DEFAULT AND INFORMATION

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Abstract

In traditional models of default it is implicitly assumed that the information used to calibrate and run the model is publicly available. In reality, model inputs and parameters are uncertain. We develop a class of structural default models in which investors are uncertain about the assets or the liability-dependent default barrier. This generalizes the analysis of Duffie & Lando (2001), who considered incomplete asset information. We find that incomplete information about the barrier only is different from all other instances of incomplete information, since it involves learning effects: The historical asset low is investors' upper bound on the barrier. This bound improves as time evolves, leading to declining term structures. We find that for other types of incomplete information the term structure is qualitatively different. We describe the uncertainty about default through the pricing trend, which is the cumulative intensity. This allows to consider all types of incomplete information from a unified perspective. We provide a reduced-form pricing framework in terms of this trend, which generalizes the classical intensity-based framework. Closed-form expressions of the trend, intensity, default probabilities, prices of default-sensitive securities, and spreads are obtained.

Key words: incomplete information, credit spreads, compensator, pricing trend, intensity. *JEL Classification:* G12; G13

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1 Introduction

Recently a number of accounting scandals shocked the business world. Management at Enron, WorldCom, and Tyco misrepresented the level of assets and liabilities on corporate statements. Auditors, possibly knowingly, approved the incorrect statements that were then released to the public. Misled investors suffered huge losses from the ensuing bankruptcies, which are among the largest ever in U.S. corporate history.

In retrospect, investors could have realized that a firm's financial figures are uncertain, even at the date of their publication. Therefore any model for default prediction should include a way to quantify the degree of uncertainty around model inputs and parameters.

How do we model a firm's default? A natural approach is to consider the *cause and effect* nature of events, i.e. the true mechanism behind default. There are many economically plausible, or "structural" definitions of default. The classical definition advanced by Black & Scholes (1973) and Merton (1974), postulates that a firm defaults at debt maturity if assets are not sufficient to pay off the debt. Another definition proposed by Black & Cox (1976) takes as premise that a firm may default at any time before debt maturity. This is described by defining the default event as the first time the firm's assets fall to some lower barrier. The asset level which triggers default can be imposed exogenously [Black & Cox (1976), Longstaff & Schwartz (1995)] or endogenously by having the shareholders optimally liquidate the firm [e.g. Leland (1994), Leland & Toft (1996), and Anderson & Sundaresan (1996)]. A less strict definition of default in this framework would allow the asset value to cross the barrier level multiple times, see François & Morellec (2002) and Moraux (2002). This leads to a model distinction of default and liquidation.

Absent from the discussion is the concept of information and how it is revealed over time. In the cause and effect model it is implicitly assumed that the information used to calibrate and run the model is available to bond investors. Model inputs and parameters are taken as known. In reality, however, asset value, volatility and growth rate are not observable. Accounting statements that purportedly contain the information required to deduce the correct default trigger level are difficult to interpret. These issues translate into model outputs that do not fit empirical data. One important difficulty is related to the credit spreads forecast by cause and effect models that are based on a continuous asset value process. Their absolute level is typically too low. Credit

	Information				
	Complete	*Incomplete*			
		Assets	Barrier		Both
			At Historical Low	Above Historical Low	
Term Structure	Hump-shaped	Hump-shaped	Decreasing	Hump-shaped	Decreasing
Short Spreads	Zero	Non-zero	Non-zero	Zero	Non-zero

Table 1: Type of information vs. credit spread term structure shape for cause and effect default models that are based on a continuous asset value process.

spreads go to zero as maturity goes to zero. This is because investors have "perfect foresight": they know the true distance of the firm to default so they are warned in advance that default is imminent. Investors are hence not willing to pay a default risk premium for short maturity bonds. This is neither intuitive nor empirically plausible, cf. Eom, Helwege & Huang (2002). A second difficulty is that forecast bond prices converge continuously to their recovery values. This is not consistent with the price jumps observed at default, see Sarig & Warga (1989) and Beneish & Press (1995).

Duffie & Lando (2001) were the first to consider a cause and effect default model in which investors cannot observe firm *assets* perfectly. In their model investors receive noisy asset reports but have complete information about the default barrier. In this paper we generalize the analysis of Duffie & Lando (2001). We consider, from a common vantage point, *all* possible situations in which investors are not completely informed. Besides the example analyzed by Duffie & Lando (2001), this includes the important situation where investors have incomplete information about the *default trigger level* only and observe assets perfectly. Another example is when investors have incomplete information about both assets and barrier. The difficulty in observing the liability-dependent barrier was highlighted in the accounting scandals.

Most interestingly, the *type* of bond investors' information determines the *shape* of the implied credit spread term structure. We can distinguish several cases; these are summarized in Table 1 for a firm whose assets follow a geometric Brownian motion. The complete information case and the case with incomplete asset information were analyzed in Duffie & Lando (2001). In this paper we consider in particular the two remaining situations. (1) If assets are completely observable but the barrier is not, then two term structure shapes are possible. (i) If assets are at their historical low to date, then the term structure is downward sloping with non-zero short spreads. (ii) If assets are above their historical low, then the term structure is hump-shaped with zero short spreads. However, here spreads do not converge to zero as fast as in the complete information case. (2) With incomplete information about both assets and default barrier, we find that the term structure is always downward sloping with non-zero short spreads. Consistent with the empirical study Yu (2002), with increasing firm transparency the short-term credit spread thus decreases; for a fully transparent firm it is zero. Assuming efficient credit markets, the qualitative differences in term structures suggest that one can gauge the market's assessment of the quality of accounting information on assets versus liabilities just by looking at the shapes of spread curves. This awaits further empirical research.

It is surprising that different types of incomplete information lead to qualitative differences in the term structure. Why is unobservability of the default trigger level any different from that of assets? One is tempted to think that if it is the distance of the firm to default, usually proxied by the normalized distance of firm assets to default barrier, that matters, then it should be irrelevant whether we are uncertain about the assets or the barrier. If either one is uncertain, then the distance to default is uncertain as well. The answer is that incomplete barrier information is different in that we *learn* over time where the barrier is: it must lie below the historical low of assets to date. This upper bound on the default barrier only improves as time evolves, which results in a declining spread curve. No such learning occurs in the case analyzed by Duffie & Lando (2001), who deal with incomplete information about assets.

This implicit information acquisition and learning over time makes a big difference if we predict default probabilities with a model based on incomplete barrier information. In Giesecke & Goldberg (2003*a*) we calibrate such a model to historical equity prices. We find that the model is superior to the otherwise equivalent model with perfect information. First, the incomplete information model reacts more quickly to changes in the asset value, since it takes account of the whole history of asset values, not just current values. It thus allows to detect credit quality deterioration earlier. Second, the implied spread term structure changes its shape upon the arrival of relevant news. As an example, we considered United Airlines before and after the attack on the World Trade

	Information			
	Complete	*Incomplete*		
		Assets	Barrier	Both
Default	Anticipated	Unexpected	Unexpected	Unexpected
Intensity	No	Yes	No	Yes
Trend	No	Yes	Yes	Yes

Table 2: Type of information vs. default event, intensity, and pricing trend for cause and effect default models with continuous asset process.

Center. On September 17, 2001, the first trading day after the attack, United stock fell more than 40 percent, resulting in a term structure much more steeply sloped in the short end. Short spreads became positive, reflecting the distress of United after the attack.

From a methodological point of view, incomplete information resolves the dichotomy between the cause and effect or structural default models and the purely statistical, or *reduced-form* models. In the latter ad-hoc models the default event is not causally modeled in terms of the firm's assets and liabilities, but is typically given exogenously. It is assumed that the default comes unexpectedly. The stochastic structure of default is directly prescribed in terms of an intensity; see for example Artzner & Delbaen (1995), Jarrow & Turnbull (1995), Duffie & Singleton (1999), Duffie, Schroder & Skiadas (1996), and Lando (1998). The intensity can be interpreted as a conditional default rate, or as the conditional density of the default time, given the information available on the bond market. In the reduced-form models the short spread is given directly by the default rate. In close analogy to ordinary default-free term structure modeling, the term structure of defaultable bonds can be conveniently represented in terms of the default rate.

In complete information cause and effect models investors have typically perfect foresight so such models do *not* admit an intensity. With incomplete information, investors are always uncertain about the distance of the firm to default, so default comes unexpectedly as in the reduced-form models. A natural question then is whether we can describe our uncertainty about the default in terms of an intensity. Duffie & Lando (2001) were able to establish such an intensity for the case with incomplete asset information. If it is the uncertain distance of the firm to default that matters, it is tempting to conclude that we can use Duffie & Lando's (2001) techniques to find the intensity for the other situations with incomplete information as well. Surprisingly, this fails for the case with incomplete information about the barrier only, because such a model does not admit an intensity (see Table 2). The reason for this is the implicit learning of investors, which is only possible in this situation. Here, whenever assets are above their historical low, the conditional probability of defaulting in the next instant (the intensity) is zero, since we learned that the default barrier must lie *below* the historical asset low.

The reduced-form framework for the pricing of default-contingent claims has proven to be extremely useful in the valuation and empirical analysis of default-sensitive securities. If there is no intensity, it cannot be applied. So a natural question is whether there is a reduced-form pricing framework that is predicated on the surprise nature of default and is general enough to deal with no-intensity cases as well. If there is an intensity then this general framework should ideally reduce to the well-understood intensity based one. We develop such a general reduced-form pricing framework. Subject to only technical conditions, it can be applied to *all* default models that are predicated on the surprise nature of default. This includes all previous reduced-form models and all cause and effect models based on incomplete information. Moreover, in some cases we are able to obtain *closed-form expressions* of the default rate, default probabilities, default-sensitive security prices, and credit spreads.

The salient concept we introduce is that of a *pricing trend*. Any default model with incomplete information admits a pricing trend, no matter of what type the available information is (see Table 2). If the trend is differentiable, then the derivative is the intensity. Hence we can think of the trend as the *cumulative* intensity. We explicitly calculate the trend for our class of default models with incomplete information in terms of the conditional distribution of assets and default barrier as assessed by bond investors. Duffie & Lando's (2001) main result arises as the special case with incomplete asset information.

While in this paper we focus on a single firm only, Giesecke (2003*b*) extends the pricing trend characterization to a multi-firm setting with correlated defaults. In the multi-firm context the learning effects lead to information-based default *contagion* effects; these are analyzed in Giesecke (2003*a*).

In Section 2, we describe a first-passage cause and effect default model. The evolution of information and its implications are considered in Section 3. In Section 4 we introduce the pricing trend and provide a reduced-form pricing framework. In Section 5 the trend is characterized more explicitly. In Section 6 we examine the spread term structure. The proofs are in the Appendix.

2 A Cause and Effect Model of Default

In this section we set up an cause and effect model for default, which will exemplify the general concepts we develop below. While there are several economically plausible default definitions, to aid comparison with the existing literature we adopt a standard first-passage definition of default. Here default is triggered by firm value falling below some specified boundary, cf. for example Black & Cox (1976), Leland (1994), or Longstaff & Schwartz (1995).

Uncertainty in the economy is modeled by some fixed complete probability space $(\Omega, \mathcal{G}, P)$. There exists some asset that pays interest at a rate r, which we assume to be deterministic. We define a risk-free numéraire security with value $\alpha_t = \exp(\int_0^t r_s ds)$ at time t. The agents in our economy are assumed to be risk-neutral, so the probability measure P is a martingale measure with respect to the numéraire α.

We take as given some stochastic process V, called *asset process*, where V_t is a sufficient statistic for our given firm's future cash flows as seen from time t. We may think of V as the market value (log-market value) of the firm; standard models for V include geometric (arithmetic) Brownian motion or more general jump-diffusion processes. We assume that all investors agree on the stochastic process followed by V, but do not adopt a specific model for V at this stage. The running minimum asset process M is defined by

$$M_t = \min\{V_s \mid 0 \leq s \leq t\},$$

and describes the evolution of the *historical asset low* over time.

We suppose that the firm is (partly) financed by a bond, which is issued at time $t = 0$ on the public bond market. When the firm stops servicing its contractually agreed payment obligations associated with this bond issue, we say it defaults. The firm then enters financial distress and some form of corporate reorganization takes place. The firm's management decides whether and when to default, and we suppose that it chooses to default if the firm's expected future cash flows are sufficiently low. Bond investors are aware of this liquidation policy. Thus there exists a *default threshold* $D < V_0$ such that debt service is ceased when the asset value V falls to D for the first time. The threshold is a random quantity which does not vary through time.[1] The firm's

[1] A default threshold that is constant through time is consistent with a stationary capital structure. This is reasonable in our frictionless continuous-time framework, where the firm has no incentive to alter its capital structure. Black & Cox (1976) consider a thresh-

default time τ is then given by

$$\tau = \inf\{t > 0 \,|\, V_t \leq D\}. \tag{1}$$

The default indicator process N is defined by $N_t = 1_{\{t \geq \tau\}}$. That is, N is zero before default and jumps to one upon default. Note that $\{\tau \leq t\} = \{M_t \leq D\}$, meaning that default by time t is equivalent to the historical asset low at t being below the default threshold.

3 Information

In the previous section, we described a *standard first passage default model*, which is fully characterized by a firm's default threshold and the parameters of its asset process. Once these parameters are set, the default probabilities depend only on the value of firm assets. Note that this model implicitly assumes that the information used to calibrate and run the model is publicly available and static. In practice, firm value, volatility and growth rate are usually not observable investors. Corporate statements are difficult to interpret, making it challenging to deduce the correct default barrier level. These issues translate into model outputs that are at odds with empirical observations.

We assume that investors have incomplete information and acquire new information over time. This leads to a generalization of the standard model, which allows to specify the *degree of confidence* that investors have around model parameters and inputs.

Our approach is formulated in terms of a mathematical model of uncertainty and the time dependent revelation of information. The probability space $(\Omega, \mathcal{G}, P)$ is our model for the uncertainty in the economy. In particular, the sigma-algebra $\mathcal{G}$ determines the resolution to which agents can distinguish different states of the world $\omega \in \Omega$. In this context information corresponds to the ability to distinguish different states of the world, and we are interested in the evolution of information over time. The mathematical model for this is a *filtration*. A filtration is a sequence $(\mathcal{G}_t)_{t \geq 0}$ of sub-sigma algebras of $\mathcal{G}$ indexed by time t. Here $\mathcal{G}_t$ stands for the set of events which can be distinguished at time t, or the information available at t. Since information is accumulated over time, a natural requirement is that the family $(\mathcal{G}_t)$ is nondecreasing, $\mathcal{G}_s \subset \mathcal{G}_t$ for $s \leq t$. We impose two additional technical conditions, often called the

old which varies deterministically through time, while Nielsen, Saa-Requejo & Santa-Clara (1993) develop a first-passage default model with stochastically varying threshold.

"usual conditions." The first is that $(\mathcal{G}_t)$ is right-continuous. The second is that $\mathcal{G}_0$ contains all P-null sets, meaning that one can always identify a sure event. Without mentioning it again, these conditions will be imposed on every filtration that we introduce in the sequel.

The filtration $(\mathcal{G}_t)$ will be our model for the evolution of publicly available information on the bond market. Below we consider different specifications of this information structure, providing us with a whole family of standard first-passage default models. The members of this family possess quite different probabilistic properties, lead to different default probability forecasts, and imply different credit spread term structures.

3.1 Complete Information

We consider the standard but fairly unrealistic situation in which public bond investors have complete information. This premise is the basis of the majority of cause and effect default models in the literature. In the context of our standard model it entails that

$$\mathcal{G}_t = \sigma(V_s : s \leq t) \vee \sigma(D), \tag{2}$$

meaning that bond investors observe the firm's asset value V as it evolves over time and, right at bond issuance at time $t = 0$, the firm's default threshold D. This corresponds to a transparent firm which updates the public bond market continuously through time about its true financial conditions.

In this situation investors know the distance of the firm to default, i.e. the nearness of the asset value to the default threshold. So they know in particular the default status of the firm. In mathematical terms, the default time τ is a stopping time: for each time t, the event $\{\tau \leq t\}$ is in the set $\mathcal{G}_t$ of observable events at t. Moreover, the default comes typically not as a surprise to the bond market; we say it is *predictable*. In mathematical terms, predictability means that there is an increasing sequence of stopping times, strictly smaller than τ, which converges to τ with probability one. Intuitively, investors can foretell the default event by observing a succession of pre-default events, such as the asset value falling dangerously close to the default threshold.

The existence of such a sequence (τ_n) of announcing pre-default events with complete information depends additionally on the properties of the asset process V on which the model is based. Suppose first that V is modeled as a continuous process such as geometric Brownian motion; the standard choice in the literature. In this case $\tau_n = \inf\{t > 0 \,|\, V_t \leq D + 1/n\}$ defines an announcing sequence (τ_n) of pre-default events, which converge to τ almost surely. Now suppose we allow for jumps in V as in Zhou (2001). Then assets can continuously "diffuse" to the default threshold or cross it with a sudden jump. In this situation the sequence (τ_n) defined above converges to τ only with a probability strictly less than one, so that τ is not predictable any more despite complete information.

The predictability of default has a significant implication for the credit spread forecasts of the model. The *credit yield spread* $S(t,T)$ on zero coupon bonds issued by the firm is the difference between the yield at time t on a credit risky zero bond and that on a credit risk-free zero bond, both maturing at T. Throughout the paper, we base the credit spread on zero recovery bonds. The term structure of credit yield spreads at t is the schedule of $S(t,T)$ against the horizon T. The *short credit spread* $\lim_{T\downarrow t} S(t,T)$ at t is the credit spread for maturity T approaching t. The short spread is the excess yield over the risk-free yield demanded by bond investors for assuming the default risk of the bond issuer over an infinitesimal time period. The predictability of defaults implies zero short spreads.

Theorem 3.1. *If the default is predictable, then for $t < \tau$ almost surely*

$$\lim_{h\downarrow 0} \frac{1}{h} \int_0^t P[\tau \leq s + h \,|\, \mathcal{G}_s] ds = 0.$$

In particular, short spreads are zero almost surely:

$$\lim_{T\downarrow t} S(t,T) = 0.$$

Most structural default models in the literature are based on the predictability of defaults, and so is our standard first-passage default model of Section 2 when investors are completely informed and assets follow some continuous process. Assuming that assets V follow a continuous Brownian motion process with drift $\mu = 6\% - \frac{1}{2}\sigma^2$, Figure 1 shows the term structure of credit spreads in the standard first-passage default model for varying asset volatilities σ. The current distance to default is $V_t - D = 0.4$. We observe that credit spreads go to zero as maturity goes to zero. This is robust under the variation of the firm's business risk, as proxied by asset volatility.

Zero short spreads imply that bond investors do not demand a risk premium for taking over the default risk of the bond issuer for sufficiently short

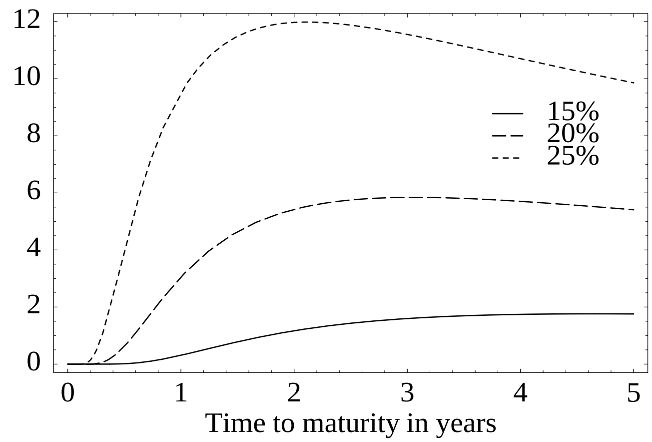

Figure 1: **Term structure of credit spreads with complete information.** We plot the credit yield spread $S(t,T)$ (in percent) as a function of time to maturity $T-t$ (in years), for firms with asset volatilities $\sigma = 15\%$, 20%, and 25%. It is assumed that assets V follow a Brownian motion with drift $\mu = 6\% - \frac{1}{2}\sigma^2$. The current distance to default is $V_t - D = 0.4$.

maturities. In our illustration in Figure 1, this would include maturities of up to 3 months, depending on the riskiness of the firm. This is economically hardly plausible, as it would correspond to investors with perfect foresight over short horizons. Empirical spread studies such as Sarig & Warga (1989) find that credit spreads remain in general bounded away from zero, and thus are at odds with the predictability of defaults.

3.2 Incomplete Information

In this section we deviate from the transparent firm concept of the previous section. We take account of the fact that bond holders are outside investors and their access to inside firm information is limited after the bonds have been issued. Investors are typically not kept fully informed about the true assets of the firm and the true asset level which triggers default.

The uncertainty about the liability-dependent default threshold D was highlighted by the recent accounting scandals at Enron, WorldCom, and Tyco. This uncertainty is described by a *prior distribution* G on D, on which all investors agree. For example, investors may choose G such that its expectation is equal to the published liabilities of the firm. Then the variance of G is a simple measure of the degree of confidence investors have around the published numbers. For simplicity we assume that D is independent of assets and that G admits a density g.

The direct observation of the firm's asset value V over time is generally difficult as well. The source and nature of the available asset information can be quite diverse, and depend on the legal status of the firm, its information policy, or whether its equity is traded publicly. The evolution of this asset information over time is modeled by the filtration $(\mathcal{A}_t)_{t\geq 0}$. The following examples describe some possible asset information scenarios.

Example 3.2 (Lambrecht & Perraudin (1996), RiskMetrics Group (2002)). *Bond investors observe assets, so*

$$\mathcal{A}_t = \sigma(V_s : s \leq t).$$

This is plausible for a public firm, where investors can infer information on the firm's assets from the price of its shares.

Example 3.3 (Duffie & Lando (2001)). *Bond investors receive at times $t_1 < t_2 < \ldots < t_m$ a noisy asset report $Y_{t_k} = V_{t_k} + U_{t_k}$, where U_{t_k} is some independent noise random variable:*

$$\mathcal{A}_t = \sigma(Y_s, s \leq t, s \in \{t_1, \ldots, t_m\}).$$

The variance of U_{t_k} can be interpreted as a measure of the degree of asset noise at time t. The U_{t_k} can be serially correlated, reflecting persistence of noise in time, or correlated with the asset value V_{t_k}.

Example 3.4 (Kusuoka (1999)). *Bond investors receive noisy asset reports continuously through time by observing some process Y whose drift $\mu = f(V_t, t)$ is modulated by the asset process V for some smooth function f:*

$$\mathcal{A}_t = \sigma(Y_s, s \leq t).$$

Example 3.5. *Bond investors receive no asset information at all: $\mathcal{A}_t = \{\Omega, \emptyset\}$. This is plausible for a private firm with restrictive information policy.*

Due to the uncertainty about the default threshold, asset information $\mathcal{A}_t$ alone is not sufficient to discern whether a default has occurred or not. That is, the event $\{\tau \leq t\}$ is not included in the set $\mathcal{A}_t$. In reality however, a default event is publicly announced. To take account of this, we define the information flow on the bond market to include also the default information:[2]

$$\mathcal{G}_t = \mathcal{A}_t \vee \sigma(N_s, s \leq t), \tag{3}$$

meaning that public bond investors' asset information evolves according to the model $(\mathcal{A}_t)$, and that they observe the default in the moment where it occurs. Now the event $\{\tau \leq t\}$ is included in the set $\mathcal{G}_t$ of observable events at t.[3]

Even with complete asset information, public investors are uncertain about the distance of the firm to default $V_t - D$, so that the default hits the bond market as a complete surprise. Indeed, here the default is an *unpredictable* phenomenon: we have $P[\tau = T] = 0$ for all predictable stopping times $T < \infty$. In contrast to the case with complete information, in this situation the default cannot be foretold by observing a sequence of pre-default events, because such an announcing sequence does not exist.

The unpredictability of default is consistent with the empirically observed fact that bond prices jump at or around the bankruptcy announcement, cf. Beneish & Press (1995) and Duffie, Pedersen & Singleton (2003), who consider sovereign bonds. Predictability is not consistent with jumps, since it implies that bond prices converge continuously to their default-contingent values.

The extent of available information and hence the probabilistic properties of default are closely related to the completeness of the financial market. If the default is predictable, then the market is complete if the firm's assets are traded. In this case a default can be perfectly hedged by dynamically trading in the riskless asset and the firm's assets. If the default is unpredictable, then the market is generally incomplete unless there are securities available for trading, such as default swaps on the bond, which provide default-contingent payoffs. In this situation there exists no perfect hedge for the unpredictable jumps in bond prices at or around default, and the bond carries intrinsic risk.[4]

[2] Here $(\mathcal{G}_t)$ is the smallest filtration that includes $(\mathcal{A}_t)$ and makes τ a $(\mathcal{G}_t)$-stopping time.

[3] To retain such a simple information model for the bond market, we maintain the following additional assumptions throughout. By insider legislation for example, management is not permitted to trade in the bond market. Otherwise management could control the firm so as to maximize the value of their debt investments, and depart from the liquidation policy defined by (1). Also, management's transactions could reveal inside firm information, for example on the true default threshold or asset value of the firm.

[4] We refer to Föllmer & Schweizer (1990) for a characterization of hedging strategies which minimize the remaining risk, when incomplete information may lead to market incompleteness.

4 Pricing Trend

In the previous section we characterized a family of standard first-passage default models parameterized by the level of available information. In this section we provide, under technical conditions, a convenient and tractable reduced-form representation of *all* those family members that are based on incomplete information. The evolution of information is prescribed by (3).

Based on the asset information model $(\mathcal{A}_t)$ and the default barrier model G, investors assess the credit quality of the firm and value the firm's bond. A measure of the credit quality is the *conditional survival probability*

$$L_t = P[\tau > t \,|\, \mathcal{A}_t]. \tag{4}$$

We assume that $L_t > 0$ for all $t > 0$. This implies the natural requirement on the asset filtration that we mentioned informally earlier: having information $\mathcal{A}_t$ allows investors not to discern whether the default has occurred by time t or not. Let furthermore $L_{t-} = \lim_{s \uparrow t} L_s$ and $L_{0-} = 1$.

A significant observation is that the process L has a downward trend: the present value L_s is at least as big as the conditional estimator $E[L_t \,|\, \mathcal{A}_s]$ of L_t at time $t > s$. In other words, given the asset information $\mathcal{A}_s$ at time s, the probability that the firms survives time s is as least as big as the probability that the firm survives until a later time $t > s$. A process L with this property is called a supermartingale. A process Z with zero downward trend is called a martingale: here the estimator $E[Z_t \,|\, \mathcal{A}_s]$ is equal to the present value Z_s for $s < t$. A martingale is "fair" in the sense that the expected gain or loss is zero.

It is possible to isolate the downward trend from L. This is a consequence of the Doob-Meyer decomposition theorem, which is a fundamental result in the theory of stochastic processes. It states that there exists an increasing predictable process K starting at zero such that the sum process $L+K$ becomes a martingale, cf. Dellacherie & Meyer (1982). The unique process K counteracts the downward trend in L; it is therefore called *compensator*. Together with its compensator, the conditional survival probability is now used to define the central object of this section.

Definition 4.1. *The increasing process A defined by*

$$A_t = \int_0^t \frac{dK_s}{L_{s-}} \tag{5}$$

is called the pricing trend *associated with the default time τ.*

The pricing trend characterizes prices of default-sensitive securities issued by the firm. A *default-contingent claim* is a security specified by a pair (T, X), where T is a maturity date and $X \in \mathcal{G}_T$ is an integrable random variable. The security pays X at T if there was no default by T and zero otherwise. We give two examples of such contracts.

Example 4.2. *Let $X = 1$. The associated default-contingent claim is a* defaultable zero bond *issued by the firm. This bond matures at time T and has zero recovery.*

Example 4.3. *Let $X = F(1-R)$. The associated default-contingent claim is the default-contingent loss payment leg in a* default swap *with maturity T, referenced on a bond with face value F, maturity T, and recovery rate R issued by the firm.*

The main result of this section is a convenient and tractable reduced-form representation of the default probability and the price of (T, X) in terms of the pricing trend. We need to impose a mild technical condition on conditional survival probabilities, which is equivalent to the pricing trend being continuous (see Proposition B.1 in the Appendix).

Condition 4.4. *For every predictable stopping time σ in investors' filtration $(\mathcal{G}_t)$, the conditional survival probability L satisfies*

$$E[L_{\sigma-}] = E[L_\sigma].$$

The unpredictability of default is necessary but not sufficient for Condition 4.4 to hold, cf. Dellacherie & Meyer (1982). Hence all models with unpredictable defaults, and in particular all standard first-passage models with incomplete information are consistent with that condition. All models with predictable defaults, and in particular all standard models with complete information violate Condition 4.4.

Theorem 4.5. *Suppose Condition 4.4 is satisfied. If the process Y given by*

$$Y_t = E[X e^{-\int_t^T r_s ds + A_t - A_T} \,|\, \mathcal{G}_t]$$

is continuous at default, then the default-contingent claim (T, X) has a value of $Y_t 1_{\{t<\tau\}}$ at time $t \leq T$. Default probabilities are then given by

$$P[\tau \leq T \,|\, \mathcal{G}_t] = 1 - E[e^{A_t - A_T} \,|\, \mathcal{G}_t], \quad t < \tau, \quad t \leq T.$$

Theorem 4.5 shows that the valuation of a default-contingent claim can be reduced to that of a non-defaultable claim by simply adjusting the numéraire $\alpha_t = \exp(\int_0^t r_s ds)$ for the prevailing default risk. The default-adjusted numéraire is given by $\alpha_t \exp(\int_0^t dA_s) = \exp(\int_0^t r_s ds + A_t)$. This underlines the interpretation of the pricing trend as a cumulative default premium. A closely related result has been proved, under an additional hypothesis, by Elliott, Jeanblanc & Yor (2000). Their price representation distinguishes between the asset filtration $(\mathcal{A}_t)$ and the bond market filtration $(\mathcal{G}_t)$.

The convenient principle of simply adjusting discount factors in order to price defaultable claims has already appeared in the *intensity-based* approach to default, cf. for example Jarrow & Turnbull (1995), Duffie & Singleton (1999), Duffie et al. (1996), and Lando (1998). This intensity-based approach is in fact a *special case* of the general reduced-form framework considered here: if there is some integrable non-negative $(\mathcal{A}_t)$-predictable process λ such that the pricing trend can be written as

$$A_t = \int_0^t \lambda_s ds, \tag{6}$$

then we say that A admits the intensity λ. In other words, if the trend is differentiable, then the derivative is the intensity. Hence we can think of the trend as the cumulative intensity. The intensity with respect to the bond market filtration $(\mathcal{G}_t)$ is given by $\lambda_t 1_{\{t<\tau\}}$, and can be interpreted as the conditional default arrival rate at time t:

$$\lambda_t = \lim_{h \downarrow 0} \frac{1}{h} P[\tau \leq t + h \,|\, \mathcal{G}_t] \quad \text{a.s.}\ , \quad t < \tau. \tag{7}$$

Condition 4.4 is a necessary but not sufficient condition for an intensity to exist: there are plausible situations where the pricing trend A is continuous but not absolutely continuous.[5] In Section 5.1 below we provide an example of such a situation. If Condition 4.4 is satisfied *and* the pricing trend admits an intensity, Theorem 4.5 leads immediately to the well-known intensity-based representations of default-contingent claim prices and default probabilities.

Corollary 4.6. *Assume that the pricing trend admits the intensity λ. If the process Y given by*

$$Y_t = E\big[X e^{-\int_t^T (r_s + \lambda_s) ds} \,|\, \mathcal{G}_t\big] \tag{8}$$

[5]We can summarize the relation between existence of an intensity, Condition 4.4, and the unpredictability of defaults as follows: (6) $\Rightarrow$ Condition 4.4 $\Rightarrow$ τ unpredictable.

is continuous at default, then the default-contingent claim (T, X) *has a value of* $Y_t 1_{\{t<\tau\}}$ *at time* $t \leq T$*. Default probabilities are then given by*

$$P[\tau \leq T \,|\, \mathcal{G}_t] = 1 - E[e^{-\int_t^T \lambda_s ds} \,|\, \mathcal{G}_t], \quad t < \tau, \quad t \leq T.$$

With predictable defaults, short spreads are zero as shown in Theorem 3.1. It turns out that for unpredictable defaults whose pricing trend admits an intensity, short spreads are generally bounded away from zero.

Proposition 4.7. *Assume that the pricing trend admits the right-continuous intensity* λ*. If the process* Y *defined in (8) does not jump at default, short spreads are given by the intensity:*

$$\lim_{T \downarrow t} S(t, T) = \lambda_t \quad a.s. \,, \quad t < \tau.$$

We summarize as follows. All default models satisfying Condition 4.4 admit a reduced-form representation in terms of their pricing trend. The ones whose trend satisfies the stronger condition (6) admit an equivalent reduced-form representation in terms of the intensity. Then implied short spreads are equal to the intensity.

5 Analyzing the Pricing Trend

The class of standard first-passage default models that are based on incomplete information can be represented through the corresponding pricing trends. Definition 4.1 constructs the trend in terms of the conditional survival probability L and its compensator. Through L, the trend is determined by (i) the definition of the default time τ, (ii) the asset information model $(\mathcal{A}_t)$, and (iii) the prior distribution G of the default threshold. Given our first-passage default definition (1), different information models $(\mathcal{A}_t)$ and G induce different trends and thus different bond prices.

In this section we take G as given and analyze the pricing trend explicitly. With incomplete information about the barrier, we can imagine two broad scenarios: complete and incomplete asset information. With complete asset information as in Example 3.2 the pricing trend satisfies Condition 4.4, but does typically not admit an intensity. With incomplete asset information as in Examples 3.3 – 3.5 an intensity does exist under technical conditions. These relationships are summarized in Table 2 in the Introduction.

We do neither put specific assumptions on asset dynamics nor on the specific information model $(\mathcal{A}_t)$. Our results are thus general and universal in that they yield immediately the pricing trend for any particular asset process and information model $(\mathcal{A}_t)$.

5.1 Incomplete Information About Barrier Only

If $\mathcal{A}_t = \sigma(V_s : s \leq t)$ and assets are observable, then the trend can be derived in terms of the historical asset low M and the barrier prior G.

Theorem 5.1. *Assume that the firm's assets are observable by bond investors. Then the pricing trend A is given by*

$$A_t = \int_0^t \frac{dK_s}{G(M_{s-})}.$$

where K is the compensator of the process $(G(M_t))$ in the filtration $(\mathcal{A}_t)$. If the asset process has continuous paths, then

$$A_t = -\log G(M_t). \tag{9}$$

If public bond investors are only uncertain about the default threshold and the asset process is continuous, then they can calculate the pricing trend simply in terms of their threshold prior G and the observable historical asset value M_t. It is remarkable that the pricing trend does not depend on the distribution of the firm's assets. The trend satisfies Condition 4.4, so investors can use this trend to estimate default probabilities and prices of default-contingent claims based on Theorem 4.5.

Can investors alternatively describe their uncertainty about default through an intensity? Consider the instructive case $A_t = -\log G(M_t)$. If the asset value process has sufficiently irregular paths,[6] then the set of times

$$\{t \geq 0 : V_t = M_t\}$$

where the asset value hits a new historical low has Lebesgue measure zero. This implies that the derivative of A_t with respect to time t is zero almost surely, and an intensity λ such that $A_t = \int_0^t \lambda_s ds$ does not exist. In other words the continuous trend A is not absolutely continuous, but *singular*.

[6]This is the case, for example, if assets follow a diffusion process such as geometric Brownian motion. It is not the case in the uninteresting situation where assets follow some deterministic (non-stochastic) process.

A standard default model with observable assets but unobservable default threshold does admit a pricing trend-based representation, but it does not admit an intensity-based representation. From an economic point of view, this is due to investors' implicit *learning* about the default barrier as time passes. Since investors observe the asset process V, they are also fully informed about the historical low of assets to date M. Under the assumption that the default barrier is constant through time, if the firm has not yet defaulted investors know that the default barrier must lie *below* M_t at time t. As time evolves, the historical asset low M can only decrease, so investors' upper bound on the barrier improves over time.

If $V_t > M_t$ and the asset value is above its historical low to date, then investors know that the firm is not in immediate danger to default unless the firm value can jump downwards. Indeed, since the barrier must lie below M_t, it takes some time until assets fall from the current V_t to M_t. During this time the firm cannot default. If $V_t = M_t$ and the asset value is at its historical low to date, then the firm is subject to immediate default: the barrier might be just below M_t. The trend measures the *cumulative* "local" default probability or default rate, i.e. the probability of defaulting in the next instant. Hence the trend does not change if the immediate default probability is zero, which is when $V_t > M_t$. The trend changes whenever the probability of immediate default is non-zero, which is only when $V_t = M_t$. In fact, the points of increase of the trend are equal to the points of decrease of the historical low, i.e. the points $\{t \geq 0 : V_t = M_t\}$. To see this we apply Itô's formula to (9) and get

$$dA_t = -\frac{g(M_t)}{G(M_t)}\, dM_t,$$

provided that G is twice continuously differentiable. Since the trend increases only on a set of measure zero, it has no intensity in the sense of (7).

If, as in Duffie & Lando (2001), firm assets are incompletely observed, then such implicit learning effects are not present as we will see below, and this paves the way to an intensity.

5.2 Incomplete Information About Barrier and Assets

In this section we assume that investors will never be certain about the firm's true asset value after the bonds have been issued: for $t > 0$, V_t is never contained in the set of observable events $\mathcal{G}_t$. The firm discloses at bond issuance ($t = 0$) only its initial value V_0, which we normalize to be zero without loss of generality. After issuance, investors may receive incomplete asset information, such as noisy accounting reports; this is described by the model $(\mathcal{A}_t)$ (see Examples 3.3 – 3.5). Together with the chosen asset process this model determines the $\mathcal{A}_t$-conditional distribution function of the historical asset low M_t, whose regular version we denote by $H(t,\cdot,\omega)$ for $t>0$. We suppose that $0<H(t,x)<1$, for all $t>0$ and $x<0$.

Proposition 5.2. *Assume that investors have incomplete asset information. If $H(t,\cdot)$ is continuous and increasing, then the pricing trend A is given by*

$$A_t = -\log\left(1-\int_{-\infty}^0 H(t,x)\,g(x)\,dx\right).$$

The conditions on H required in this result are satisfied, for example, when assets cannot be observed at all (Example 3.5). Under assumptions on the distribution of threshold and assets, in Section 6 we derive a closed form solution for the trend in this case. Stronger conditions on H are necessary to obtain an intensity. To make these precise we introduce the difference quotient

$$F(t,h,x,\omega)=\frac{1}{h}E\big[H(t+h,x)-H(t,x)\,|\,\mathcal{A}_t\big](\omega),\quad t,h>0,\quad x\le 0.$$

Theorem 5.3. *Assume that investors have incomplete asset information and that the following conditions are satisfied:*

(1) For $t,h>0$ and almost every ω, $|F(t,h,x,\omega)|$ has an upper bound $U(t,h,x,\omega)$ such that $\int_{-\infty}^0 U(t,h,x,\omega)g(x)dx$ is finite.

(2) For $x\le 0$, $t>0$ and almost every ω, $F(t,h,x,\omega)\to f(t,x,\omega)$ as $h\to 0$.

(3) For $x\le 0$, $t>0$ and almost every ω, $H(t,x,\omega)$ and $f(t,x,\omega)$ are bounded.

(4) For $x\le 0$, the processes $(H(t,x))_{t\ge 0}$ and $(f(t,x))_{t\ge 0}$ are $(\mathcal{A}_t)$-predictable.

Then the pricing trend admits the intensity

$$\lambda_t=\frac{\int_{-\infty}^0 f(t,x)\,g(x)\,dx}{1-\int_{-\infty}^0 H(t,x)\,g(x)\,dx},\quad t>0.$$

With incomplete asset information, public bond investors can calculate the default intensity and thus the pricing trend in terms of their threshold prior and the conditional distribution of the historical asset low, given the incomplete asset information ($\mathcal{A}_t$) available to the market. This remains true even if the default threshold is disclosed to investors at bond issuance (i.e. when $D \in \mathcal{G}_0$). Under the conditions of Theorem 5.3, we then have

$$\lambda_t = \frac{f(t,D)}{1 - H(t,D)}, \quad t > 0. \tag{10}$$

Regardless of whether the default threshold is observable or not, a standard first-passage default model with incomplete asset information admits an intensity-based representation of default-contingent claim prices and default probabilities. Here the intensity is explicitly linked to fundamental firm variables. In that sense, such a model provides an economic underpinning for the ad-hoc intensity-based reduced form models in the literature, where intensities are typically given exogenously.

At first glance, it is not obvious that with incomplete information about assets an intensity exists, while with incomplete information about the default barrier only it does not. In both cases, the distance of default, measured by the distance of assets to barrier, is uncertain. It is tempting to think that it does not matter what type of incomplete information actually leads to this uncertainty. But incomplete barrier observation is different, since it involves implicit learning effects as we have seen. These effects are *not* present in the current informational setting. Here, learning about the barrier via the historical asset low M is impossible, since M is not perfectly observable. Learning about assets via the barrier is also impossible, since no information about the barrier is revealed over time. The same is true if the barrier is known a priori. It follows that investors must be prepared to suffer a firm default at any time. There are no time intervals where a default over the next instant, given current information, is impossible. Hence, from investors' perspective, the "local" probability of default over the next instant is generally non-zero. The trend A measures the cumulative local default probability, and therefore increases on a set of positive measure. Under the technical conditions described in Theorem 5.3, it is differentiable, and then the derivative is the intensity, or conditional local default probability.

The standard model for asset dynamics in the literature is (geometric) Brownian motion. For this specific choice we now clarify the structure of the intensity by calculating $f(t,x)$ in terms of the conditional density of assets, given the incomplete asset information. This quantity can often be computed explicitly, see Appendix A for some examples.

Proposition 5.4. *Assume that investors have incomplete asset information. Suppose investors agree that the firm's asset value follows a Brownian motion with drift $\mu \in \mathbb{R}$ and volatility $\sigma > 0$,*

$$dV_t = \mu dt + \sigma dB_t, \tag{11}$$

where B is a standard Brownian motion. Assume that the following conditions are satisfied:

(1) For $t > 0$, the asset value V_t admits a conditional density $a(t, x, \cdot, \omega)$ given $\mathcal{A}_t$ and $M_t > D := x < 0$ with support $[x, \infty)$.

(2) For each (t, x, ω), $a(t, x, \cdot, \omega) = 0$ on $(-\infty, x]$ and $a(t, x, \cdot, \omega)$ is continuously differentiable on (x, ∞) and differentiable from the right at x.

(3) For almost every ω, the derivative $|a_z(s, x, z, \omega)|$ is bounded on sets of the form $\{(s, x, z) : \ 0 \leq s \leq t, \ -\infty < x < 0, \ x \leq z < \infty\}$.

Then

$$f(t, x) = \frac{1}{2}\sigma^2 a_z(t, x, x)(1 - H(t, x)), \quad t > 0, \quad x \leq 0.$$

This result allows us to link our intensity Theorem 5.3 to the result of Duffie & Lando (2001). They assumed that the default threshold D is a known non-random quantity and that investors receive incomplete asset information through noisy reports at discrete times (see Example 3.3). Duffie & Lando (2001) established the $(\mathcal{G}_t)$-intensity λ directly as the limit (7) and showed that[7]

$$\lambda_t = \lim_{h \downarrow 0} \frac{1}{h} P[\tau \leq t + h \,|\, \mathcal{G}_t] = \frac{1}{2}\sigma^2 k_z(t, D) 1_{\{t<\tau\}}, \quad t > 0,$$

where $k(t, \cdot)$ is the conditional density of V_t given $\mathcal{A}_t$ and survivorship, and $k_z(t, z)$ is the derivative of $k(t, z)$ with respect to z from the right. This is clearly consistent with our results. From Theorem 5.3, the intensity with observable threshold is given by (10). Together with Proposition 5.4 we then obtain for the $(\mathcal{A}_t)$-intensity $\lambda_t = \frac{1}{2}\sigma^2 a_z(t, D, D)$. The corresponding $(\mathcal{G}_t)$-intensity is

[7] Using the fact that the intensity does not depend on the asset's drift, Duffie & Lando (2001) extend to the case where the asset value solves the SDE $dV_t = \mu(V_t, t)dt + \sigma(V_t, t)dW_t$ for μ and σ satisfying technical conditions. Then the intensity is for $t > 0$ given by $\lambda_t = \frac{1}{2}\sigma^2(D, t)\, k_z(t, D) 1_{\{t<\tau\}}$.

$\lambda_t 1_{\{t<\tau\}}$. Both results coincide: for given default threshold $D \in \mathcal{G}_0$ we have $a(t, D, z) = k(t, z)$ for all $t < \tau$ and $z \in [D, \infty)$ by definition.

We finally stress the difference in the applied methodologies that lead to the intensity. We characterize the pricing trend, which in turn defines the default arrival intensity if it exists. Duffie & Lando (2001), in contrast, calculate the intensity directly as the limit (7). We have seen that both approaches lead to equivalent results if an intensity does indeed exist. However, if an intensity does not exist as with complete asset information, only a reduced-form representation based on the always existing pricing trend can be given.

6 Term Structure of Credit Spreads

6.1 Overview

In this section we illustrate the significant implications of incomplete information on the term structure of credit spreads. Within the family of standard first-passage default models, we compare four cases:

(1) Complete information

(2) Incomplete information

  (a) Incomplete information about the barrier only

  (b) Incomplete information about assets only

  (c) Incomplete information about both barrier and assets

The case of complete information (1) was analyzed in Section 3.1. We found, in line with the conclusions in the literature, that the term structure is hump-shaped with zero short spreads. Below we consider the incomplete information cases (2). We find that each specific *type* of *in*complete information (a), (b), or (c) corresponds to a *qualitatively different* term structure shape. The results are robust with respect to perturbations in the business risk of the firm, proxied by asset volatility. Our conclusions are summarized in Table 1.

That the complete information case (1) and the incomplete information case (2) differ in their corresponding term structures is unsurprising given the fundamentally different probabilistic properties of default in these cases. In fact, Duffie & Lando (2001) compared already the complete information case (1) with the incomplete asset information case (b). They found that incomplete information about assets leads to hump-shaped term structures with non-zero short spreads. So from a qualitative point of view, it is in this case the non-zero short spread that is different from the complete information case (1).

That the term structures in the other incomplete information cases (a) and (c) are different from the case (b) analyzed by Duffie & Lando (2001) is surprising at first glance. In fact, all types of incomplete information (a), (b), and (c) lead to uncertainty about the distance of the firm to default, and hence unexpected default. So we would not expect any differences in the corresponding term structures.

The answer is related to the revelation of information, which is different in the incomplete information cases. In the case (a) with incomplete information about the barrier only, we have seen earlier that investors *learn* over time about the location of the barrier by observing the historical asset low as an upper bound. This learning results in a quite singular behavior of spreads; they can even be very similar to those with complete information. No such learning is present in the other cases (b) and (c). These two differ in that (c) is obviously worse from investors' perspective. Having at least barrier information as in (b) allows superior short-term predictions of the credit quality dynamics of the firm. Hence spread curves are different over shorter horizons.

Assuming efficient credit markets, these qualitative differences suggest that we can gauge the market's assessment of the quality of accounting information on assets versus liabilities just by looking at the shapes of spread curves. This awaits further empirical research. A first step in this direction is Yu (2002), who finds that with increasing firm transparency the short-term credit spread decreases. This is consistent with the predictions of our model.

6.2 Specific Assumptions

To analyze the incomplete information cases (a), (b), and (c) in detail, we specialize in the general setup of the previous sections by making concrete assumptions about assets and default barrier. We suppose that the total market value Z of the firm follows a geometric Brownian motion with constant drift $m \in \mathbb{R}$ and volatility $\sigma > 0$. That is, $Z_t = Z_0 \, e^{V_t}$ with initial value $Z_0 > 0$. Here $V_t = \mu t + \sigma B_t$ is a Brownian motion with drift $\mu = m - \frac{1}{2}\sigma^2$ and B is a standard Brownian motion. In the sequel we take V to be our "asset process" in the sense of Section 2. Then the distribution function $\Psi(t, x) = P[M_t \leq x]$

of the historical asset low M_t is

$$\Psi(t,x) = \Phi\left(\frac{x-\mu t}{\sigma\sqrt{t}}\right) + \exp\left(\frac{2\mu x}{\sigma^2}\right)\Phi\left(\frac{x+\mu t}{\sigma\sqrt{t}}\right), \tag{12}$$

for $x \leq 0$ and $t > 0$. Φ is the standard normal distribution function. Unless noted otherwise, our base case parameters for calculations are

$$m = 6\% \quad \text{and} \quad \sigma = 20\%.$$

Furthermore, the a priori default threshold distribution with respect to Z is assumed to be independent and uniform on $(0, Z_0)$. This choice corresponds to uninformed investors not having any specific knowledge on the default barrier. It implies that investors' threshold prior with respect to the asset process V is represented by the distribution function

$$G(x) = g(x) = e^x, \quad x < 0. \tag{13}$$

With these concrete assumptions on our standard default model we obtain *closed-form expressions* for pricing trends, default probabilities, bond prices, and credit spreads from the general results of the previous sections.

6.3 Incomplete Information About Barrier Only

We start with an analysis of case (a). With observable assets but unobservable barrier, Theorem 5.1 implies that the pricing trend is simply given by the negative historical asset low:

$$A_t = -M_t.$$

Clearly A does not admit a default intensity. According to Theorem 4.5, the conditional default probability at time $t < \tau$ for the horizon $T > t$ is then given in terms of the pricing trend as follows:

$$P[\tau \leq T \,|\, \mathcal{G}_t] = 1 - E[e^{M_T - M_t} \,|\, \mathcal{G}_t] = p(T-t, V_t - M_t), \tag{14}$$

where for $s > 0$ and $v \geq 0$ we define

$$p(s,v) = \int_{-\infty}^{-v} \Psi(s,y) e^{y+v} dy.$$

Recalling that $m = \mu + \sigma^2/2$, defining the constants $\nu = \mu + \sigma^2$, $\gamma = 1 + 2\mu/\sigma^2$, $\delta = \mu - \gamma\sigma^2$, and $\beta = -\mu\gamma + \gamma^2\sigma^2/2$, integration by parts yields the closed-form formula

$$\begin{aligned} p(s,v) = {} & \Phi\left(\frac{-v-\mu s}{\sigma\sqrt{s}}\right) - e^{v+sm}\Phi\left(\frac{-v-\nu s}{\sigma\sqrt{s}}\right) \\ & + \frac{1}{\gamma}e^{(1-\gamma)v}\Phi\left(\frac{\mu s - v}{\sigma\sqrt{s}}\right) - \frac{1}{\gamma}e^{v+s\beta}\Phi\left(\frac{\delta s - v}{\sigma\sqrt{s}}\right). \end{aligned} \tag{15}$$

According to Theorem 4.5, the price $B(t,T)$ at time $t < \tau$ of a zero coupon bond maturing at $T > t$ with zero recovery is given by

$$\begin{aligned} B(t,T) &= E[e^{-\int_t^T r_s ds + M_T - M_t} \,|\, \mathcal{G}_t] \\ &= b(t,T)\big[1 - p(T-t, V_t - M_t)\big], \end{aligned}$$

where $b(t,T) = e^{-\int_t^T r_s ds}$ is the price of a default-free zero coupon bond maturing at T. The credit spread is available in closed-form as well:

$$S(t,T) = -\frac{1}{T-t}\log\big[1 - p(T-t, V_t - M_t)\big], \quad t < T, \quad t < \tau.$$

It is interesting to note that default probabilities, bond prices, and credit spreads depend upon the historical asset low M, i.e. they are asset value *path-dependent*. With certainty about the default threshold as in Black & Cox (1976) these quantities depend only on current asset values; the asset value path does not matter. As shown in Giesecke & Goldberg (2003*a*), this has important consequences for the predictive power of the default model. There we calibrate both the incomplete and the complete information model to historical asset value data and compare the default probability and credit spread forecasts. We find that the incomplete information model reacts more quickly to changes in the asset value, since it takes account of the whole history of asset values. Loosely, a change in the asset value is not considered as "absolute," but as relative to the historical asset low. This means that the incomplete information model detects a deterioration in the credit quality earlier than the corresponding complete information model. In the latter historical asset information is not taken into account.

The path-dependence is also reflected in credit spread curves. In Figure 2 we graph the term structure of credit spreads for varying distance $V_t - M_t$ of current assets to their historical low. Two distinct shapes appear. If $V_t = M_t$,

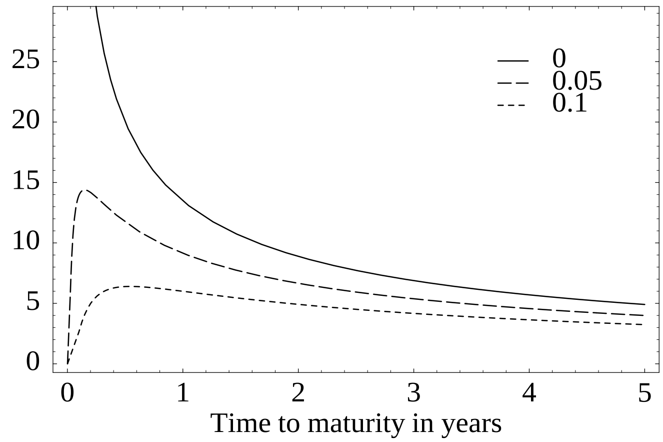

Figure 2: **Term structure of credit spreads with perfect asset observation but unobservable default threshold.** We plot the credit yield spread $S(t,T)$ (in percent) as a function of time to maturity $T-t$ (in years), for firms with current distance to historical asset low $V_t - M_t = 0$, 0.05, and 0.1. It is assumed that assets V follow a Brownian motion with drift $\mu = 4\%$ and volatility $\sigma = 20\%$. The prior distribution of the default threshold is $G(x) = e^x$.

then the term structure is decreasing and credit spreads are strictly positive for all maturities. Credit spreads tend to infinity if maturity T approaches t:

$$\lim_{T \downarrow t} S(t,T) = \infty \quad \text{for} \quad V_t = M_t,$$

at a rate asymptotically proportional to $1/\sqrt{T-t}$. As soon as the credit quality of the firm improves and the current asset value V_t increases above its historical low M_t, the spread curve becomes hump shaped. If maturity T approaches t, spreads tend to zero:

$$\lim_{T \downarrow t} S(t,T) = 0 \quad \text{for} \quad V_t > M_t,$$

at a rate asymptotically proportional to $\exp(-\frac{1}{T-t})$. Despite the fact that default is unpredictable with unobservable default threshold, short spreads are zero. The convergence rate to zero is however smaller than in the case with perfect information, cf. Figure 1. We single out the effect of the distance $V_t - M_t$ on the short spread in Figure 3 for varying asset volatilities.

The economic intuition behind these spread curves is closely related to the implicit *learning* that takes place with incomplete information about the

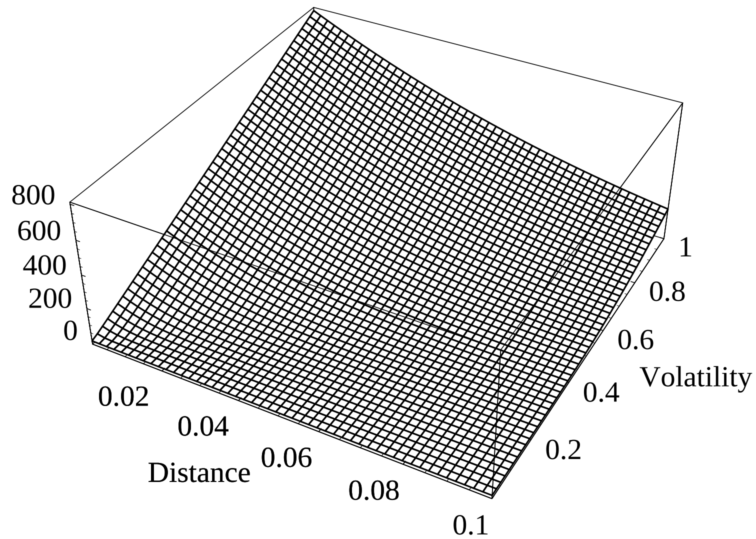

Figure 3: **Short-term credit spreads with perfect asset observation but unobservable default threshold.** We plot the short-term credit spread $S(t, t+0.01)$ (in percent) as a function of distance of assets to their historical low $V_t - M_t > 0$ and asset volatility σ. It is assumed that assets V follow a Brownian motion with drift $\mu = 4\%$. The prior distribution of the default threshold is $G(x) = e^x$.

barrier only. We discussed this earlier in Section 5.1 in connection with the non-existence of a default intensity. We saw that investors learn that the default threshold must lie below the historical asset low to date if the firm has not yet defaulted. If the assets are at their historical low, then investors have to be prepared to observe the firm defaulting in the next instance of time. Indeed, if the unknown barrier is just below M_t, then a slight deterioration in assets can lead to an immediate default. Investors will therefore demand a non-trivial short-term default risk premium, so short spreads are non-zero.

Whenever $V_t > M_t$ the firm is not subject to immediate default. As the asset value cannot jump, it takes time until assets can fall to the level M_t. During that time short-term or “local” default probabilities are zero. This is quite similar to the case with complete information. Thus investors do not demand a risk premium for short-term default risk, and short spreads are zero. The bigger the distance of assets to their historical low, the longer extends the

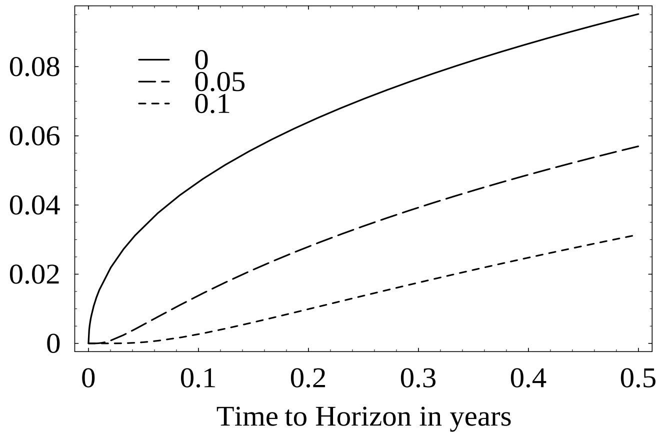

Figure 4: **Term structure of default probabilities with perfect asset observation but unobservable default threshold.** We plot the conditional default probability $p(T-t, V_t - M_t)$ (in percent) as a function of time to horizon $T-t$ (in years), for firms with current distance to historical asset low $V_t - M_t = 0$, 0.05, and 0.1. It is assumed that assets V follow a Brownian motion with drift $\mu = 4\%$ and volatility $\sigma = 20\%$. The prior distribution of the default threshold is $G(x) = e^x$.

horizon for which default probabilities are zero, see Figure 4. On the level of the spread, the bigger $V_t - M_t$ the faster the spread converges to zero as maturity goes to zero. This relation is clearly observable in Figure 5: the bigger $V_t - M_t$, the closer the term structure gets to the one that obtains with complete information. In this sense incomplete information about the barrier looses its effect on the term structure the higher the quality of the firm, measured in terms of $V_t - M_t$.

Learning is also the reason for the downward slope in spread curves. Since the historical asset low M can only decrease over time, investors' upper bound M for the default barrier can only improve as time evolves. This leads to a declining term structure.

Recall from Proposition 4.7 that short spreads are equal to the default intensity if such an intensity exists. With incomplete information about the barrier only, the short spread can only take on two values. Depending on the distance of firm assets to their historical low, it is either zero or infinity. In line with the conclusion we draw in Section 5.1, this is clearly not consistent with an intensity in the sense of (7).

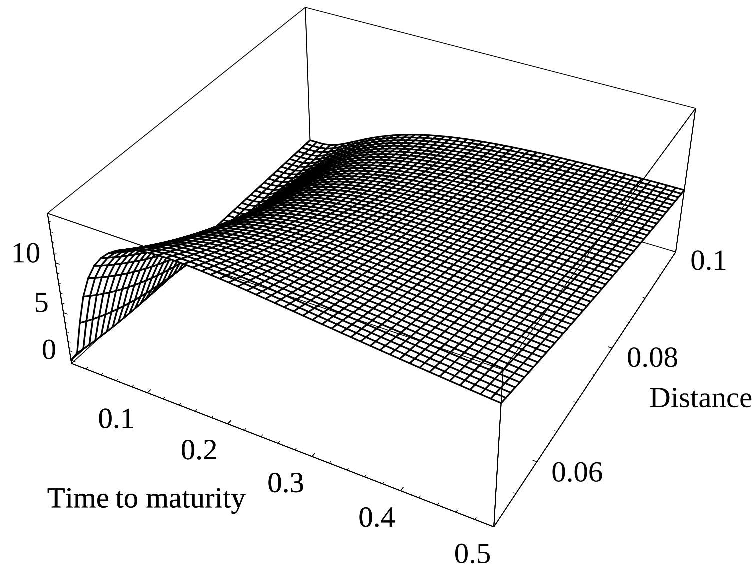

Figure 5: **Credit spreads with perfect asset observation but unobservable default threshold.** We plot the credit spread $S(t,T)$ (in percent) as a function of time to maturity $T-t$ (in years) and distance of assets to their historical low $V_t - M_t > 0$. It is assumed that assets V follow a Brownian motion with drift $\mu = 4\%$ and volatility $\sigma = 20\%$. The prior distribution of the default threshold is $G(x) = e^x$.

There is a basic analogy in the term structure properties in relation to the firm's riskiness between our standard first-passage default model with unobservable threshold and the classic complete information model of Merton (1974). Merton measures the firm's riskiness in terms of the ratio of the risklessly discounted bond face value to firm value. If the firm is quite risky and this ratio is equal or larger than one, then the term structure is decreasing with infinite short spreads. If the firm is less risky and the ratio is less than one, then the term structure is hump shaped with zero short spreads. This corresponds to the spread properties in our setup, the difference being that with unobservable threshold the firm's riskiness is measured by the distance of current asset value to its historical low.

6.4 No Information about Barrier and Assets

As a special but instructive case of situation (c), in this section we consider the term structure if no information about assets and barrier is available. Investors observe only the default. Then the conditional distribution of M_t is given by $H(t,x) = \Psi(t,x)$. By Proposition 5.2, the pricing trend is in this situation

$$A_t = -\log\big[1 - p(t,0)\big], \tag{16}$$

where $p(t,0)$ is given by the closed form formula (15). The derivative $p_s(s,0)$ of $p(s,0)$ with respect to s is easily found:

$$\begin{aligned} p_s(s,0) = {} & \phi\left(-\frac{\mu\sqrt{s}}{\sigma}\right)\left(\frac{-\mu}{2\sigma\sqrt{s}}\right) - me^{ms}\Phi\left(-\frac{\nu\sqrt{s}}{\sigma}\right) - e^{ms}\phi\left(-\frac{\nu\sqrt{s}}{\sigma}\right)\left(\frac{-\nu}{2\sigma\sqrt{s}}\right) \\ & + \frac{1}{\gamma}\phi\left(\frac{\mu\sqrt{s}}{\sigma}\right)\left(\frac{\mu}{2\sigma\sqrt{s}}\right) - \frac{\beta e^{\beta s}}{\gamma}\Phi\left(\frac{\delta\sqrt{s}}{\sigma}\right) - \frac{e^{\beta s}}{\gamma}\phi\left(\frac{\delta\sqrt{s}}{\sigma}\right)\left(\frac{\delta}{2\sigma\sqrt{s}}\right), \end{aligned}$$

where ϕ is the standard normal density function. From Theorem 5.3 or by directly differentiating the pricing trend A with respect to time we obtain the default intensity λ in closed form:

$$\lambda_t = \frac{p_s(t,0)}{1 - p(t,0)}, \qquad t > 0.$$

In Figure 6 we graph λ_t as a function of time t and asset volatility σ. Since by Proposition 5.3 $\lim_{T\downarrow t} S(t,T) = \lambda_t$ almost surely, this gives the profile of short credit spreads over time and asset volatility. In line with intuition, the default intensity is increasing in the degree of business risk, as proxied by asset volatility σ. With a positive asset value drift μ, the intensity is decreasing in time: conditional on survivorship, the "local" probability of hitting the default threshold decreases with the passage of time as assets increase on average.

Based on the continuous deterministic pricing trend A given by (16), by Theorem 4.5 the probability at time $t \in (0,\tau)$ of default before $T \geq t$ is

$$P[\tau \leq T \,|\, \mathcal{G}_t] = 1 - e^{A_t - A_T} = 1 - e^{-\int_t^T \lambda_s ds} = \frac{p(T,0) - p(t,0)}{1 - p(t,0)}.$$

At time $t \in (0,\tau)$, defaultable zero recovery zero bond prices are thus given by

$$B(t,T) = b(t,T)e^{A_t - A_T} = b(t,T)\frac{1 - p(T,0)}{1 - p(t,0)}, \quad T \geq t,$$

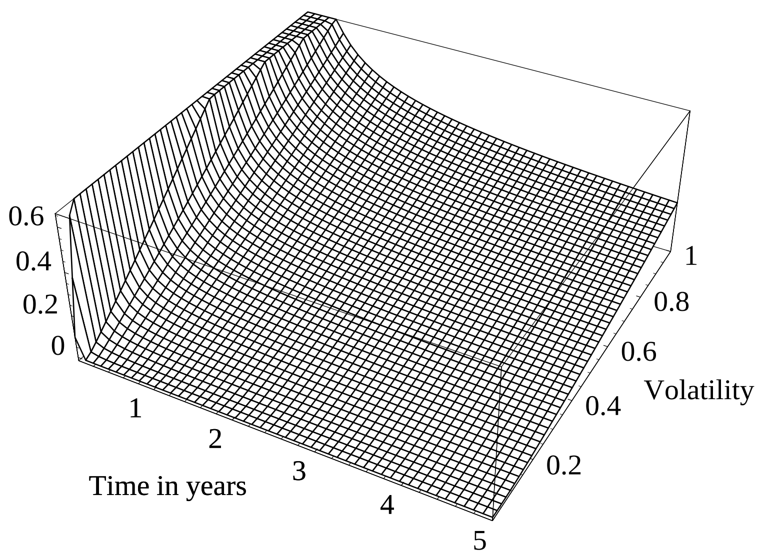

Figure 6: **Default arrival intensity (short credit spread) when both assets and default threshold are unobservable.** We plot the default arrival intensity λ_t (in events per year) as a function of time t (in years) and asset volatility σ. It is assumed that assets V follow a Brownian motion with drift $\mu = 6\% - \frac{1}{2}\sigma^2$. The prior distribution of the default threshold is $G(x) = e^x$.

and for credit spreads we obtain

$$S(t,T) = \frac{A_T - A_t}{T-t} = -\frac{1}{T-t} \log\left(\frac{1-p(T,0)}{1-p(t,0)}\right), \quad T > t.$$

The term structure of credit spreads is plotted in Figure 7, where we vary the asset volatility. Spreads are bounded away from zero for all maturities. Short spreads $S(t,T)$ for $T \downarrow t$ are given by the intensity λ_t, cf. Proposition 5.3. With incomplete asset information there are no learning effects as in the case with incomplete information about the barrier only. Learning is impossible since the historical low of assets is not observable, so investors cannot derive an upper bound on the barrier. It follows that the admissible terms structures are qualitatively different from those in the case with incomplete information about the barrier only. Variation in the level of the spreads is due to variation in asset volatility. As expected, the higher the volatility the higher the spread.

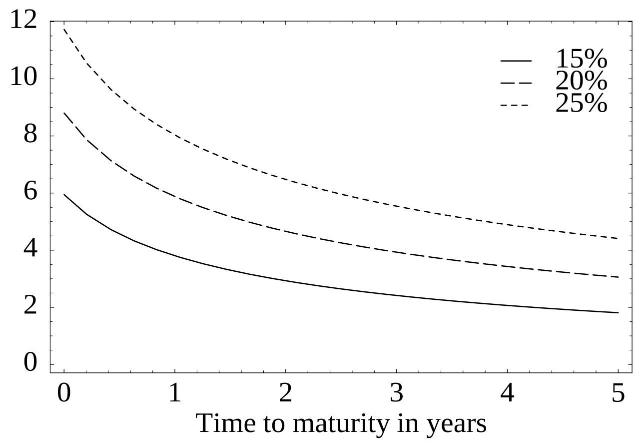

Figure 7: **Term structure of credit spreads when both assets and default threshold are unobservable.** We plot the credit yield spread $S(t,T)$ (in percent) as a function of time to maturity $T-t$ (in years), for firms with asset volatilities $\sigma = 15\%$, 20%, and 25%. We set $t = 0.5$ years, i.e. the bonds were issued 6 months ago. It is assumed that assets V follow a Brownian motion with drift $\mu = 6\% - \frac{1}{2}\sigma^2$. The prior distribution of the default threshold is $G(x) = e^x$.

6.5 No Information About Assets

As a special but instructive case of the situation (b) analyzed by Duffie & Lando (2001), in this section we consider the term structure if the barrier is observed but the assets are not. Suppose the default threshold is revealed to the bond market at $t = 0$ when the bonds are issued. Before default, we then get

$$S(t,T) = -\frac{1}{T-t}\log\left(\frac{1-\Psi(T,D)}{1-\Psi(t,D)}\right), \quad T > t > 0.$$

Figure 8 displays the term structure of credit spreads in that case (we set $D = -0.3$). In line with the case analyzed by Duffie & Lando (2001), the term structure is slightly hump shaped with non-zero short spreads. As in the previously considered case with unobservable barrier, learning effects are not present. Indeed, all uncertainty about the barrier is fully resolved at time zero, so the barrier itself acts merely as a lower bound on possible asset values. As time evolves no new information on the barrier is revealed. This explains the qualitative difference to the spread curves in case (a). In comparison with the

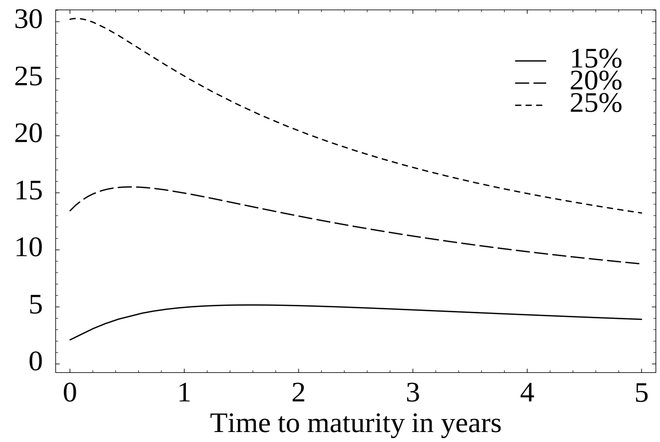

Figure 8: **Term structure of credit spreads when the default threshold is observable but assets are not.** We plot the credit yield spread $S(t, T)$ (in percent) as a function of time to maturity $T - t$ (in years), for firms with asset volatilities $\sigma = 15\%$, 20%, and 25%. We set $t = 0.5$ years, i.e. the bonds were issued 6 months ago. It is assumed that assets V follow a Brownian motion with drift $\mu = 6\% - \frac{1}{2}\sigma^2$. The default threshold is $D = -0.3$.

spread curves in case (c) analyzed previously, here spreads are slightly humped, whereas in (c) there are downward sloping. This difference over shorter horizons is due to the fact that investors are better off in case (b) than in case (c). Indeed, knowing the exact location of the barrier allows an improved estimate of default probabilities and spreads over shorter terms. This is due to the fact that, in the short run the asset value can move up or down only relatively short distances from its initial value. Over longer horizons, the uncertainty about the location of the assets relative to earlier times increases, so the spread curve becomes qualitatively similar to that in case (c).

Given the results of the previous section, it appears that incomplete asset information is sufficient for the generation of strictly positive and bounded spreads. The short spreads are given by the intensity, which is (Theorem 5.3)

$$\lambda_t = \frac{\psi(t, D)}{1 - \Psi(t, D)}, \qquad t > 0, \tag{17}$$

where Ψ is defined in (12), and the derivative $\psi(t, x)$ of Ψ with respect to t is

$$\psi(t, x) = \frac{1}{2\sigma} \left[\left(\frac{\mu}{\sqrt{t}} - \frac{x}{\sqrt{t^3}} \right) e^{\frac{2\mu x}{\sigma^2}} \phi \left(\frac{x + \mu t}{\sigma \sqrt{t}} \right) - \left(\frac{x}{\sqrt{t^3}} + \frac{\mu}{\sqrt{t}} \right) \phi \left(\frac{\mu t - x}{\sigma \sqrt{t}} \right) \right]$$

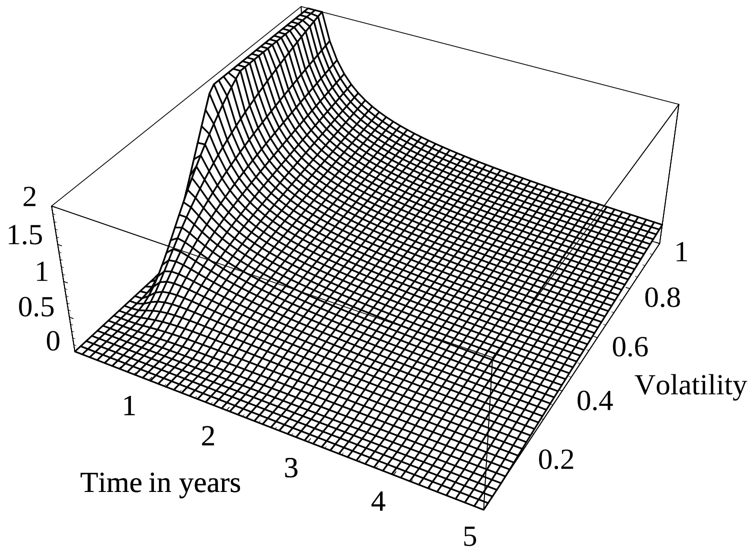

Figure 9: **Default arrival intensity (short credit spread) when the default threshold is observable but assets are not.** We plot the default arrival intensity λ_t (in events per year) as a function of time t (in years) and asset volatility σ. It is assumed that assets V follow a Brownian motion with drift $\mu = 6\% - \frac{1}{2}\sigma^2$. The default threshold is $D = -0.3$.

for $x \leq 0$ and $t > 0$. In Figure 9 the intensity is plotted as a function of time t and asset volatility σ. In comparison to the intensity curve in the previously analyzed case (c), the humps reflect the informational advantage of investors if they know the exact location of the default barrier. This is similar to the humps in spreads curves.

7 Conclusion

Cause and effect models of default postulate that a firm defaults when its assets fall below some liability-dependent trigger level. In these models it is typically assumed that the information used to calibrate and run the model is publicly available. In practice, however, asset value, volatility and growth rate are difficult to observe directly. As highlighted by the recent accounting scandals, corporate statements are difficult to interpret, making it hard to deduce the correct default trigger level.

In response to that, Duffie & Lando (2001) considered the case where investors have incomplete asset information. In this paper we generalize the analysis of Duffie & Lando (2001). We consider, from a common vantage point, *all* possible situations in which investors are not completely informed. Besides the example analyzed by Duffie & Lando (2001), this includes the situation where investors have incomplete information about the default barrier only. Another example is when investors have incomplete information about both assets and barrier.

We find that different types of incomplete information lead to qualitatively different credit spread term structures. A particularly interesting case is when there is incomplete information about the barrier only, a case that has not been considered previously. Here investors *learn* over time where the unknown barrier is by observing the historical asset low as an upper bound. This bound improves over time, which leads to decreasing term structures. Such learning does not occur with other instances of incomplete information, which is the reason for the differences in spread term structures.

With incomplete information the default comes unexpectedly, as in reduced-form models of default. In these models the uncertainty about default is typically prescribed by an exogenously given intensity, or conditional default rate. Duffie & Lando (2001) established such an intensity for a default model with incomplete asset information. This is economically satisfying, as the intensity is now defined endogenously in terms of fundamental firm variables. It is also methodologically satisfying, since it shows that the two very different approaches to default lead to equivalent results. In contrast to what we would expect, we find that there is no such intensity if investors have incomplete information about the default barrier only. The reason for this is the implicit learning of investors, which is only possible in this situation.

We generalize the concept of an intensity for the description of our uncertainty about default. We introduce the pricing trend, which we can think of as the cumulative intensity. For *any* default model that is predicated on the surprise nature of default we can find such a trend. We calculate the trend for our incomplete information model in terms of fundamental firm variables. This covers all possible situations with incomplete information from a common vantage point. Based on the trend we develop a reduced-form pricing framework, which generalizes the well-known intensity based pricing formulas in the literature. Our framework allows for closed-form expressions of the trend, intensity, default probabilities, defaultable security prices, and spreads.

A Conditional Asset Density

The conditional asset density $a(t, x, \cdot)$ of V_t given $\mathcal{A}_t$ and $M_t > x$ used in Proposition 5.4 can be computed explicitly in some cases.

Consider the information structure described in Example 3.3 and assume that the firm's asset value V satisfies (11). Suppose that bond investors receive a noisy asset report $Y_t = V_t + U$ at time t, where U is a noise variable independent of V and D, with given density q. First note that

$$P[M_t \in dy, V_t \in dz \,|\, Y_t] = \frac{q(Y_t - z)\, \varphi(t, y, z)\, dy\, dz}{p(t, Y_t)},$$

where $\varphi(t, \cdot, \cdot)$ is the joint density of (M_t, V_t), which is available explicitly, cf. Borodin & Salminen (1996). $p(t, \cdot)$ is the density of Y_t, which can be obtained via convolution of q and density of V_t, given by $\int_{-\infty}^{0} \varphi(t, y, \cdot) dy$ (explicit as well, cf. Borodin & Salminen (1996)). For $t < \tau$ we get by Bayes' rule

$$\begin{aligned} a(t, x, z)dz = P[V_t \in dz \,|\, Y_t, M_t > D := x] &= \frac{P[M_t > x, V_t \in dz \,|\, Y_t]}{P[M_t > x \,|\, Y_t]} \\ &= \frac{q(Y_t - z) \int_x^0 \varphi(t, y, z)\, dy\, dz}{\int_x^0 \int_x^\infty q(Y_t - v) \varphi(t, y, v)\, dv\, dy}. \end{aligned}$$

Now consider the information structure described in Example 3.5 and assume again that the firm's asset value V satisfies (11). We get

$$a(t, x, z)dz = P[V_t \in dz \,|\, M_t > D := x] = \frac{\int_x^0 \varphi(t, y, z)\, dy\, dz}{1 - H(t, x)},$$

which is also available explicitly, cf. Borodin & Salminen (1996).

B Proofs

Proof of Theorem 3.1. The compensator C of any positive càdlàg submartingale Y with respect to the filtration $(\mathcal{G}_t)$ can be written as

$$C_t = \lim_{h \downarrow 0} \frac{1}{h} \int_0^t \Big(E[Y_{s+h} \,|\, \mathcal{G}_s] - Y_s \Big) ds, \tag{18}$$

see Meyer (1966). If τ is $(\mathcal{G}_t)$-predictable, then the $(\mathcal{G}_t)$-compensator C of its indicator process N is given by $C = N$ itself. For $t < \tau$ we then have

$$\lim_{h \downarrow 0} \frac{1}{h} \int_0^t \Big(E[N_{s+h} \,|\, \mathcal{G}_s] - N_s \Big) ds = 0.$$

Due to the fact that for $t < \tau$

$$\int_0^t P[\tau \leq s \,|\, \mathcal{G}_s] ds = \int_0^t 1_{\{\tau \leq s\}} ds = 0, \tag{19}$$

we get

$$\lim_{h \downarrow 0} \frac{1}{h} \int_0^t P[\tau \leq s + h \,|\, \mathcal{G}_s] ds = 0. \tag{20}$$

Under our assumptions the credit spread is given by

$$S(t,T) = -\frac{1}{T-t} \log P[\tau > T \,|\, \mathcal{G}_t], \quad t < T, \quad t < \tau, \tag{21}$$

and the short spread at time t is

$$\lim_{h \downarrow 0} S(t, t+h) = \lim_{h \downarrow 0} \frac{1}{h} P[\tau \leq t + h \,|\, \mathcal{G}_t], \quad t < \tau. \tag{22}$$

Now the second statement is easily seen to be implied by (20). □

Proposition B.1. *The pricing trend A associated with the default time τ is continuous if and only if Condition 4.4 is satisfied.*

Proof of Proposition B.1. We observe that (5) is a Stieltjes integral because K is predictable and of bounded variation. It follows that A is continuous if and only if K is continuous. As proved by Giesecke & Goldberg (2003*b*), the compensator of a càdlàg supermartingale L that admits a Doob-Meyer decomposition is continuous if and only if L satisfies

$$\lim_{n \uparrow \infty} E[L_{\sigma_n}] = E[L_\sigma]$$

for every $(\sigma_n) \uparrow \sigma$. Since L is bounded, this is equivalent to Condition 4.4. □

Let A^τ denote the pricing trend stopped at default: $A_t^\tau = A_{t \wedge \tau}$ where $a \wedge b = \min(a, b)$.

Theorem B.2. *The process $N - A^\tau$ is a martingale in the filtration $(\mathcal{G}_t)$.*

Proof of Theorem B.2. See Jeulin & Yor (1978). □

Theorem 4.5 is a corollary to the following result.

Proposition B.3. *Let Condition 4.4 be satisfied. Let, for a fixed time T, Z be some bounded $\mathcal{G}_T$-measurable random variable. If the process Y defined by*

$$Y_t = E[Ze^{A_t - A_T} \,|\, \mathcal{G}_t], \quad t \leq T,$$

is continuous at τ, then on the set $\{\tau > t\}$ we have a.s. that

$$E[Z(1 - N_T) \,|\, \mathcal{G}_t] = E[Ze^{A_t - A_T} \,|\, \mathcal{G}_t], \quad t \leq T.$$

Proof of Proposition B.3. Letting $K_t = E[Ze^{-A_T} \,|\, \mathcal{G}_t]$, we can write $Y_t = e^{A_t} K_t$. By Proposition B.1, A is continuous and by virtue of Itô's product rule we have

$$dY_t = e^{A_t} dK_t + Y_{t-} dA_t.$$

Denote by $\Delta W_t = W_t - W_{t-}$ the jump of some càdlàg process W at t. Defining $U_t = (1 - N_t) Y_t$, we find again with the aid of the product rule that

$$\begin{aligned} dU_t &= -Y_{t-} dN_t + (1 - N_{t-}) dY_t + \Delta(1 - N_t) \Delta Y_t \\ &= (1 - N_{t-}) e^{A_t} dK_t - Y_{t-}(dN_t - (1 - N_{t-}) dA_t) \\ &= (1 - N_{t-}) e^{A_t} dK_t - Y_{t-}(dN_t - dA^\tau_t), \end{aligned} \tag{23}$$

where we have used our assumption that Y is continuous at τ to set $\Delta(1 - N_t)\Delta Y_t = 0$. Now integration of both sides of (23) yields

$$U_T - U_t = \int_t^T (1 - N_{s-}) e^{A_s} dK_s - \int_t^T Y_{s-} d(N_s - A^\tau_s).$$

Note that $(K_t)_{0 \leq t \leq T}$ and $N - A^\tau$ are martingales (for the latter see Theorem B.2). Since the integrands are bounded and predictable, U is a martingale and

$$U_t = Y_t(1 - N_t) = E[U_T \,|\, \mathcal{G}_t] = E[Z(1 - N_T) \,|\, \mathcal{G}_t],$$

which is our assertion. □

Proof of Theorem 4.5. The statement follows directly from Proposition B.3 by setting $Z = Xe^{-\int_t^T r_s ds}$ for $X \in \mathcal{G}_T$. □

Proof of Proposition 4.7. If τ admits an intensity λ, then $A_t = \int_0^t \lambda_s ds$. Now we can apply Proposition B.3 to (22) to see that on the set $\{\tau > t\}$ a.s.

$$\lim_{T \downarrow t} S(t, T) = -\frac{\partial}{\partial T} E[e^{A_t - A_T} \,|\, \mathcal{G}_t] \Big|_{T=t}$$

By dominated convergence,

$$\lim_{T\downarrow t} S(t,T) = -E[\frac{\partial}{\partial T}\, e^{A_t - A_T} \,|\, \mathcal{G}_t]\Big|_{T=t}$$
$$= E[\lambda_T\, e^{A_t - A_T} \,|\, \mathcal{G}_t]\Big|_{T=t}$$
$$= \lambda_t,$$

which completes the proof. □

Proof of Theorem 5.1. Noting that the threshold D is independent of assets, the survival process L can be written as

$$L_t = P[\tau > t \,|\, \sigma(V_s : s \le t)] = P[D < M_t \,|\, \sigma(V_s : s \le t)] = G(M_t).$$

Now the first statement follows from the definition of the trend.

If the asset process has continuous paths almost surely, then the running minimum asset process M is continuous as well. It follows that survival process L is continuous. By the uniqueness of the Doob-Meyer decomposition, the $(\mathcal{A}_t)$-compensator K of L is then given by $K = 1 - L$. In this case A is almost surely continuous and given by

$$A_t = -\int_0^t \frac{dL_s}{L_s} = -\log L_t. \tag{24}$$

This yields the second statement. □

Proof of Proposition 5.2. We can write for the survival process

$$L_t = P[\tau > t \,|\, \mathcal{A}_t] = E[1 - H(t,D)] = 1 - \int_{-\infty}^{0} H(t,x)\, g(x)\, dx. \tag{25}$$

By assumption, L is continuous and decreasing, so that its compensator is simply $K = 1 - L$. From (24), $A_t = -\log L_t$, and the result follows. □

Proof of Theorem 5.3. From Meyer (1966), we know that the $(\mathcal{A}_t)$-compensator K of the positive càdlàg $(\mathcal{A}_t)$-supermartingale L can be written as

$$K_t = \lim_{h\downarrow 0} \frac{1}{h} \int_0^t \Big(L_s - E[L_{s+h} \,|\, \mathcal{A}_s]\Big) ds.$$

If for $t, h \geq 0$ versions of the conditional expectation $E[L_{t+h} \,|\, \mathcal{A}_t]$ exist such that the limit

$$k_t = \lim_{h \downarrow 0} \frac{1}{h} \Big(L_t - E[L_{t+h} \,|\, \mathcal{A}_t] \Big)$$

exists almost surely for all $t \geq 0$, and (k_t) is progressively measurable with $E[\int_0^t |k_s| ds] < \infty$ for all $t \geq 0$, then

$$K_t = \int_0^t k_s ds. \tag{26}$$

With (25), we have

$$L_t - E[L_{t+h} \,|\, \mathcal{A}_t] = \int_0^t \Big(E[H(t+h, x) \,|\, \mathcal{A}_t] - H(t, x) \Big)\, g(x)\, dx,$$

and under our assumptions we get by dominated convergence

$$k_t = \int_{-\infty}^0 f(t, x)\, g(x)\, dx.$$

The statement then follows from (25) and (26) together with the definition of the trend. □

Proof of Proposition 5.4. First observe that, for $t > 0$ and $x \leq 0$,

$$\begin{aligned} f(t, x) &= \lim_{h \downarrow 0} \frac{1}{h} E[H(t+h, x) - H(t, x) \,|\, \mathcal{A}_t] \\ &= \lim_{h \downarrow 0} \frac{1}{h} P[M_{t+h} \leq x < M_t \,|\, \mathcal{A}_t] \\ &= m(t, x)(1 - H(t, x)), \end{aligned}$$

where we define $m(t, x) = \lim_{h \downarrow 0} \frac{1}{h} P[M_{t+h} \leq x \,|\, \mathcal{A}_t, M_t > x]$. From the Markov property of Brownian motion and by substituting $y = (x - z)/\sigma\sqrt{h}$ we obtain

$$\begin{aligned} m(t, x) &= \lim_{h \downarrow 0} \frac{1}{h} \int_x^\infty P[M_h \leq x - z] a(t, x, z)\, dz \\ &= -\sigma \lim_{h \downarrow 0} \int_{-\infty}^0 P[M_h \leq y\sigma\sqrt{h}] \frac{1}{\sqrt{h}} a(t, x, x - y\sigma\sqrt{h}) dy. \end{aligned}$$

The probability $P[M_t \le x] = \Psi(t,x)$ is explicitly given in (12) and we get

$$\lim_{h\downarrow 0} P[M_h \le y\sigma\sqrt{h}] = \lim_{h\downarrow 0} \Big(\Phi\Big(y - \frac{\mu\sqrt{h}}{\sigma}\Big) + e^{\frac{2\mu y\sqrt{h}}{\sigma}}\Phi\Big(y + \frac{\mu\sqrt{h}}{\sigma}\Big)\Big)$$
$$= 2\Phi(y),$$

where Φ is the standard normal distribution function. Now, since $a(t,x,x) = 0$, we obtain

$$\lim_{h\downarrow 0} \frac{1}{\sqrt{h}} a(t,x,x - y\sigma\sqrt{h}) = -y\sigma a_z(t,x,x),$$

where the derivative is taken from the right. By our hypotheses, for all $h < 1$

$$|P[M_h \le y\sigma\sqrt{h}] \frac{1}{\sqrt{h}} a(t,x,x - y\sigma\sqrt{h})|$$

has an integrable upper bound $U(y)$ that decreases to zero exponentially fast as $y \to -\infty$. Now by dominated convergence we get

$$m(t,x) = 2\sigma^2 a_z(t,x,x) \int_{-\infty}^{0} \Phi(y)\, y\, dy = \frac{1}{2}\sigma^2 a_z(t,x,x),$$

which completes the proof. □

References

Anderson, Ronald & Suresh Sundaresan (1996), 'Design and valuation of debt contracts', *Review of Financial Studies* **9**, 37–68.

Artzner, Philippe & Freddy Delbaen (1995), 'Default risk insurance and incomplete markets', *Mathematical Finance* **5**, 187–195.

Beneish, M. & E. Press (1995), 'Interrelation among events of default', *Contemporay Accounting Research* **12**, 299–327.

Black, Fischer & John C. Cox (1976), 'Valuing corporate securities: Some effects of bond indenture provisions', *Journal of Finance* **31**, 351–367.

Black, Fischer & Myron Scholes (1973), 'The pricing of options and corporate liabilities', *Journal of Political Economy* **81**, 81–98.

Borodin, A. & P. Salminen (1996), *Handbook of Brownian Motion: Facts and Formulae*, Birkhäuser, Basel.

Dellacherie, C. & P.A. Meyer (1982), *Probabilities and Potential*, North Holland, Amsterdam.

Duffie, Darrell & David Lando (2001), 'Term structures of credit spreads with incomplete accounting information', *Econometrica* **69**(3), 633–664.

Duffie, Darrell & Kenneth J. Singleton (1999), 'Modeling term structures of defaultable bonds', *Review of Financial Studies* **12**, 687–720.

Duffie, Darrell, Lasse Heje Pedersen & Kenneth J. Singleton (2003), 'Modeling sovereign yield spreads: A case study of Russian debt', *Journal of Finance* **58**(1), 119–159.

Duffie, Darrell, Mark Schroder & Costis Skiadas (1996), 'Recursive valuation of defaultable securities and the timing of resolution of uncertainty', *Annals of Applied Probability* **6**, 1075–1090.

Elliott, Robert, Monique Jeanblanc & Marc Yor (2000), 'On models of default risk', *Mathematical Finance* **10**(2), 179–195.

Eom, Y., Jean Helwege & Jay Huang (2002), Structural models of corporate bond pricing: An empricial analysis. To appear in Review of Financial Studies.

Föllmer, Hans & Martin Schweizer (1990), Hedging of contingent claims under incomplete information, *in* M.Davis & R.Elliot, eds, 'Applied Stochastic Analysis', Gordon and Breach, London, pp. 389–414.

François, Pascal & Erwan Morellec (2002), Capital structure and asset prices: Some effects of bankruptcy procedures. To appear in Journal of Business.

Giesecke, Kay (2003*a*), Correlated default with incomplete information. To appear in Journal of Banking and Finance.

Giesecke, Kay (2003*b*), Successive correlated defaults: Pricing trends and simulation. Working Paper, Cornell University.

Giesecke, Kay & Lisa Goldberg (2003*a*), Forecasting default in the face of uncertainty. Working Paper, Cornell University.

Giesecke, Kay & Lisa Goldberg (2003*b*), Trends and compensation. Working Paper, Cornell University.

Jarrow, Robert A. & Stuart M. Turnbull (1995), 'Pricing derivatives on financial securities subject to credit risk', *Journal of Finance* **50**(1), 53–86.

Jeulin & Marc Yor (1978), Grossisment d'une filtration et semimartingales: Formules explicites, *in* 'Séminaire de Probabilités XII, Lecture Notes in Mathematics 649', Springer-Verlag, Berlin, pp. 78–97.

Kusuoka, Shigeo (1999), 'A remark on default risk models', *Advances in Mathematical Economics* **1**, 69–82.

Lambrecht, Bart & William Perraudin (1996), 'Creditor races and contingent claims', *European Economic Review* **40**, 897–907.

Lando, David (1998), 'On cox processes and credit risky securities', *Review of Derivatives Research* **2**, 99–120.

Leland, Hayne E. (1994), 'Corporate debt value, bond covenants, and optimal capital structure', *Journal of Finance* **49**(4), 1213–1252.

Leland, Hayne E. & Klaus Bjerre Toft (1996), 'Optimal capital structure, endogenous bankruptcy, and the term structure of credit spreads', *Journal of Finance* **51**(3), 987–1019.

Longstaff, Francis A. & Eduardo S. Schwartz (1995), 'A simple approach to valuing risky fixed and floating rate debt', *Journal of Finance* **50**(3), 789–819.

Merton, Robert C. (1974), 'On the pricing of corporate debt: The risk structure of interest rates', *Journal of Finance* **29**, 449–470.

Meyer, Paul-Andre (1966), *Probability and Potentials*, Blaisdell, London.

Moraux, Franck (2002), Valuing corporate liabilities when the default threshold is not an absorbing barrier. Working Paper, Université de Rennes I.

Nielsen, Lars, Jesus Saa-Requejo & Pedro Santa-Clara (1993), Default risk and interest rate risk: The term structure of credit spreads. Working Paper, ISEAD.

RiskMetrics Group (2002), Creditgrades. Technical Document, New York.

Sarig, Oded & Arthur Warga (1989), 'Some empirical estimates of the risk structure of interest rates', *Journal of Finance* **44**, 1351–1360.

Yu, Fan (2002), Accounting transparency and the term structure of credit spreads. Working Paper, University of California at Irvine.

Zhou, Chunsheng (2001), 'The term structure of credit spreads with jump risk', *Journal of Banking and Finance* **25**, 2015–2040.