discounted expected value of a sure dollar, where the discount factor is the spot rate plus the loss rate.

C. Pricing

We now derive a pricing formula for defaultable bonds. First, let r_t denote the spot rate process adapted to $\mathcal{F}_t^X$.[8] The spot rate process could come from any arbitrage-free default-free term structure model, such as Heath, Jarrow, and Morton (1992, HJM hereafter).

Since P is an equivalent martingale measure, the money market account and bond prices are given by

$$B(t) = \exp\left(\int_0^t r_s\,ds\right), \tag{7}$$

$$p(t,T) = E_t\left(\frac{B(t)}{B(T)}\right), \tag{8}$$

and
$$v^i(t,T) = E_t\left(\frac{B(t)}{B(T)}\left(\delta^i 1_{\{\tau^i \le T\}} + 1_{\{\tau^i > T\}}\right)\right), \tag{9}$$

where $E_t(\cdot)$ denotes the expectation conditional on time-t information $\mathcal{F}_t$. The following proposition expresses equation (9) in terms of the default intensity.

PROPOSITION 1: *The defaultable bond price is given by*

$$v^i(t,T) = \delta^i p(t,T) + 1_{\{\tau^i > t\}}(1-\delta^i)E_t \exp\left(-\int_t^T (r_s + \lambda_s^i)\,ds\right), \quad T \ge t. \tag{10}$$

Proof: This is an extension of Lando (1994, Proposition 3.3.1). Details are in Appendix A.

Equation (10) is very intuitive. It states that the risky bond's price can be divided into two components. The first component is the recovery rate δ^i that is received for sure, discounted to time t. The second component is the residual $1 - \delta^i$ in the event of no default, also discounted to time t. The second term reflects the default risk since it is discounted with an adjusted spot rate. Pricing under the equivalent martingale measure thus depends critically on the evaluation of the expectation in equation (10).

II. Construction of Default Processes

The previous section assumes the existence of a collection of doubly stochastic Poisson processes, whose intensities satisfy a special measurability condition. Using this and only this knowledge, the distribution of the first

[8] The spot rate process is defined by $r_t = -\lim_{T\to t} + \partial \log p(t,T)/\partial T$.

jump times can be derived. The bond price of a firm whose default probability is affected only by macroeconomic conditions and not by the default of other firms can be easily calculated using equation (10). For a firm, A, whose default probability is strongly affected by the default of firm B, the calculation of its bond price entails knowing the distribution of the default time for B. However, if B holds a significant amount of debt issued by A, the distribution of the default time for B would then depend on that of A. In general, whenever such a relationship forms a loop, the calculation of bond prices is a difficult task. In the following, we first use a simple example to illustrate the complexity involved when looping default is introduced into the model. We then impose a restriction on the structure of our model in order to eliminate looping default.

A. Looping Default

Consider the case where each firm holds the other firm's debt, so that when A defaults, B's default probability will jump, and vice versa. Specifically, the default intensities can be described by

$$\lambda_t^A = a_1 + a_2 1_{\{t \geq \tau^B\}}, \tag{11}$$

$$\text{and} \qquad \lambda_t^B = b_1 + b_2 1_{\{t \geq \tau^A\}}, \tag{12}$$

where a_1, a_2, b_1, and b_2 are positive constants.

The pricing of these bonds requires knowledge of the distribution of the first jump times τ^A and τ^B. However, these distributions are defined recursively through each other and can be obtained explicitly only in special cases. In a full-fledged model of looping default among three firms, the calculation of bond prices would require joint distributions of pairs of default times, and among four firms, triples of default times, and so forth. Working out these distributions is more difficult, and it is why only two firms are used in this illustration.

Let $F(t) = P(\tau^A \leq t)$ and $G(t) = P(\tau^B \leq t)$ be the marginal distribution functions for τ^A and τ^B. By equations (6) and (11),

$$\begin{aligned} P(\tau^A > t) &= E \exp\left(- \int_0^t (a_1 + a_2 1_{\{s \geq \tau^B\}})\, ds \right) \\ &= E \exp\left(- \int_0^t (a_1 + a_2)\, ds + \int_0^t a_2 1_{\{s < \tau^B\}} ds \right) \\ &= e^{-(a_1 + a_2)t} E\, e^{a_2 \min(\tau^B, t)}. \end{aligned} \tag{13}$$

Hence

$$1 - F(t) = e^{-(a_1 + a_2)t} \left(\int_0^t ds\, g(s) e^{a_2 s} + \int_t^\infty ds\, g(s) e^{a_2 t} \right), \tag{14}$$

where $g(s) = dG(s)/ds$. Multiplying both sides by $e^{(a_1+a_2)t}$, we obtain

$$e^{(a_1+a_2)t}(1 - F(t)) = \int_0^t ds\, g(s)e^{a_2 s} + e^{a_2 t}(1 - G(t)). \tag{15}$$

Differentiating the above, we arrive at an ordinary differential equation for F in terms of G and a similar equation for G in terms of F:

$$(a_1 + a_2)(1 - F) - f = a_2 e^{-a_1 t}(1 - G), \tag{16}$$

$$\text{and} \qquad (b_1 + b_2)(1 - G) - g = b_2 e^{-b_1 t}(1 - F), \tag{17}$$

where $f(t) = dF(t)/dt$.

In general, equations (16) and (17) must be solved with numerical methods. But when the firms have identical default intensities, they can be solved analytically. Let $a_1 = b_1 = a$ and $a_2 = b_2 = b$; then $F = G$ and they are given by

$$F(t) = 1 - e^{-(a+b)t+(b/a)(1-e^{-at})}. \tag{18}$$

Once the distribution functions are derived, bond prices are calculated using numerical integration.

Equation (18) shows how counterparty risk ($b > 0$) increases the probability of default prior to time t in contrast to the no counterparty risk case ($b = 0$). The change is generally nonlinear as illustrated here.

B. Primary–Secondary Framework

Although looping default generates an interesting complexity in our model, it is unlikely to occur in practical applications of our model. In a typical application one considers the default probability of a commercial bank. If bank A holds a significant amount of production firm B's debt, it is unlikely that B is also holding bank A's debt or equity, let alone amounts large enough to influence B's default probability. Thus we impose the following restriction.

ASSUMPTION 1: *Let I be the set of firms in the economy. A subset $S_1 \subset I$ contains primary firms whose default intensities depend only on $\mathcal{F}_t^X$. Its complement, $S_2 \subset I$, contains secondary firms whose default intensities depend only on $\mathcal{F}_t^X$ and the status of the primary firms.*

This assumption splits firms into one of two mutually exclusive types: primary firms and secondary firms. Primary firms' default processes only depend on macrovariables. Secondary firms' default processes depend on macrovariables and the default processes of the primary firms.

This assumption makes it easy to construct the doubly stochastic Poisson processes used in Section I. To achieve this, consider the state variables X_t that describe the evolution of the default-free term structure. For instance, X_t could be the independent Brownian motions generating the HJM model. One could also add to X_t additional factors thought to be important determinants of the default process, such as GNP indices.

The probability space on which X_t is defined is then enlarged to accommodate a set of independent unit exponential random variables, which are also independent of X_t. If we call these variables $\{E^i, 1 \leq i \leq S_1\}$,[9] then the default times of the set of S_1 primary firms can be defined as

$$\tau^i = \inf\left\{t : \int_0^t \lambda_s^i ds \geq E^i\right\}, \quad 1 \leq i \leq S_1, \tag{19}$$

where the intensity λ_t^i is adapted to $\mathcal{F}_t^X$. Since E^i is independent of X_t, the distribution of τ^i is given by

$$P(\tau^i > t \,|\, \mathcal{F}_{T^*}^X) = \exp\left(-\int_0^t \lambda_s^i ds\right), \quad t \in [0, T^*]. \tag{20}$$

Then, for each i, define $N_t^i = 1_{\{\tau^i \leq t\}}$ as firm i's default process and $\mathcal{F}_t^i = \sigma(N_s^i, 0 \leq s \leq t)$ as the information set generated by the default process of firm i. This first step of our construction is directly from Lando (1994).

In the second step, we construct the secondary default processes. We enlarge the probability space further by introducing another set of independent unit exponential random variables, which are independent of both X_t and $\{\tau^i, 1 \leq i \leq S_1\}$. If we call them $\{E^j, S_1 + 1 \leq j \leq I\}$, then the default times of the set of S_2 secondary firms can be similarly defined as

$$\tau^j = \inf\left\{t : \int_0^t \lambda_s^j ds \geq E^j\right\}, \quad S_1 + 1 \leq j \leq I, \tag{21}$$

where the intensity λ_t^j is adapted to $\mathcal{F}_t^X \vee \mathcal{F}_t^1 \vee \cdots \vee \mathcal{F}_t^{S_1}$. Since E^j is independent of both X_t and the set of primary default times, the distribution of τ^j is given by

$$P(\tau^j > t \,|\, \mathcal{F}_{T^*}^X \vee \mathcal{F}_{T^*}^1 \vee \cdots \vee \mathcal{F}_{T^*}^{S_1}) = \exp\left(-\int_0^t \lambda_s^j ds\right), \quad t \in [0, T^*]. \tag{22}$$

[9] When it does not cause any ambiguities, we use the name of a set to refer to the number of elements in it. This use should be clear from its context. Furthermore, without loss of generality, the set of firms is reordered so that the first S_1 firms are primary firms and the next S_2 firms are secondary.

For secondary firms, we assume that their default intensities have the following form:

$$\lambda_t^j = a_{0,t}^j + \sum_{k=1}^{S_1} a_{k,t}^j 1_{\{t \geq \tau^k\}}, \quad S_1 + 1 \leq j \leq I. \tag{23}$$

We assume that firm j holds only the assets of primary firms, and that $a_{k,t}^j$ is adapted to $\mathcal{F}_t^X$ for all k. Depending upon the sign of $a_{k,t}^j$, the default of firm k increases ($a_{k,t}^j > 0$) the probability that firm j defaults or decreases ($a_{k,t}^j < 0$) it.[10] This is not the most general form of secondary firm default intensity. In particular, it can be made more general by including interactions among the indicator functions. Its linear form could also be generalized.

Under the above assumption, the joint distribution of primary default times conditional on the history of the state variables, which is required in order to calculate secondary bond prices, simply becomes the product of the conditional distribution of the individual primary default times. This is a result of the definition of a primary firm. Without the primary–secondary structure, the point processes governing default are defined recursively (through their intensities) and there is no simple procedure for either their construction or the derivation of the joint distribution of the first jump times.

III. Bond Pricing When Default Is Independent of the Default-free Term Structure

In this and the next section, we use the primary–secondary framework developed in Section II to study examples of bond pricing in which explicit pricing formulas can be derived. The examples, although simple, capture the qualitative impact of counterparty risk on default and the pricing of bonds. In this section, we assume that the point processes governing default are independent of the risk-free spot interest rate. We also assume that the risk-free spot rate is the only state variable, and as such, default is not influenced by any state variables. These simplifying assumptions allow the effects of counterparty risk to be studied separately from those of interest rate risk, and therefore provide a good starting point for the analysis.

A. Single Counterparty

The simplest and perhaps most relevant case in which counterparty risk occurs is where one firm's position in another firm constitutes a significant fraction of its total assets.

[10] Some of the coefficients will be zero since firm j does not necessarily hold all primary assets. We also note that the coefficients must collectively satisfy a set of regularity conditions resulting from two requirements mentioned in Section I.A. First, the intensity must be strictly positive. Second, the expected number of jumps must not diverge within finite intervals. In the interest of brevity, we will not present the regularity conditions here.

Consider firms A and B. Assume that the spot interest rate process r_t is the result of an arbitrage-free default-free term structure model. Let firm A represent the primary firm and B the secondary firm. Let the recovery rates be δ^A and δ^B, respectively. Since no state variable influences default, let A's intensity of default be $\lambda_t^A = a > 0$. Assume that B's intensity of default depends on whether A has defaulted in the past, perhaps because B holds a significant amount of A's liabilities in its portfolio. Specifically, let

$$\lambda_t^B = b_1 + 1_{\{t \geq \tau^A\}} b_2, \tag{24}$$

where $b_1 > 0$ and b_2 could be any value that preserves the positiveness of the intensity.[11]

There are zero-coupon bonds issued by A and B. The time-t prices of zero-coupon bonds maturing at $T \leq T^*$ are given by the following proposition.

PROPOSITION 2: *The time-t prices of zero-coupon bonds issued by A and B with maturity T are*

$$\frac{v^A(t,T)}{p(t,T)} = \delta^A + (1 - \delta^A) 1_{\{\tau^A > t\}} e^{-a(T-t)} \tag{25}$$

and

$$\frac{v^B(t,T)}{p(t,T)} = \begin{cases} \delta^B + (1 - \delta^B) 1_{\{\tau^B > t\}} \dfrac{b_2 e^{-(a+b_1)(T-t)} - a e^{-(b_1+b_2)(T-t)}}{b_2 - a} & \text{if } b_2 \neq a \\ \delta^B + (1 - \delta^B) 1_{\{\tau^B > t\}} (a(T-t) + 1) e^{-(a+b_1)(T-t)} & \text{if } b_2 = a \end{cases} \tag{26}$$

if firm A has not defaulted by time t, and

$$\frac{v^B(t,T)}{p(t,T)} = \delta^B + (1 - \delta^B) 1_{\{\tau^B > t\}} e^{-(b_1+b_2)(T-t)} \tag{27}$$

if firm A has defaulted by time t.

[11] As discussed earlier, an alternative interpretation of (24) is in terms of product market interaction—think of A as a supplier of B (if $b_2 > 0$). The default of A will then disrupt the production process of B, reducing its profitability and possibly causing default to occur sooner. Firm A is treated as a primary firm, perhaps because it supplies many firms, and is therefore insulated from the default of any of its counterparties.

Proof: See Appendix B.

To see the effect of counterparty risk on firm B's bond prices, we focus on the yield spread as a function of maturity. In a continuous-time context, the yield spread between defaultable bond v and default-free bond p is defined as

$$y(t,T) = -\frac{1}{T-t}\ln\frac{v(t,T)}{p(t,T)}. \tag{28}$$

To further simplify the matter, we assume that the recovery rate is zero. Viewed in this light, the yield spread on firm B's bond is simply B's martingale default probability averaged over the time interval $T - t$.[12] In the absence of counterparty risk, b_2 is equal to 0, and the yield spread is a constant b_1 that does not depend on maturity. When $b_2 > 0$, firm B's default is "accelerated" by firm A's default, and its yield spread exhibits an increasing pattern. The opposite, a "deceleration," holds when $b_2 < 0$, so that the yield spread on B-bond is a decreasing function of maturity. For a set of reasonable parameters, these effects are illustrated in Figure 1. It is interesting to note that, although completely different by source, these effects are graphically analogous to the credit spread curves in Jarrow et al. (1997, Figure 2) when there is the possibility of credit downgrade or upgrade.

B. Multiple Counterparties

The monotonic relation between yield spread and maturity exhibited in Figure 1 suggests that perhaps more interesting shapes can be attained when counterparty risk is not limited to a single party. An example with two counterparties is given below.

[12] Using the pricing formula (10), the zero recovery assumption and the independence assumption that is valid in this section imply that

$$\begin{aligned} v(t,T) &= p(t,T)E_t \exp\left(-\int_t^T \lambda_s\, ds\right) \\ &= p(t,T)P(\tau > T). \end{aligned}$$

Hence

$$\begin{aligned} -\frac{1}{T-t}\ln\frac{v(t,T)}{p(t,T)} &= -\frac{1}{T-t}\ln P(\tau > T) \\ &= -\frac{1}{T-t}\ln(1 - P(\tau \le T)) \\ &\approx \frac{1}{T-t}P(\tau \le T) \end{aligned}$$

when the default probability is reasonably small.

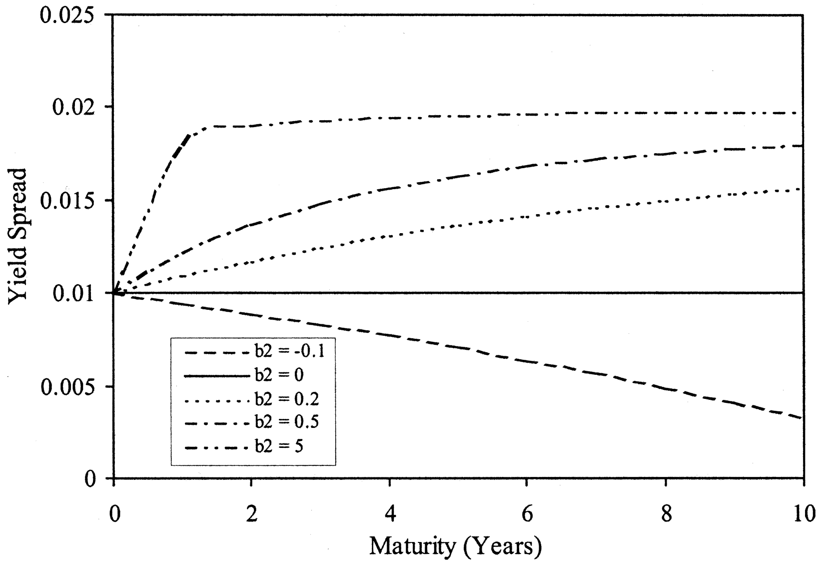

Figure 1. Term structure of yield spreads of the secondary firm *B*. This figure illustrates the term structure of yield spreads of a secondary firm with different exposures to a single counterparty. The default intensity of the secondary firm B is given in equation (24) where $b_1 = 0.01$ and b_2 can take on various values in the diagram. The primary counterparty, firm A, is assumed to have a constant default intensity of $a = 0.01$.

Consider firms A, B, and C. As in the last section, we assume that the spot interest rate is given by the process r_t, recovery rates are constants, δ^A, δ^B, δ^C, respectively, and furthermore, that no state variables influence default.

Let A and B be primary firms and C be a secondary firm. The default intensities of A and B are $\lambda_t^A = a$ and $\lambda_t^B = b$, respectively. The default intensity of C is

$$\lambda_t^C = c_0 + 1_{\{t \geq \tau^A\}} c_1 + 1_{\{t \geq \tau^B\}} c_2 + 1_{\{t \geq \tau^A,\, t \geq \tau^B\}} c_3, \tag{29}$$

where $c_0 > 0$, $c_1 > 0$ and $c_2 < 0$. The interpretation of equation (29) is that firm C holds a portfolio that contains a significant amount of long positions of A-bonds and short positions of B-bonds. Thus C's default probability increases in the event A defaults and decreases in the event B defaults. Here we allow an interaction term, so that equation (29) is a generalization of the formulation in equation (23).

The pricing of the primary A- and B-bonds is given by (25) in Section III.A. To price C-bonds, we assume that N^A and N^B are conditionally independent. This assumption says that each primary firm's default is systematically influenced only by economy-wide state variables, that is, the residual default risk is idiosyncratic once the systematic part of default has

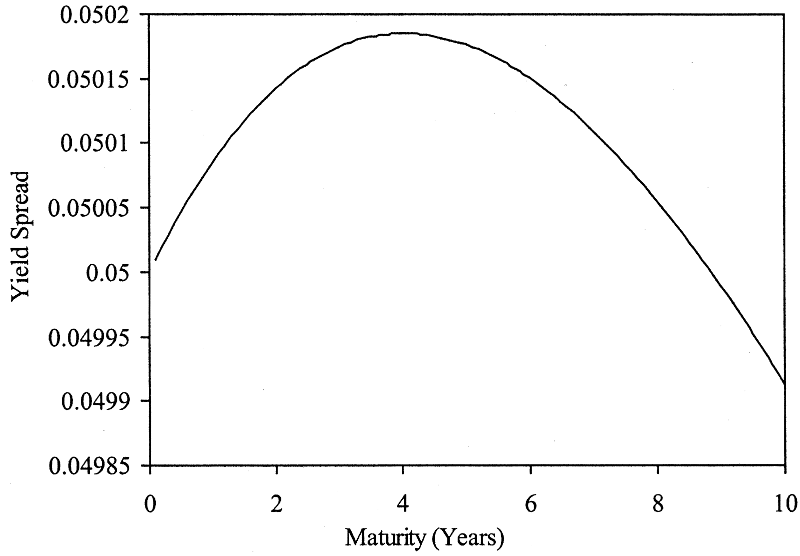

Figure 2. Term structure of yield spreads of the secondary firm *C*. This figure illustrates the term structure of yield spreads of a secondary firm with two counterparties. The default intensity of the secondary firm C is given in equation (29) where $c_0 = 0.05$, $c_1 = 0.1$, $c_2 = -0.02$, and $c_3 = 0$. The primary counterparties are firm A and firm B with constant default intensities $a = 0.01$ and $b = 0.04$, respectively.

been removed. It is consistent with the primary–secondary construction in Section II.B. Since the intensities are constants, this is equivalent to assuming unconditional independence between N^A and N^B.

If it is assumed that neither firm A nor firm B has defaulted before time t, the price of a C-bond is given by

$$\frac{v^C(t,T)}{p(t,T)} = \delta^C + (1-\delta^C)1_{\{\tau^C>t\}}E_t \exp\left(-\int_t^T \lambda_s^C\,ds\right), \tag{30}$$

where

$$E_t \exp\left(-\int_t^T \lambda_s^C\,ds\right) = e^{-c_0(T-t)}E_t\, e^{-c_1(T-\tau^A)1_{\{\tau^A\le T\}}-c_2(T-\tau^B)1_{\{\tau^B\le T\}}-c_3(T-\tau^A\vee\tau^B)1_{\{\tau^A\vee\tau^B\le T\}}}. \tag{31}$$

The expectation in the above expression can be calculated, albeit rather tediously. For details, see Appendix D.

Again assuming zero recovery rates, for a set of reasonable parameters, the yield spread on C-bonds is plotted as a function of maturity in Figure 2. It is possible for the function to be hump-shaped. This is due to the fact that the "acceleration" from the long position dominates the "deceleration" from the short position early on, but the default probability of the short party grows at a faster rate, and prevails when the maturity becomes sufficiently long.

IV. Bond Pricing When Default Is Correlated with the Default-free Term Structure

In this section we allow the default process to depend upon the spot rate of interest and consider random spot rates that follow a Gaussian process. Specifically, we use the extended Vasicek model:

$$dr(t) = a(\bar{r}(t) - r(t))\,dt + \sigma_r\,dW(t), \tag{32}$$

where $W(t)$ is a Wiener process under the equivalent martingale measure P, and $\bar{r}(t)$ is a deterministic function chosen to fit an initial term structure, with a and σ_r constants. Therefore, a one-factor model for the default-free term structure is used, where the factor is the spot rate.

Given that one may reasonably expect an interaction between credit risk and market interest rates, the introduction of a nontrivial default-free term structure into our model may enhance our ability to identify the effect of counterparty risk embedded in bond prices. In the following, we again consider the pricing of defaultable bonds issued by two firms, one primary and one secondary, that are related by the simplest counterparty structure.

A. Primary Bonds

Suppose that firm A is a primary firm whose default is independent of the default risk of other firms, but is otherwise dependent on the prevailing short-term interest rate in the economy. This is perhaps because a large portion of firm A's cash outflow is needed to service its debt. To model this, we assume a linear relation below:[13]

$$\lambda_t^A = \lambda_0^A + \lambda_1^A r_t, \tag{33}$$

Assuming a zero recovery rate and no default before t, the price of A-bond is

$$\begin{aligned} v^A(t,T) &= E_t \exp\left(-\int_t^T (r_s + \lambda_s^A)\,ds\right) \\ &= \exp\left(-\lambda_0^A(T-t) - (1+\lambda_1^A)\mu_{t,T} + \frac{1}{2}(1+\lambda_1^A)^2\sigma_{t,T}^2\right), \end{aligned} \tag{34}$$

[13] Depending on the signs of the coefficients, some values of the short rate will imply a negative intensity. One can get around this by assuming that $\lambda_t^A = \max(\lambda_0^A + \lambda_1^A r_t, 0)$, but this complicates the subsequent algebra. Longstaff and Schwartz (1995) argue that negative interest rates are unlikely for realistic parameters of the Vasicek model. We adopt a similar argument here.

where $\mu_{t,T}$ and $\sigma_{t,T}^2$ are the mean and variance of $\int_t^T r_u \, du$ defined in Appendix C. Then the yield spread is

$$y(t,T) = \lambda_0^A + \lambda_1^A \frac{\mu_{t,T}}{T-t} - \lambda_1^A \left(1 + \frac{1}{2}\lambda_1^A\right) \frac{\sigma_{t,T}^2}{T-t}, \tag{35}$$

which depends on the time-t spot rate through its linear relation with $\mu_{t,T}$. One can show that $\mu_{t,T}$ is related to $r(t)$ by

$$\mu_{t,T} = \frac{1 - e^{-a(T-t)}}{a} r(t) + A(t,T), \tag{36}$$

where $A(t,T)$ is a function that does not depend on the spot rate (see Appendix E for its derivation). Hence, when λ_1^A is positive, a higher spot rate will lead to a higher yield spread. The finding by Duffee (1998) that yield spreads on noncallable corporate bonds are negatively related to three-month Treasury yields suggests that λ_1^A should be negative. This observation has also been predicted by the theoretical model of Longstaff and Schwartz (1995).[14] However, Duffee's (1998) finding applies only to portfolios of noncallable corporate bonds formed by their credit ratings. Whether this finding holds for individual bonds in general is still an open question that awaits empirical studies.[15]

To see how a spot rate dependent intensity changes the term structure of credit spreads, we rewrite equation (35) using equation (C12) from Appendix C:

$$y(t,T) = \lambda_0^A - \lambda_1^A \frac{\ln p(t,T)}{T-t} - \frac{1}{2}\lambda_1^A (1 + \lambda_1^A) \frac{\sigma_{t,T}^2}{T-t}. \tag{37}$$

Using parameters of the extended Vasicek model estimated from U.S. data, the third term is an order of magnitude less than the second term for maturities up to 30 years (Appendix C shows that $\sigma_{t,T}^2/(T-t)$ is an increasing function of maturity). Ignoring the third term for the moment, the above equation suggests that the shape of the credit spread curve is determined by the contemporaneous default-free term structure, since the second term is simply λ_1^A multiplying the yield on a default-free zero-coupon bond with identical maturity. When the default-free term structure is upward sloping, a positive λ_1^A implies an upward sloping credit spread curve.

[14] The level of default boundary is assumed to be a constant in their model. However, a more careful treatment should allow it to migrate upward when interest rates increase, reflecting the increase of debt burden. This effect can dominate the increase of drift of the risk-neutral firm value process, causing default probability to increase instead.

[15] Some of Duffee's (1998) empirical results on portfolios have been confirmed at the individual bond level by Collin-Dufresne, Goldstein, and Martin (1999).

Since the expectations hypothesis holds in our risk-neutral world, an upward sloping default-free term structure indicates that future spot rate would increase, and so would the credit spread.

In models where the interaction between credit and market risk is ignored, such as the rating-based Markov model of Jarrow et al. (1997), investment-grade issuers usually have upward sloping credit spread curves, while speculative-grade issuers have downward sloping credit spread curves. Here we show that there is an additional dimension to this problem when economy-wide risk factors influence the likelihood of default.

We illustrate these points with Figure 3. With the exception of Panel A where the default-free term structure is flat, it is clear that the term structure of credit spreads is determined by the shape of the default-free term structure in conjunction with the coefficient λ_1^A.

B. Secondary Bonds

Suppose that firm B is a secondary firm whose default risk depends on the prevailing interest rate in the economy as well as the default risk of firm A. We assume that

$$\lambda_t^B = \lambda_0^B + \lambda_1^B r_t + c \cdot 1_{\{t \geq \tau^A\}}, \tag{38}$$

A more general formulation would require that the default of firm A could also cause a shift in the slope of the intensity function. We only illustrate the simpler case.

Assuming a zero recovery rate and no default before t, the price of a B-bond is (assuming that firm A has not defaulted before t):

$$\begin{aligned} v^B(t,T) &= E_t \exp\left(-\int_t^T (r_s + \lambda_s^B)\, ds\right) \\ &= E_t \exp(-\lambda_0^B(T-t) - (1+\lambda_1^B)R_{t,T} - c(T-\tau^A)1_{\{\tau^A \leq T\}}) \\ &= E_t(\exp(-\lambda_0^B(T-t) - (1+\lambda_1^B)R_{t,T})E_t(e^{-c(T-\tau^A)1_{\{\tau^A \leq T\}}} \mid \mathcal{F}_t \vee \mathcal{F}_{T^*}^r)), \end{aligned} \tag{39}$$

where $R_{t,T} = \int_t^T r_u\, du$. The last step above uses the law of iterated conditional expectations.

From equation (5) we recall that the conditional distribution of primary default time τ^A is

$$P_t(\tau^A > s \mid \mathcal{F}_t \vee \mathcal{F}_{T^*}^r) = \exp\left(-\int_t^s \lambda_u^A\, du\right) = e^{-\lambda_0^A(s-t) - \lambda_1^A R_{t,s}}, \quad s \in [t, T^*]. \tag{40}$$

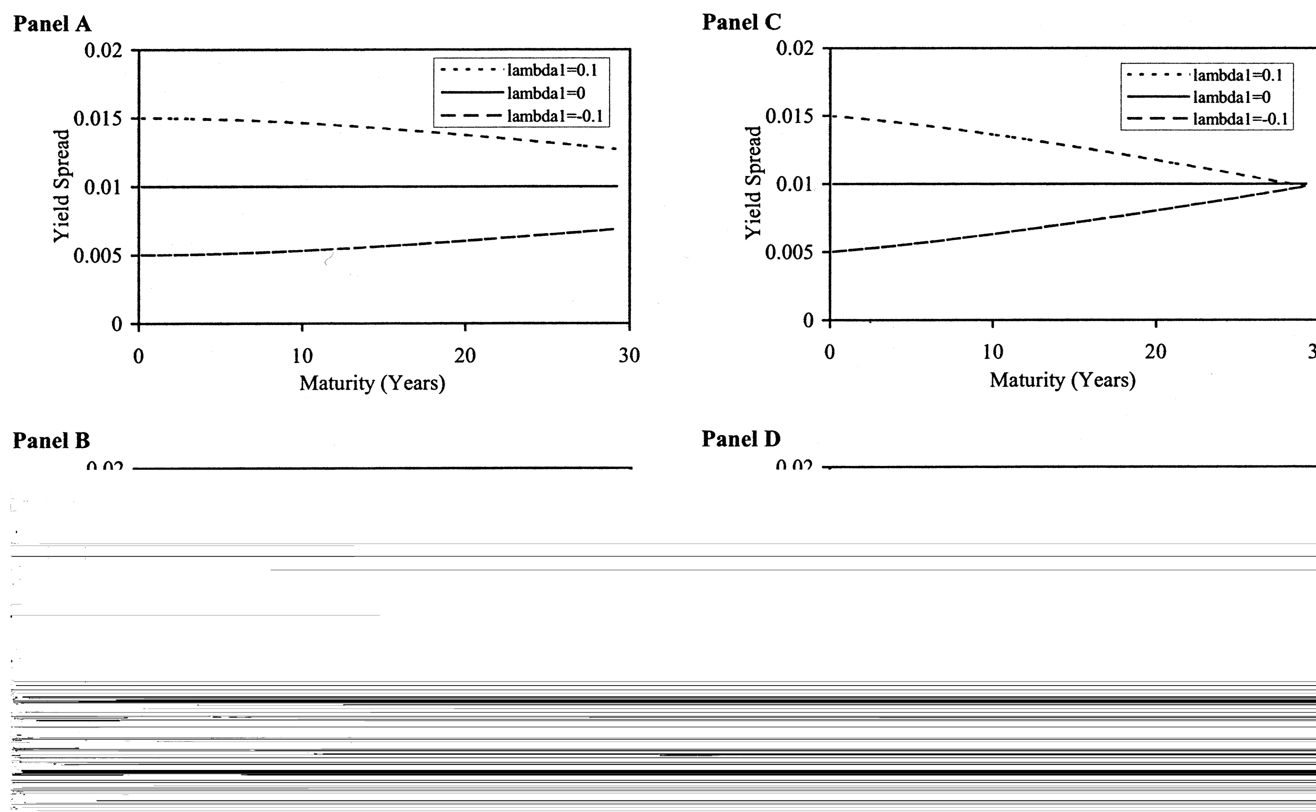

Figure 3. Term structure of yield spreads of the primary firm A when default is correlated with the spot interest rate. Panel A assumes a flat default-free term structure. Panel B assumes an upward sloping default-free term structure with a slope of 0.2 percent per year. Panel C assumes a downward sloping default-free term structure with a slope of -0.1 percent per year. Panel D uses the default-free term structure constructed from Treasury coupon strips on 02/28/97. The default intensity of firm A is given in equation (33) where $\lambda_0^A = 0.01$ and λ_1^A can be -0.1, 0, or 0.1. The parameters for the extended Vasicek model are taken from Yu (1999) as $a = 0.0254$ and $\sigma_r = 0.0157$.

Let V denote the conditional expectation embedded in equation (39); we can then proceed to evaluate it as follows:

$$\begin{aligned} V &= \left(\int_t^T + \int_T^\infty\right) e^{-c(T-s)1_{\{s\le T\}}} d\left(1 - e^{-\lambda_0^A(s-t)-\lambda_1^A R_{t,s}}\right) \\ &= e^{-c(T-t)}\left(1 + c\int_t^T e^{-(\lambda_0^A - c)(s-t)-\lambda_1^A R_{t,s}}\, ds\right). \end{aligned} \tag{41}$$

Hence

$$\begin{aligned} v^B(t,T) &= E_t\left(e^{-(\lambda_0^B + c)(T-t)-(1+\lambda_1^B)R_{t,T}}\left(1 + c\int_t^T e^{-(\lambda_0^A - c)(s-t)-\lambda_1^A R_{t,s}}\, ds\right)\right) \\ &= e^{-(\lambda_0^B + c)(T-t)-(1+\lambda_1^B)\mu_{t,T}+(1+\lambda_1^B)^2\sigma_{t,T}^2/2} \\ &\quad \times \left(1 + c\int_t^T ds \cdot e^{-(\lambda_0^A - c)(s-t)-\lambda_1^A\mu_{t,s}+(\lambda_1^A)^2\sigma_{t,s}^2/2+\lambda_1^A(1+\lambda_1^B)\rho(t,s,T)}\right), \end{aligned} \tag{42}$$

where

$$\begin{aligned} \rho(t,s,T) &= \mathrm{cov}_t(R_{t,T}, R_{t,s}) \\ &= \mathrm{cov}_t\left(\int_t^T dW(u)\, b(u,T), \int_t^s dW(u)\, b(u,s)\right) \\ &= \sigma_r^2\int_t^s du\, b(u,T)\, b(u,s), \end{aligned} \tag{43}$$

due to the independent increment property of the Wiener process.

The evaluation of equation (42) involves an interchange of the expectation and the integral, and also uses the following formula:

$$E\exp(A+B) = \exp\left(E(A) + E(B) + \frac{1}{2}\left(\mathrm{var}(A) + 2\,\mathrm{cov}(A,B) + \mathrm{var}(B)\right)\right), \tag{44}$$

where A and B are normal random variables. Equation (42) is useful in numerical procedures designed to calculate model parameters from market prices.

If the default probability of the secondary firm depends on the default processes of two or more primary firms, we assume that the default processes of the primaries are independent conditional on the path of the spot rate. The embedded conditional expectation in equation (39) then becomes a product of terms similar to that contained in equation (41). The expectation

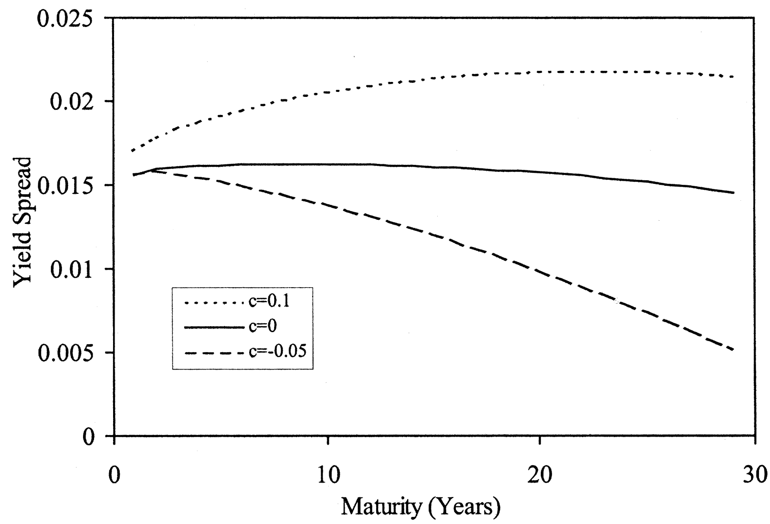

Figure 4. Term structure of yield spreads of the secondary firm *B* when default is correlated with the spot interest rate. The default-free term structure is constructed from Treasury coupon strips on 02/28/97. The default intensity of the primary firm A is given in equation (33) where $\lambda_0^A = 0.01$ and $\lambda_1^A = 0$. The default intensity of the secondary firm B is given in equation (38) where $\lambda_0^B = 0.01$, $\lambda_1^B = 0.1$, and the level of counterparty risk c can be -0.05, 0, or 0.1.

and multiple integrals can then be interchanged and evaluated. Again, it should be emphasized that this is based on the primary–secondary firm methodology adopted in Section II.B.

Equation (42) is complicated and does not permit a direct computation of the yield spread as in the previous section. Moreover, the level of the yield spread also depends on the shape of a contemporaneous default-free term structure. However, we expect the patterns exhibited in Figure 1 to persist. Namely, the default risk from the long (short) position causes yield spread to widen (narrow) over time.

Figure 4 not only confirms this conjecture, but it also shows how information about counterparty risk can be empirically extracted. The effect of counterparty risk results in a "twist" of the term structure of credit spreads for a primary firm (see Panel D of Figure 3), which is essentially the rescaled default-free term structure. Indeed, any deviation of the credit spread curve from the shape of the default-free term structure will be interpreted as counterparty risk by the estimation procedure. This limitation is due to the use of a simplistic one-factor term structure model and our specification of the default process as depending only on the contemporaneous level of the spot rate. It is easy to extend our model to a multifactor setting where other macroeconomic factors can also influence default. For example, a similar framework that relies on purely statistical factors is used in Duffee's (1999) modeling of corporate bond yield spreads.

V. Credit Derivatives

An investigation of counterparty risk is incomplete without studying its impact on the pricing of credit derivatives. Credit derivatives are securities whose payoffs depend on the value of credit-sensitive assets. Depending on the specifics of the derivative instrument, credit risk can enter in a variety of ways. In the case of an over-the-counter options contract, credit sensitivity could be associated with the underlying asset on which the contract is written, or it could be associated with the writer of the contract.[16] In the case of an interest rate swap, credit risk is two-sided, since both parties in the swap can default.[17] In the case of a default swap, the default risk of the two counterparties and the reference asset must be considered simultaneously.

Despite these complexities, the pricing of credit derivatives is easily handled by the martingale pricing technique. In the rest of this section we focus on default swaps. When it comes to pricing default swaps, it seems that practitioners are most concerned with the default correlation between the swap seller and the reference asset. Intuitively, when this correlation is positive and large, the buyer's fixed rate payment must be smaller than when the correlation is absent. A failure to consider this will result in an overpayment on the part of the insured. The writings in the popular press clearly reflect the need among practitioners to have a unified framework for the pricing of default swaps—specifically, one that takes proper care of default correlations.[18] The primary–secondary framework developed in this paper can do just that, by considering possible counterparty relations between the default swap seller and the reference asset. In the following subsections, we first derive a general pricing formula and then use a numerical example to illustrate the role of default correlation.

A. Default Swaps

In a default swap (sometimes also called a credit swap), party A holds bonds with some long maturity T_2. These bonds are issued by a reference party R that is subject to default. To hedge this risk, party A agrees to make a stream of payments to party B at a fixed rate (again called the "swap rate") from $t = 0$ to $T_1 < T_2$, in exchange for B's promise to compensate A for its loss up to a certain amount in the event of R's default. B's payment is

[16] When the writer is subject to default, the options contract is said to be "vulnerable." For the literature on pricing vulnerable options, see Jarrow and Turnbull (1995).

[17] For reduced-form models on two-sided default risk applied to swaps, see Duffie and Huang (1996) and Jarrow and Turnbull (1997).

[18] For example, see the recent article in the *Economist* (Is there money in misfortune, 1998) on the proliferation of default swaps in Japan and the lack of accurate pricing methodologies therein.

contingent on some kind of credit event occurring to R, such as a missed interest payment or a credit downgrade, and is payable at the expiration of the default swap. Not only can the reference asset default, but the two counterparties, A and B, can default as well. Thus the pricing of a default swap, or the determination of the swap rate, has to take into account the credit risk from all three sources.

To simplify the calculation, we assume that each party is obligated to pay until its own default, regardless of whether the other party has defaulted or not.[19] Moreover, assume that all relevant recovery rates are zero. Let the default times of A, B, and R be denoted by τ^A, τ^B, and τ^R, and the default intensities λ^A, λ^B, and λ^R. Let the swap rate needed to fully insure one dollar of reference asset from time 0 to T_1 be denoted by c. The market value of A's fixed rate payment at time 0 is

$$E\left(\int_0^{T_1} \exp\left(-\int_0^s r_u\,du\right) c 1_{\{\tau^A > s\}}\,ds\right) = c\int_0^{T_1} v^A(0,s)\,ds. \tag{45}$$

The time-0 market value of B's promised payment in the event of R's default is

$$E\left(1_{\{\tau^R \le T_1\}} \exp\left(-\int_0^{T_1} r_u\,du\right) 1_{\{\tau^B > T_1\}}\right). \tag{46}$$

Therefore the swap rate is

$$c = \frac{E\left(1_{\{\tau^R \le T_1\}} \exp\left(-\int_0^{T_1} r_u\,du\right) 1_{\{\tau^B > T_1\}}\right)}{\int_0^{T_1} v^A(0,s)\,ds}. \tag{47}$$

Further simplification of equation (47) is possible within the primary–secondary framework. For instance, assuming that the reference asset is primary (hence λ^R is adapted to $\mathcal{F}_t^X$) and party B is secondary, the above can be simplified by conditioning on the complete history of the state variables and all primary default processes. This yields

$$c = \frac{v^B(0,T_1) - E\left(\exp\left(-\int_0^{T_1} (r_u + \lambda_u^B)\,du\right) 1_{\{\tau^R > T_1\}}\right)}{\int_0^{T_1} v^A(0,s)\,ds}. \tag{48}$$

[19] Again, this may not be the case in some default swaps. Also, a default swap may stipulate that B's payment to A occur immediately after R's default, rather than at the maturity of the swap. Due to these simplifying assumptions, we call this an "idealized default swap."

B. An Example

Assume the following parameterization. Firm B's default process is

$$\lambda_t^B = b_1 + b_2 1_{\{t \geq \tau^R\}}, \tag{49}$$

where $b_1 = 0.01$ and b_2 can be freely varying. As such, B is a secondary firm whose default depends on the status of the reference asset R. The reference asset R has a default process of

$$\lambda_t^R = d, \tag{50}$$

where $d = 0.01$. For simplicity, we assume that the firm buying the default swap on R, firm A, is default free. We also assume a constant interest rate of $r = 0.05$.

With these parameters, it is shown (in Appendix F) that equation (48) implies the following default swap rate:

$$c = \frac{rd(e^{-(b_1+d)T_1} - e^{-(b_1+b_2)T_1})}{(b_2 - d)(e^{rT_1} - 1)}. \tag{51}$$

This correct swap rate is shown in Panel A of Figure 5 as a function of the maturity of the swap for different values of b_2. A successively higher b_2 indicates an increasing default correlation between the swap seller, B, and the reference asset R. As b_2 becomes very large, we expect the fair swap rate to drop to zero since the default swap is ineffective.

In practice, it is possible for a financial institution to have the right marginal distributions to price all of its assets correctly, and yet end up with the wrong evaluation of its credit exposure.[20] For instance, firm A could estimate a model such as equation (49) correctly based on the prices of securities issued by firm B. However, if it ignores the fact that the jump term in equation (49) is caused by the reference asset R, it will still misprice its default swap. In effect, firm B is being treated as a secondary firm whose default depends on an unidentified counterparty whose default is independent of the reference asset. In this case one can show (also in Appendix F) that the default swap rate is

$$c = \frac{r(1 - e^{-dT_1})(b_2 e^{-(b_1+d)T_1} - de^{-(b_1+b_2)T_1})}{(b_2 - d)(e^{rT_1} - 1)}. \tag{52}$$

This swap rate as a function of maturity is shown in Panel B of Figure 5. A comparison between the two panels shows that the default swap is severely overpriced due to the ignored default correlation.

[20] We thank the referee for mentioning this point.

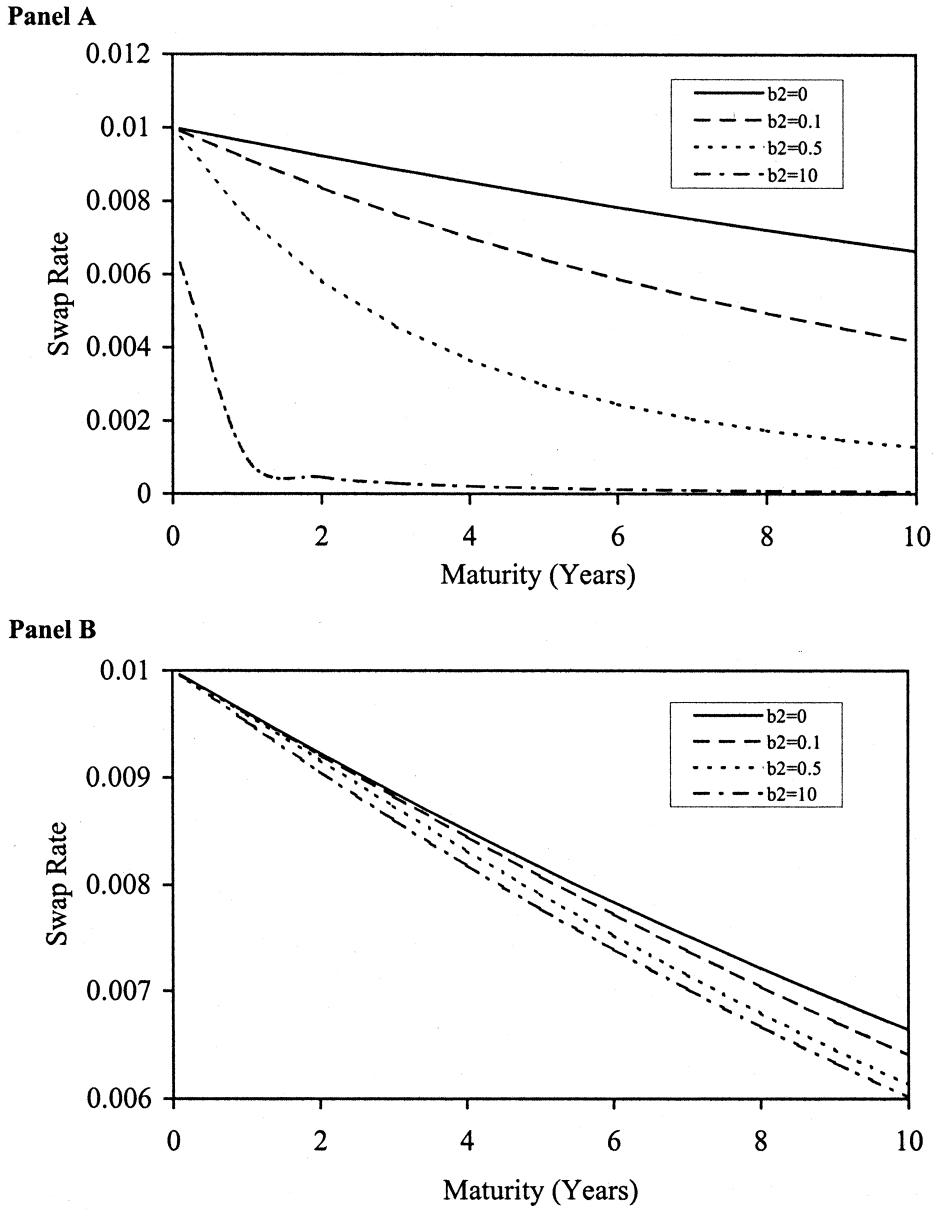

Figure 5. Term structure of default swap rates. The swap buyer, firm A, is assumed to be default-free. The seller, firm B, has a one percent default probability per annum, which can jump up by b_2 depending on the status of the reference asset. The reference asset R has a one percent default probability per annum. Panel A assumes that the counterparty relation between B and R is properly identified by A. Therefore the default swap rate is generated with equation (51). In Panel B, firm A correctly estimates the intensity parameters for firm B but is not aware of the counterparty relation between B and R. The default swap rate is therefore generated with equation (52). The interest rate is assumed to be a constant five percent. The level of counterparty risk, b_2, varies in the diagram.

VI. Conclusion

In this paper we present a reduced-form model that jointly describes defaultable bonds issued by many firms, where a counterparty relationship among these firms could affect their default probabilities. This is a new contribution to the literature. We provide examples with explicit bond pricing formulas and show that market-wide risk factors and firm-specific counterparty risks interact to generate a variety of shapes for the term structure of credit spreads. We also show how credit derivatives are priced within such a framework, and explain how mispricings are produced by models that ignore counterparty relations.

The implications of this model can be empirically investigated. For instance, in cases where the market perceives a counterparty relation, one can estimate its impact on default using bond prices and see whether the said effect exists. When this is confirmed, more elaborate bond pricing models can be constructed, and their parameters can be implicitly estimated and used to price credit derivatives. A candidate of this sort, based on the extended Vasicek model, can be found in Section IV of this paper.

Appendix A: Proof of Proposition 1

Proof: First it can be shown that

$$E(1_{\{\tau^i>T\}}|\mathcal{G}_t^i) = 1_{\{\tau^i>t\}} \exp\left(-\int_t^T \lambda_s^i\, ds\right). \tag{A1}$$

To show this, first note that

$$\begin{aligned} E(1_{\{\tau^i>T\}}|\mathcal{G}_t^i) &= P(\tau^i > T|\mathcal{G}_0^i \vee \mathcal{F}_t^i) \\ &= 1_{\{N_t^i=0\}} \frac{P(\tau^i > T, N_t^i = 0|\mathcal{G}_0^i)}{P(N_t^i = 0|\mathcal{G}_0^i)} \\ &= 1_{\{\tau^i>t\}} \frac{P(\tau^i > T|\mathcal{G}_0^i)}{P(\tau^i > t|\mathcal{G}_0^i)} \\ &= 1_{\{\tau^i>t\}} \frac{\exp\left(-\int_0^T \lambda_s^i\, ds\right)}{\exp\left(-\int_0^t \lambda_s^i\, ds\right)} \\ &= 1_{\{\tau^i>t\}} \exp\left(-\int_t^T \lambda_s^i\, ds\right). \end{aligned} \tag{A2}$$

which proves (A1). To show the pricing relation in the proposition, iterate the expectation:

$$\begin{aligned} v^i(t,T) &= E_t\left(\frac{B(t)}{B(T)}(\delta^i 1_{\{\tau^i \le T\}} + 1_{\{\tau^i > T\}})\right) \\ &= E_t\left(\frac{B(t)}{B(T)}(\delta^i + (1-\delta^i)1_{\{\tau^i > T\}})\right) \\ &= E_t\left(E\left(\frac{B(t)}{B(T)}(\delta^i + (1-\delta^i)1_{\{\tau^i > T\}})|\mathcal{G}_t^i\right)\right) \\ &= E_t\left(\frac{B(t)}{B(T)}\left(\delta^i + (1-\delta^i)1_{\{\tau^i > t\}}\exp\left(-\int_t^T \lambda_s^i\, ds\right)\right)\right) \\ &= \delta^i p(t,T) + 1_{\{\tau^i > t\}}(1-\delta^i)E_t \exp\left(-\int_t^T (r_s + \lambda_s^i)\, ds\right). \quad \text{Q.E.D.} \end{aligned} \tag{A3}$$

Appendix B: Proof of Proposition 2

Proof: Because of the independence between default and the spot rate, A-bond price is given by

$$\begin{aligned} v^A(t,T) &= \delta^A p(t,T) + (1-\delta^A)1_{\{\tau^A > t\}}E_t \exp\left(-\int_t^T (r_s + a)\, ds\right) \\ &= p(t,T)(\delta^A + (1-\delta^A)1_{\{\tau^A > t\}}e^{-a(T-t)}). \end{aligned} \tag{B1}$$

This proves equation (25).

B-bond price is (assuming $\tau^A > t$)

$$v^B(t,T) = \delta^B p(t,T) + (1-\delta^B)1_{\{\tau^B > t\}}E_t \exp\left(-\int_t^T (r_s + b_1 + 1_{\{s \ge \tau^A\}}b_2)\, ds\right), \tag{B2}$$

or

$$\frac{v^B(t,T)}{p(t,T)} = \delta^B + (1-\delta^B)1_{\{\tau^B > t\}}E_t \exp(-b_1(T-t) - b_2(T-\tau^A)1_{\{\tau^A \le T\}}). \tag{B3}$$

Note that conditional on $\tau^A > t$, τ^A has the following distribution:

$$P_t(\tau^A > s) = e^{-a(s-t)}. \tag{B4}$$

It is then sufficient to show the following calculation:

$$
\begin{aligned}
E_t\, e^{-b_2(T-\tau^A)1_{\{\tau^A\le T\}}} &= \int_t^\infty ds\, a e^{-a(s-t)} e^{-b_2(T-s)1_{\{s\le T\}}} \\
&= \int_t^T ds\, a e^{-a(s-t)-b_2(T-s)} + \int_T^\infty ds\, a e^{-a(s-t)} \\
&= a e^{at-b_2T} \frac{1}{b_2-a}\left(e^{(b_2-a)T} - e^{(b_2-a)t}\right) + e^{-a(T-t)} \\
&= \frac{a\left(e^{-a(T-t)} - e^{-b_2(T-t)}\right)}{b_2-a} + e^{-a(T-t)} \\
&= \frac{b_2 e^{-a(T-t)} - a e^{-b_2(T-t)}}{b_2-a}.
\end{aligned}
\tag{B5}
$$

Implicitly assumed above is that $b_2 \neq a$. When $b_2 = a$, the following holds instead:

$$
E_t\, e^{-b_2(T-\tau^A)1_{\{\tau^A\le T\}}} = a(T-t)e^{-a(T-t)} + e^{-a(T-t)} = (a(T-t)+1)e^{-a(T-t)}. \tag{B6}
$$

If firm A has already defaulted by time t ($\tau^A \le t$), firm B's default intensity becomes $b_1 + b_2$ and similar to the calculation of A-bond price, B-bond price is given by

$$
v^B(t,T) = p(t,T)(\delta^B + (1-\delta^B)1_{\{\tau^B>t\}} e^{-(b_1+b_2)(T-t)}). \quad \text{Q.E.D.} \tag{B7}
$$

Appendix C: Specifying the Extended Vasicek Model within the HJM Framework

To fully utilize the tools developed for the HJM term structure model, we specify a forward rate process that is consistent with the extended Vasicek spot rate model.[21] To do this, first solve equation (32) using methods described in Duffie (1995, page 293). The solution to the stochastic differential equation is

$$
r(t) = r(0)e^{-at} + \int_0^t e^{a(s-t)} a\bar{r}(s)\, ds + \int_0^t e^{a(s-t)} \sigma_r\, dW(s), \tag{C1}
$$

[21] For a detailed analysis of the relation between the Vasicek model or the CIR model and the HJM model, see Brenner and Jarrow (1993). The extended Vasicek model first appeared in Hull and White (1990).

where the volatility process is given by

$$\sigma(t,T) = e^{a(t-T)}\sigma_r. \tag{C2}$$

Using the HJM drift restriction under the risk-neutral measure, the drift process is related to the volatility by

$$\begin{aligned}\alpha(t,T) &= \sigma(t,T)\int_t^T \sigma(t,s)\,ds \\ &= \frac{\sigma_r^2}{a}e^{a(t-T)}(1-e^{a(t-T)}).\end{aligned} \tag{C3}$$

Using the time-0 forward rate curve, one can give an alternative expression for the spot rate:

$$r(t) = f(0,t) + \int_0^t \alpha(v,t)\,dv + \int_0^t \sigma(v,t)\,dW(v) \tag{C4}$$

$$= f(0,t) + \frac{\sigma_r^2}{2a^2}(1-e^{-at})^2 + \int_0^t e^{a(v-t)}\sigma_r\,dW(v). \tag{C5}$$

Comparing equation (C1) with equation (C5), the following is derived:

$$\int_0^t e^{a(s-t)} a\bar{r}(s)\,ds = f(0,t) - e^{-at}f(0,0) + \frac{\sigma_r^2}{2a^2}(1-e^{-at})^2. \tag{C6}$$

Differentiating this with respect to t yields

$$\bar{r}(t) = f(0,t) + \frac{1}{a}\frac{\partial f(0,t)}{\partial t} + \frac{\sigma_r^2}{2a^2}(1-e^{-2at}). \tag{C7}$$

The requirement that the spot rate process be consistent with the time-0 forward rate curve thus uniquely determines $\bar{r}(t)$.

It is also possible to compute the integral of $r(t)$, which can then lead to explicit formulas for the discount factor and default-free zero coupon bond prices. To calculate $\int_t^T r(u)\,du$, it would be more convenient to use time-t forward rates instead of time-0 forward rates in equation (C5). Using equation (C4) and the following:

$$f(t,u) = f(0,u) + \int_0^t \alpha(v,u)\,dv + \int_0^t \sigma(v,u)\,dW(v), \tag{C8}$$

we have

$$
\begin{aligned}
\int_t^T r(u)\,du &= \int_t^T f(t,u)\,du + \int_t^T du \int_t^u \alpha(v,u)\,dv + \int_t^T du \int_t^u \sigma(v,u)\,dW(v) \\
&= \int_t^T f(t,u)\,du + \int_t^T du\, \frac{\sigma_r^2}{2a^2}\,(1 - e^{-a(u-t)})^2 \\
&\quad + \int_t^T dW(v) \int_v^T \sigma(v,u)\,du \\
&= \int_t^T f(t,u)\,du + \int_t^T du\, \frac{\sigma_r^2}{2a^2}\,(1 - e^{-a(T-u)})^2 \\
&\quad + \int_t^T dW(v)\, \frac{\sigma_r}{a}\,(1 - e^{-a(T-v)}) \\
&= \int_t^T f(t,u)\,du + \int_t^T du\, \frac{b(u,T)^2}{2} + \int_t^T dW(u)\, b(u,T),
\end{aligned}
\tag{C9}
$$

where $b(u,T) = (\sigma_r/a)(1 - e^{-a(T-u)})$. Thus its mean and variance are

$$
\mu_{t,T} \equiv E_t \int_t^T r(u)\,du = \int_t^T f(t,u)\,du + \int_t^T du\, \frac{b(u,T)^2}{2}, \tag{C10}
$$

$$
\text{and} \qquad \sigma_{t,T}^2 \equiv \text{var}_t \int_t^T r(u)\,du = \int_t^T b(u,T)^2\,du. \tag{C11}
$$

The default-free zero coupon bond price is then

$$
\begin{aligned}
p(t,T) &= E_t \exp\left(-\int_t^T r(u)\,du\right) \\
&= \exp\left(-\mu_{t,T} + \frac{\sigma_{t,T}^2}{2}\right) \\
&= \exp\left(-\int_t^T f(t,u)\,du\right),
\end{aligned}
\tag{C12}
$$

which merely asserts the definition of forward rates.

A result that would be useful in Section IV.A is that $\sigma_{t,T}^2/(T-t)$ is an increasing function of $T-t$. This can be shown as follows:

$$
\begin{aligned}
\frac{d}{d(T-t)}\left(\frac{\sigma_{t,T}^2}{T-t}\right) &= -\frac{\partial}{\partial t}\left(\frac{\sigma_{t,T}^2}{T-t}\right) \\
&= -\frac{\partial}{\partial t}\left(\frac{\int_t^T (1-e^{-a(T-u)})^2\,du}{T-t}\right) \\
&= \frac{(1-e^{-a(T-t)})^2(T-t) - \int_t^T (1-e^{-a(T-u)})^2\,du}{(T-t)^2} > 0,
\end{aligned}
\tag{C13}
$$

since $(1-e^{-a(T-u)})^2$ is decreasing in u.

Appendix D: Derivation of Equation (31)

Since N^A and N^B are independent, conditioning on $\tau^A > t$ and $\tau^B > t$, the joint probability density of τ^A and τ^B is

$$
P_t[\tau^A \in (u, u+du), \tau^B \in (v, v+dv)] = ae^{-a(u-t)}be^{-b(v-t)}\,du\,dv. \tag{D1}
$$

The region of integration is then appropriately divided into five pieces. The first piece is where $t \le \tau^A \le T$ and $\tau^A \le \tau^B \le T$:

$$
\begin{aligned}
I &= ab\int_t^T du\, e^{-a(u-t)}\int_u^T dv\, e^{-b(v-t)}e^{-c_1(T-u)-c_2(T-v)-c_3(T-v)} \\
&= ab\, e^{(a+b)t-(c_1+c_2+c_3)T}\int_t^T du\, e^{(c_1-a)u}\int_u^T dv\, e^{(c_2+c_3-b)v} \\
&= \frac{ab}{c_2+c_3-b}\, e^{(a+b)t-(c_1+c_2+c_3)T}\int_t^T du\, e^{(c_1-a)u}\left(e^{(c_2+c_3-b)T} - e^{(c_2+c_3-b)u}\right) \\
&= \frac{ab}{c_2+c_3-b}\left(\frac{e^{-(a+b)(T-t)} - e^{-(b+c_1)(T-t)}}{c_1-a} - \frac{e^{-(a+b)(T-t)} - e^{-(c_1+c_2+c_3)(T-t)}}{c_1+c_2+c_3-a-b}\right).
\end{aligned}
\tag{D2}
$$

The fifth piece is where $t \le \tau^B \le T$ and $\tau^B \le \tau^A \le T$:

$$
\begin{aligned}
V &= ab\int_t^T dv\, e^{-b(v-t)}\int_v^T du\, e^{-a(u-t)}e^{-c_1(T-u)-c_2(T-v)-c_3(T-u)} \\
&= \frac{ab}{c_1+c_3-a}\left(\frac{e^{-(a+b)(T-t)} - e^{-(a+c_2)(T-t)}}{c_2-b} - \frac{e^{-(a+b)(T-t)} - e^{-(c_1+c_2+c_3)(T-t)}}{c_1+c_2+c_3-a-b}\right).
\end{aligned}
\tag{D3}
$$

The second piece is where $t \leq \tau^A \leq T$ and $\tau^B \geq T$:

$$\begin{aligned} II &= ab\int_t^T du\, e^{-a(u-t)}\int_T^\infty dv\, e^{-b(v-t)}e^{-c_1(T-u)} \\ &= \frac{a}{c_1-a}\left(e^{-(a+b)(T-t)} - e^{-(b+c_1)(T-t)}\right). \end{aligned} \tag{D4}$$

The third piece is where $t \leq \tau^B \leq T$ and $\tau^A \geq T$:

$$\begin{aligned} III &= ab\int_t^T dv\, e^{-b(v-t)}\int_T^\infty du\, e^{-a(u-t)}e^{-c_2(T-v)} \\ &= \frac{b}{c_2-b}\left(e^{-(a+b)(T-t)} - e^{-(a+c_2)(T-t)}\right). \end{aligned} \tag{D5}$$

Finally the fourth piece is where $\tau^B \geq T$ and $\tau^A \geq T$:

$$\begin{aligned} IV &= ab\int_T^\infty du\, e^{-a(u-t)}\int_T^\infty dv\, e^{-b(v-t)}1 \\ &= e^{-(a+b)(T-t)}. \end{aligned} \tag{D6}$$

Adding up the pieces and by substitution, one obtains the price of C-bonds. Note that if either firm A or B has defaulted before time t, then the pricing formula will collapse to equation (26) in Section III.A. If both A and B have defaulted before t, then C-bond will be priced like primary bonds. Furthermore, conditions need to be imposed on the default intensities to ensure that none of the denominators in the above expressions is equal to zero. However, it is reasonable to assume that primary firms' default probabilities are much smaller than those of secondary firms. This implies that a and b are much smaller than c_1 and c_2, and that the denominators will never go to zero.

Appendix E: Derivation of Equation (36)

From equation (C1), one can solve for $r(0)$:

$$r(0) = r(t)e^{at} - \int_0^t e^{as}a\bar{r}(s)\,ds - \int_0^t e^{as}\sigma_r\,dW(s). \tag{E1}$$

This can then be substituted into the following expression:

$$r(u) = r(0)e^{-au} + \int_0^u e^{a(s-u)}a\bar{r}(s)\,ds + \int_0^u e^{a(s-u)}\sigma_r\,dW(s), \quad u \geq t, \tag{E2}$$

to yield

$$r(u) = r(t)e^{a(t-u)} + \int_t^u e^{a(s-u)} a\bar{r}(s)\,ds + \int_t^u e^{a(s-u)}\sigma_r\,dW(s), \quad u \ge t. \tag{E3}$$

Hence,

$$\begin{aligned}\int_t^T r(u)\,du = r(t)\int_t^T e^{a(t-u)}\,du + \int_t^T\left(\int_t^u e^{a(s-u)} a\bar{r}(s)\,ds\right)du \\ + \int_t^T\left(\int_s^T e^{a(s-u)}\sigma_r\,du\right)dW(s),\end{aligned} \tag{E4}$$

where the last term results from an application of Fubini's Theorem.

Therefore,

$$\mu_{t,T} = \frac{1-e^{-a(T-t)}}{a}r(t) + \int_t^T\left(\int_t^u e^{a(s-u)} a\bar{r}(s)\,ds\right)du, \tag{E5}$$

where the second term depends only on $\bar{r}(s)$, which is determined by the time-t forward rate curve. The dependence of $\mu_{t,T}$ on $r(t)$ thus comes solely from the first term in equation (E5).

Appendix F. Derivation of Equations (51) and (52)

First, we compute $v^B(0,T_1)$. This is given by the result in Proposition 2:

$$v^B(0,T_1) = \frac{b_2 e^{-(b_1+d)T_1} - d e^{-(b_1+b_2)T_1}}{b_2 - d} e^{-rT_1}. \tag{F1}$$

We then calculate the second term in the numerator of equation (48). However, this is simply

$$\begin{aligned}&E\left(\exp\left(-\int_0^{T_1}(r_u+\lambda_u^B)\,du\right)1_{\{\tau^R>T_1\}}\right)\\ &\quad = E(\exp(-(r+b_1)T_1 - b_2(T_1-\tau^R)1_{\{\tau^R\le T_1\}})1_{\{\tau^R>T_1\}})\\ &\quad = P(\tau^R>T_1)\exp(-(r+b_1)T_1)\\ &\quad = e^{-(r+b_1)T_1}e^{-dT_1}.\end{aligned} \tag{F2}$$

Combining the above results we obtain equation (51):

$$
\begin{aligned}
c &= \frac{\dfrac{b_2 e^{-(b_1+d)T_1} - d e^{-(b_1+b_2)T_1}}{b_2 - d}\, e^{-rT_1} - e^{-(r+b_1)T_1} e^{-dT_1}}{\dfrac{1}{r}\,(1 - e^{-rT_1})} \\
&= \frac{rd\,(e^{-(b_1+d)T_1} - e^{-(b_1+b_2)T_1})}{(b_2 - d)(e^{rT_1} - 1)}.
\end{aligned}
\tag{F3}
$$

To obtain equation (52), we note that equation (48) becomes

$$
c = \frac{v^B(0,T_1) - E\left(\exp\left(-\int_0^{T_1} (r_u + \lambda_u^B)\, du\right) 1_{\{\tau^Y > T_1\}}\right)}{\int_0^{T_1} v^A(0,s)\, ds}, \tag{F4}
$$

where Y is the unidentified counterparty to firm B. However, since we assume that the default of Y is independent of that of the reference asset R, the above is

$$
c = \frac{v^B(0,T_1)(1 - P(\tau^Y > T_1))}{\int_0^{T_1} v^A(0,s)\, ds} = \frac{v^B(0,T_1)(1 - e^{-dT_1})}{\int_0^{T_1} v^A(0,s)\, ds}. \tag{F5}
$$

Using previous results, equation (52) is easily obtained.

Note that the unidentified counterparty Y has the same default intensity as that of the reference asset R. This is to be expected as we assume that firm B's bonds are priced correctly.

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