DEFAULT RISK IN ASSET PRICING

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Abstract

Default risk in asset pricing

This paper provides an analytical solution for the impact of default risk on the valuation of realistically intricate claims on time dependent uncertain income streams. It can be used to price defaultable bonds and credit derivatives. Its modular structure allows us to adjust the set of assumptions concerning the event of default to the specificity of the environment which surrounds the asset. The importance of such a flexibility is illustrated in the context of corporate debt, examining the case of finite lived coupon paying corporate bonds with principal repayment at maturity. The magnitude of risk premia, as well as the term structure of credit spreads, are largely determined by the assumed default scenario.

Le risque de defaut dans l'evaluation d'actifs

Ce papier fournit une solution analytique d'evaluation de l'impact du risque de defaut sur la valeur d'actifs. Le actifs sont ici definis comme une promesse a une suite realistiquement compliquee de revenus incertains dependant du temps. Elle peut etre employee pour evaluer des obligations ainsi que des produits derives comportant un risque de defaut. Sa structure modulaire permet d'ajuster les suppositions concernant l'evenement de default aux specificites de l'environement. L'importance d'une telle flexibilite est illustree dans le contextes de la dette d'entreprise, en examinant le cas d'une obligation payant une suite de coupons jusqu'a la date de maturite ou un principal est alors repaye. L'ampleur de la prime de risque ainsi que sa structure dans le temps sont largement determinees par le scenario de default suppose.

1 Introduction

In recent years, a contingent-claims-based approach to asset valuation has become an important tool in incorporating the effects of real options affecting the life of assets. These options are typically decisions that an agent holding residual control rights will find optimal to take as the state of the world evolves. Such open decisions largely affect the value of the claims held by the different asset holders. The most obvious application in finance concerns the decision to default on contracted obligations.[1]

Although the risk of default decision is very influential, it has proved difficult, for sufficiently realistic contracts, to assess its impact on asset values: The exercise is relatively simple for a perpetual entitlement to a time homogeneous payoff function of the state variable. However, (i) most claims are not infinitely lived and (ii) their attached payoffs are not just state dependent, but contractually (or are expected to) depend on time.

This paper will develop, within a continuous time framework, closed-form solutions for the value of a very wide range of contracts, taking into account the possibility of default: Our contribution here is to solve for the value of the asset not only (i) when it has an overall finite maturity, but also when (ii) the payoffs are intricate functions of an economic fundamental and time.

The contracts we consider can possibly involve (i) a series of state dependent streams of income flow, each applicable for specified lengths of time. This encompasses a share on an earnings flow, a complex interest payment scheme, or simply a fixed coupon. The claim can additionally include (ii) series of lump sum payments expected to occur at different dates, such as expected dividend payments, or principal repayments.

Our framework therefore gives a rich tool that may be applied to many financial assets. We actually provide a series of examples to illustrate how this framework is to be applied to value not only corporate debt but also different credit derivatives. The methodology employed is fairly simple. The influence of the different terms of the contract remains easily separable, which enables the intuition to be developed.

Our framework is also very flexible in that it allows for different descriptions of the environment surrounding the asset: The structure of the framework is modular and we are able to adjust the set of assumptions which reflect (i) our expectations concerning the "scenario" that is most likely to trigger default and (ii) the specificity of the firm and economic environment.

This does not give us just generality, but it enables us to assess whether differences in the way the event of bankruptcy is modeled affect not only the value of a defaultable security, but also the implied term structure of credit spreads. Although the literature is very silent on this issue, as each models makes its own assumptions, we obtain that the

[1] This is the most substantially discussed application in the literature: Black and Cox (1976) derived the pricing of perpetual interest paying bonds. Leland (1994) extended the results in a capital structure model where equity and perpetual debt are priced with an endogenously determined bankruptcy rule. Bartolini and Dixit (1991) employed similar models to price the secondary market value of perpetual sovereign debt when the country has a limited ability to pay. Fries, Mella-Barral and Perraudin (1997) applied these techniques in banking, to determine the fair pricing of deposit guarantees.

conjecture concerning the expected scenario triggering default drastically modifies the value of the same asset.

We establish this as follows: First, we outline the different descriptions of the event of bankruptcy encountered in the literature and discuss their implementation in our framework. Then, we consider *two* different structural assumptions, and examine the resulting value of a debt contract with (i) a fixed unit-time coupon payment which is promised until (ii) the lump-sum repayment of the principal at the date of maturity.[2]

The bottom line result of our analysis is that the magnitude of risk premia, as well as the term structure of credit spreads, are largely determined by the initially expected bankruptcy scenario. Adjusting the expected bankruptcy scenario to the specificity of the firm and its economic context is therefore of crucial importance.

The paper is organized as follows. Section 2 defines the general class of assets we consider and provides the pricing formula. To illustrate the flexibility of the method, Section 3 applies the framework to the case of defaultable coupon paying bonds and to credit derivatives such as default swaps and default puts. Section 4 discusses possible structural assumptions and the generality of our framework. Section 5 illustrates how important differences in these structural assumptions are, implementing two different possible descriptions of the same firm. We then examine the impact on the resulting term structure of credit spreads and carry out a sensitivity analysis. Section 6 suggests extensions of the model.

2 Pricing the Risk of Default

Let x_t designate a state variable which reflects economic fundamentals and captures all the uncertainty affecting the valuation of the asset we consider. We assume that the *dynamics of economic fundamentals* follow a geometric Brownian motion

$$dx_t \;=\; x_t\,\mu\,dt \;+\; x_t\,\sigma\,dB_t, \tag{1}$$

where μ , $\sigma$are constants. Consider an asset $\mathcal{A}$, which is an entitlement

1. to a unit-period cash flow function $a(x,t)$. This can be a share on an earnings flow, a complex interest payment scheme or simply a fixed coupon. This cash flow function possibly encompasses a series of state dependent streams of income flow, each applicable for specified lengths of time. That is there exists J "starting" dates $\{T'_1,...T'_J\}$, and J "finishing" dates $\{T''_1,...T''_J\}$, such that the $a(x,t)$ takes a different form of payoff in each one of the J intervals: $a(x,t)=a_j(x)\ ,\ \ \forall t\in(T'_j;T''_j]\ .$
2. to a series of K lump sum payments $\{p_k(x)\}$, expected to respectively occur at dates $\{T_1,...T_K\}$. This can be a series of expected dividend payments, or principal repayments. These payments are function of the economic fundamental, in order to reflect, for example, the influence of an expected dividend policy.

[2]In this case, some terms in our pricing formula resemble the expression of Leland and Toft (1996) who considered this type of contract for the purpose of an optimal capital structure model.

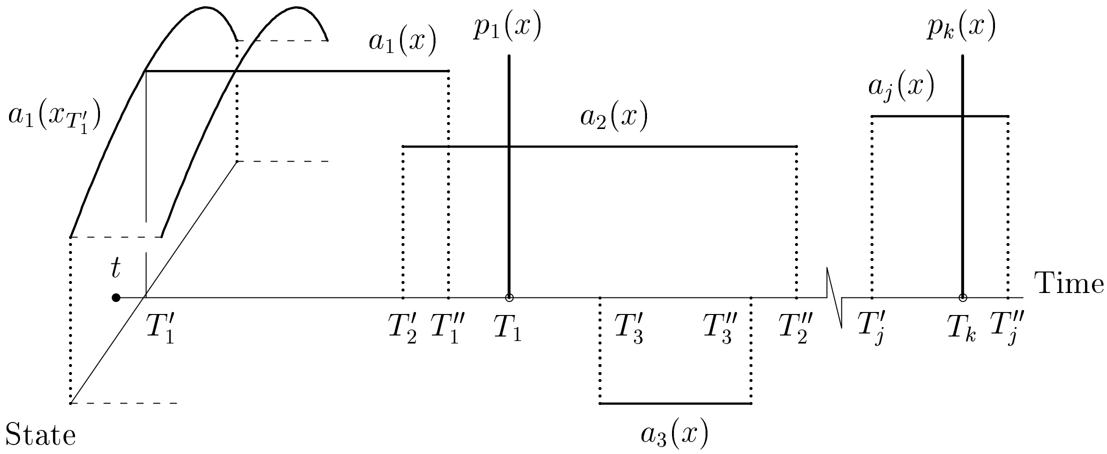

Figure 1: Time Structure of Expected Payoffs

3. to a residual claim in the event of default. Default occurs the first time the economic fundamental reaches a lower *default trigger* level $\underline{x}$.[3] In this event, the above promised payments $a(x,t)$ and $\{p_k(x)\}$ are interrupted, and we denote $\underline{V}(x)$ the *residual value of the asset in default.*

The contract has an overall maturity date T, which is just the largest of all previously cited payment dates: $T = \max\{T_J''; T_K\}$. Figure 1 provides a graphical interpretation of the time dependency of expected payoffs in such contracts.

We will now determine the present value of asset $\mathcal{A}$, which we will denote $V(x_t|t)$. We start introducing the following probability-weighted discount factor:

Let $T_{\underline{x}} \equiv \inf\{\, \tau \mid x_\tau = \underline{x} \,\}$ be the first time at which x_t hits the level $\underline{x}$, and denote $f_t(T_{\underline{x}})$ the density of $T_{\underline{x}}$ conditional on information at t (that the state is x_t). The random time $T_{\underline{x}}$ is well-defined since the sample paths of x_t are continuous almost surely. Assuming risk neutrality, and a constant identical borrowing and lending safe interest rate, ρ,[4] the Laplace transform of $f_t(T_{\underline{x}})$, which we will denote

$$\mathcal{P}(x_t \rhd \underline{x}) \;\equiv\; \int_t^{\infty} e^{-\rho(T_{\underline{x}}-t)} f_t(T_{\underline{x}})\, dT_{\underline{x}} \;, \tag{2}$$

can be interpreted very intuitively as follows: $\mathcal{P}(x_t \rhd \underline{x})$ is a probability-weighted discount factor for payoffs accruing the first time the state variable reaches a level $\underline{x}$, given that it is currently equal to x_t. For a geometric Brownian motion, $\mathcal{P}(x_t \rhd \underline{x})$ has the following very

[3]As we will discuss in Section 4, this threshold level is either (i) determined endogenously and results from the optimization of the value of agent's claim or (ii) simply exogenously given if as for protected debt, this is part of a contract.

[4]Harrison and Kreps (1979) show how to extend the results of the paper to a non risk neutral world, with risk adjusted probabilities i.e. under an equivalent martingale measure.

simple expression:

$$\mathcal{P}(x_t \rhd \underline{x}) \quad = \quad (x_t/\underline{x})^{\lambda_0} \quad , \tag{3}$$

where the constant λ_0 is the negative root of the quadratic equation $\rho(\lambda) = 0$, where $\rho(\lambda) \equiv r - \lambda[\mu + (\lambda - 1)\sigma^2/2]$.

We will now derive an expression of $V(x_t|t)$ where the contribution to value of the different terms of the contract remains easily separable. Generating such a flexible formula is essential for the exercise we intend to conduct later.

• To start with, let us denote $\overline{A}_j(x_t|t : +\infty)$, the contribution to value of a *default-riskless and perpetual* flow of income, $a_j(x)$:

$$\overline{A}_j(x_t|t : +\infty) \;\equiv\; E_t \int_t^{+\infty} e^{-r\tau} \; a_j(x_\tau) \; d\tau \; . \tag{4}$$

Given that we know that the event of default will actually occur the first time the state variable x_t reaches the level $\underline{x}$, we can directly express the contribution to value of a *perpetual* entitlement to a flow of income, $a_j(x)$, prior to default:

$$A_j(x_t|t : +\infty) \;=\; \overline{A}_j(x_t|t : +\infty) \;-\; \overline{A}_j(\underline{x}|T_{\underline{x}} : +\infty) \; \mathcal{P}(x_t \rhd \underline{x}) \; . \tag{5}$$

This decomposition of value is similar to Black and Cox (1976): The first term on the right hand side is the value of a perpetual entitlement to the flow of income, $a_j(x)$, if default was never to occur. The second term is the product of (i) the change in contribution to asset value that will intervene when default occurs, $\left[0 - \overline{A}_j(\underline{x}||T_{\underline{x}} : +\infty)\right]$, and (ii) the probability-weighted discount factor for this event $\mathcal{P}(x_t \rhd \underline{x})$.

We now derive the value, $A_j(x_t|t : T_j'')$, of the payoff stream $a_j(x)$ from now to the date T_j'', with risk of default. To do this we express it as the difference between its value if the maturity was infinite, $A_j(x_t|t : +\infty)$, and the discounted expected value of an infinite maturity claim written at the date T_j'' against the same income flow, conditional on the process not being absorbed in the meanwhile.

$$A_j(x_t|t : T_j'') \;=\; A_j(x_t|t : +\infty) \;-\; e^{-r(T_j''-t)} \; \underline{E}_{T_j''} \left[A_j(x_{T_j''}|T_j'', +\infty) \right] \; . \tag{6}$$

$\underline{E}_T$ is the expected value operator conditional on the state variable not hitting the absorbing barrier $\underline{x}$ before T. Finally the value, $A_j(x_t|T_j' : T_j'')$, of a stream of payoff $a_j(x)$, starting at date T_j' and finishing at date T_j'', is obtained subtracting $A_j(x_t|t : T_j')$ to $A_j(x_t|t : T_j'')$.

$$A_j(x_t|T_j' : T_j'') \;=\; e^{-r(T_j'-t)} \; \underline{E}_{T_j'} \left[A_j(x_{T_j'}|T_j', +\infty) \right] \;-\; e^{-r(T_j''-t)} \; \underline{E}_{T_j''} \left[A_j(x_{T_j''}|T_j'', +\infty) \right] \; . \tag{7}$$

• Similarly, $P_k(x_t|T_k)$, the present value of a lump-sum payment $p_k(x)$ expected at date T_k if default does not occur in the meantime is also easily expressed using the conditional expected value operator $\underline{E}_T$,

$$P_k(x_t|T_k) \;=\; e^{-r(T_k-t)} \; \underline{E}_{T_k'} \left[p_k(x_{T_k'}) \right] \; . \tag{8}$$

• Finally, the current value of an infinite maturity guarantee to receive $\underline{V}(\underline{x})$ in the event of default is $\underline{V}(\underline{x})\ (x_t/\underline{x})^{\lambda_0}$. Given that the maturity of the contract T is finite, $\underline{V}(\underline{x})$, contributes to the overall value of the asset by

$$\underline{V}(x_t|t:T) \;=\; e^{-r(T-t)}\ \underline{E}_T\,[\ \underline{V}(\underline{x})\ \mathcal{P}(x_t \rhd \underline{x})\ ]\ . \tag{9}$$

We now develop separately a central pricing operator, which is applicable to any elementary value function $\alpha x_T{}^{\lambda}$ of the realization at time T of the economic fundamental.

Lemma 1 *The expected discounted value of a payment $\alpha x_T{}^{\lambda}$ at time T, accruing if and only if the underlying process whose current level is x_t does not reach $\underline{x}$ in the meantime, is*

$$I(x_t|\alpha,\lambda,\underline{x},T) \;\equiv\; e^{-r(T-t)}\ \underline{E}_T\left[\alpha\, x_T{}^{\lambda}\right]\ , \tag{10}$$

$$I(x_t|\alpha,\lambda,\underline{x},T) \;=\; \alpha\, x_t{}^{\lambda}\ e^{-\rho(\lambda)\ (T-t)}\left[\ \Phi\,[d_1] - \Phi\,[d_2]\,(x_t/\underline{x})^{\gamma(\lambda)}\ \right]\ , \tag{11}$$

$$\textit{where}\quad \rho(\lambda) \;\equiv\; r-\lambda[\mu+(\lambda-1)\sigma^2/2]\ , \qquad \gamma(\lambda) \;\equiv\; \frac{\mu+(\lambda-1/2)\sigma^2}{-\sigma^2/2}\ ,$$

$$d_1 \;\equiv\; \frac{ln(x_t/\underline{x})\ -\ \gamma(\lambda)\,\sigma^2(T-t)/2}{\sigma\sqrt{T-t}}\ , \qquad d_2 \;\equiv\; d_1-\frac{2ln(x_t/\underline{x})}{\sigma\sqrt{T-t}}\ ,$$

and $\Phi[.]$ is the cumulative normal distribution.

Now, with a fairly very weak restriction on the functional forms of $a_j(x)$ and $\{p_k(x)\}$, we can use the versatile operator $I(x_t|\alpha,\lambda,\underline{x},T)$ to obtain closed-form solutions for the value of the asset $\mathcal{A}$, as a sum of such operators. We simply need the arguments inside the expectations operators in equations (4) and (8) to be linear combinations of power functions of x.

Assumption 1 Each $a_j(x)$ and $\{p_k(x)\}$ can be expressed in terms of

$$a_j(x) \;=\; \sum_{i=1}^{I}\ \alpha_{ij}x^{\lambda_i}\ , \qquad p_k(x) \;=\; \sum_{l=1}^{L}\ \alpha_{lk}x^{\lambda_{lk}}\ , \tag{12}$$

for all $j \in \{1,...J\}$, and $k \in \{1,...K\}$, where all α_{ij}, λ_{ij}, α_{lk}, and λ_{lk} are constants.

Proposition 1 *Under Assumption 1, the asset $\mathcal{A}$ is currently worth*

$$\begin{aligned} V(x_t|t) \;=\; & \sum_{j=1}^{J}\sum_{i=1}^{I} I(x_t|\frac{\alpha_{ij}}{r-\lambda_{ij}\mu},\lambda_{ij},\underline{x},T_j') \;-\; \sum_{j=1}^{J} I(x_t|\sum_{i=1}^{I}\frac{\alpha_{ij}\,\underline{x}^{\lambda_{ij}-\lambda_0}}{r-\lambda_{ij}\mu},\lambda_0,\underline{x},T_j') \\ & -\sum_{j=1}^{J}\sum_{i=1}^{I} I(x_t|\frac{\alpha_{ij}}{r-\lambda_{ij}\mu},\lambda_{ij},\underline{x},T_j'') \;+\; \sum_{j=1}^{J} I(x_t|\sum_{i=1}^{I}\frac{\alpha_{ij}\,\underline{x}^{\lambda_{ij}-\lambda_0}}{r-\lambda_{ij}\mu},\lambda_0,\underline{x},T_j'') \\ & +\sum_{k=1}^{K}\sum_{l=1}^{L} I(x_t|\alpha_{lk},\lambda_{lk},\underline{x},T_k) \;+\; I(x_t|\underline{V}(\underline{x})\,\underline{x}^{-\lambda_0},\lambda_0,\underline{x},T)\ . \end{aligned} \tag{13}$$

To obtain the formula, simply notice that with Assumption 1,

$$\overline{A}_j(x_t|t:+\infty) \;\equiv\; E_t\int_t^{+\infty} e^{-r\tau}\ a_j(x_\tau)\ d\tau \;=\; \sum_{i=1}^{I}\ \frac{\alpha_{ij}\ x_t{}^{\lambda_{ij}}}{r-\lambda_{ij}\mu}\ . \tag{14}$$

3 Examples of Applications

The generality of our set-up can prove useful not only for the valuation of corporate debt but also for the valuation of credit derivatives such as default swaps, default puts and default digital puts. In this section, we illustrate its use for these different securities:

A *coupon paying bond* with default risk and finite maturity entitles its holder to a stream of constant interest payments, the coupon c, each period until the lump sum repayment of the principal p, at the date of maturity T. Default corresponds to an interruption of the debt service. Using the previous section's notations, this case corresponds to $J = 1$ and $a_1(x) = c$ $(I = 1)$; $K = 1$ and $p_1(x) = p$ $(L = 1)$; Debt service obligation is not deferred, hence $T_1' < t$; The fact that the principal is repaid at maturity implies $T_1 = T_1'' = T$. The value of this asset is therefore

$$\begin{aligned} V(x_t \mid t) &= \frac{c}{r} - \left[\frac{c}{r} - \underline{V}(\underline{x})\right]\left(\frac{x_t}{\underline{x}}\right)^{\lambda_0} \\ &\quad - I(x_t|[c/r - p], 0, \underline{x}, T) \;+\; I(x_t|[c/r - \underline{V}(\underline{x})]\underline{x}^{-\lambda_0}, \lambda_0, \underline{x}, T)\ . \end{aligned} \tag{15}$$

A *default swap* on a given defaultable security is a swap contract where the counterpart A pays the swap rate s until default or maturity of the default swap, and the counterpart B pays in the event of default the difference between the recovery value of the defaulted bond and its nominal value p. Setting this loss quota to θp, then the value of the default swap is given by

$$V_A(x_t \mid t) = \frac{s}{r} - \left[\frac{s}{r}\right]\left(\frac{x_t}{\underline{x}}\right)^{\lambda_0} - I(x_t|s/r, 0, \underline{x}, T) \;+\; I(x_t|[s/r]\underline{x}^{-\lambda_0}, \lambda_0, \underline{x}, T) \tag{16}$$

$$V_B(x_t \mid t) = I(x_t|[\theta p]\underline{x}^{-\lambda_0}, \lambda_0, \underline{x}, T)\ . \tag{17}$$

The swap rate s is determined by solving

$$V_A(x_t \mid t) = V_B(x_t \mid t)\ . \tag{18}$$

A *default put* is very similar to a default swap contract with the difference that counterpart A pays a lump sum payment instead of a regular rate until the event of default. Therefore the value of the default put is

$$V(x_t \mid t) = I(x_t|[\theta p]\underline{x}^{-\lambda_0}, \lambda_0, \underline{x}, T\ )\ . \tag{19}$$

A *default digital put* differs from the default put in that the payment made by counterpart B at default is fixed to \$1, so that the value of the default digital put is equal to

$$V(x_t \mid t) = I(x_t|\underline{x}^{-\lambda_0}, \lambda_0, \underline{x}, T\ )\ . \tag{20}$$

4 Structural Assumptions Encountered in the Literature

In our framework, the set of parameter functions describing (i) the choice and dynamics of the economic fundamental $\{\mu, \sigma\}$, (ii) the expected default trigger $\underline{x}$, and (iii) the residual value of the asset in default $\underline{V}(x)$, are distinct inputs hence the formula is flexibly modular: The above triple choice can be adjusted and justified separately. This not only provides great latitude to accommodate for the specificity of each asset, but importantly enables an understanding of the relative impact of alternative sets of assumptions on the pricing of default risk.

It is hard for any particular description of the environment to prove universally acceptable. However adjusting such assumptions to the particularities of the environment surrounding the claim is very important for pricing purposes. We now show how strikingly true this is in the context of different default-prone securities. Parameterizations we have encountered in the finance literature are as follows:

(i) Dynamics of the Economic Fundamental: $\{\mu, \sigma\}$. Merton (1974), Black and Cox (1976), Brennan and Schwartz (1978) Kim, Ramaswamy and Sundaresan (1993), Longstaff and Schwartz (1995), Leland (1994) and Leland and Toft (1996) all employ the *total value of the firm's assets* as economic fundamental.[5] In the absence of arbitrage and if investors are assumed risk neutral, the expected drift rate μ of this process must equal r, the discount factor.

Alternatively, cash flow based models such as Mello and Parsons (1992), Mella-Barral and Perraudin (1997) and Fries, Miller and Perraudin (1997) consider the firm's *operating earnings*, hence the selling price of production as the driving process. This choice is particularly suitable when liquidity problems are an issue. Furthermore, data concerning the cash flows generated by a firm is often easier to obtain than assessing the total value of its assets.

(ii) Default Trigger: $\underline{x}$. Black and Cox (1976) and Brennan and Schwartz (1978) consider debt with a positive *net-worth covenant* written in the contract. The existence of this sort of covenant implies that bankruptcy is triggered when the market value of specific assets falls below a pre-established level. For example, using Leland's (1994) terminology, protected debt corresponds to the case where contractually the overall value of the firm cannot fall below the debt principal.

Alternatively, Kim, Ramaswamy and Sundaresan (1993) assume that bankruptcy occurs as soon as current earnings do not suffice to cover the firm's contractual debt service obligations. This *strict liquidity constraint* probably yields an excessively early closure rule. It is nevertheless attractive because liquidity problems often determine the timing of bankruptcy.

Finally, in Black and Cox (1976), Leland (1994), Leland and Toft (1996), and Mella-Barral (1999), bankruptcy is declared at the *equity holders' optimal abandonment point.* That is, the optimal (random) time to exercise their limited liability option is such that the derivative of the equity value with respect to x_t equals zero at the optimal closure trigger level $\underline{x}$. The endogenous nature of this closure rule is attractive.[6] Equity holders are

[5] The models of Kim, Ramaswamy and Sundaresan (1993) and Longstaff and Schwartz (1995) have the very important additional ability to handle a stochastic riskless interest rate.

[6] It can be applied not only when the driving economic fundamental is the total value of the firm's assets but also with a cash flow model. It is nevertheless only applicable considering a stationary debt structure environment, where the firm continuously sells a constant amount of new debt, and always with the same maturity. The coupon written on newly issued debt is furthermore assumed independent of expected changes in the level of the state variable. Equity holders are basically guaranteed to be able to roll over the debt with

nevertheless assumed to have deep-pockets, as they are assumed always able, when willing, to cover operating losses. Therefore, as far as liquidity problems are concerned, these models correspond to the softest possible constraint, i.e. the opposite of Kim, Ramaswamy and Sundaresan (1993).

(iii) Residual value of the Asset in Default: $\underline{V}(x)$. Most of the corporate finance pricing literature assumes that defaulting is sanctioned by a liquidation of the firm. The bankruptcy procedure is then assumed to involve some exogenously given costs. The total value of the firm once default occurs is then calculated subtracting these costs to the total value if bankruptcy were avoided. In the papers previously cited, the exogenous parameter quantifying bankruptcy costs consists either of a proportional reduction factor θ, or of a fixed lump sum.[7]

This aggregate residual value of the firm is shared among the different asset holders, according to pre-established sharing-rules that are more or less respected. For example, if the Absolute Priority Rule is respected, debt holders are paid first out of the proceeds of the piece-meal liquidation sale, because debt is senior to equity. Then, the equity has only a non zero residual value if debt holders fully recover their principal. Notice that expected departures from absolute priority are easily handled introducing an additional parameter to denote the fraction of the total value of the firm in default that equity holders are expected to ultimately get hold of.

5 Importance of the Structural Assumptions

In this section we illustrate the impact of a particular choice of (i) underlying economic fundamental and (ii) default trigger rule, on (a) the value of debt, and (b) the term structure of credit spreads.

We consider a firm which has issued a number of identical coupon paying bonds. Overall, the debt entitles bondholders to a stream of constant interest payments, the aggregate coupon obligations contracted upon at entry c, each period until the lump sum repayment of the principal p, at the date of maturity T. The aggregate value of the debt is therefore, as in equation (15),

$$\begin{aligned} V(x_t \mid t) &= \frac{c}{r} - \left[\frac{c}{r} - \underline{V}(\underline{x})\right]\left(\frac{x_t}{\underline{x}}\right)^{\lambda_0} \\ &\quad - I(x_t|[c/r-p],0,\underline{x},T) \; + \; I(x_t|[c/r-\underline{V}(\underline{x})]\underline{x}^{-\lambda_0},\lambda_0,\underline{x},T) \; . \end{aligned} \tag{21}$$

In this Section, our purpose is to contrast the results obtained with the following two different sets of modeling assumptions:

the same maturity, always paying the same coupon. This assumption generates a debt service burden which is constant throughout the existence of the firm, hence creates a time-independent willingness to service it. This is necessary, given the requirement of a time-independent $\underline{x}$.

[7]In stochastic interest models, the reduction factor θ can also be a proportion of a riskless discount bond, as in Kim, Ramaswamy and Sundaresan (1993), or a proportion of the principal, as in Longstaff and Schwartz (1995).

- **Set of Assumptions 1:**
  1. The economic fundamental x_t is the *total value of the firm's assets.* Therefore $\mu = r$.
  2. The debt is *protected*, and bankruptcy is triggered the first time the total firm value, x_t, reaches the debt principal level, p.
  3. No departure from the absolute priority rule is expected. The debt residual value in bankruptcy $\underline{V}(x)$ is therefore equal to $\min\{(1-\theta)x \ ; \ p\}$, where θ represents a proportional bankruptcy cost.
- **Set of Assumptions 2:**
  1. The economic fundamental x_t is the *firm's operating earnings.* The drift rate μ is related to the expected inflation rate. We set μ equal to zero, assuming no inflation.
  2. Bankruptcy is triggered by a strict *liquidity constraint*, hence $\underline{x} = c$.
  3. No departures from the absolute priority rule is expected. Debt value in bankruptcy $\underline{V}(x)$ is therefore $\min\{(1-\theta)\ x/r \ ; \ p\}$.

Figures 2(a) and 3(a) exhibit the value of corporate debt $V(x|0)$, at the date of entry $t = 0$, for these two descriptions of the environment[8]. These graphs are constructed in the following manner: For each assumed maturity of the debt, the level of the state variable at the time the security is issued, x_0, corresponds in both cases to the same total value of the firm. The value of the debt $V(x|0)$ is an increasing function of the state variable x. The higher the value of x, the further the firm is from the default threshold $\underline{x}$ and the smaller the discount for default risk. As x increases, the debt value is asymptotic to the value of riskless debt. Conversely, when x decreases, debt value tends to the value of the firm if default occurs $\underline{V}(\underline{x})$.

We considered different debt maturities. However, to isolate the influence of our assumptions, we make sure that although contracts differ in their maturity, they represent comparable entities. That is, they all have the same face value and are sold at par, regardless of their maturity. The coupon written at entry is then a function of the maturity: It is such that the initial value of the debt equals the debt principal, p. In other words, for a given maturity date T, the aggregate coupon contracted upon at entry, $c(T)$, is such that

$$V(x_0|0) \quad = \quad p \ . \tag{22}$$

Figures 2(a) and 3(a) show the debt value that results from the set of assumptions 1 and 2, respectively. We see that the shape of the value function $V(x|0)$, strongly depends on the maturity of the debt. For $x < x_0$, debt value is decreasing with maturity, and conversely when $x > x_0$. This is the result of two competing effects:

[8]For our simulations we chose the following baseline parameters: a riskless interest rate r of 6 % reflects approximately the current US prime rate. The asset price volatility σ is set to 0.15 and we adopt a figure of 30% for bankruptcy costs. Altman (1984) reports an average writedown rate of 40% for secured debt considering a sample of defaulted bonds and Alderson and Betker (1995) report a 36.5% mean liquidation cost calculated as the ratio of going-concern value less liquidation value to going-concern value.

1. For all levels of x , the probability of reaching a given $\underline{x}$ before the contract ends is lower for shorter maturities. This induces a lower weighted probability discount factor associated with the event of bankruptcy. Shorter maturities therefore correspond to lower probabilities of default.

2. For a high x, the debt value asymptotes to its riskless value $c(T)/r - [c(T)/r - p]e^{-rT}$, which is increasing in maturity (as bonds are sold at par at entry). Shorter maturities are therefore associated with lower riskless values.

Notice that with the set of assumptions 1, and in particular, the strict liquidity constraint closure scenario, bankruptcy points depend on the maturity of the contract (Figure 3(a)). This is because the coupon written at entry is a function of the maturity and $\underline{x} = c(T)$. Clearly, this is not the case with a net-worth covenant, as in the set of assumptions 2.

We also examined the implied term structures of credit spreads at entry, i.e. $c/p - r$ function of the maturity. The term structure corresponding to the set of assumptions 1 and 2, is plotted in figures 2(b) and 3(b), respectively. The second one is much more hump-shaped. The spreads increase until an intermediate maturity of either 8 or 4 years and then decrease as the maturity increases. This is the result of a trade-off between the exposure to credit risk and the expected present value of the coupon payoff stream, which both increase with maturity.

Observe that the magnitude of credit spreads is substantially different depending on the description of the environment. The first set of assumptions generates much smaller credit spreads than the second.[9] This is not surprising: When the total value of the firm's assets is the economic fundamental, the chances that the process hits a barrier from above are lower than when the dynamics are driven by the operating earnings. This is because the expected drift rate μ, is equal to r instead of 0, thus the expected time of default is further.

Earlier empirical work by Jones, Mason and Rosenfeld (1984) and Sarig and Warga (1989) showed that structural models of default were typically unable to generate credit spreads consistent with those observed. Here our research suggests that models with operating earnings as the fundamental can generate higher credit spreads than firm's asset value based models, hence that the former assumption may be a more suitable one.

6 Future Research and Extensions

It is interesting to notice in Figures 2(a) and 3(a), how substantially the value of the bonds exceeds the value of the firm in default for levels of the state variable in the interval just above $\underline{x}$. This just stems from the fact that whenever bankruptcy is costly, there is a surplus to be renegotiated prior to triggering the procedure.[10] Most importantly, these plots show

[9]Notice that the magnitudes of credit spreads obtained are consistent with the average levels observed in debt markets. Kim, Ramaswamy and Sundaresan (1993) report 77 basis points as the average spread for investment-grade corporate bonds, and Litterman and Iben (1991) report historical ranges for par spreads between 20 and 130 basis points.

[10]Empirical studies by Franks and Torous (1989 and 1994) suggest that bankruptcy procedures give scope for opportunistic behaviour by the parties involved. If they have substantial bargaining power, debtors are in

that this is even more the case when the maturity is short: The surplus to be gained from renegotiation, hence the scope for renegotiation is larger.[11]

Consequently, models which do not allow for renegotiation exaggerate more the debt value for short maturities than they do for long ones. This is not only because creditors are more willing to make self interested concessions to the debtors, but also because debtors can expect to strategically extract more concessions from creditors. Now, none of the existing finite maturity models allows for the firm in financial distress to renegotiate its debt, and this modeling short fall yields larger errors with short maturity contracts.

Our research suggests that the next step in future research consists in incorporating debt renegotiation in a globally justified pricing model. This is a difficult step, because it requires constructing a recursive pricing structure that preserves the set of renegotiation options open after a first renegotiation, all of this in a non time-homogeneous setting where optimization is difficult.

Appendix

Proof of Lemma 1: A simple way to obtain $I(x_t \mid \alpha, \lambda, \underline{x}, T)$ consists in first applying Itô's lemma to equation (1):

$$\begin{aligned} dln(x_t) &= (\mu - \sigma^2/2)dt + \sigma dB_t \ , \\ ln(x_T/x_t) &\sim N\left[(\mu - \sigma^2/2)\,(T-t)\ ;\ \sigma\sqrt{T-t}\right] \ . \end{aligned}$$

Cox and Miller (1965) p.221 give the probability density function of an arithmetic Brownian motion with an absorbing barrier:

$$\begin{aligned} f(ln(x_T/x_t)) &= \frac{1}{\sigma\sqrt{2\pi(T-t)}} \left\{ exp\left[\frac{-\left[ln(x_T/x_t) - (\mu-\sigma^2/2)(T-t)\right]^2}{2\sigma^2(T-t)}\right] \right. \\ &\quad - \left. exp\left[\frac{2(\mu-\sigma^2/2)\,ln(\underline{x}/x_t)}{\sigma^2} - \frac{\left[ln(x_T/x_t) - 2ln(\underline{x}/x_t) - (\mu-\sigma^2/2)(T-t)\right]^2}{2\sigma^2(T-t)}\right] \right\} . \end{aligned}$$

Secondly change variable in the expression of $I \equiv I(x_t \mid \alpha, \lambda, \underline{x}, T)$:

$$\begin{aligned} I &= exp\left[-r(T-t)\right] \int_{ln(\underline{x}/x_t)}^{+\infty} \alpha\, x_t^{\lambda}\, exp\left[\lambda ln(x_T/x_t)\right]\ f(ln(x_T/x_t))\ dln(x_T/x_t) \\ I &= exp\left[-r(T-t)\right] \int_{ln(\underline{x}/x_t)}^{+\infty} \alpha\, x_t^{\lambda}\, \frac{1}{\sigma\sqrt{2\pi(T-t)}} \end{aligned}$$

a position to extract any such surplus. This debtors' opportunistic behaviour in financial distress is analysed in Anderson and Sundaresan (1996) and Mella-Barral and Perraudin (1997).

[11] This is in accordance with the intuition: Creditors willingness to reduce debtors' debt service obligations, in order to reduce the likelihood of formal bankruptcy and increase the market value of their bonds, is greater when the lump sum principal repayment is due in the near future (but will only intervene if bankruptcy is avoided today).

$$
\begin{aligned}
& .exp\,[\lambda ln(x_T/x_t)]\ exp\left[\frac{-\left[ln(x_T/x_t)-(\mu-\sigma^2/2)(T-t)\right]^2}{2\sigma^2(T-t)}\right] dln(x_T/x_t) \\
- \quad & exp\,[-r(T-t)]\ \int_{ln(\underline{x}/x_t)}^{+\infty} \alpha\, x_t^{\lambda} \frac{1}{\sigma\sqrt{2\pi(T-t)}} exp\left[\frac{(\mu-\sigma^2/2)\,ln(\underline{x}/x_t)}{\sigma^2/2}\right] \\
& .exp\,[\lambda ln(x_T/x_t)]\ exp\left[\frac{-\left[ln(x_T/x_t)-2ln(\underline{x}/x_t)-(\mu-\sigma^2/2)(T-t)\right]^2}{2\sigma^2(T-t)}\right] dln(x_T/x_t) \\
I \quad = \quad & exp\,[-r(T-t)]\,\alpha\, x_t^{\lambda}\left(\int_{ln(\underline{x}/x_t)}^{+\infty} \frac{1}{\sigma\sqrt{2\pi(T-t)}}\, exp\,[\psi(0)]\ dln(x_T/x_t)\right. \\
- \quad & \left. exp\left[\frac{(\mu-\sigma^2/2)\,ln(\underline{x}/x_t)}{\sigma^2/2}\right]\int_{ln(\underline{x}/x_t)}^{+\infty} \frac{1}{\sigma\sqrt{2\pi(T-t)}}\, exp\,[\psi(ln(\underline{x}/x_t))]\ dln(x_T/x_t)\right). \\
& \text{where} \qquad \psi(y) \ = \ -\frac{\left[ln(x_T/x_t)-\{2y+(\mu-\sigma^2/2)(T-t)\}\right]^2}{2\sigma^2(T-t)} \ + \ \lambda\, ln(x_T/x_t).
\end{aligned}
$$

Rearranging terms yields

$$
\psi(y) \ = \ -\frac{\left[ln(x_T/x_t)-\{2y+[\mu+(\lambda-1/2)\sigma^2](T-t)\}\right]^2}{2\sigma^2(T-t)} + 2\lambda y + \lambda[\mu+(\lambda-1)\sigma^2/2](T-t).
$$

After replacing $\psi(.)$ in $I \ \equiv \ I(x_t \mid \alpha, \lambda, \underline{x}, T)$ we directly obtain the expression in the lemma.

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