Estimating Structural Bond Pricing Models*

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Abstract

A difficulty which arises when implementing structural bond pricing models is the estimation of the value and risk of the firm's assets – neither of which is directly observable. We perform a simulation experiment in order to evaluate a maximum likelihood method applicable to this problem. The properties of the bond price estimators are examined using four theoretical bond pricing models: the Black & Scholes (1973) / Merton (1974) model, the Leland & Toft (1996) model, the Briys & de Varenne (1997) model, as well as the Ericsson & Reneby (2001) model. We contrast the performance of the maximum likelihood estimators to that of estimators traditionally used in academia and industry. The results are strongly supportive of the maximum likelihood approach. In fact, the inefficiency of the traditional estimator may explain the failure of past attempts to implement structural bond pricing models.

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1 Introduction

Corporate bond markets have more than doubled in size over the last ten years to reach a size exceeding that of the Treasury markets. The growth of the corporate debt sector to a dominant source of finance for US corporations underlines, by itself, the importance of accurate bond pricing models.[1] In addition, the market for credit derivatives is growing rapidly and accurate risk management and valuation tools will become necessary. Moreover, banks and regulators have recently taken a marked interest in credit risk modelling for risk management purposes. An important issue in this context is whether banks should be permitted to use in-house credit risk management models to determine capital requirements. A number of different approaches have been suggested, among them KMV Corporation's *PortfolioManager* which is based on a structural bond pricing model following Merton (1974).

The objective of this paper is to perform a simulation study to evaluate two distinct approaches to estimating structural bond pricing models. The performance of the currently most popular method is contrasted to a maximum likelihood approach developed by Duan (1994), which to date has been largely ignored. We believe that a simulation study is a valuable first step before bringing a theoretical or empirical model to bear on market data.

The traditional approach to implementing structural models has been to solve a system of equations that match the observed stock price and estimated stock volatility with model outputs (see Ronn & Verma (1986)). However, as pointed out by Duan, in theory one of the equations is redundant and no unique solution exists – except if, as in practice, the model is misspecified. Nevertheless, the approach is simple to implement and may have merit from a practical perspective if it provides sufficiently precise estimates. It has been applied in academic studies, adapted for commercial purposes by the KMV corporation and is often the only estimation approach considered in major finance textbooks (such as Hull (2000)). However, we demonstrate that the maximum likelihood approach, which circumvents the theoretical problem, exhibits markedly superior performance.

Structural bond pricing models value debt as a contingent claim on the firm's assets. This approach was pioneered by Black & Scholes (1973) and Merton (1974) and has since drawn considerable attention from practitioners and academics alike. An important feature of structural bond pricing models is that since all securities of a firm are treated as derivatives on the firm's assets, it is possible to use price information for one security – typically equity – to infer the value of another – typically debt.

Perhaps as a result of the failure of initial attempts to implement structural bond pricing models (see Jones et al. (1984) and Ogden (1987)), little progress was made in the empirical validation of the contingent claims approach. During the 90's, a number of stylized facts were incorporated into models – among them violations of the absolute priority rule in bankruptcy, taxes, costly financial distress, debt renegotiation and stochastic interest rates.[2] The more recent models are often better able than their predecessors to generate prices in line with market quotes with reasonable inputs. However, this alone does not guarantee that they will actually do well on market data given that the problem of estimating the unobserved asset value and its volatility remains.

Lately, the interest in empirical validation of structural models has also been rekindled. From the perspective of the information set used to estimate the models, one can distinguish between two approaches. One is to implement a model by relying only on stock prices and capital structure information. The other, more recent, method is to extend the information set to include bond prices.[3] No doubt the motivation for this can, at least in part, be attributed to the past poor performance of the traditional estimation technique.[4]

In this paper, we study the case when debt is priced using stock price and balance sheet data only.[5] This choice was made for the following reasons: first, it is a natural point of departure which, when evaluated, will serve as a benchmark for studying an extended information set. Second, many firms do not have traded bonds, or only thinly traded bonds, in which case using bond price information is not even an alternative. Finally, it is not clear how one would adjust the traditional estimation technique to include bond prices and therefore a comparison of the two estimation methods would be rendered difficult.

The evaluation is based on four theoretical bond pricing frameworks: the classic Black & Scholes (1973) / Merton (1974) model, the Briys & de Varenne (1997) model, the Leland & Toft (1996) model and the Ericsson & Reneby (2001) model.[6] The Black & Scholes / Merton model is the first, simplest and most well known of the structural models. It has also been implemented recently in the academic literature.[7] The Briys & de Varenne model is similar to the Longstaff & Schwartz (1995) and Nielsen et al. (1993) models in that it allows for stochastic interest rates and the possibility of default prior to debt maturity. Leland & Toft retain a constant term structure but, by incorporating taxes and default costs, are able to study the link between optimal capital structure and the cost of debt financing. Finally, while allowing firms to increase their leverage over time, the Ericsson & Reneby (2001) model distinguishes between aggregate debt and individual bonds, thus allowing salient features of a firm's financial structure to be captured while retaining the flexibility to account for the precise cash flows of a given corporate bond issue. These models all share a common (and for our purposes necessary), feature in that they allow a closed form solution for the value of a firm's equity. In contrast, for some structural models such as Nielsen et al. (1993) and Longstaff & Schwartz (1995), it is not immediately obvious how to value the firm's equity.

We first examine to which degree estimators for asset risk, firm value and bond prices are unbiased and efficient. Second, we investigate whether the asymptotic distributions of estimators carry over to small samples. The maximum likelihood approach is then contrasted to the traditional method of estimating structural bond pricing models.

To evaluate the performance of the two methods we perform a series of Monte Carlo experiments. We simulate sample paths for the asset value of firms that differ along the dimensions of operating risk and financial leverage. The corresponding stock price paths are then used to estimate, using both methods, the prices and credit spreads of different corporate bonds. A similar set of experiments was carried out by Lo (1986) to study the performance of maximum likelihood estimators of option prices. However, in that study, the state variable (the underlying stock price) is directly observable, whereas in what follows we use stock prices to estimate the level and parameters of our state variable – the firm's asset value. Many of our results will be directly related to this added complexity.

We demonstrate that the maximum likelihood approach clearly outperforms the traditional method both in terms of unbiasedness and efficiency. The errors of the latter approach are of a magnitude that can help to explain the consistent failure of attempts to implement structural bond pricing models. In contrast, maximum likelihood bond price estimators are unbiased and efficient, even for very risky bonds. The performance of the traditionally used approach, on the other hand, deteriorates as the spread increases. Moreover, the asymptotic distributions of the maximum likelihood estimators turn out to provide useful approximations in small samples.

The structure of the paper is the following. The next section provides a brief overview of the theoretical models. Section 3 reviews the maximum likelihood as well as the traditional estimation approach, and section 4 describes the simulation experiment. Section 5 reports and discusses the results and section 6 concludes.

2 Structural bond pricing models

In this section, we review the four bond pricing models: the Black & Scholes / Merton (BSM) model, the Briys & de Varenne (BV) model, the Leland & Toft (LT) model and the Ericsson & Reneby (ER) model. Since the focus of this paper is on the performance of an estimation approach rather than on the theoretical properties of any given model, we provide only brief descriptions of the models we study.

In all four models, the same fundamental assumptions are made regarding financial markets. Arbitrage opportunities are ruled out and investors are price takers. Furthermore, for at least some large investors, there are no restrictions on short selling stocks or risk free bonds and these can be traded costlessly and continuously in time. In addition, when we analyze the BV model we consider the special case where the term structure is driven by a Vasicek-model for the short rate r_t under the risk-adjusted pricing measure

$$dr_t = a\left(\overline{r} - r_t\right)dt + \gamma\, dW_t^r$$

where a denotes the mean reversion speed, $\overline{r}$ the mean reversion level of the short rate and γ its standard deviation.[8] The variable W_t^r is a Wiener process. The other three models are based on a constant interest rate r.

Furthermore, we assume that at least one class of the firm's securities, such as common stock, is traded and consequently completes the market; we do not need to assume that the assets of the firm are traded (see Ericsson

& Reneby (1999) for a discussion of this issue).

The value of assets is denoted ω_t and its changes are taken to obey a geometric Brownian motion. In the LT and ER models, assets generate revenue which is not reinvested at a rate β. This cash flow is used to service debt before being paid out as dividends to shareholders. Thus the evolution of the asset value can be described by

$$\begin{cases} d\omega_t = (r_t + \lambda\sigma - \beta)\, \omega_t \, dt + \sigma \, \omega_t \, dW_t^\omega \\ \omega_0 = \underline{\omega} \end{cases} \tag{1}$$

The term $(r_t + \lambda\sigma)$ is the expected return from holding the firm's assets – including accumulating the cash flow $\beta\omega$. The *growth rate* of the assets is $(r_t + \lambda\sigma - \beta)$. The parameter σ is the volatility of the asset value and λ can be interpreted as the market price of risk associated with the operations of the firm. Finally, W_t^ω is the Wiener process that generates asset value uncertainty. When interest rates are stochastic, we denote by ρ the instantaneous correlation between W_t^r and W_t^ω.

Next, consider the firm's securities. In particular, we need a formula for the stock price in order to estimate the asset value and volatility, as well as a formula for the bond which we ultimately want to price. We distinguish between two "layers" of debt: the firm's total debt (the sum of bank loans, bonds, accounts payable, salaries due, accrued taxes etc.) and the specific bond we are interested in. We will simply refer to the former as *debt* ($\mathcal{D}$) and the latter as the *bond* ($\mathcal{B}$). For future reference, denote the value of the corresponding riskfree debt with D. The value of the firm ($\mathcal{F}$) is the sum of the value of debt and equity ($\mathcal{E}$).

The four models differ in how the capital structure is set up. In the BSM and BV models, the firm issues a single discount bond which therefore also constitutes the firm's total debt.

In the LT model, the firm continuously issues and redeems bonds of a given maturity. The bonds are serviced by a continuous coupon stream until the principal repayment. The firm's debt is made up of all previously issued but unredeemed bonds, and total debt service is consequently the sum of payments to all those bonds. The coupon and the principal of the bonds are designed to establish a *constant* aggregate debt service flow which provides the basis for a closed form solution for equity. Thus, the model elegantly combines finite maturity debt with a tractable pricing function for the firm's stock.

In the ER model, the flexibility to model bonds' actual discrete coupon payments and other contractual features is retained. By assuming that, as an approximation, the overall debt service is unaffected by the particular structure of the single bond issue, a closed form solution for the value of equity can be derived. The firm is allowed to issue additional debt in the future, which prevents the firm's expected leverage ratio and associated default probability from converging to zero at long horizons.

Furthermore, there are differences across models in the way that financial distress is triggered. In the BSM model, default occurs at the maturity of the single bond issue if the value of the assets is insufficient to repay the principal amount. In such a case, the creditors take over the firm and recover the value of the assets.

In the other three models, default can occur at any time. In the LT and ER models, financial distress is triggered when shareholders no longer find it profitable, given the revenue produced by the assets, to continue servicing debt. Finally, in the BV model, default occurs when the asset value crosses an exogenous lower threshold.

We now turn to a more detailed description of the valuation formulae implied by the models (a list of notation is provided in table 1).

2.1 The Black & Scholes / Merton (BSM) Model

The value of equity in the Black & Scholes (1973) / Merton (1974) model is computed using the standard call option formula, with the exercise price set equal to the nominal amount (N) of the discount bond with maturity T:

$$\mathcal{E}(\omega_t, t) = \omega_t \cdot \phi(d_1) - e^{-r(T-t)} N \cdot \phi(d_2) \tag{2}$$

where $\phi[\cdot]$ represents the standard normal distribution function with d_1 and d_2 given in the appendix. All revenue generated by the assets is reinvested and thus $\beta = 0$ in (1). The value of the bond is, of course, equal to the value of the assets less the value of equity:

$$\mathcal{B}(\omega_t, t) = \omega_t - \mathcal{E}(\omega_t, t) \tag{3}$$

Since there are no taxes nor bankruptcy costs in this model, the value of the assets equals the value of the firm. Note that $(1 - \phi(d_2))$ represents the risk-adjusted probability of default up to date T.

2.2 The Briys & de Varenne (BV) Model

The Briys & de Varenne (1997) model differs from the Black & Scholes / Merton model in two respects: default can occur prior to the maturity of the single bond issue and interest rates are stochastic. More precisely, default occurs if the value of the firm's assets at any time falls below

$$L_t = \delta \cdot N \cdot P(t, T)$$

where $0 < \delta < 1$ and $P(t, T)$ is the value at t of a unit riskfree bond maturing at T. The value of the unit bond was derived by Vasicek (1977) and is reported in the appendix.

In case of default prior to maturity, bondholders receive $f_1 \cdot L_t$ and equity holders $(1 - f_1) \cdot L_t$; in case of default at maturity, they receive $f_2 \cdot \omega_T$ and

$(1-f_2)\cdot\omega_T$, respectively. The value of the bond is

$$\mathcal{B}(\omega_t,t)=N\cdot P(t,T)\cdot\left[\begin{array}{c}1-P_E(l_0,1)+P_E\left(q_0,\frac{l_0}{q_0}\right)\\ \\ -(1-f_1)\,l_0\left[\phi(-d_3)+\frac{\phi(-d_4)}{q_0}\right]\\ \\ -(1-f_2)\,l_0\left[\phi(d_3)-\phi(d_1)+\frac{\phi(d_4)-\phi(d_6)}{q_0}\right]\end{array}\right] \tag{4}$$

where the P_E, d_1-d_6, l_0 and q_0 are given in the appendix. The bond expression consists of five terms. The first captures the value of an otherwise identical credit riskfree bond. The second reflects the loss in value at the maturity of the bond, corresponding to the short put present in the BSM model. The third captures the recovery in case of a default prior to maturity and the last two capture the fact that sharing of any surplus in financial distress is assumed to deviate from the absolute priority rule ($f_1=f_2=1$ would correspond to the case of no such deviations).

Given the assumed absence of bankruptcy costs or taxes, equity is, as in the BSM model, simply the residual claim to the firm's assets:

$$\mathcal{E}(\omega_t,t)=\omega_t-\mathcal{B}(\omega_t,t) \tag{5}$$

The model is a direct extension of Nielsen et al. (1993) which differs from the model of Longstaff & Schwartz (1995) only by assuming that the default threshold, rather than being a constant, is linked to the value of a riskfree bond. In contrast to these two models, the BV model readily allows the derivation of an equity formula for a firm with discount debt.

2.3 The Leland & Toft (LT) Model

Leland & Toft (1996) allow for taxes and bankruptcy costs, and thus the value of the firm differs from the value of the assets (the unlevered firm).

The continuously paid coupons C are tax deductible at a rate τ and the realized costs of financial distress amount to a fraction k of the value of the assets in default. In this setting, the value of the firm is equal to the value of assets *plus* the tax shield *less* the costs of financial distress:

$$\mathcal{F}(\omega_t) = \omega_t + \frac{\tau C}{r}\left[1 - \left(\frac{\omega_t}{L}\right)^{-x}\right] - kL \cdot \left(\frac{\omega_t}{L}\right)^{-x}$$

We let L denote the default barrier, i.e. the critical asset value at which the equity holders voluntarily declare bankruptcy. The formulae for L and x are given in the appendix.

Shareholders are residual claimants to the value of the firm and so

$$\mathcal{E}(\omega_t) = \mathcal{F}(\omega_t) - \mathcal{D}(\omega_t) \tag{6}$$

where the value of debt is given by

$$\mathcal{D}(\omega_t) = \frac{C}{r} + \left(N - \frac{C}{r}\right)\left(\frac{1 - e^{-rT}}{rT} - I(\omega_t)\right) + \left((1-k)L - \frac{C}{r}\right)J(\omega_t)$$

The formulae for the functions $I(\omega_t)$ and $J(\omega_t)$, as well as for $i(\omega_t)$ and $j(\omega_t)$ used below, are given in the appendix. Note that the value of equity, the firm and debt are independent of time.

The value of a bond with maturity T, principal $P = \frac{N}{T}$ and coupon $c = \frac{C}{T}$ is:

$$\mathcal{B}(\omega, t) = \frac{c}{r} + e^{-r(T-t)}\left[P - \frac{c}{r}\right][1 - i(\omega_t)] + \left[(1-k)L - \frac{c}{r}\right]j(\omega_t) \tag{7}$$

This particular choice of bond principal and coupon is required to derive the tractable equity formula (6) above.

2.4 The Ericsson & Reneby (ER) Model

In this model, the capital structure of the firm is modelled in the spirit of Black & Cox (1976) and Leland (1994). However, unlike in those models, the firm is allowed to increase its total debt over time.[9] Although the increase in total debt is the result of many small debt issues, each floated at a fair price, it is assumed that this growth can be approximated by a continuous increase at a rate α

$$dN_t = \alpha \, N_t \, dt$$

Shareholders service debt continuously. Total debt service at time t is denoted C_t which also increases at rate α so that $C_t = C_0 e^{\alpha t}$. The coupon increases because new loans are taken up and need to be serviced, and not because the coupon to a single loan increases. The coupons are tax-deductible.

The distress barrier is determined by equity holders as in the LT model, i.e. as the level of asset value L at which shareholders voluntarily declare bankruptcy. In contrast to their model, the barrier grows (exponentially) with time along with the total nominal amount. Financial distress is costly and deviations from the absolute priority rule may take place. Shareholders receive a fraction ε of the assets, leaving creditors with $(1 - \varepsilon - k) \cdot L$, where k denotes the fractional default cost.

In this setup, the value of equity is given by

$$\begin{aligned} \mathcal{E}\left(\omega_t, t; \cdot\right) &= \omega_t - L_t \cdot G^{\alpha}\left(\omega_t\right) \\ &\quad - N_t \cdot \left(1 - G\left(\omega_t\right)\right) \\ &\quad + \tau \frac{C_t}{r - \alpha} \cdot \left(1 - G^{\alpha}\left(\omega_t\right)\right) \\ &\quad + \left(1 - \varepsilon - k\right) \cdot L_t \cdot \left(G^{\alpha}\left(\omega_t\right) - G\left(\omega_t\right)\right) \\ &\quad + \varepsilon \cdot L_t \cdot G^{\alpha}\left(\omega_t\right) \end{aligned} \tag{8}$$

where the formulae for $\{G, G^{\alpha}, L_t\}$ are given in appendix.[10] G is the value of a contingent claim that pays off one dollar when default occurs, whereas G^{α} is the value of receiving one dollar compounded at a rate α until the date of default. The interpretations of the five components of the equity price are respectively: (i) the value of the firm's earnings, (ii) the cost of debt service to current creditors, (iii) the value of the tax shield, (iv) the value of future borrowing using the assets as collateral, and finally (v) the value of payouts to shareholders in reorganization.

Next, consider a bond issued by this firm. We denote the size of the coupon with c and the principal with P. The coupons are due at times $\{t_i : i = 1...M\}$ where $T = t_M$ represents maturity. Furthermore, in case of default a fraction ψ of the principal is recovered. The value of such a bond is

$$\begin{aligned} \mathcal{B}(\omega_t, t; \cdot) &= \sum_{i=1}^{M-1} c \cdot H(\omega_t, t; t_i) \\ &\quad + (c + P) \cdot H(\omega_t, t; T) \\ &\quad + \psi \cdot P \cdot G(\omega_t, t; T) \end{aligned} \tag{9}$$

where $H(\omega_t, t; t_i)$ is a contingent claim that pays off one dollar at maturity unless the firm has defaulted prior to that date. The claim $G(\omega_t, t; T)$ pays off one dollar if (and when) the firm defaults prior to T. The formulae for these claims are given in the appendix.

3 Estimation

In this section we turn to a description of the two empirical approaches we wish to evaluate: first, the maximum likelihood approach developed by Duan (1994), and then the traditional method used by Ronn & Verma (1986) and others.

We do not consider estimation of the parameters of the short rate process in the Briys & de Varenne model; these are assumed to be known. When implementing a structural bond pricing model with stochastic interest rates, one generally calibrates the model of the riskfree term structure first using treasuries, before turning to the credit risk model itself (see e.g. Eom et al. (2000)). Moreover, it is not clear how to adjust the traditional method to estimate the parameters of an interest rate model.

3.1 Maximum Likelihood Estimation

The problem at hand is thus maximum likelihood estimation of the unobserved asset value process (1). This problem was first studied by Duan (1994) in the context of deposit insurance. The estimation is carried out using a time series of stock prices, $\mathcal{E}^{obs} = \left\{\mathcal{E}_i^{obs} : i = 1...n\right\}$. We use subscript '$i$' to index daily observations, in contrast to subscript 't', which refers to a point in time in years. For example, $\mathcal{E}_n^{obs}$ is the current stock price and $\mathcal{E}_1^{obs}$ is the stock price $(n-1)$ days ago.

We need the likelihood function of the observed price variable. Defining $f(\cdot)$ as the conditional density for $\mathcal{E}_i^{obs}$ gives us the following log-likelihood function for the observed equity vector $\mathcal{E}^{obs}$

$$L_{\mathcal{E}}\left(\mathcal{E}^{obs};\theta\right) = \sum_{i=2}^{n} \ln f\left(\mathcal{E}_i^{obs}\left|\mathcal{E}_{i-1}^{obs}\right.;\theta\right) \tag{10}$$

where θ is a vector of parameters to be estimated. The choice of individual parameters to include in θ will depend on the model and the data set.

To derive the density function for equity, a change of variables is made. This allows us to work with the well known density function g for a normally distributed variable – the log of the asset value:

$$g\left(\ln\omega_i \left|\ln\omega_{i-1}\right.;\theta\right) = \frac{1}{\sqrt{2\pi s_i^2}} \exp\left\{-\frac{(\ln\omega_i - m_i)^2}{2s_i^2}\right\} \tag{11}$$

The (one-period) conditional moments of the asset value distribution, m_i and s_i, are, for each model, given in the appendix.

The change of variables results in

$$f\left(\mathcal{E}_i^{obs}\left|\mathcal{E}_{i-1}^{obs}\right.;\theta\right) = g\left(\ln\omega_i\left|\ln\omega_{i-1}\right.;\theta\right)\Big|_{\omega_i=w\left(\mathcal{E}_i^{obs},t_i;\underline{\theta}\right)} \times\left(\left.\frac{\partial\,\mathcal{E}_i}{\partial\,\ln\omega_i}\right|_{\omega_i=w\left(\mathcal{E}_i^{obs},t_i;\underline{\theta}\right)}\right)^{-1} \tag{12}$$

where $\underline{\theta}\subset\theta$ is the subset of the parameter vector necessary to price equity. Note that $\mathcal{E}_i=\mathcal{E}_i\left(\omega_i,t_i;\cdot\right)$ refers to the equity *formula*, whereas $\mathcal{E}_i^{obs}$ denotes an observed market value. The function transforming equity to asset value is defined as $w\left(\mathcal{E}_i^{obs},t_i;\underline{\theta}\right)\equiv\mathcal{E}^{-1}\left(\mathcal{E}_i^{obs},t_i;\underline{\theta}\right)$, the inverse of the equity value function. Hence, there is a one-to-one correspondence between the stock price $\mathcal{E}_i^{obs}$ and the asset value ω_i (given $\underline{\theta}$). By inserting (12) into (10) we obtain the log-likelihood of the vector $\mathcal{E}^{obs}$ for a given choice of θ as

$$L_{\mathcal{E}}\left(\mathcal{E}^{obs};\theta\right) = \sum_{i=2}^{n}\ln g\left(\ln\omega_i\left|\ln\omega_{i-1}\right.;\theta\right)\Big|_{\omega_i=w\left(\mathcal{E}_i^{obs},t_i;\underline{\theta}\right)} -\sum_{i=2}^{n}\ln\left.\frac{d\,\mathcal{E}\left(\omega_i,t_i;\underline{\theta}\right)}{d\ln\omega_i}\right|_{\omega_i=w\left(\mathcal{E}_i^{obs},t_i;\underline{\theta}\right)} \tag{13}$$

Differentiating the equity formulae (2), (5), (6) or (8) with respect to the (log-) asset value yields the desired results. The parameter vector, $\widehat{\theta}$, is estimated by maximizing equation (13) with respect to θ. The market price of risk is estimated as a consequence of the chosen estimation method, even though it is not relevant for pricing. Finally, an estimate of the value of assets is simply obtained using the inverse equity function: : $\hat{\omega}_t=w\left(\mathcal{E}_n^{obs},t_n;\widehat{\underline{\theta}}\right)$.

Once we have obtained the pair $\left(\widehat{\omega}_t,\widehat{\theta}\right)$, we can compute the corresponding bond prices and credit spreads, $(\mathcal{B},\,\mathcal{S})$. Following Lo (1986), we can calculate the asymptotic distributions of these estimators. For any function of a variable, it holds that the ML estimator of that function is the function evaluated at the ML estimator of the variable. In this case it also holds that the estimators are asymptotically normally distributed:

$$\begin{aligned}\sqrt{n}\left(w\left(\mathcal{E}_n^{obs},t;\widehat{\theta}\right)-\omega_t\right)&\xrightarrow{L}N\left(0,\Sigma_{\widehat{\theta}}\frac{dw}{d\theta}\right)\\\sqrt{n}\left(\mathcal{B}\left(\widehat{\omega}_t,t;\widehat{\theta}\right)-\mathcal{B}\left(\omega_t,t;\theta\right)\right)&\xrightarrow{L}N\left(0,\Sigma_{\widehat{\theta}}\frac{d\mathcal{B}}{d\theta}\right)\\\sqrt{n}\left(\mathcal{S}\left(\widehat{\omega}_t,t;\widehat{\theta}\right)-\mathcal{S}\left(\omega_t,t;\theta\right)\right)&\xrightarrow{L}N\left(0,\Sigma_{\widehat{\theta}}\frac{d\mathcal{S}}{d\theta}\right)\end{aligned}\tag{14}$$

where $\Sigma_{\widehat{\theta}}$ is the asymptotic standard deviation of $\widehat{\theta}$. It is estimated using a Taylor-series expansion of the likelihood function (see appendix).

3.2 The Volatility Restriction Method

We will now review the traditional method of estimating the model's parameters from stock prices (see for example Jones et al. (1984) and Ronn & Verma (1986), Ogden (1987), Delianedis & Geske (1999), Eom et al. (2000) and Hull (2000)). For reasons that will become clear below we will refer to this as the *volatility restriction (VR) method.* Only two parameters can be estimated - the current asset value, ω_n, and its volatility σ. The following steps are typically carried out:

1. The stock price volatility $\sigma_{\mathcal{E}}$ is estimated using historical data. Denote this estimate by $\widehat{\sigma}_{\mathcal{E}}^{VR}$.

2. The asset value and volatility are estimated by solving the following system of equations

$$\left.\begin{array}{c}\sigma\cdot\frac{\partial\mathcal{E}(\omega_n,t_n;\sigma)}{\partial\omega}\cdot\omega_n=\widehat{\sigma}_{\mathcal{E}}^{VR}\cdot\mathcal{E}_n^{obs}\\ \\ \mathcal{E}\left(\omega_n,t_n;\sigma\right)=\mathcal{E}_n^{obs}\end{array}\right\}\rightarrow\left(\widehat{\omega}_n^{VR},\widehat{\sigma}^{VR}\right)$$

The first equality is implied by the application of Itô's lemma to equity as a function of ω and t and the second from matching the theoretical equity price with the observed market price.

There are several theoretical problems with this method, as pointed out by Duan (1994). The stock price volatility $\sigma_{\mathcal{E}}^{VR}$ is typically estimated assuming that it is constant[11] – even though it is a known function of ω and t. Furthermore, the first equation is redundant since it was used to derive the equity price formula in the *second* equation.[12] Another disadvantage of this approach is that it does not allow the straightforward calculation of the distributions of the estimators for ω and σ.

4 The Simulation Experiment

As noted above, one major drawback of the volatility restriction approach is that it does not allow estimation of parameters such as the debt growth rate, α, and the earnings rate, β. To allow us to compare the maximum likelihood approach to the volatility restriction approach, we estimate only asset value and risk and thus set $\underline{\theta} = \sigma$ (and $\theta = (\lambda, \sigma)$).

4.1 Experiment Design

To measure the performance of the two estimation approaches, we implement them for various firm scenarios. Firms are defined along two dimensions, *financial risk* and *business risk*. Following Merton (1974), we measure financial risk with the quasi-debt ratio, $\frac{D(N)}{\omega_n}$, the ratio of riskfree debt to asset value. Scenarios are constructed keeping ω_n constant and changing N, the nominal amount of debt. Business risk is measured by the riskiness of the firm as captured by the instantaneous volatility of the asset value, σ.

Four base case scenarios are created by setting financial and business risk to be either "high" or "low". A firm is considered to have high financial risk if its quasi-debt ratio of 1, and low if it is 0.75. Business risk is deemed high if $\sigma = 40\%$ and low if $\sigma = 20\%$. A complete list of the parameters used to construct the base scenarios in the four models can be found in table 1. The values are chosen so as to be representative of values used in previous studies and reported empirical evidence.[13]

The four base case scenarios are characterized in terms of leverage, equity volatility and spreads in table 2. Leverage is calculated using the book value of debt, and volatility is the instantaneous volatility of the stock price process. Leverage and equity volatility are the observable market equivalents of the more fundamental characteristics we use to define the scenarios in the first place ($\frac{D(N)}{\omega_n}$ and σ). For example, a firm with high financial but low business risk in the ER-model displays a leverage of 69% and a current equity volatility of 42%, whereas a low financial but high business risk firm has a leverage of 54% and an equity volatility of 58%. The resulting spreads are 194 and 292 basis points, respectively. In table 2, spreads range from 80 to over 600 basis points – a bond issued by a firm with high risk along both dimensions. This ensures that our study covers a wide array of bonds ranging from investment grade to speculative grade. We have chosen relatively high levels of financial and business risk. We expect estimation to be more difficult in such conditions and thus an evaluation of estimator performance to be more informative. Intuitively, the higher the uncertainty, the more difficult it should be to indirectly "observe" asset value using equity values.

To evaluate the performance of estimators in a given scenario, we place ourselves at time t_n and price the bond using historical stock prices. Clearly, numerous stock price histories may have led to the current leverage and equity volatility. In the Monte Carlo experiment, we take this into account by simulating 1000 historical asset value (and interest rate) paths consistent with the chosen scenario at time t_n. A similar procedure is used by Lo (1986) in order to assess the small sample properties of the maximum likelihood estimators of stock price volatility and option prices. A detailed description of the generation of sample paths is provided in the appendix.

For each asset value path, we first compute the corresponding stock price path using the relevant equity value formula. Second, we use this equity path to estimate the current asset value and the parameter vector θ. Finally, we use the estimates $(\widehat{\sigma}, \widehat{\omega}_n)$ to price the bond (using equations (2), (4), (7) or (9)), compute the standard error of the estimates (using equation (14)) and construct the corresponding confidence intervals. These steps are repeated for each sample path in order to assess the sampling distribution for a given model and scenario.

4.2 Output

In this section, we discuss the outputs produced and the tests performed. Since the ultimate use of an implemented model is to price bonds, the first question to address is whether the price and spread estimators are unbiased in small samples. The metric reported in the tables is the *mean error* of an estimator. Second, to measure how efficient they are, we report the *standard deviation* of the estimators as well as the corresponding average *absolute error*.

A third issue is whether the asymptotic distributions are useful approximations in small samples. We measure the skewness (*Sk.*) and kurtosis (*Ku.*) of the sampled distributions and perform Jarque-Bera (*JB*) tests for normality.[14] Even if the normality of a particular estimator can be rejected, its estimated standard deviation might be useful for hypothesis testing and to compute confidence intervals in small samples. Therefore, we report the mean estimated standard deviation (*mean estimated std.*) and as a measure of its efficiency we used *its* standard deviation (*std. of estimated std.*).[15]

To further pursue this issue we carry out size tests – i.e., we count how often the true value of an estimated price parameter falls outside the confidence interval calculated using the estimated standard deviation, i.e. the population size. If its size is close to the chosen nominal size (we use 1%, 5% and 10%), one may conclude that the asymptotic distribution is useful for purposes of calculating confidence intervals and conducting t-tests. An asterisk indicates that we can reject the null hypothesis of the population size equalling the nominal size.[16]

The price estimates ultimately depend on the estimates of the volatility and value of the assets. To help understand the results, we therefore report output for $(\widehat{\sigma}, \widehat{\omega}_n)$ as well.[17]

No asymptotic distribution has been suggested for the volatility restriction method. Therefore, only bias and efficiency are reported for this approach.

5 Results

Tables 3-6 display the results of the Monte Carlo simulations for the base case scenarios. Note that each table is the result of a different experiment with distinct set of asset value and, if applicable, interest rate paths.

Table 7 summarizes tables 3-6: tables 7a-7b show results organized by scenario, averaged over models, and table 7c reports average results across models *and* scenarios. Although the economic interpretation of these averages may be somewhat ambiguous, table 7 nevertheless provides an accessible overview of the overall performance of the volatility restriction and maximum likelihood methods. We do not, however, test the size for the summary tables.

Table 8 provides results for some alternative scenarios, and tables 9-

10, finally, show the effect on estimators of varying sample size and bond maturity.

5.1 Result Overview

First, we note in table 7 that the bias of the maximum likelihood approach is negligible for practical purposes. This result holds for all models and scenarios. In contrast, the volatility restriction approach exhibits an average pricing error of -4.5% (table 7c). However, the bias varies across models and scenarios which will be discussed in detail below.

It is also evident from table 7 that the ML approach is much more efficient than the VR approach; the standard deviation and mean absolute error of the latter estimators are multiple times higher (e.g. the standard deviation of the ML estimator of the spread is 35 b.p., whereas the VR estimator is 486 b.p.). Again, this is true for all models and scenarios. Figure 1 provides a visual summary of the relative efficiency of the two empirical approaches.

The explanation for the failure of the VR approach is intuitive. A highly volatile historical stock price series translates into a high estimated asset volatility and vice versa. This is a direct effect of solving the system of equations, as illustrated by the straight line in figure 2 which plots the estimated asset risk as a function of historical stock volatility. However, high stock volatility is not necessarily the result of high asset volatility – it could be the result of a historically high leverage. Typically, the higher the leverage, the higher the stock volatility. Thus, in a situation where asset value and hence stock prices have risen over the sample period, leverage and stock volatility have fallen. Historical stock volatility, computed as the average of realized volatilities, will therefore be higher than the current level. This translates into an excessive asset volatility estimate and, thus, a too low bond price estimate (figure 3). In sum, the described *volatility restriction effect* implies that increasing stock prices result in underpriced bonds, whereas decreasing stock prices produce overpriced bonds.

The reason the VR approach systematically underprices is straightforward: first, because stock prices increase on average in our simulations, and second, because the positive impact of leverage on stock volatility is more pronounced at high leverages. Both effects are visible in figure 4 which plots the value of the assets at the start of the sample period against historical stock volatility. To interpret the diagram, remember that the current value of assets is 1000 and that the current stock volatility is 65% in this scenario. The positive average growth in asset values (and thus equity values) is illustrated by the concentration of initial starting values below 1000 in the plot. A low initial stock price implies a high initial leverage, and the impact of leverage on stock volatility is visible from the negative correlation between asset values and stock volatility. In addition, the effect is stronger at high leverages, which is confirmed by the steeper slope to the left in the diagram.

This effect can also explain why estimation is less successful when the financial risk is high (tables 7a-7b). The higher the leverage, the more pronounced the effect on stock volatility. In a low leverage firm, the assumption of constant equity volatility is less severe.

The problem just described is not present when applying the maximum likelihood approach. As illustrated by figures 2 and 3, the ML estimators are able to disentangle the effect of leverage on stock volatility from the impact of business risk.

We now turn to the small sample distributions of the maximum likelihood-estimators. As is evident from a comparison of columns *Standard deviation* and *Mean estimated std.* in any scenario or model, the standard deviation estimator is unbiased and quite efficient: its own standard deviation (*Std.*

of estimated std.) is on average six times lower than the mean estimate (see table 7c). This means that the estimated standard deviation can be trusted upon to provide an accurate apprehension of the precision of a price or spread estimate.

However, by the Jarque-Bera test, we often reject the hypothesis of normally distributed estimators. Yet the size tests are quite successful, in the sense that the population size is close to the nominal size. This indicates that one can still use the estimated standard deviations to construct reliable confidence intervals and t-tests. In this sense, then, the estimators appear to be "sufficiently normal".

5.2 Sample Size

In this section, we examine how the distributions of the maximum likelihood estimators depend on the length of the equity price sample (n). We investigate four sample sizes : 90, 250, 500 and 750 days. Results are displayed in table 9: in panels a-b for the Black & Scholes/Merton model and in panels c-d for the Ericsson & Reneby model.

Using a three-month sample size, the ML estimators are quite inefficient; a slight bias is even introduced in the estimation of the Black & Scholes model. Increasing the sample size to 250 days leads to improved efficiency and normality. As noted above, estimators are unbiased, and efficiency is improved by roughly one third. Moreover, the Jarque-Bera test statistic has decreased; we can not reject the hypothesis that the estimators of the Ericsson & Reneby model are normally distributed, and we are less confident in rejecting normality for the Black & Scholes/Merton estimators. The size tests leave a mixed impression; most population sizes approach the nominal ones, but in some cases the convergence does not appear to be monotonic (e.g. the 10% size in the Ericsson & Reneby model seems to oscillate).

Nevertheless we cannot detect any statistically significant deviation from the nominal sizes.

Turning to longer samples still, the same pattern can be observed. Efficiency (and the efficiency of the efficiency estimate – the *Std. of estimated std.*) improves monotonically, whereas the effect of the measures of normality is less clear cut.[18] However, overall, the small sample distributions of estimators become more normal as the sample size increases.

5.3 Maturity

Table 10 presents results on the significance of bond maturity T in the Black & Scholes/Merton model. It is constructed keeping the quasi-debt ratio constant for maturities 5, 10, 20 and 30 years. The general result is that performance of estimators is worse for shorter maturities, but that the deterioration is more significant for VR than ML estimators. Increasing maturity beyond 10 years does not seem to have a very large effect for either estimator. Again, the explanation is tied to stock volatility: the shorter the maturity the more unstable the stock volatility.[19] As noted above, this is detrimental for the VR approach. The effect of changing the maturity is similar in the Briys & de Varenne model, since the capital structure is identical.

The effect of shifting bond maturity in the Leland & Toft model is not straightforward, since the technical conditions $C = c \cdot T$ and $N = P \cdot T$ tie the bond's coupon (c) and principal (P) to the firm's aggregate debt coupon (C) and nominal amount (N). A change in the amount of debt the equity holders have to service shifts the endogenous default barrier, the level of which has a strong impact on estimators – however, this "maturity" effect is due to the conditions above and is therefore perhaps more appropriately thought of as a barrier effect, which we will discuss in the next section.

In the Ericsson & Reneby model, as opposed to in the other models, the maturity of the bond does not enter the equity formula (eq. (8)). Consequently, the empirical performance of the ER model will not be influenced by the maturity of the bond one attempts to price.

5.4 Further Results

As noted above, the bias of the volatility restriction approach varies across models and scenarios. A closer look at tables 3-6 reveals that though never unbiased, it is mainly the implementation of the risky scenarios for the Leland & Toft model that accounts for the bias. For example, pricing a bond in the "Low Business Risk & High Financial Risk" scenario results in an average bias of -17.6% (the spread is overestimated by 892 basis points). The reason is that the (instantaneous) stock volatility in this model becomes extreme as asset values approach the barrier. This accentuates the volatility restriction effect. In this scenario, the distress barrier is 687 which, when assets are worth 1000, leads to a stock volatility of 95% (table 2). When assets are down to 800, volatility is 246%, at 750, it is 439% and at 700 stock volatility reaches a staggering 2046%. Thus, asset value paths that at some point during the sample period were near the barrier, lead to a severe overestimation of the current stock volatility and thus underpriced bonds. In a scenario with a very low barrier, on the other hand, no or very few paths would be likely to pass close to the barrier, few extreme stock volatilities would be observed and the effect just described would be mitigated. This is shown in table 8d, which reports results from the LT model in a scenario with a distress barrier equal to 303.

This particular problem for the VR approach does not arise when estimating the Ericsson & Reneby model since that model incorporates deviations from the absolute priority rule in financial distress. A lower bound on the payoff reduces volatility in general; in particular, at the brink of bankruptcy, shareholders do not face an all-or-nothing situation which, in turn, prevents stock volatility from attaining extreme values. Consider a scenario in the ER model, which is comparable to the scenario in the LT model studied above; setting the nominal amount of debt to 1710 we arrive at the same default barrier.[20] Table 8c displays Monte Carlo results from this scenario. It is evident that even with a very high barrier, and consequently with as many asset paths close to the barrier as in the LT scenario, the volatility restriction approach does not fail to the same extent for the ER model.

Of the four theoretical models, the implementation of the Briys & de Varenne model seems to be the most successful for the maximum likelihood as well as the volatility restriction approach. This can e.g. be seen by comparing the performance of either approach using this model (table 4) to the performance using the similar but simpler Black & Scholes/Merton model (table 3). The VR approach is approximately twice as efficient for the former model. Recall that the two models have the same capital structure, but that in addition, the BV model allows for stochastic interest rates and the possibility of early default. The explanation for estimator performance in this model is again related to the default barrier, although not through its effect in stock volatility.[21] In the BV model, the barrier L is an exogenous fraction (δ) of riskfree debt. The recovery to bondholders in financial distress is therefore also exogenous: $(1 - f_1) \cdot L$. This implies that one crucial component of the bond price – the payoff in financial distress – is fixed and independent of the estimated value and risk of assets. Naturally, this benefits both estimation approaches, as evidenced by the low standard deviations, and in particular the VR approach which otherwise would produce biased estimates. For example in table 4a, the VR approach overestimates business risk by more than 5%; yet prices are essentially unbiased.[22]

An effect particular to the BV model is that the performance of some estimators actually improves as the risk and hence the spread increases: this can be observed in table 4a by comparing the efficiency of spreads and prices in scenarios with low and high business risk. The explanation is that the riskier the scenario, the more important the (exogenously specified) distress component of the bond price. Finally, note that a small (and arguably economically insignificant) bias in prices and spreads, combined with a very low standard error, explains the failed size tests for the BV model.

Turning to the errors of the maximum likelihood approach, we have already noted in figures 2 and 3 that they appear independent of historical leverage. Errors do, however, depend on the realized volatility of the asset value path; an unusually high asset volatility results in an unusually high stock volatility which, in turn, translates into an excessive asset volatility estimate. However, after controlling for realized asset volatility, no further variables describing the realized stock value path have any explanatory power on the errors.[23]

The maximum likelihood approach works well also for very risky firms. As an example, table 8b displays the results from estimating the price of a bond issued by a "start-up" firm with low financial risk but extreme business risk. Even though the bond spread exceeds 1300 basis points, ML estimators are unbiased and efficient: the mean absolute spread error is 106. The VR approach, on the other hand, overestimates the spread by 178 basis points and has an absolute error of 366.

6 Concluding remarks

We have evaluated a maximum likelihood approach, originally developed by Duan (1994), for implementing structural bond pricing models. We have run Monte Carlo simulations for four different models (the classical Black & Scholes / Merton model, the Briys & de Varenne model, the Leland & Toft model and the Ericsson & Reneby model) to gauge the small sample properties of the estimators and contrasted the method to the traditional "volatility restriction" approach.

The studied maximum likelihood approach has several advantages over the volatility restriction approach beyond avoiding theoretical inconsistencies. First, it allows the straightforward derivation of the distributions of estimators and thus also bounds around estimates of bond prices, spreads and potentially any other metric that can be inferred from the model. Notably, it would allow the computation of confidence intervals for default probabilities, which would clearly be useful in credit risk management applications. Second, it readily allows several model parameters, as opposed to just the asset volatility, to be estimated.

We demonstrate that the maximum likelihood approach clearly dominates the traditionally used alternative. In fact, the latter performs so poorly that it may explain the failure of attempts to implement structural bond pricing models in the past. No matter how satisfactory the theoretical features of a model, its empirical use may have been limited by the chosen implementation method. The maximum likelihood approach analyzed in this paper, on the other hand, appears well suited for model testing.

The maximum likelihood bond price and spread estimators are unbiased and relatively efficient – absolute price errors lie in the range 1%-5%. In many instances we can reject the hypothesis that the asymptotic (normal) distributions of estimators are carried over to small (250 day) samples. Nevertheless, we show that standard deviations of estimators are often useful for calculating confidence bounds and conducting hypothesis tests. Thus even if the estimators are generally non-normal, they are "sufficiently normal" to be useful in applied work.

Notes

[1]During the period 1997-2000, about 70% of new capital raised by US corporations was in the form of debt. There are now approximately 3.5 trillion dollars worth of corporate debt outstanding.

[2]Models of corporate debt along these lines include Kim et al. (1993), Nielsen et al. (1993), Leland (1994), Longstaff & Schwartz (1995), Anderson & Sundaresan (1996), Leland & Toft (1996), Briys & de Varenne (1997), Mella-Barral & Perraudin (1997), Ericsson & Reneby (1998), Mella-Barral (1999), Fan & Sundaresan (2000), Duffie & Lando (2000) and Collin-Dufresne et al. (2001).

[3]Wei & Guo (1997) and Anderson & Sundaresan (2000) use this extended information set, whereas Jones et al. (1984), Ronn & Verma (1986), Ogden (1987) and Delianedis & Geske (1999) base their analysis on stock prices and balance sheets only. Eom et al. (2000) use both approaches.

[4]This is also likely to have been a driving force behind the development of reduced form credit risk models, where the explicit link between the firm's securities, asset value and capital structure is foregone. The information set typically used to estimate these models does not include equity values but consists only of bond prices. See for example Duffee (1999).

[5]However, the maximum likelihood approach can readily be applied with the extended information set – see Ericsson & Reneby (2001).

[6]Our choice of models is similar to that of Eom et al. (2000) who implement the models of Merton (1974), Geske (1977), Longstaff & Schwartz (1995) and Leland & Toft (1996).

[7]For example in Delianedis & Geske (1999) and Eom et al. (2000).

[8]The model in Briys & de Varenne (1997) allows for time dependent parameters but retains the Gaussian framework.

[9]Other papers that consider time varying leverage levels include Fischer et al. (1989), Collin-Dufresne & Goldstein (2001) and Taurén (1999).

[10]When $\alpha = r$ the third line is given by its limit value: $\tau C_t \ln \left(\frac{\omega_t}{L_t} \right)^{\frac{1}{\beta + 0.5\sigma^2}}$.

[11]The method used in the cited studies is identical to the estimation of the instantaneous (and constant) stock return volatility assuming the stock obeys a geometric Brownian motion. We will follow this approach when we implement this method in our Monte Carlo study.

[12]It is interesting to note that this implies that if the estimation of $\sigma_{\mathcal{E}}$ would produce the *correct* estimate ($\widehat{\sigma}_{\mathcal{E}}^{VR} = \sigma_{\mathcal{E}}(\omega_n, t; \sigma)$), one of the equations would be redundant. Thus, the first theoretical inconsistency (assuming constant stock price volatility) is necessary to find a unique solution to a system of equations which, otherwise, would have an infinite number of solutions.

[13]For example we use 20% as an approximation of the effective tax rate faced by US corporations (see Leland & Toft (1996)), and a 31% recovery rate for corporate debt (see Altman & Kishore (1996)).

[14]The null hypothesis of a normally distributed estimator can be rejected with 5% significance if the JB-statistic exceeds 6.

[15]When using the term “estimate”, we refer to the estimate for a particular sample path. The expected value of an estimate is calculated as the mean of estimates across generated sample paths.

[16]The standard error of the population size p, given 1000 replications, is

$\sqrt{\frac{p(1-p)}{1000}}$. This yields relatively wide confidence bounds. Suppose that the population sizes are 1%, 5% and 10% respectively, then standard errors are approximately 0.3%, 0.7% and 0.9%.

[17]For brevity, we do not report results for the estimation of the market price of risk, λ, since the estimates are too weak to make them at all useful; this is related to the result that it is next to impossible to estimate the expected return of an asset whose dynamics can be described by an Itô process (see Merton (1980)). Fortunately, we don't need the market price of risk to price bonds.

[18]Lo (1986) reports similar results.

[19]The formula for the equity volatility contains the first derivative of equity with respect to asset value. When maturity is short this derivative is more sensitive to changes in the state variable. An analogous observation is that the gamma of stock options that are close to the money tends to be higher the shorter the remaining time to expiration.

[20]In the ER model, the stock volatility at asset value $\omega = 1000$ is 80%, which is a bit lower than in the LT model, but the striking difference is noticable as asset values decline: at asset value 800, volatility is 98%, at 750, 84%, and at 700 only 38%. In the limit, as the value of assets approaches the barrier, stock volatility converges to the volatility of the payoff $\varepsilon \cdot \omega$ (i.e. the asset volatility σ).

[21]That the perfomance is not linked to the presence of stochastic interest rates is evident from table 8a, which presents results from the BV model with constant interest rates. The performance of the empirical approaches is in line with the previously presented results using stochastic interest rates.

[22]The reason for the VR approach overestimating business risk in the BV, but not in the BSM, model is the presence of the barrier which strengthens the previously discussed volatility restriction effect.

[23]Regression results available upon request.

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7 Appendix

For notational convenience, we assume all pricing takes place at $t = 0$ and drop related subscripts (e.g., we use N to denote N_0).

7.1 The Black & Scholes Model

The integration limits are standard:

$$\begin{cases} d_1 = \frac{\ln \frac{\omega}{N} + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \\ d_2 = d_1 - \sigma\sqrt{T} \end{cases}$$

7.2 The Briys & de Varenne model

The integration limits are

$$\begin{aligned} d_1 &= \frac{\ln l_0 + \frac{1}{2}\Sigma(T)}{\sqrt{\Sigma(T)}} = d_2 + \sqrt{\Sigma(T)} \\ d_3 &= \frac{\ln q_0 + \frac{1}{2}\Sigma(T)}{\sqrt{\Sigma(T)}} = d_4 + \sqrt{\Sigma(T)} \\ d_5 &= \frac{\ln \frac{q_0^2}{l_0} + \frac{1}{2}\Sigma(T)}{\sqrt{\Sigma(T)}} = d_6 + \sqrt{\Sigma(T)} \end{aligned}$$

with

$$\begin{aligned} l_0 &= \frac{A_0}{FP(0,T)} \\ q_0 &= \frac{A_0}{\alpha FP(0,T)} \end{aligned}$$

and

$$\begin{aligned} P_E(l_0, 1) &= -l_0 N(-d_1) + N(-d_2) \\ P_E\left(q_0, \frac{l_0}{q_0}\right) &= -q_0 N(-d_5) + \frac{l_0}{q_0} N(-d_6) \end{aligned}$$

Finally, the volatility term in the Vasicek case is given by

$$\begin{aligned}\Sigma(T) &= 2\rho\sigma\frac{\gamma}{a}\left(T-\frac{1-e^{-aT}}{a}\right) \\ &+\frac{\gamma^2}{a^2}\left(T-\frac{1}{2a}\left(e^{-2a\Delta t}-4e^{-a\Delta t}+3\right)\right) \\ &+\sigma^2 T\end{aligned} \tag{15}$$

The value of a risk free unit bond is

$$P(0,T)=A(0,T)\,e^{-B(0,T)\cdot r_t}$$

with

$$\begin{aligned}B(0,T) &= \frac{1-e^{aT}}{a} \\ A(0,T) &= \exp\left[\frac{(B(0,T)-T)\left(a^2\overline{r}-\frac{1}{2}\gamma^2\right)}{a^2}-\frac{\gamma^2 B(0,T)^2}{4a}\right]\end{aligned}$$

7.3 The Leland & Toft Model

The default barrier is

$$L=\frac{\frac{C}{r}\left(\frac{A}{rT}-B\right)-\frac{AP}{rT}-\frac{\tau Cx}{r}}{1+\alpha x-(1-\alpha)B}$$

where

$$\begin{aligned}A &= 2ye^{-rT}\phi\left[y\sigma\sqrt{T}\right]-2z\phi\left[z\sigma\sqrt{T}\right] \\ &-\frac{2}{\sigma\sqrt{T}}n\left[z\sigma\sqrt{T}\right]+\frac{2e^{-rT}}{\sigma\sqrt{T}}n\left[y\sigma\sqrt{T}\right]+(z-y)\end{aligned}$$

$$B=-\left(2z+\frac{2}{z\sigma^2T}\right)\phi\left[z\sigma\sqrt{T}\right]-\frac{2}{\sigma\sqrt{T}}n\left[z\sigma\sqrt{T}\right]+(z-y)+\frac{1}{z\sigma^2T}$$

and $n[\cdot]$ is the standard normal density function.

The components of the debt and bond formulae are

$$I(\omega)=\frac{1}{rT}\left(i(\omega)-e^{-rT}j(\omega)\right)$$

$$i(\omega) = \phi[h_1] + \left(\frac{\omega}{L}\right)^{-2a} \phi[h_2]$$

$$j(\omega) = \left(\frac{\omega}{L}\right)^{-y+z} \phi[q_1] + \left(\frac{\omega}{L}\right)^{-y-z} \phi[q_2]$$

and

$$J(\omega) = \frac{1}{z\sigma\sqrt{T}} \left(\begin{array}{c} -\left(\frac{\omega}{L}\right)^{-a+z} \phi[q_1]\, q_1 \\ \\ +\left(\frac{\omega}{L}\right)^{-a-z} \phi[q_2]\, q_2 \end{array} \right)$$

Finally

$$\begin{aligned} q_1 &= \frac{-b - z\sigma^2 T}{\sigma\sqrt{T}} \\ q_2 &= \frac{-b + z\sigma^2 T}{\sigma\sqrt{T}} \\ h_1 &= \frac{-b - y\sigma^2 T}{\sigma\sqrt{T}} \\ h_2 &= \frac{-b + y\sigma^2 T}{\sigma\sqrt{T}} \end{aligned}$$

and

$$\begin{aligned} y &= \frac{r - \beta - 0.5\sigma^2}{\sigma^2} \\ z &= \frac{\sqrt{y^2\sigma^4 + 2r\sigma^2}}{\sigma^2} \\ x &= y + z \\ b &= \ln\frac{\omega}{L} \end{aligned}$$

7.4 The Ericsson & Reneby Model

The dollar-in-default claim

$$G = \left(\frac{\omega}{L}\right)^{-\theta}$$

$$\theta = \frac{\sqrt{\left(\frac{r-\beta-\alpha-0.5\sigma^2}{\sigma}\right)^2 + 2r} + \frac{r-\beta-\alpha-0.5\sigma^2}{\sigma}}{\sigma}$$

The "α-increasing" dollar-in-default claim:

$$G^\alpha = \left(\frac{\omega}{L}\right)^{-\theta^\alpha}$$

$$\theta^\alpha = \frac{\sqrt{\left(\frac{r-\beta-\alpha-0.5\sigma^2}{\sigma}\right)^2 + 2\left(r-\alpha\right)} + \frac{r-\beta-\alpha-0.5\sigma^2}{\sigma}}{\sigma}$$

The default barrier (for $r \neq \alpha$):

$$L_t = \frac{\tau \frac{r}{r-\alpha}\theta^\alpha - \theta}{(\varepsilon - 1)\cdot(1+\theta^\alpha) + \delta\cdot(\theta^\alpha - \theta)} \cdot N_t$$

As can be seen, the endogenous barrier grows, along with the nominal amount of debt, at a rate of α.

The down-and-out binary option is worth

$$H\left(\omega, 0; t_i\right) = e^{-rt_i} \cdot Q^B\left[\omega; t_i\right]$$

where the risk-adjusted probability (under the probability measure where prices normalized by the money market account are martingales) of no default until t_i is given by

$$Q^B\left[\omega; t_i\right] = \phi\left[d^B\right] - \left(\frac{\omega}{L}\right)^{-2\frac{r-\beta-\alpha-0.5\sigma^2}{\sigma^2}} \cdot \phi\left[D^B\right]$$

$$d^B = \frac{\ln\frac{\omega}{L} + \left(r-\beta-\alpha-0.5\sigma^2\right)t_i}{\sigma\sqrt{t_i}}$$

$$D^B = \frac{\ln\frac{L}{\omega} + \left(r-\beta-\alpha-0.5\sigma^2\right)t_i}{\sigma\sqrt{t_i}}$$

The value of the finite maturity dollar-in-default claim is

$$G(\omega,0;t_i) = G(\omega)\cdot\left(1 - Q^G[\omega;t_i]\right)$$

where the risk-adjusted probability (under the probability measure where prices normalized with G are martingales) of no default until t_i is given by

$$\begin{aligned} Q^G[\omega;t_i] &= \phi\left[d^G\right] \\ &\quad - \left(\frac{\omega}{L}\right)^{-2\frac{r-\beta-\alpha-\left(\frac{1}{2}+\theta\right)\sigma^2}{\sigma^2}}\phi\left[D^G\right] \end{aligned}$$

with

$$\begin{aligned} d^G &= \frac{\ln\frac{\omega}{L} + \left(r-\beta-\alpha-\left(\frac{1}{2}+\theta\right)\sigma^2\right)t_i}{\sigma\sqrt{t_i}} \\ D^G &= \frac{\ln\frac{L}{\omega} + \left(r-\beta-\alpha-\left(\frac{1}{2}+\theta\right)\sigma^2\right)t_i}{\sigma\sqrt{t_i}} \end{aligned}$$

7.5 Monte Carlo Simulation

First consider the integral equations for the state parameters. For the constant interest rate models, the integral equation for the value of (logged) assets simply is

$$\ln\omega_t = \ln\omega_0 + \left(r + \lambda\sigma - \beta - \frac{1}{2}\sigma^2\right)t + \sigma W_t^\omega \tag{16}$$

(In the Black & Scholes model, $\beta = 0$). In the Briys & de Varenne model, the integral equation for the short rate is

$$r_t = r_0 e^{-at} + \overline{r}(1-e^{-at}) + \gamma\int_0^t e^{a(s-t)}dW_s^r \tag{17}$$

The integral equation for the log asset value is

$$\begin{aligned} \ln\omega_t &= \ln\omega_0 + \left(\overline{r} + \lambda\sigma - \frac{1}{2}\sigma^2\right)t + \sigma W_t^\omega \\ &\quad + (r_0 - \overline{r})\left(\frac{1-e^{-at}}{a}\right) + \frac{\gamma}{a}\int_0^t\left(1-e^{-as}\right)dW_s^r \end{aligned} \tag{18}$$

7.5.1 Simulating data

In the constant interest rate models, we first generate an $(n-1)$ vector of discrete Wiener increments

$$\Delta \mathtt{W}^{\boldsymbol{\omega}} = \{\Delta W_i^\omega : i = 1...n-1\}$$

which are normally i.i.d.

$$\Delta W_i^\omega \sim N\left(0, \sqrt{\Delta t}\right)$$

These are used to construct the corresponding log-asset value path from (16)

$$\ln \omega_{i+1} = \ln \omega_i + \left(r + \lambda\sigma - \beta - \frac{1}{2}\sigma^2\right)\Delta t + \sigma \Delta W_i^\omega : i = 1...n-1$$

where $\ln \omega_n$ is defined by the scenario.

In the Briys & de Varenne model, we first generate an $(n-1) \times 2$ matrix of discrete Wiener increments

$$\Delta \mathtt{W}^{\boldsymbol{\omega}} = \{\Delta W_i^\omega, \Delta W_i^r : i = 1...n-1\}$$

which have (pair-wise) correlation ρ and distribution as above. These are used to construct the corresponding short rate path from (17)

$$r_{i+1} = r_i e^{-a\Delta t} + \overline{r}(1 - e^{-a\Delta t}) + \gamma e^{a\Delta t}\Delta W_i^r : i = 1...n-1$$

and log-asset value path from (18)

$$\begin{aligned} \ln \omega_{i+1} \quad &= \quad \ln \omega_i + \left(\overline{r} + \lambda\sigma - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\Delta W_i^\omega \qquad (19) \\ &\quad + (r_i - \overline{r})\left(\frac{1 - e^{-a\Delta t}}{a}\right) + \gamma\left(\frac{1 - e^{-a\Delta t}}{a}\right)\Delta W_i^r : i = 1...n-1 \end{aligned}$$

with r_n and $\ln \omega_n$ defined by the scenario.

7.5.2 Estimating with maximum likelihood

The maximum likelihood function (13) depends on the model in two ways: first, through the derivative of the equity function and second, through the conditional moments of the asset value distribution. The former are straightforward to calculate from the respective equity formulae (2), (5), (6) and (8). The latter are, for the three constant interest rate models, as follows:

$$\left\{\begin{array}{l} m_i = E\left[\ln\omega_i \left|\ln\omega_{i-1}\right.\right] = \ln\omega_{i-1} + \left(r - \beta + \lambda\sigma - \frac{1}{2}\sigma^2\right)\Delta t \\ \\ s^2 = E\left[\left(\ln\omega_i - m_i\right)^2 \left|\ln\omega_{i-1}\right.\right] = \sigma^2\Delta t \end{array}\right. \quad : i = 2...n$$

For the Briys & de Varenne model, the conditional moments are (from (19) above)

$$\left\{\begin{array}{l} m_i = E\left[\ln\omega_i \left|\ln\omega_{i-1}\right.\right] = \ln\omega_{i-1} + \left(\overline{r} + \lambda\sigma - \frac{1}{2}\sigma^2\right)\Delta t + B\left(0,\Delta t\right)\left(r_{i-1} - \overline{r}\right) \\ \\ s^2 = \Sigma\left(\Delta t\right) \end{array}\right. \quad : i = 2...n$$

where $B\left(0,\Delta t\right)$ and $\Sigma\left(\Delta t\right)$ were given in section 7.2 above.

7.6 The standard deviation of the estimated parameter vector

We use the GAUSS Constrained Maximum Likelihood Application to estimate the standard deviation of the estimated parameter vector, $\sum_{\underline{\widehat{\theta}}}$. The estimates this application provides are based on a Taylor-series approximation to the likelihood function (see e.g. Amemiya (1985), page 111).

$$\sqrt{n}\left(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}\right) \xrightarrow{L} N\left(0, A^{-1}BA^{-1}\right)$$

where

$$
\begin{aligned}
A &= E\left[\frac{\partial^2 L}{\partial \boldsymbol{\theta}\,\partial \boldsymbol{\theta'}}\right] \rightarrow \widehat{A} = \frac{1}{n}\sum_{i=2}^{n} \frac{\partial^2 L_i}{\partial \boldsymbol{\theta}\,\partial \boldsymbol{\theta'}} \\
B &= E\left[\left(\frac{\partial L}{\partial \boldsymbol{\theta}}\right)'\left(\frac{\partial L}{\partial \boldsymbol{\theta}}\right)\right] \rightarrow \widehat{B} = \frac{1}{n}\sum_{i=2}^{n} \left(\frac{\partial L_i}{\partial \boldsymbol{\theta}}\right)'\left(\frac{\partial L_i}{\partial \boldsymbol{\theta}}\right)
\end{aligned}
$$

From *Gauss Applications - Constrained Maximum Likelihood* (1995). Specifically, in the this study, where $\boldsymbol{\theta} = (\sigma\,\lambda)$, the estimated standard deviation for the estimated asset volatility is

$$
\widehat{\sum}_{\hat{\sigma}} = \begin{pmatrix} 1 & 0 \end{pmatrix} \left(\widehat{A}^{-1}\widehat{B}\widehat{A}^{-1}\right) \begin{pmatrix} 1 \\ 0 \end{pmatrix}
$$

TABLE 1: List of Notation/Base Case Scenario Parameters

Parameter	Notation	Base Case Value: Black & Scholes/ Merton	Briys & de Varenne	Leland & Toft	Ericsson & Reneby
Current asset value	ω_n	1000	1000	1000	1000
Riskfree rate	r_n	5%	5%	5%	5%
Market price of risk	λ	0.5	0.5	0.5	0.5
Asset volatility (low/high)	σ	20%/40%	20%/40%	20%/40%	20%/40%
Revenue rate	β	–	–	2%	2%
Debt increase rate	α	–	–	–	4%
Nominal debt (low/high)	N	1237/1649	1237/1649	665/887	750/1000
Debt coupon	C	–	–	$c \cdot T$	$r \cdot N$
Violations of Absolute Priority	$(1-f_1) = (1-f_2)$; ε	–	40%	–	5%
Bankruptcy costs	k	–	–	15%	15%
Tax rate	τ	–	–	20%	20%
Maturity	T	10	10	10	10
Bond principal	P	N	N	$\frac{N}{T}$	100
Bond coupon	c	–	–	$8\% \times P$	$8\% \times P$
Default threshold	δ	–	60%	–	–
Recovery rate of bond	ψ	–	–	–	31%
Asset value/short rate correlation	ρ	–	-0.25	–	–
Mean reversion speed of short rate	a	–	0.2	–	–
Mean level of short rate	$\overline{r}$	–	5%	–	–
Standard deviation of short rate	γ	–	2%	–	–

TABLE 2: Base Case Scenario Characteristics

Model	Business Risk	Financial Risk	Default Barrier	Equity Volatility (%)	Leverage (%)	Bond Spread (b.p.)
Black & Scholes	Low	Low	-	43	67	164
	Low	High	-	50	80	285
	High	Low	-	59	58	506
	High	High	-	62	68	640
Briys & de Varenne	Low	Low	459	45	79	158
	Low	High	612	59	82	270
	High	Low	459	76	62	351
	High	High	612	104	76	415
Leland & Toft	Low	Low	515	52	65	104
	Low	High	687	95	83	268
	High	Low	383	88	64	419
	High	High	511	122	79	600
Ericsson & Reneby	Low	Low	301	34	56	82
	Low	High	401	42	69	194
	High	Low	178	58	54	292
	High	High	237	65	64	413

TABLE 3a: Base Scenario Results - Black & Scholes/Merton Model

	Empirical Approach	True value	Bias & Efficiency: Mean error	Standard deviation	Mean absolute error	Distribution: Mean estimated std.	Std. of estimated std.	Sk.	Ku.	JB	Size test 1%	5%	10%
Low Business Risk & Low Financial Risk													
Asset volatility	ML	20%	0.1%	1.6%	1.3%	1.7%	0.3%	0.1	3.0	1	1.4%	5.0%	9.1%
	VR		0.5%	2.8%	2.3%								
Asset value	ML	1000	-2	20	16	20	4	-0.1	3.0	3	2.2%*	5.4%	9.6%
	VR		-6	34	27								
Spreads	ML	164	3	31	25	32	7	0.2	3.0	9	2.2%*	5.2%	9.4%
	VR		12	55	44								
Prices	ML	637	-0.3%	3.1%	2.5%	3.2%	0.6%	-0.1	3.0	3	2.0%*	5.4%	9.5%
	VR		-1.1%	5.4%	4.3%								
Low Business Risk & High Financial Risk													
Asset volatility	ML	20%	0.1%	2.2%	1.7%	2.1%	0.4%	0.4	3.3	28	2.0%*	5.5%	9.8%
	VR		0.5%	3.5%	2.8%								
Asset value	ML	1000	-1	41	33	41	7	-0.3	3.2	20	2.1%*	5.9%	10.4%
	VR		-8	66	53								
Spreads	ML	285	3	56	44	55	12	0.5	3.5	54	2.0%*	5.6%	10.3%
	VR		15	92	72								
Prices	ML	752	-0.1%	5.5%	4.4%	5.4%	0.9%	-0.3	3.2	20	2.0%*	6.1%	10.3%
	VR		-1.1%	8.8%	7.0%								

TABLE 3b: Base Scenario Results - Black & Scholes/Merton Model (continued)

			Bias & Efficiency			Distribution							
	Empirical Approach	True value	Mean error	Standard deviation	Mean absolute error	Mean estimated std.	Std. of estimated std.	Sk.	Ku.	JB	Size test 1%	Size test 5%	Size test 10%
High Business Risk & Low Financial Risk													
Asset volatility	ML	40%	-0.1%	3.1%	2.5%	3.1%	0.4%	0.0	2.9	0	1.7%	5.6%	10.0%
	VR		0.0%	4.2%	3.4%								
Asset value	ML	1000	2	34	27	34	4	0.1	2.9	3	1.9%*	5.9%	10.4%
	VR		1	45	36								
Spreads	ML	506	-1	74	59	75	12	0.1	2.9	2	2.2%*	5.9%	10.6%
	VR		3	102	81								
Prices	ML	452	0.4%	7.4%	5.9%	7.5%	0.9%	0.1	2.9	3	1.9%*	5.9%	10.4%
	VR		0.3%	10.0%	8.1%								
High Business Risk & High Financial Risk													
Asset volatility	ML	40%	0.0%	3.5%	2.8%	3.5%	0.5%	0.3	3.3	17	1.8%	5.1%	9.6%
	VR		-0.1%	4.8%	3.8%								
Asset value	ML	1000	2	49	39	49	6	0.0	3.2	2	1.8%	5.4%	9.4%
	VR		4	66	53								
Spreads	ML	640	1	94	74	94	15	0.4	3.4	29	1.8%	5.2%	9.8%
	VR		1	127	101								
Prices	ML	527	0.4%	9.3%	7.4%	9.3%	1.1%	0.0	3.2	2	1.8%	5.5%	9.5%
	VR		0.7%	12.5%	10.0%								

TABLE 4a: Base Scenario Results - Briys & de Varenne Model

	Empirical Approach	True value	Bias & Efficiency: Mean error	Standard deviation	Mean absolute error	Distribution: Mean estimated std.	Std. of estimated std.	Sk.	Ku.	JB	Size test 1%	Size test 5%	Size test 10%
Low Business Risk & Low Financial Risk													
Asset volatility	ML	20%	0.2%	1.1%	0.9%	1.2%	0.1%	0.0	3.0	0	0.8%	4.0%	8.2%*
	VR		-0.3%	2.7%	2.1%								
Asset value	ML	1000	-1	11	9	12	1	0.1	3.0	2	1.1%	4.7%	8.9%
	VR		4	26	21								
Spreads	ML	158	3	17	14	18	2	0.0	3.0	0	0.9%	4.2%	8.5%
	VR		-4	41	32								
Prices	ML	652	-0.2%	1.7%	1.4%	1.8%	0.2%	0.1	3.0	1	0.9%	4.3%	8.8%
	VR		0.6%	4.0%	3.3%								
Low Business Risk & High Financial Risk													
Asset volatility	ML	20%	-0.3%	0.9%	0.8%	0.9%	0.2%	0.2	2.9	5	3.0%*	9.9%*	14.9%*
	VR		1.4%	6.3%	3.6%								
Asset value	ML	1000	6	12	11	11	2	0.0	2.7	3	5.2%*	12.5%*	18.4%*
	VR		-4	47	38								
Spreads	ML	270	-6	15	13	15	3	0.0	2.8	2	3.5%*	10.0%*	15.4%*
	VR		9	63	50								
Prices	ML	778	0.6%	1.5%	1.3%	1.5%	0.3%	0.0	2.8	2	3.3%*	9.7%*	15.3%*
	VR		-0.7%	6.1%	4.9%								

TABLE 4b: Base Scenario Results - Briys & de Varenne Model (continued)

			Bias & Efficiency			Distribution							
	Empirical Approach	True value	Mean error	Standard deviation	Mean absolute error	Mean estimated std.	Std. of estimated std.	Sk.	Ku.	JB	Size test 1%	Size test 5%	Size test 10%
High Business Risk & Low Financial Risk													
Asset volatility	ML	40%	-0.6%	1.6%	1.4%	1.7%	0.2%	0.1	3.0	1	2.1%*	7.1%*	12.5%*
	VR		5.1%	17.2%	9.0%								
Asset value	ML	1000	6	8	8	7	1	0.4	3.2	24	4.6%*	13.6%*	19.6%*
	VR		-3	32	26								
Spreads	ML	351	-6	13	11	13	1	-0.2	2.9	4	1.5%	6.3%	11.6%
	VR		12	59	47								
Prices	ML	538	0.6%	1.3%	1.1%	1.4%	0.2%	0.2	2.9	6	1.4%	6.3%	11.6%
	VR		-1.0%	5.7%	4.6%								
High Business Risk & High Financial Risk													
Asset volatility	ML	40%	-0.7%	1.6%	1.4%	1.6%	0.2%	0.1	3.1	1	2.5%*	9.6%*	15.9%*
	VR		5.6%	18.6%	11.8%								
Asset value	ML	1000	7	9	8	6	1	1.8	8.9	2025	11.8%*	22.7%*	31.2%*
	VR		5	39	31								
Spreads	ML	415	-5	9	8	8	1	-0.3	3.3	22	2.4%*	8.8%*	16.3%*
	VR		-1	52	43								
Prices	ML	673	0.5%	0.9%	0.8%	0.8%	0.1%	0.4	3.3	27	2.4%*	8.9%*	16.2%*
	VR		0.2%	5.2%	4.3%								

TABLE 5a: Base Scenario Results - Leland & Toft Model

	Empirical Approach	True value	Bias & Efficiency: Mean error	Standard deviation	Mean absolute error	Distribution: Mean estimated std.	Std. of estimated std.	Sk.	Ku.	JB	Size test 1%	5%	10%
Low Business Risk & Low Financial Risk													
Asset volatility	ML	20%	0.0%	1.1%	0.9%	1.1%	0.2%	0.2	3.0	7	0.9%	4.0%	9.3%
	VR		6.5%	13.4%	7.6%								
Asset value	ML	1000	0	3	3	3	1	-0.3	2.9	20	2.0%*	5.8%	10.0%
	VR		-6	34	13								
Spreads	ML	104	1	15	12	15	3	0.3	3.1	17	1.3%	4.2%	9.5%
	VR		111	270	125								
Prices	ML	76	0.0%	1.0%	0.8%	1.0%	0.2%	-0.3	3.1	12	1.3%	4.1%	9.6%
	VR		-6.1%	11.3%	7.1%								
Low Business Risk & High Financial Risk													
Asset volatility	ML	20%	0.0%	1.6%	1.3%	1.7%	0.4%	0.3	3.2	22	1.4%	5.3%	8.5%
	VR		29.5%	63.4%	32.2%								
Asset value	ML	1000	-1	2	1	2	1	-2.8	12.2	4820	4.7%*	5.9%	6.8%*
	VR		-117	186	117								
Spreads	ML	268	-1	30	24	31	8	0.4	3.3	27	1.4%	5.2%	8.5%
	VR		892	2666	941								
Prices	ML	91	0.1%	2.1%	1.6%	2.1%	0.5%	-0.3	3.2	16	1.3%	5.1%	8.5%
	VR		-17.6%	29.6%	22.8%								

TABLE 5b: Base Scenario Results - Leland & Toft Model (continued)

	Empirical Approach	True value	Bias & Efficiency: Mean error	Standard deviation	Mean absolute error	Distribution: Mean estimated std.	Std. of estimated std.	Sk.	Ku.	JB	Size test 1%	5%	10%
High Business Risk & Low Financial Risk													
Asset volatility	ML	40%	-0.1%	2.7%	2.1%	2.7%	0.5%	0.4	4.1	74	1.4%	4.8%	9.7%
	VR		18.7%	44.8%	22.8%								
Asset value	ML	1000	0	8	6	8	2	-0.9	5.4	379	3.0%*	6.6%*	11.7%
	VR		-59	108	68								
Spreads	ML	419	-1	50	39	50	11	0.5	4.4	124	1.5%	4.6%	9.5%
	VR		492	1444	563								
Prices	ML	61	0.1%	3.3%	2.6%	3.4%	0.6%	-0.3	3.9	52	1.3%	4.9%	9.9%
	VR		-13.9%	24.6%	19.0%								
High Business Risk & High Financial Risk													
Asset volatility	ML	40%	-0.2%	3.3%	2.6%	3.3%	0.7%	0.1	3.3	4	2.0%*	6.2%	10.4%
	VR		31.7%	62.7%	37.6%								
Asset value	ML	1000	0	18	14	18	4	-0.3	3.2	14	2.3%*	6.5%	11.6%
	VR		-126	196	153								
Spreads	ML	600	-3	69	54	69	15	0.1	3.3	7	2.0%*	5.9%	10.5%
	VR		984	2331	1100								
Prices	ML	73	0.3%	4.5%	3.5%	4.5%	0.9%	0.1	3.3	3	2.0%*	6.1%	10.3%
	VR		-19.9%	32.9%	28.1%								

TABLE 6a: Base Scenario Results - Ericsson & Reneby Model

	Empirical Approach	True value	Bias & Efficiency: Mean error	Standard deviation	Mean absolute error	Distribution: Mean estimated std.	Std. of estimated std.	Sk.	Ku.	JB	Size test 1%	5%	10%
Low Business Risk & Low Financial Risk													
Asset volatility	ML	20%	0.0%	1.1%	0.9%	1.1%	0.1%	0.0	2.8	2	1.0%	4.5%	8.1%*
	VR		0.9%	2.3%	1.9%								
Asset value	ML	1000	0	2	2	2	0	-0.5	3.1	37	2.9%*	5.7%	10.3%
	VR		-2	5	4								
Spreads	ML	82	0	11	9	11	1	0.1	2.8	4	1.1%	4.8%	8.4%
	VR		19	41	32								
Prices	ML	116	0.0%	0.8%	0.6%	0.8%	0.1%	-0.1	2.8	4	1.1%	4.8%	8.3%
	VR		-1.3%	2.9%	2.3%								
Low Business Risk & High Financial Risk													
Asset volatility	ML	20%	0.0%	1.3%	1.0%	1.3%	0.1%	0.2	3.0	4	1.3%	5.8%	10.3%
	VR		1.7%	3.8%	3.0%								
Asset value	ML	1000	0	7	5	7	1	-0.2	3.1	10	1.4%	5.8%	10.4%
	VR		-9	21	16								
Spreads	ML	194	0	18	15	18	2	0.2	3.0	4	1.3%	5.9%	10.5%
	VR		56	114	88								
Prices	ML	107	0.0%	1.3%	1.1%	1.3%	0.1%	-0.1	3.0	2	1.2%	5.8%	10.5%
	VR		-3.6%	7.3%	5.9%								

TABLE 6b: Base Scenario Results - Ericsson & Reneby Model (continued)

	Empirical Approach	True value	Bias & Efficiency: Mean error	Standard deviation	Mean absolute error	Distribution: Mean estimated std.	Std. of estimated std.	Sk.	Ku.	JB	Size test 1%	5%	10%
High Business Risk & Low Financial Risk													
Asset volatility	ML	40%	0.1%	2.3%	1.8%	2.3%	0.3%	0.1	2.9	2	0.8%	4.2%	8.1%*
	VR		2.2%	5.8%	4.4%								
Asset value	ML	1000	0	9	7	9	1	-0.1	2.9	3	0.9%	4.5%	8.5%
	VR		-9	23	18								
Spreads	ML	292	1	28	23	29	4	0.1	2.9	3	0.8%	4.4%	8.4%
	VR		50	119	89								
Prices	ML	99	0.0%	2.0%	1.6%	2.1%	0.2%	-0.1	2.9	1	0.9%	4.2%	7.8%*
	VR		-3.1%	7.6%	5.9%								
High Business Risk & High Financial Risk													
Asset volatility	ML	40%	-0.1%	2.6%	2.1%	2.5%	0.3%	0.2	3.0	5	1.3%	5.6%	10.0%
	VR		3.5%	8.8%	6.6%								
Asset value	ML	1000	0	15	12	15	2	-0.1	3.0	3	1.4%	5.8%	10.0%
	VR		-18	46	36								
Spreads	ML	413	-1	35	28	34	4	0.2	3.0	5	1.3%	5.7%	9.8%
	VR		91	207	159								
Prices	ML	91	0.1%	2.4%	1.9%	2.4%	0.3%	-0.1	3.0	1	1.3%	5.4%	9.7%
	VR		-5.0%	12.1%	9.9%								

TABLE 7a: Summaries across Models

	Empirical Approach	Bias & Efficiency: Mean error	Standard deviation	Mean absolute error	Distribution: Mean estimated std.	Std. of estimated std.	Sk.	Ku.	JB	Size test 1%	Size test 5%	Size test 10%
Low Business Risk & Low Financial Risk												
Asset volatility	ML	0.1%	1.2%	1.0%	1.3%	0.2%	0.1	3.0	2	1.0%	4.4%	8.7%
	VR	1.9%	5.3%	3.5%								
Asset value	ML	-1	9	7	9	1	-0.2	3.0	16	2.1%	5.4%	9.7%
	VR	-3	25	16								
Spreads	ML	2	18	15	19	3	0.2	3.0	8	1.4%	4.6%	9.0%
	VR	34	102	58								
Prices	ML	-0.1%	1.6%	1.3%	1.7%	0.3%	-0.1	3.0	5	1.3%	4.7%	9.1%
	VR	-2.0%	5.9%	4.2%								
Low Business Risk & High Financial Risk												
Asset volatility	ML	-0.1%	1.5%	1.2%	1.5%	0.3%	0.3	3.1	15	1.9%	6.6%	10.9%
	VR	8.3%	19.2%	10.4%								
Asset value	ML	1	15	12	15	3	-0.8	5.3	1213	3.4%	7.5%	11.5%
	VR	-35	80	56								
Spreads	ML	-1	30	24	30	6	0.3	3.1	22	2.1%	6.7%	11.2%
	VR	243	733	288								
Prices	ML	0.1%	2.6%	2.1%	2.6%	0.5%	-0.2	3.0	10	2.0%	6.7%	11.2%
	VR	-5.7%	13.0%	10.1%								

TABLE 7b: Summaries across Models (continued)

	Empirical Approach	Bias & Efficiency: Mean error	Bias & Efficiency: Standard deviation	Bias & Efficiency: Mean absolute error	Distribution: Mean estimated std.	Distribution: Std. of estimated std.	Sk.	Ku.	JB	Size test 1%	Size test 5%	Size test 10%
High Business Risk & Low Financial Risk												
Asset volatility	ML	-0.2%	2.4%	2.0%	2.5%	0.4%	0.1	3.2	20	1.5%	5.4%	10.1%
	VR	6.5%	18.0%	9.9%								
Asset value	ML	2	15	12	15	2	-0.1	3.6	102	2.6%	7.7%	12.6%
	VR	-17	52	37								
Spreads	ML	-2	41	33	42	7	0.1	3.3	33	1.5%	5.3%	10.1%
	VR	139	431	195								
Prices	ML	0.3%	3.5%	2.8%	3.6%	0.5%	0.0	3.1	16	1.4%	5.3%	10.0%
	VR	-4.4%	12.0%	9.4%								
High Business Risk & High Financial Risk												
Asset volatility	ML	-0.2%	2.8%	2.2%	2.7%	0.4%	0.1	3.2	7	1.9%	6.6%	11.5%
	VR	10.2%	23.7%	14.9%								
Asset value	ML	2	23	18	22	3	0.3	4.6	511	4.3%	10.1%	15.6%
	VR	-34	87	68								
Spreads	ML	-2	52	41	51	9	0.1	3.3	16	1.9%	6.4%	11.6%
	VR	269	679	351								
Prices	ML	0.3%	4.3%	3.4%	4.3%	0.6%	0.1	3.2	8	1.9%	6.5%	11.5%
	VR	-6.0%	15.7%	13.1%								

TABLE 7c: Summary across Models and Scenarios (averages of Tables 7a-7b)

		Bias & Efficiency			Distribution							
	Empirical Approach	Mean error	Standard deviation	Mean absolute error	Mean estimated std.	Std. of estimated std.	Sk.	Ku.	JB	Size test 1%	Size test 5%	Size test 10%
Asset volatility	ML	-0.1%	2.0%	1.6%	2.0%	0.3%	0.2	3.1	11	1.6%	5.8%	10.3%
	VR	6.7%	16.6%	9.7%								
Asset value	ML	1	15	12	15	2	-0.2	4.1	460	3.1%	7.7%	12.3%
	VR	-22	61	44								
Spreads	ML	-1	35	28	35	6	0.2	3.2	20	1.7%	5.8%	10.5%
	VR	171	486	223								
Prices	ML	0.1%	3.0%	2.4%	3.0%	0.4%	-0.1	3.1	10	1.6%	5.8%	10.4%
	VR	-4.5%	11.6%	9.2%								

TABLE 8a: Alternative Scenarios - Briys & de Varenne Model with Constant Interest Rates

	Empirical Approach	True value	Bias & Efficiency: Mean error	Standard deviation	Mean absolute error	Distribution: Mean estimated std.	Std. of estimated std.	Sk.	Ku.	JB	Size test 1%	5%	10%
Asset volatility	ML	40%	-0.6%	1.5%	1.3%	1.6%	0.2%	0.0	3.2	1	2.1%*	7.8%*	13.1%*
	VR		7.2%	23.3%	12.6%								
Asset value	ML	1000	7	9	8	6	1	2.0	10.7	3116	11.3%*	21.7%*	27.4%*
	VR		2	37	29								
Spreads	ML	415	-4	8	7	8	1	-0.3	3.2	17	2.0%*	7.0%*	12.9%*
	VR		3	49	41								
Prices	ML	660	0.5%	0.8%	0.7%	0.8%	0.1%	0.3	3.3	21	2.0%*	7.5%*	13.4%*
	VR		-0.2%	4.9%	4.1%								

Parameter values: $\sigma = 40\%, N = 1649$ and the rest as in base case scenario except: $\overline{r} = 5\%, \gamma = 0$.
Scenario characteristics: default barrier 600, equity volatility 101%, leverage 75%, bond spread 415

TABLE 8b: Alternative Scenarios - A Start-Up Firm (Ericsson & Reneby Model)

	Empirical Approach	True value	Mean error	Standard deviation	Mean absolute error	Mean estimated std.	Std. of estimated std.	Sk.	Ku.	JB	Size test 1%	5%	10%
Asset volatility	ML	100%	-0.2%	5.3%	4.2%	5.6%	0.8%	0.0	3.0	0	1.0%	4.6%	8.7%
	VR		4.6%	15.3%	10.4%								
Asset value	ML	1000	1	11	9	12	1	0.3	3.1	13	1.2%	3.8%*	7.9%*
	VR		-6	25	19								
Spreads	ML	1342	-4	133	106	141	21	0.1	3.0	1	1.1%	4.7%	8.9%
	VR		178	532	366								
Prices	ML	49	0.4%	6.0%	4.8%	6.4%	0.8%	0.1	3.0	3	1.2%	4.4%	8.8%
	VR		-5.2%	18.6%	14.3%								

Parameter values: $\sigma = 100\%, N = 750$ and the rest as in base case scenario except $c = 3\% \cdot P, \alpha = 4\%, \beta = 0\%, \psi = 10\%$.
Scenario characteristics: default barrier 53, equity volatility 117%, leverage 48%, bond spread 1342.

TABLE 8c: Alternative Scenarios - Ericsson & Reneby Model with High Barrier

	Empirical Approach	True value	Bias & Efficiency: Mean error	Standard deviation	Mean absolute error	Distribution: Mean estimated std.	Std. of estimated std.	Sk.	Ku.	JB	Size test 1%	5%	10%
Asset volatility	ML	20%	-0.1%	1.5%	1.2%	1.5%	0.3%	0.2	3.1	7	1.7%	5.2%	10.2%
	VR		0.6%	4.0%	3.4%								
Asset value	ML	1000	2	24	19	24	4	-0.1	3.0	1	1.4%	4.9%	9.5%
	VR		-5	65	54								
Spreads	ML	507	-1	12	10	13	2	0.1	3.0	1	1.5%	4.9%	9.7%
	VR		18	152	131								
Prices	ML	86	0.1%	0.8%	0.7%	0.9%	0.1%	-0.1	3.0	1	1.5%	5.0%	9.6%
	VR		-0.6%	10.6%	8.9%								

Parameter values: σ = 20%, N = 1710 and the rest as in base case scenario.
Scenario characteristics: default barrier 687, equity volatility 83%, leverage 92%, bond spread 507.

TABLE 8d: Alternative Scenarios - Leland & Toft Model with Low Barrier

	Empirical Approach	True value	Mean error	Standard deviation	Mean absolute error	Mean estimated std.	Std. of estimated std.	Sk.	Ku.	JB	1%	5%	10%
Asset volatility	ML	20%	0.0%	1.1%	0.9%	1.1%	0.1%	0.0	3.0	0	1.2%	4.4%	8.8%
	VR		1.3%	3.0%	2.3%								
Asset value	ML	1000	0	7	5	7	1	-0.1	3.0	1	1.8%	4.4%	8.8%
	VR		-9	19	14								
Spreads	ML	303	-1	27	22	28	3	0.1	3.0	0	1.5%	4.4%	8.8%
	VR		34	76	58								
Prices	ML	6	0.0%	2.0%	1.6%	2.1%	0.2%	0.0	3.0	0	1.2%	4.5%	8.7%
	VR		-2.4%	5.3%	4.2%								

Parameter values: σ = 20%, N = 750 and the rest as in base case scenario except T = 100, β = 6%.
Scenario characteristics: default barrier 333, equity volatility 39%, leverage 62%, bond spread 303.

TABLE 9a: Sample size - Black & Scholes/Merton Model (Low Business Risk & High Financial Risk)

	Empirical Approach	True value	Bias & Efficiency: Mean error	Standard deviation	Mean absolute error	Distribution: Mean estimated std.	Std. of estimated std.	Sk.	Ku.	JB	Size test 1%	5%	10%
Sample size 90 days													
Asset volatility	ML	20%	0.3%	3.4%	2.7%	3.6%	0.9%	0.4	3.5	37	2.4%*	6.1%	8.9%
	VR		0.4%	3.8%	3.0%								
Asset value	ML	1000	-4	64	51	69	15	-0.3	3.2	17	2.7%*	6.4%	9.6%
	VR		-7	71	57								
Spreads	ML	285	10	88	69	94	28	0.6	3.9	95	3.3%*	6.8%*	9.8%
	VR		14	99	77								
Prices	ML	752	-0.6%	8.5%	6.8%	9.1%	2.0%	-0.3	3.2	17	2.7%*	6.4%	9.6%
	VR		-0.9%	9.5%	7.6%								
Sample size 250 days													
Asset volatility	ML	20%	0.1%	2.2%	1.7%	2.1%	0.4%	0.4	3.3	28	2.0%*	5.5%	9.8%
	VR		0.5%	3.5%	2.8%								
Asset value	ML	1000	-1	41	33	41	7	-0.3	3.2	20	2.1%*	5.9%	10.4%
	VR		-8	66	53								
Spreads	ML	285	3	56	44	55	12	0.5	3.5	54	2.0%*	5.6%	10.3%
	VR		15	92	72								
Prices	ML	752	-0.1%	5.5%	4.4%	5.4%	0.9%	-0.3	3.2	20	2.0%*	6.1%	10.3%
	VR		-1.1%	8.8%	7.0%								

Parameter values and scenario characteristics as in base case scenario.

TABLE 9b: Sample size - Black & Scholes/Merton Model (Low Business Risk & High Financial Risk) (continued)

	Empirical Approach	True value	Bias & Efficiency: Mean error	Standard deviation	Mean absolute error	Distribution: Mean estimated std.	Std. of estimated std.	Sk.	Ku.	JB	Size test 1%	5%	10%
Sample size 500 days													
Asset volatility	ML	20%	0.0%	1.6%	1.3%	1.6%	0.3%	0.3	3.2	22	1.7%	3.9%	9.6%
	VR		1.0%	4.3%	3.4%								
Asset value	ML	1000	0	31	24	30	5	-0.3	3.2	17	1.8%	4.1%	9.7%
	VR		-17	80	64								
Spreads	ML	285	1	41	32	40	8	0.4	3.4	37	1.7%	4.4%	9.6%
	VR		30	114	89								
Prices	ML	752	-0.1%	4.1%	3.2%	4.0%	0.7%	-0.3	3.2	17	1.8%	4.1%	9.7%
	VR		-2.4%	10.6%	8.5%								
Sample size 750 days													
Asset volatility	ML	20%	0.0%	1.3%	1.1%	1.3%	0.2%	0.2	3.2	9	1.2%	5.9%	10.9%
	VR		1.1%	4.8%	3.8%								
Asset value	ML	1000	0	26	20	25	4	-0.2	3.1	6	1.2%	5.9%	11.2%
	VR		-19	88	71								
Spreads	ML	285	0	34	27	33	7	0.3	3.3	17	1.3%	6.1%	11.1%
	VR		34	126	99								
Prices	ML	752	0.0%	3.4%	2.7%	3.3%	0.6%	-0.2	3.1	6	1.2%	5.9%	11.2%
	VR		-2.6%	11.7%	9.4%								

Parameter values and scenario characteristics as in base case scenario.

TABLE 9c: Sample size - Ericsson & Reneby Model (High Business Risk & Low Financial Risk)

	Empirical Approach	True value	Bias & Efficiency: Mean error	Standard deviation	Mean absolute error	Distribution: Mean estimated std.	Std. of estimated std.	Sk.	Ku.	JB	Size test 1%	5%	10%
Sample size 90 days													
Asset volatility	ML	40%	0.0%	3.8%	3.0%	4.0%	0.6%	0.1	2.7	6	1.8%	5.2%	9.8%
	VR		0.9%	4.7%	3.8%								
Asset value	ML	1000	0	15	12	16	3	-0.1	2.6	8	2.2%*	5.9%	10.3%
	VR		-4	19	15								
Spreads	ML	292	1	47	38	49	8	0.1	2.7	7	1.9%*	5.5%	10.2%
	VR		20	75	61								
Prices	ML	99	0.0%	3.3%	2.7%	3.5%	0.5%	0.0	2.7	5	1.6%	5.4%	9.6%
	VR		-1.2%	5.2%	4.2%								
Sample size 250 days													
Asset volatility	ML	40%	0.1%	2.3%	1.8%	2.3%	0.3%	0.1	2.9	2	0.8%	4.2%	8.1%*
	VR		2.2%	5.8%	4.4%								
Asset value	ML	1000	0	9	7	9	1	-0.1	2.9	3	0.9%	4.5%	8.5%
	VR		-9	23	18								
Spreads	ML	292	1	28	23	29	4	0.1	2.9	3	0.8%	4.4%	8.4%
	VR		50	119	89								
Prices	ML	99	0.0%	2.0%	1.6%	2.1%	0.2%	-0.1	2.9	1	0.9%	4.2%	7.8%*
	VR		-3.1%	7.6%	5.9%								

Parameter values and scenario characteristics as in base case scenario.

TABLE 9d: Sample size - Ericsson & Reneby Model (High Business Risk & Low Financial Risk) (continued)

	Empirical Approach	True value	Bias & Efficiency: Mean error	Standard deviation	Mean absolute error	Distribution: Mean estimated std.	Std. of estimated std.	Sk.	Ku.	JB	Size test 1%	5%	10%
Sample size 500 days													
Asset volatility	ML	40%	0.1%	1.7%	1.4%	1.7%	0.2%	0.1	2.8	2	1.2%	6.1%	11.0%
	VR		5.3%	10.3%	7.6%								
Asset value	ML	1000	0	7	6	7	1	-0.1	2.8	3	1.3%	6.0%	11.1%
	VR		-20	38	29								
Spreads	ML	292	1	22	17	21	3	0.1	2.9	3	1.2%	5.7%	11.0%
	VR		119	222	164								
Prices	ML	99	-0.1%	1.5%	1.2%	1.5%	0.2%	-0.1	2.8	2	1.1%	6.3%	11.2%
	VR		-6.8%	12.5%	10.0%								
Sample size 750 days													
Asset volatility	ML	40%	-0.1%	1.4%	1.1%	1.4%	0.2%	0.1	3.3	6	1.9%*	5.3%	9.5%
	VR		7.1%	12.6%	9.4%								
Asset value	ML	1000	0	6	4	6	1	-0.1	3.3	7	1.9%*	5.4%	9.6%
	VR		-27	45	36								
Spreads	ML	292	-1	17	13	17	2	0.1	3.3	7	1.9%*	5.1%	9.5%
	VR		161	270	206								
Prices	ML	99	0.1%	1.2%	0.9%	1.2%	0.1%	-0.1	3.3	4	2.0%*	5.4%	9.6%
	VR		-8.8%	14.6%	12.2%								

Parameter values and scenario characteristics as in base case scenario.

TABLE 10a: Maturity - Black & Scholes/Merton Model (Low Business Risk & High Financial Risk)

	Empirical Approach	True value	Mean error	Standard deviation	Mean absolute error	Mean estimated std.	Std. of estimated std.	Sk.	Ku.	JB	Size test 1%	Size test 5%	Size test 10%
			Bias & Efficiency			Distribution							
Maturity 5 years													
Asset volatility	ML	20%	0.0%	2.3%	1.8%	2.3%	0.5%	0.4	3.2	28	1.6%	5.9%	10.4%
	VR		0.4%	4.9%	3.8%								
Asset value	ML	1000	0	34	27	34	7	-0.4	3.1	31	2.0%*	6.3%	10.9%
	VR		-6	71	56								
Spreads	ML	389	3	83	66	83	21	0.6	3.4	56	2.2%*	6.0%	10.8%
	VR		24	183	139								
Prices	ML	823	0.0%	4.1%	3.3%	4.1%	0.9%	-0.4	3.1	31	2.1%*	6.3%	11.0%
	VR		-0.8%	8.7%	6.8%								

Parameter values: $\sigma = 20\%$, $N = 1649$ and the rest as in base case scenario except $T = 5$.
Scenario characteristics: equity volatility 67%, leverage 85%, bond spread 389.

	Empirical Approach	True value	Mean error	Standard deviation	Mean absolute error	Mean estimated std.	Std. of estimated std.	Sk.	Ku.	JB	Size test 1%	Size test 5%	Size test 10%
Maturity 10 years													
Asset volatility	ML	20%	0.1%	1.6%	1.3%	1.7%	0.3%	0.1	3.0	1	1.4%	5.0%	9.1%
	VR		0.5%	2.8%	2.3%								
Asset value	ML	1000	-2	20	16	20	4	-0.1	3.0	3	2.2%*	5.4%	9.6%
	VR		-6	34	27								
Spreads	ML	164	3	31	25	32	7	0.2	3.0	9	2.2%*	5.2%	9.4%
	VR		12	55	44								
Prices	ML	637	-0.3%	3.1%	2.5%	3.2%	0.6%	-0.1	3.0	3	2.0%*	5.4%	9.5%
	VR		-1.1%	5.4%	4.3%								

Parameter values and scenario characteristics as in base case scenario.

TABLE 10b: Maturity - Black & Scholes/Merton Model (Low Business Risk & High Financial Risk) (continued)

	Empirical Approach	True value	Bias & Efficiency: Mean error	Standard deviation	Mean absolute error	Distribution: Mean estimated std.	Std. of estimated std.	Sk.	Ku.	JB	Size test 1%	5%	10%
Maturity 20 years													
Asset volatility	ML	20%	-0.1%	1.9%	1.5%	1.9%	0.3%	0.2	3.0	10	1.7%	5.8%	10.1%
	VR		0.2%	2.5%	2.0%								
Asset value	ML	1000	2	46	37	46	6	-0.1	2.9	3	1.8%	5.7%	9.9%
	VR		-4	60	48								
Spreads	ML	212	-1	35	28	36	6	0.3	3.1	20	2.4%*	6.2%	10.4%
	VR		5	47	37								
Prices	ML	655	0.3%	7.0%	5.6%	7.1%	0.8%	-0.1	2.9	3	1.8%	5.6%	10.0%
	VR		-0.6%	9.1%	7.3%								

Parameter values: σ = 20%, N = 1649 and the rest as in base case scenario except T = 20.
Scenario characteristics: equity volatility 39%, leverage 74%, bond spread 212.

	Empirical Approach	True value	Mean error	Standard deviation	Mean absolute error	Mean estimated std.	Std. of estimated std.	Sk.	Ku.	JB	1%	5%	10%
Maturity 30 years													
Asset volatility	ML	20%	0.0%	1.8%	1.5%	1.8%	0.2%	0.0	2.8	2	1.6%	5.2%	10.1%
	VR		0.3%	2.2%	1.8%								
Asset value	ML	1000	1	48	39	49	5	0.2	2.8	5	1.4%	5.3%	10.4%
	VR		-5	58	46								
Spreads	ML	179	1	27	22	28	4	0.1	2.8	3	2.0%*	5.5%	10.4%
	VR		5	34	27								
Prices	ML	584	0.2%	8.2%	6.6%	8.3%	0.8%	0.2	2.8	5	1.4%	5.3%	10.4%
	VR		-0.9%	9.9%	7.9%								

Parameter values: σ = 20%, N = 1649 and the rest as in base case scenario except T = 30.
Scenario characteristics: equity volatility 34%, leverage 58%, bond spread 179.

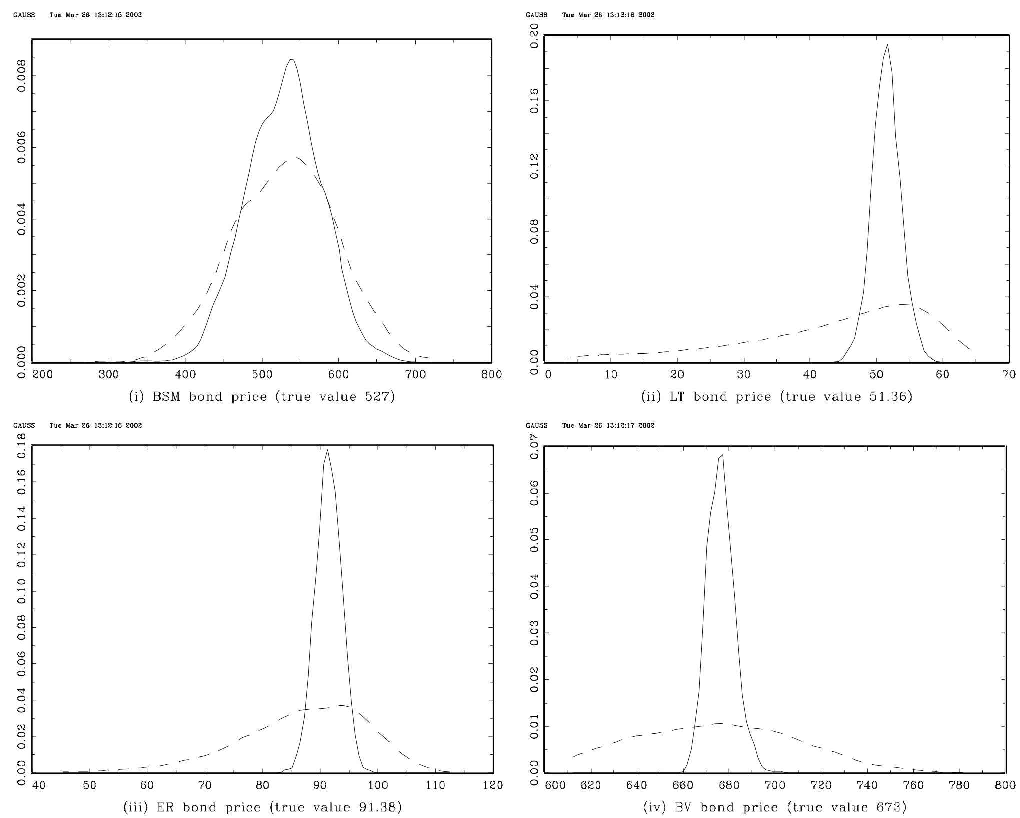

FIG. 1. Efficiency of the two estimation approaches. The four figures display kernel density plots for simulated distributions of the bond price estimators, using the maximum likelihood (solid line) and volatility restriction (dashed line) methods. For all four models, results are based on the high financial and business risk scenarios.

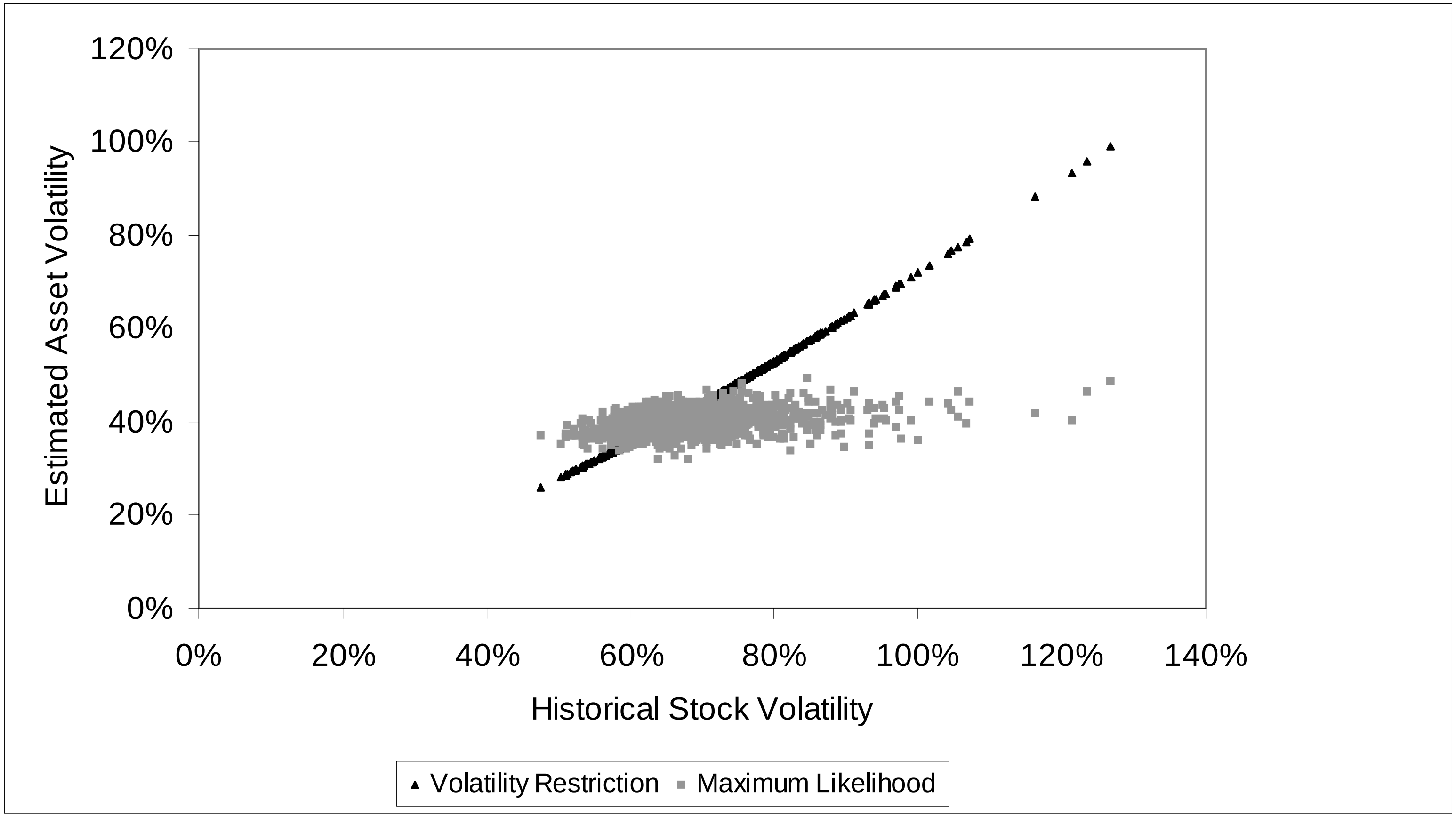

FIG. 2. Estimating business risk. The figure plots pairs of asset risk (σ) estimates and historical equity volatility for the corresponding sample path. The black dots represent results for the volatility restriction method whereas the grey dots show asset / equity volatility pairs for the maximum likelihood approach. The results are based on the high financial and business risk scenario for the Ericsson & Reneby model.

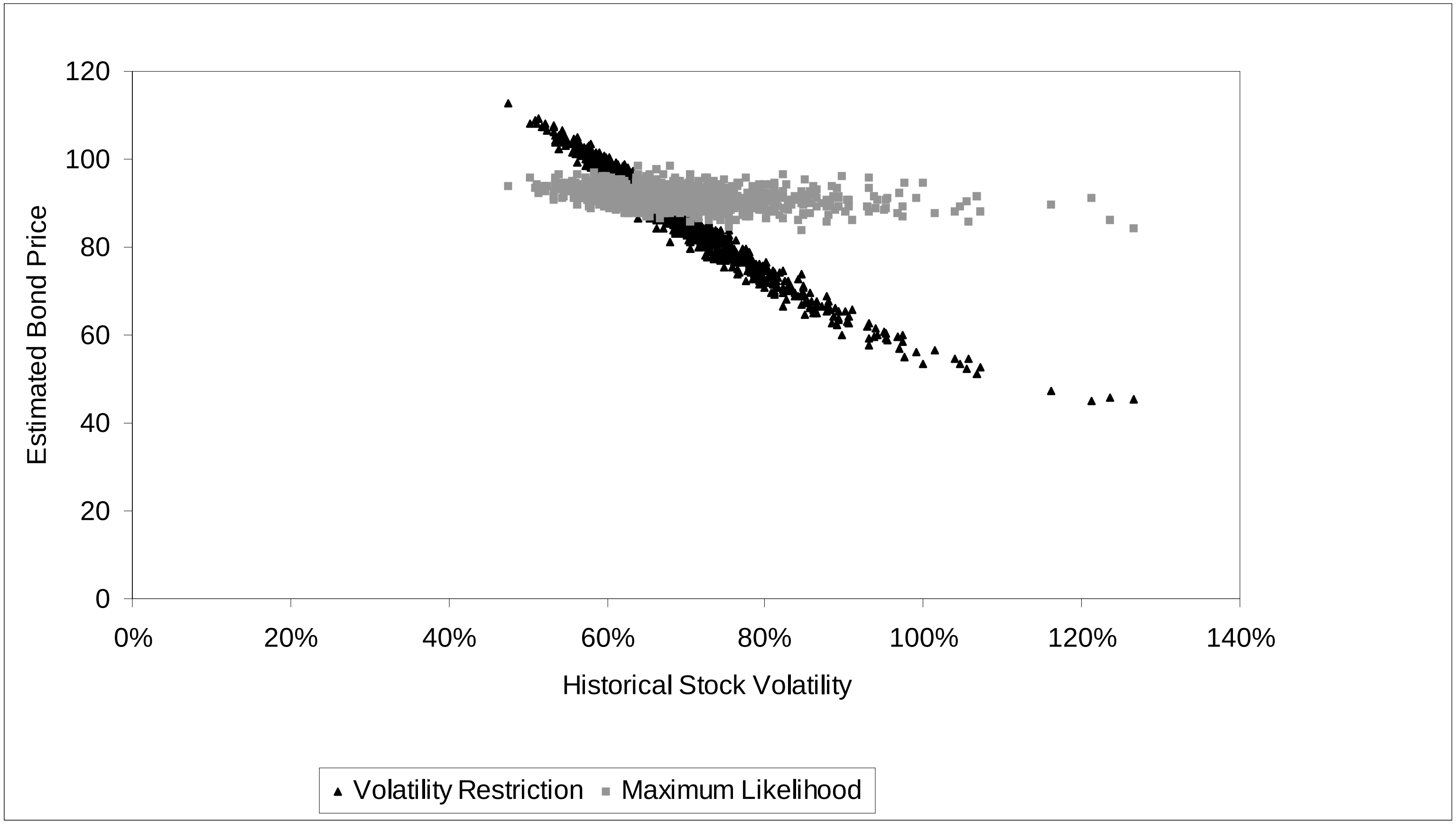

FIG. 3. Estimating bond prices. The figure plots pairs of bond price estimates and historical equity volatility for the corresponding sample path. The black dots represent results for the volatility restriction method whereas the grey dots show bond price / equity volatility pairs for the maximum likelihood approach. The results are based on the high financial and business risk scenario for the Ericsson & Reneby model.

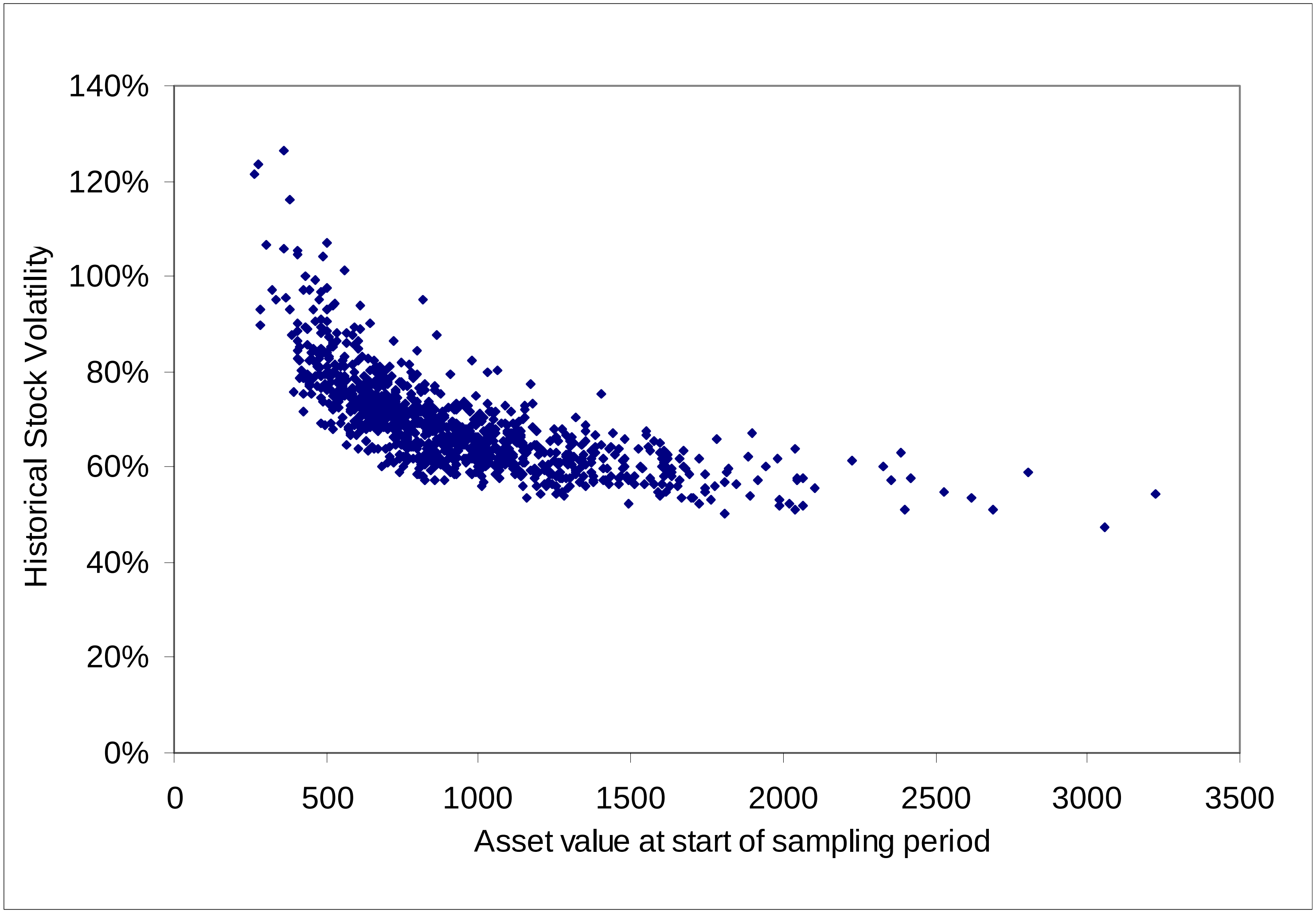

FIG. 4. The relationship between initial asset value and historical equity volatility. The figure plots the realized historical equity volatility for a given sample path against the level of asset value at which that path started. The plot is based on the Ericsson & Reneby model in the high financial and business risk scenario.