On the term structure of default premia in the swap and LIBOR markets[1]

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Abstract

Existing theories of the term structure of swap rates provide an analysis of the Treasury-swap spread based on either a liquidity convenience yield in the Treasury market, or default risk in the swap market. While these models do not focus on the relation between corporate yields and swap rates (the LIBOR-Swap spread), they imply that the term structure of corporate yields and swap rates should be identical. As documented previously (e.g. in Sun, Sundaresan and Wang (1993)) this is counter-factual. Here, we propose a model of the default risk imbedded in the swap term structure that is able to explain the LIBOR-swap spread. Whereas corporate bonds carry default risk, we argue that swap **contracts** are free of default risk. Because swaps are indexed on “refreshed”-credit-quality LIBOR rates, the spread between corporate yields and swap **rates** should capture the market’s expectations of the probability of deterioration in credit quality of a corporate bond issuer. We model this feature and use our model to estimate the likelihood of future deterioration in credit quality from the LIBOR-swap spread. The analysis is important because it shows that the term structure of swap rates does not reflect the borrowing cost of a standard LIBOR credit quality issuer. It also has implications for modeling the dynamics of the swap term structure.

1 Introduction

Existing models of swap rates focus on the spread between swap rates and Treasury yields. In this article, we provide a direct comparison of the term structures of swap rates and of corporate bond yields.

An interest rate swap is a contract by which a fixed payment stream is exchanged against a floating payment stream. The floating leg of the swap is usually set at the interbank interest rate for the relevant currency (typically the 6-month LIBOR for dollar swaps). Once the floating leg is specified, the market rate for a swap is simply the coupon rate on the fixed leg of the swap. The generic swap rate applies to a top-quality client rated AA or better. Dealers use this market rate as a reference when they quote an actual swap rate to a client and adjust for default risk and other characteristics of the client. In this paper we only consider generic swaps quoted for top-quality counterparties. We do **not** study the adjustment made to the generic swap rate for a more risky counterparty.[1] Swaps are quoted for various maturities; hence there exists a term structure of swap rates that can be compared to the term structures of Treasury yields and of defaultable corporate bond yields. For illustrative purposes we present the "average" term structures for the period 10/12/88 to 01/29/97 in figure 1.

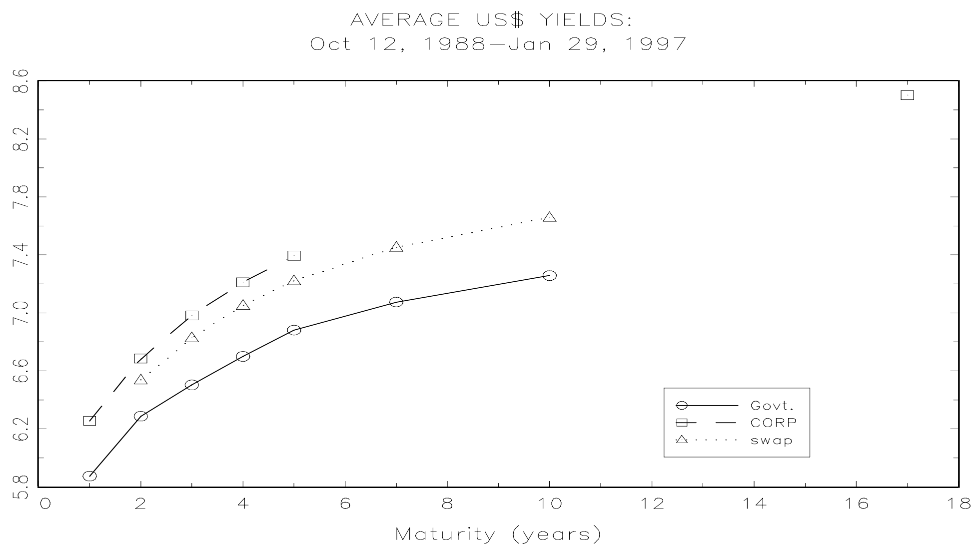

Figure 1: The average term structure of swap rates, corporate and Treasury yields: December 1998 to January 1997. All term structures are expressed in semi-annual, actual/365 convention. Data is taken from Datastream and is described in section 4.

As expected, the swap curve is well above the Treasury curve. More interestingly, casual observation suggests that the swap curve is below the corporate curve,[2] and that the LIBOR-Swap spread

[1] Studies of the adjustment in the swap rate done to reflect the credit quality of different counterparties can be found in Sorensen and Bollier (1995), Sun, Sundaresan and Wang (1993), Duffie and Huang (1996).

[2] In the empirical work, we use data on LIBOR bonds as measures of the yields on defaultable corporate bonds. LIBOR bonds are fixed-coupon bonds negotiated OTC and issued by top-quality corporate bond issuers (usually banks and financial institutions) rated AA or better. Hence, we will call "LIBOR-swap" the spread between yields on LIBOR-quality bonds and LIBOR swap rates for all quoted maturities; for example, the 5-year LIBOR-swap spread is the spread between the yield on a 5-year LIBOR bond and the fixed rate on a 5-year

increases with maturity. The average spread (across maturities and dates) between LIBOR bond yields and swap rates is around 15 basis points. It is, by construction, zero at 6-months to maturity. Our intent is to develop a model that explains the spread between corporate bond yields and swap rates (LIBOR-swap spread) and its dynamics.

Although swap rates are often quoted relative to Treasury yields for practical reasons (the Treasury term structure is widely-available and continuously-updated), the important comparison for swap rates is with corporate bond yields of similar credit quality. While the LIBOR-swap spread only amounts to a few basis points[3] it can be of significant financial importance. Corporate issuers measure their spreads relative to the swap curve rather than to the Treasury curve which is different in terms of credit quality, and exhibits significant institutional and regulatory distortions (such as repo specials, taxes and perhaps liquidity). Swaps are often used by corporate issuers in complex financing packages involving corporate bonds in order to gain some financing cost reduction compared to issuing plain-vanilla bonds. Bankers use the swap curve, in lieu of the corporate curve, as the basic tool for pricing corporate assets and liabilities. This practice originates from the observation that swap rates are continuously quoted (and traded) for a wide range of maturities and therefore more readily updated than corporate yields. Yet it is justified only if the swap term structure truly reflects the cost of financing of a top-rated corporate issuer for the various maturities. In this paper we argue that the LIBOR-swap spread, is not to be dismissed as simply resulting form data problems (or liquidity), but that it should exist on purely theoretical grounds.

The focus of extant models of the swap term structure[4] is the analysis of the spread between Treasury yields and swap rates. Little has been done to explain the spread between corporate bond yields and swap rates.[5]

Recent papers (Grinblatt (1995) and Duffie and Singleton (1997)) provide models where the term structure of swap rates can be modeled using a traditional two-factor model of the term structure. In both models the term structure of swap rates is equal to a term structure of corporate par-bond rates.

Grinblatt (1995) proposes a model where both swap contracts and Treasury bonds are free of default risk. The swap-Treasury spread arises because of a “liquidity convenience yield[6]” accruing to the holder of a government-issued security. As a result, swap rates are equal to a risk-free par bond rates in his model.

In Duffie and Singleton (1997) the swap-Treasury spread arises because swap contracts carry default risk. In their model the swap rate is equivalent to a par-bond yield on a credit risky bond, which they model using a two-factor model as in Duffie and Singleton (1999).

In this paper we provide a model of the spread between par-bond rates and swap rates. We relax two assumptions explicitly or implicitly made in previous literature, namely: (i) “homogeneous LIBOR-swap credit quality” and (ii) “refreshed credit quality” of LIBOR counter-parties which we discuss below.[7]

LIBOR swap, with all parties of top-credit quality.

[3] The LIBOR-swap spread is usually well above the swap bid-ask spread, which only amounts to a couple of basis points for generic swaps.

[4] Earlier work include Solnik (1990), Sundaresan (1991), Cooper and Mello (1991).

[5] A notable exception is Cooper and Mello (1991) who analyze spreads between risky swap and bond rates in a structural framework similar to Merton (1974). Their model differs from ours as they focus on wealth transfers between bond, swap and equity holders of a firm, and thus assume that swap contracts are credit risky. As a result, in their model corporate and swap rates are functions of the highly stylized liability structure of the firm analyzed.

[6] Grinblatt models this as an exogenous factor, similarly to convenience yields in the forward contract literature.

[7] See also Duffie and Singleton (1997) for a discussion of these assumptions.

The assumption of “homogeneous LIBOR-swap credit quality” implies that swap contracts and LIBOR bonds have the same default risk, and hence that all cash flows pertaining to either contract should be discounted under the risk-neutral measure using the same risk-adjusted rate.

However, it is very likely that swaps be not impacted at all by default risk so that they should be treated as default risk-free, unlike LIBOR bonds which carry AA default risk. It is now widely recognized[8] that corporate bonds bear more credit risk than swaps written by the same counterparties. The nature of the swap contract makes default on swaps much less costly than on bonds. The potential loss on a swap does not include the principal but only an interest rate differential (e.g. fixed minus floating), and only in the case where this difference is positive for the non-defaulting party (i.e. if interest rate movements have led to a positive swap market value for the non-defaulting party). Furthermore, this potential loss is often reduced or eliminated by the posting of collateral or marking-to-market provisions, as well as other contractual provisions in case of credit downgrading of a party. Some further argue that a swap between two parties of similar credit quality should entail no default risk premium in either direction because of the symmetric nature of the contract.[9] So the impact of credit risk on the pricing of a generic swap should at best be minimal.[10] Hence it seems essential to use different risk-adjusted rates for corporate bonds and swap contracts issued by the same party. In this article we assume that the payoffs of a generic swap are basically priced as if free of default risk: the discount factor adjusted for default risk to be used under the risk-neutral measure to price swap contracts for AA parties is the risk-free interest rate. However, the swap term structure will be different from (and above) the risk-free term structure, because the swap rate payments are indexed on 6-month LIBOR which is a default-risky rate. Hence, the swap rate will be higher than the risk-free rate even though the swap contract is free of default risk. On the other hand, the fact that swap contracts are less risky than LIBOR bonds, does **not** necessarily imply that swap rates be lower than LIBOR bond yields. This may at first sound counterintuitive, but is, in fact, just a result of the swap payments being indexed on the short end of the LIBOR bond yield curve. For example, the swap rate on a swap with a 6-month maturity is always equal to the 6-month LIBOR rate by design of the contract, no matter what the difference in credit risk is between the swap contract and the 6-month LIBOR bond.

The assumption of “refreshed credit quality” of LIBOR counter-parties presumes that the counterparties will maintain the same credit quality over time.

Our subsequent analysis shows that this assumption may be inappropriate to understand the LIBOR-swap spread. The swap contract is contractually indexed on the 6-month LIBOR rate, which is a refreshed top-credit-quality rate. On the other hand, long-term LIBOR bonds are priced to reflect the likelihood that the credit quality of a top-rated issuer may deteriorate over the life of the bond. Thus, our analysis implies that the LIBOR-swap spread captures the likelihood that an issuer’s credit quality may change over time.

We show that a model that accounts for (1) the difference in credit risk between swap contracts and top-quality corporate bonds and (2) the difference in credit quality of a constantly updated, refreshed credit quality index and that of a specific top-rated issuer that may experience a depreciation in credit-quality, can reasonably explain the observed spread and its dynamics.

Of course there may be other factors, which could further explain the dynamics of the LIBOR-swap spread, such as liquidity. Although we are not aware of documented liquidity events in the LIBOR-swap market (e.g. comparable to the repo specialness in the Treasury market), it is possible

[8] See Litzenberger (1992) and Solnik(1990),

[9] See Sorensen and Bollier (1994) and Duffie and Huang (1996).

[10] We do not study the issue of swap pricing when one of the counter-parties is of lesser credit quality. Duffie and Huang (1996) show that such a difference in credit risk has little impact on swap rates.

that the greater notional transaction volume of the swap market is an indicator for greater liquidity and that this may affect pricing. A pragmatic answer could be to reinterpret our results and consider that our instantaneous credit spread, which enters the adjusted rate used to discount under the risk-neutral measure, reflects both credit risk and swap-LIBOR liquidity differential (in the spirit of Duffie and Singleton (1997)). But the two effects cannot be disentangled. Absent a theory for liquidity, and in light of the widespread use of the swap term structure in lieu of a top-quality corporate-bond term structure, it seems useful to provide an explanation of the LIBOR-swap spread, based solely on a realistic default-risk model. The task of isolating and quantifying the impact of liquidity relative to default-risk is left to further research.

Our paper is structured as follows. In section 2, we present our model for corporate bond yields and swap rates. We examine some of the implications of our model for LIBOR-swap spreads in section 3. An empirical validation is provided in section 4. We conclude in section 5. Proofs are provided in an appendix.

2 The model

Our intent is to develop a model that provides some qualitative and realistic quantitative implications about the (relative) pricing of two securities: the zero-coupon defaultable LIBOR bonds for all maturities $T \equiv t + \tau$ with price $P_L^\tau(t)$ at time t and the swap contract (initiated at time t) to exchange the (preset) 6-month LIBOR, $Y_L^{.5}(t + 0.5(i-1))$, against fixed payments of $Y_S^\tau(t)$ every 6-months for τ years (i.e. at every $t + 0.5i \; \forall i = 1, 2, \ldots, 2\tau$). As is usual in this literature, we denote by "**risk-free**," securities that are **free of default risk**, but not necessarily of interest-rate risk. We denote by "**risky**" securities with **default risk**. Hence, with our previous assumptions, swap contracts are "risk-free," but LIBOR bonds are "risky." They all clearly carry interest-rate risk.[11]

2.1 The LIBOR bond term structure

Although top quality, these corporate LIBOR bonds carry default risk. We adopt the so-called reduced form to default-risk modeling discussed by Duffie and Singleton (1999). In this framework, default is an unpredictable stopping time modeled by the first occurrence of a point process with stochastic intensity, not necessarily related to the value of the corporate bond or the value of the firm's assets. In other words, we implicitly "assume" that the bond is small relative to the overall portfolio of assets of the firm.

As shown in Duffie and Singleton (1999) the price of the risky zero-coupon bond is given by:

$$P_L^\tau(t) = \mathrm{E}_t^{\mathcal{Q}} \left[e^{-\int_t^T R(s)ds} \right] \tag{1}$$

This formula states that the present value of risky cash flows may be found by discounting them at a risk-adjusted interest rate under the equivalent martingale measure. The risk adjusted rate $R(t)$ is equal to the instantaneous risk-free rate $r(t)$ plus an instantaneous credit spread, which is the instantaneous expected loss rate under the risk-neutral measure.

Notice that formula (1) implies that risky bonds can be priced as risk-free bonds by "expanding" the number of factors driving the term structure. For example, if we chose to model $R(t)$ as the sum of

[11] Since we focus on pricing securities in this section all processes are specified under the risk-neutral measure. We take a risk-neutral measure $\mathcal{Q}$ as given, and discuss the issue of risk-premia in the empirical section.

two independent factors, the risky term structure of interest rates would become a traditional two-factor model of the term structure.

However, it seems unlikely that such a specification of the instantaneous credit spread for top-rated credit quality issuers be appropriate. Indeed, theoretical (Merton (1974), Jarrow, Lando and Turnbull (1997)) as well as empirical (Sarig and Warga (1989) and Fons (1994)) evidence shows that the term structure of credit spreads exhibit systematic patterns, which are not well-captured by standard processes used for modeling the risk-free term structure.[12] In light of this evidence, we put more structure on our model of the instantaneous credit-spread process to allow for possible deterioration of credit quality of the LIBOR bond issuers.

We model the risk-adjusted discount rate for an issuer, who is top-rated at time-t, as $R(s) = r(s) + \delta^t(s) \ \forall s \geq t$, and assume the instantaneous credit spread of an issuer that is top-rated at time t evolves according to (for $s \geq t$):

$$d\delta^t(s) = \kappa_\delta(s)\left(\overline{\delta}^t(s) - \delta^t(s)\right)ds + \sigma_\delta(s)dw_\delta(s) + \nu_1(s)dN^t(s) \tag{2}$$

$$d\overline{\delta}^t(s) = \nu_2(s)dN^t(s) \tag{3}$$

$$\overline{\delta}^t(t) = \overline{\delta} \tag{4}$$

$w_\delta(t)$ is a brownian under the risk-neutral measure. $\kappa_\delta, \sigma_\delta, \nu_1, \nu_2$ are deterministic functions of time and $\overline{\delta}$ is a constant. In words, deterioration in credit quality is triggered by a point process with intensity $\lambda^t(s)$ and associated counting process $N^t(s)$. $N^t(s)$ is equal to the number of jumps in credit quality between t and s ($N^t(t) = 0$).[13] The intensity $\lambda^t(s)$ may be stochastic. This model implies that when the credit quality of the issuer deteriorates, his credit spread jumps up by a discrete amount ν_1. At the same time, there is an adjustment in the long term mean of the credit spread which jumps up by a discrete amount ν_2.[14]

A so-called **refreshed** top-rated issuer which is guaranteed to remain top-rated forever, has $\nu_1 = \nu_2 = 0$. The dynamics of his instantaneous credit spread $\delta(t) \equiv \delta^t(t)$ are:

$$d\delta(t) = \kappa_\delta(t)\left(\overline{\delta} - \delta(t)\right)dt + \sigma_\delta(t)dw_\delta(t) \tag{5}$$

We make the further assumption that the short-term risk-free rate $r(t)$ follows a gaussian process:

$$dr(t) = \kappa_r(t)\left(\theta(t) - r(t)\right)dt + \sigma_r(t)dw_r(t) \tag{6}$$

where $w_r(t)$ is a $\mathcal{Q}$-brownian motion. κ_r, σ_r are deterministic functions of time. Furthermore, we assume that the long-term mean is itself stochastic and mean-reverting:

$$d\theta(t) = \kappa_\theta(t)\left(\overline{\theta}(t) - \theta(t)\right)dt + \sigma_\theta(t)dw_\theta(t) \tag{7}$$

where $w_\theta(t)$ is a $\mathcal{Q}$-brownian motion, and $\kappa_\theta, \overline{\theta}, \sigma_\theta$ are deterministic functions of time. All brownian motions are possibly correlated with deterministic correlation coefficients given by: $dw_r dw_\delta =$

[12] For example, the term structure of credit spread for top-rated issuers should always be increasing with maturity.

[13] This point process is assumed to have no common jumps with the point process that triggers default. This is a technical assumption which is necessary for expression (1) to be valid. It merely states that default and deterioration cannot occur at exactly the same instant of time. Of course, any deterioration in credit quality implies that the probability of a default increases.

[14] Since we focus on top-quality counter-parties, we consider only deterioration of credit quality. The model can easily be extended to include possible appreciation in credit quality. In the appendix we derive the results for a more general model which allows also for appreciation in credit quality. We show that all our results go through as long as the expected depreciation in credit quality is higher than the expected appreciation for a top-rated credit quality issuer.

$\rho_{r\delta} dt \;\; dw_r dw_\theta = \rho_{r\theta} dt \;\; dw_\theta dw_\delta = \rho_{\theta\delta} dt$. The Gaussian processes used to model r and δ present some well-known shortcomings (negative values and homoskedasticity). We choose the Gaussian framework mainly for tractability reasons as our goal is to derive closed-form solutions that provide intuition about the relative impact of the refreshed credit quality and non-homogeneous credit quality assumption on the LIBOR-swap spread.[15]

One can show (e.g. using standard techniques developed in, for example, Duffie and Kan (1996), Das and Foresi (1996)) that the risky zero-coupon bond prices of a at time t top-rated issuer, are given by the following formula:

$$P_L^\tau(t) = P^\tau(t) P_\delta^\tau(t) e^{\int_t^T -\mu^t(s,T)ds} \tag{8}$$

where:

$$\begin{aligned} P^\tau(t) &= e^{A_r(t,T) - B_r(t,T)r(t) - C(t,T)\theta(t)} & (9)\\ P_\delta^\tau(t) &= e^{A_\delta(t,T) - B_\delta(t,T)\delta(t)} & (10)\\ \mu^t(s,T) &= \lambda^t(s)\left(1 - e^{-\nu_2(s)*(T-s) - (\nu_1(s) - \nu_2(s))B_\delta(s,T)}\right) \end{aligned}$$

And $A_r, B_r, A_\delta, B_\delta, C$ are the standard deterministic functions appearing in the computation of a zero-coupon bond (see appendix). Notice that P_t^τ is the price of a risk-free zero-coupon bond paying \$1 at time $t+\tau$, which, when coefficients are constant, is the special case of Langetieg's (1980) model analyzed by Jegadeesh and Pennacchi (1996), and which reduces to the standard Vasicek-bond price for constant θ, κ_r, σ_r.

Loosely speaking, $\mu^t(\cdot, T)$ can be viewed as the marginal increase in the yield on a defaultable zero-coupon bond, issued at t by a top-rated firm and maturing at T, due to possible deterioration in credit quality between t and T.

As for most existing models of credit-risk, in our framework, a coupon-paying bond can be priced as a sum of "risky"-zero-coupon bonds.[16] Hence the coupon, $Y_L^\tau(t)$ paid semi-annually by a corporate bond issued at par at time t and maturing at time $t+\tau$ is given by:

$$Y_L^\tau(t) = \frac{1 - P_L^\tau(t)}{\sum_{i=1}^{i=2\tau} P_L^{.5i}(t)}. \tag{11}$$

2.2 The swap term structure

We consider a plain-vanilla or generic swap indexed on 6-month LIBOR, with the three usual characteristics **C1** *the payments are indexed on a lagged floating-index value*, **C2** *the reset lag of the floating index has the same length as the payment period*, and **C3** *payment dates correspond exactly to reset dates.*

Let us define $Y_S^\tau(t)$ as the fixed rate to be paid semi-annually for τ years in a generic swap entered at date t against the six-month-LIBOR rate of $Y_L^{.5}(t)$. 6-month LIBOR is defined by the short end of the top-quality corporate term structure according to equation (11):

$$Y_L^{.5}(t) = \frac{1 - P_L^{.5}(t)}{P_L^{.5}(t)} \tag{12}$$

[15] Notice also that negative credit spreads can be interpreted as (presumably rare) situations in which default is expected to result in recovery of more than the market value of the bond just prior to bankruptcy. For example, when bankruptcy negotiation is done on the grounds of outstanding principal values, the proportion of outstanding principal reimbursed may be higher than the market value of the bond.

[16] All widely used credit risk models share the feature that coupon bonds can be priced from zero-coupon bonds, e.g. Duffie Singleton (1999), Jarrow, Lando and Turnbull (1997).

In a generic swap the floating leg payment at date $t_i \equiv t + 0.5i$ is $Y_L^{.5}(t_{i-1})$.

As discussed above, the swap contract is considered as risk-free. Consequently, the discount rate to use under the risk-neutral measure is the risk-free rate $r(t)$ defined above. By definition of the swap, Y_S^τ is the annuity that achieves a zero value for the contract at initiation, such that:

$$\mathrm{E}_\mathrm{t}^{\mathcal{Q}}\left[\sum_{i=1}^{i=2n} e^{-\int_t^{t_i} r(s)ds} Y_S^\tau(t)\right] = \mathrm{E}_\mathrm{t}^{\mathcal{Q}}\left[\sum_{i=1}^{i=2n} e^{-\int_t^{t_i} r(s)ds} Y_L^{.5}(t_{i-1})\right] \tag{13}$$

Substituting from (12), using the formulas derived above, and after some calculations, we find:

$$1 + Y_S^\tau(t) = \sum_{i=1}^{i=2n} \omega_i \frac{P_L^{.5(i-1)}(t)}{P_L^{.5i}(t)} * \mathcal{C}(t, t_{i-1}, t_i) * \mathcal{C}'(t, t_{i-1}, t_i) \tag{14}$$

with $\omega_i = P^{.5i}(t) / \sum_{i=1}^{i=2n} P^{.5i}(t)$, $\mathcal{C}$ and $\mathcal{C}'$ are given in equations A.7 and A.8 in the appendix. The expression derived above for the fixed rate on a swap looks complicated. However, it is simple to interpret. First, consider a swap with only one payment date, i.e. with a 6-month maturity ($n = 0.5$). The fixed rate to be paid $Y_S^{.5}(t)$ simplifies to the LIBOR rate:

$$Y_S^{.5}(t) = Y_L^{.5}(t) \tag{15}$$

The fixed rate paid on longer-term swaps can be interpreted as a weighted average of forward LIBOR rates corrected for default risk. $P_L^{.5(i-1)}(t)/P_L^{.5i}(t)$ is the implicit (one plus) LIBOR forward rate between t_{i-1} and t_i (since $\sum_i \omega_i = 1$). There are two correction factors $\mathcal{C}$ and $\mathcal{C}'$. The former is essentially a "Jensen-inequality effect," the latter $\mathcal{C}'(t, t_{i-1}, t_i)$ accounts for the possibility of jumps in the instantaneous credit spread of the LIBOR rates that serve as a reference for the floating leg of the swap.

The link between the fixed rate on a swap and a weighted average of forward rates has been noted in previous literature.[17] Our formula differs from previous models because it accounts for (1) differences in credit risk between swap contracts and LIBOR bonds, and (2) the difference between a continuously upgraded refreshed credit quality LIBOR rate and the yield on a typical LIBOR counter-party which reflects possible future jumps in credit quality.

Before we turn to the discussion of these issues, we would like to briefly mention the swap-Treasury spread. The swap spread is often quoted with respect to the yield on a government bond with equivalent maturity. Although both contracts are free of default risk in our model, the swap rate is different from the Treasury rate. As we have seen, the 6-month swap rate is equal to the 6-month LIBOR rate by definition of the swap contract (equation 15). So the swap term structure is "anchored" at the 6-month LIBOR, which is clearly higher than the 6-month Treasury yield, because the LIBOR rate reflects credit risk. More generally, the swap term structure depends on the credit-risk process since the floating leg of the swap contract is indexed on the 6-month LIBOR rate. Even though the swap contract is free of default risk, the swap rate depends on the credit-risk process through the floating leg indexation (it is a risk-free contract written on a risky underlying rate). As a consequence, the dynamics of the swap rates depend on the dynamics of the credit-risk process and, hence, differ from the dynamics of the Treasury rates. The swap-Treasury spread is, typically, not constant across maturities in our model.

[17] See for example Sundaresan (1991) and Duffie and Singleton (1997).

3 A better picture of the LIBOR-swap spread?

In this section we provide some intuition for the respective impact on the LIBOR-swap spread of our two main assumptions (as defined in the introduction): (1) "homogeneous vs. non homogeneous LIBOR-swap credit quality" and (2)"refreshed vs. non refreshed credit quality" in the LIBOR market.

If we were to assume that there is both "homogeneous LIBOR-swap credit quality" and "refreshed credit quality" of the LIBOR counter-parties, then the swap rate would be given by the following formula:

$$1 + Y_S^\tau(t) = \sum_{i=1}^{i=2n} \omega_i^L \frac{P_L^{.5(i-1)}(t)}{P_L^{.5i}(t)} \quad \Rightarrow \quad Y_S^\tau(t) = \frac{1 - P_L^\tau(t)}{\sum_{i=1}^{2n} P_L^{.5i}(t)} \tag{16}$$

with $\omega_i^L = P_L^{.5i}(t) / \sum_{i=1}^{i=2n} P_L^{.5i}(t)$. In this case the swap rate is equal to the LIBOR-bond yield for all maturities (see Duffie and Singleton (1997) and Sun, Sundaresan and Wang (1993)). But, we observe on average a positive LIBOR-swap spread. Since, as discussed previously, swap contracts carry less default risk than corporate bonds, it seems natural to first investigate whether relaxing the assumption of "homogeneous LIBOR-swap credit quality," can explain the observed LIBOR-Swap spread.

3.1 Non-homogeneous credit quality between swap and LIBOR markets

First we consider the case where the swap contract is risk-free whereas LIBOR bonds are risky, but where there is no possibility of jumps in the credit spread. Thus, the corporate bond is assumed to always remain of "refreshed credit quality." Then our formula (14) for swap rates reduces to:

$$1 + Y_S^\tau(t) = \sum_{i=1}^{i=2n} \omega_i \frac{P_L^{.5(i-1)}(t)}{P_L^{.5i}(t)} * \mathcal{C}(t, t_{i-1}, t_i) \tag{17}$$

with $\omega_i = P^{.5i}(t) / \sum_{i=1}^{i=2n} P^{.5i}(t)$ and $\mathcal{C}$ is as defined previously. The factor $\mathcal{C}$ is in fact just a "Jensen-inequality effect" which in practice is very close to 1.[18] Thus, the major effect of introducing non-homogeneity between swap and LIBOR bond markets is to change the weighting of forward LIBOR rates in computing the swap rate. Indeed a comparison of equations (17) with (16) for the case where $\mathcal{C} = 1$ shows that the only impact of introducing non-homogeneous credit quality is to change the weighting from $\omega_i^L = P_L^{.5i}(t) / \sum_{i=1}^{i=2n} P_L^{.5i}(t)$ in the homogeneous case to $\omega_i = P^{.5i}(t) / \sum_{i=1}^{i=2n} P^{.5i}(t)$ in the non-homogeneous case. Some algebra reveals that the slope of the forward-LIBOR curve dictates the relation between LIBOR bond yields and swap rates. We summarize this relation in the following proposition:

Proposition 1 *Assume (1) the swap contract is (default-) risk-free, (2) the LIBOR bond is default risky, (3) LIBOR bonds are sure to maintain their credit quality (refreshed credit quality), and (4) $\mathcal{C}$ is negligible (i.e. 'close' to 1). Then, when the forward-LIBOR curve is upward-sloping (downward-sloping), the swap rate curve should be above (below) the LIBOR bond yield curve.*

For example, the swap rate curve will be above the LIBOR bond yield curve when the forward-LIBOR rates are increasing with maturity. This result is purely a consequence of the indexation mechanism of swap contract.

[18] This statement is easily checked for reasonable parameter values. For example with parameter values as estimated in section 4, $\mathcal{C}(0, 5, 5.5) \approx 1 + 2 * 10^{-6}$.

The proposition above shows that relaxing the "homogeneous swap-LIBOR credit quality" alone will not explain the observed LIBOR-swap credit spreads. Since on average we observe upward-sloping LIBOR curves and increasing forward-LIBOR curves, the above *proposition* implies that the swap curve should be mostly **above** the corporate rate curve. Empirically, however, we observe the opposite as documented in Sun, Sundaresan and Wang (1993).

3.2 Relaxing the "refreshed-credit-quality" assumption

We claim that the LIBOR-swap spread reflects the probability of credit deterioration in a top-quality LIBOR counter-party. Indeed, by contractual definition, the swap contract is indexed on a refreshed LIBOR rate index, which is continuously updated so as to maintain its credit quality. On the other hand a typical LIBOR bond issuer may experience a deterioration in credit quality at anytime which is priced into the bond yield. Comparing equations (14) and (17), we see that this is reflected by the factor $\mathcal{C}'$, which captures the possible change in credit quality over time and can be viewed as the difference between two credit risks. The first applies to an issuer with refreshed top-credit quality on all reset dates (as implicit in the swap rate) and the second applies to an issuer who was of top-credit quality at time of issue, t (as implicit in the LIBOR bond yield).[19]

3.3 Model-implied spreads between refreshed-quality-LIBOR yields and LIBOR yields

In this section, we use our model to provide some insights into the cost paid by a typical LIBOR counter-party for the likelihood of beeing downgraded over the future life of the bond. This cost can be measured within our framework as the difference between the yield paid by a top-rated issuer computed using equation (8), and that paid by a refreshed credit quality issuer computed using the same formula, but setting the intensity of credit-deterioration to zero ($\lambda = 0$).[20] The non-refreshed bond corresponds to a typical top-quality corporate bond, while the refreshed-quality bond is fictitious. The refreshed-credit quality bond does not carry any credit-deterioration risk, but may be defaulted upon anytime. The standard corporate bond reflects both: it may default at anytime and it may experience deterioration in its credit quality.

Figure 2 shows the spread in bond yields between top-rated issuers with constant expected instantaneous downgrading set to 10 basis points (i.e. in our previous notations $\nu_1 = \nu_2 = \nu$, $\lambda^t(s) = \lambda$ and $\lambda\nu = 10bp$) and top-rated issuers with refreshed credit quality ($\lambda\nu = 0$). Of course, the constant expected instantaneous downgrading can result from different combinations of jump size and intensity of credit depreciation. We show two cases: a high size/low intensity case ($\nu = 100bp$, $\lambda = 0.1$) and a low size/high intensity case ($\nu = 10bp$, $\lambda = 1$). All other parameter values correspond to those estimated in the next section, Table 1. The values of the instantaneous risk-free rate and of the credit spread are set at their long-term means.

Figure 2 shows that the spread between non-refreshed and refreshed credit quality bond yields is economically significant, increasing with maturity, reaching 60 bp at a 20-year maturity. Interestingly, the figure also reveals that the spread has a slightly different sensitivity to size and intensity of credit

[19] To understand the intuition, consider the exposure on a 10-year-maturity corporate bond versus a 10-year swap. Compare the default spread on the cash flow of one particular maturity, say in 7 years. Holding a 10-year corporate bond entitles one to receive a coupon in 7 years if there has not been any previous default. The value of that coupon depends on the expected recovery rate of a cash-flow received in 7 years by a today top-rated firm. On the other hand, the cash flow to be received in 7 years in a (default-risk-free) swap contract incorporates default risk only through the floating index, which depends on the expected recovery rate on a 6-month defaultable bond issued by a firm that will be top-rated in 6.5 years.

[20] We thank a referee for suggesting this analysis.

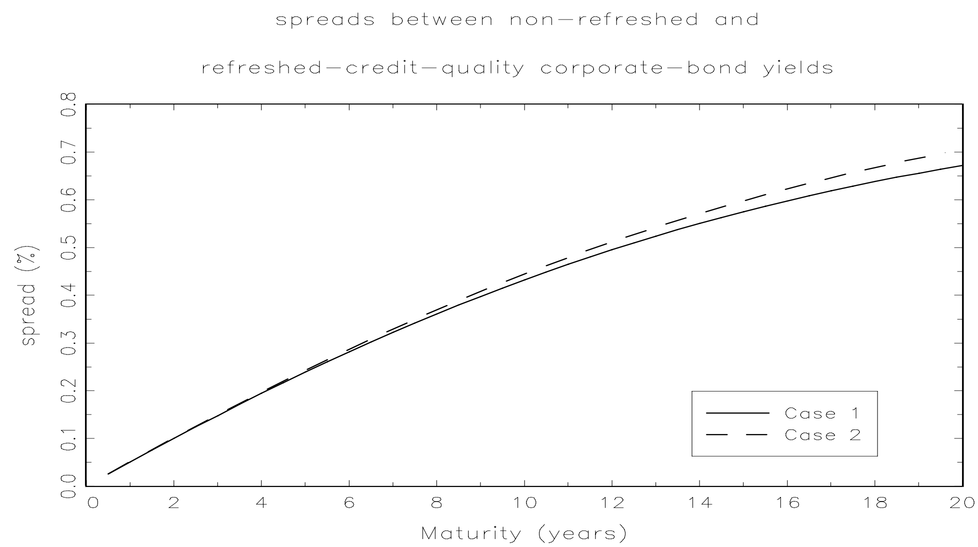

Figure 2: Term structures of spreads between yields of non-refreshed and refreshed credit quality corporate bonds as implied by the model. All parameter values are taken from the estimated values in Table 1 below. We use two different values for jump size and jump intensity. Case 1 has $\nu = 100bp$ and $\lambda = 0.1$, case 2 has $\nu = 10bp$ and $\lambda = 1$. Notice that in both cases we keep the expected instantaneous downgrading constant to $\lambda\nu = 10bp$.

depreciation risk. For a constant expected depreciation in credit quality, the credit spread is actually increasing in jump intensity but decreasing in size. In other words, credit spreads are more sensitive to changes in intensity than to changes in the size of the jump in credit spreads.

4 Some empirical results

Using data on Treasury bond yields, LIBOR bond yields and swap rates we now estimate the parameters of our model. This allows us to determine the significance of the deterioration in credit quality of top-rated issuers implicit in the LIBOR-swap spread.[21] We shortly describe the data and econometric methodology used and discuss the empirical results.

4.1 Data and econometric methodolgy

We use weekly data for Treasury, LIBOR par-bond and swap rates from October 12, 1988 to January 29 1997. The data were obtained from Datastream. Datastream reports the mid swap rates[22] quoted by a

[21] We use Treasuries as a proxy for the "true" risk-free rate even though they are often claimed to offer advantages over and above the risk-free asset, such as liquidity and taxes. This allows us to isolate the different component of the LIBOR-swap spread and give some economic interpretation to our results. Notice that since we estimate the LIBOR-swap spread, we may reasonably hope this will not have a big impact on our estimation of the instantaneous credit-risk process.

[22] The bid and ask swap rates quoted depend on the credit quality of the customer. The bid-mid and mid-ask spreads for a generic swap quoted to a AAA or AA customer are generally equal to one basis point over the period. As mentioned in Sun, Sundaresan and Wang (1993) and Cossin and Pirotte (1997), the spreads increase by a few basis points for a lesser-rated customer.

major swap dealer for maturities of 2, 3, 4, 5, 7 and 10 years. Treasury bond data covers the maturities: 1, 2, 3, 4, 5, 7 and 10 years. Finally we use the LIBOR yields reported by Datastream for maturities 0.5, 1, 2, 3, 4, and 5 years. These are quoted yields for fixed-coupon par-bonds negotiated OTC and issued by corporate issuers (usually banks and financial institutions) rated AA or better.[23]

In order to subject our model to empirical scrutiny, we make a few simplifying assumptions. We assume that all parameters are constant. To reduce the number of parameters to be estimated, we assume that $\nu_1 = \nu_2 = \nu$ and that $\lambda^t(s) = \lambda$. In words, we assume that when the credit quality deteriorates, both the long-term mean and the level of the credit spreads jump by an equal amount, and that the probability of credit deterioration is constant. Because of a well-known indeterminacy arising in such models (Duffee (1999), Duffie and Singleton (1999)) we cannot estimate λ, the intensity of the jump, separately from ν the size of the jump. We thus estimate the joint product $\mu \equiv \lambda\nu$.[24]

We also need to make assumptions about the risk premia associated with our three stochastic factors, because our data is observed under the historical $\mathcal{P}$-measure whereas we have specified the processes under the risk-neutral measure. For the empirical implementation we assume risk-premia to be constant.[25] We thus have three additional parameters to estimate, $\lambda_r,\ \lambda_\theta,\ \lambda_\delta$ the risk premia associated with interest-rate risk and generic "refreshed credit quality" credit spread. The risk premia capture the shift in distribution going from the physical measure $\mathcal{P}$ to the risk-neutral measure $\mathcal{Q}$.[26]

To further reduce the number of parameters to be estimated, we constrain the autocorrelation coefficient for all the error terms to be the same. We thus have a total of 16 parameters to estimate.

We use maximum-likelihood estimation using both time-series and cross-sectional data in the spirit of Chen and Scott (1993). The approach consists of using three arbitrarily chosen yields, e.g. a swap rate and a LIBOR bond yield to determine the state (r, θ, δ) using formulas in (8) and (14) and given a vector of parameter values. The remaining yields, which, at any point in time, are also deterministic functions of the state variables are then over identified. Following Chen and Scott (1993), we assume these other yields are priced or measured with 'error.'[27] Given the known transition density for the state variables and some assumed distribution for the error terms, the likelihood can be derived.

4.2 Results

Estimated parameters are reported in Table 1.

They are reasonable and statistically significant except for the risk-premia on central tendency and

[23] The market is pretty liquid, see Sun, Sundaresan and Wang (1993) for a discussion of the LIBOR bond market and comparisons of the Datastream-data with alternative data sets. Further details on our data set can can be found in an appendix.

[24] With these assumptions $\mu^t(s)$ reduces to: $\mu^t(s,T) = \lambda\left(1 - e^{-\nu(T-s)}\right) \approx \lambda\nu(T-s)$ for small ν (empirically it is of the order of 10^{-4}). We thus estimate the parameter $\mu \equiv \lambda\nu$ using the approximation: $\mu^t(s,T) = \mu(T-s)$. If there is also a possibility for appreciation in credit quality than μ is equal to the expected depreciation in credit quality net of expected appreciation (as shown in the appendix).

[25] Since we do not observe actual jumps in the jump process, we cannot estimate the change of measure (i.e. of intensity). In other words, we can only estimate the risk-neutral expected credit-risk depreciation.

[26] In the gaussian framework, risk-premia have a nice interpretation. In our notation, λ_θ is the amount which must be added to the risk-neutral long-term mean $\overline{\theta}$ to obtain the long-term mean of the short rate under the historical measure, i.e $\overline{\theta} = \overline{\theta}^{\mathcal{Q}} + \lambda_\theta$. Similar interpretations apply for the $\lambda_r, \lambda_\delta$ process. Except of course, that λ_r denotes the amount by which the whole path of θ has to be shifted. Notice that our definition is slightly different from the traditional risk-premium, because we find the adjustment in terms of the change in long-term means more intuitive. Of course, Girsanov's theorem gives the relation between the brownian motions and the traditional market price of risk: $dw^{\mathcal{P}} = dw^{\mathcal{Q}} - \frac{\lambda\kappa}{\sigma}dt$.

[27] Duffie and Singleton (1997) use a similar method. Alternatively, we could have used a Kalman-filter to avoid making an arbitrary assumption on which yields are priced without errors.

Table 1: Parameter estimates resulting from the Maximum Likelihood described in section 4.2. all parameters are presented for state variables of the form $dz = \kappa_z(\theta_z - z)dt + \sigma_z dw^{\mathcal{P}}$ for $z = r, \theta, \delta$.

Mean log-likelihood	128.1957	
Parameters	Estimates	Std. err.
κ_r	0.1028	0.0136
σ_r	0.0097	0.0003
$\overline{\theta}$	0.0878	0.0025
κ_θ	0.4851	0.0430
σ_θ	0.0872	0.0141
$\overline{\delta}$	0.0038	0.0002
κ_δ	1.4248	0.0956
σ_δ	0.0131	0.0006
$\rho_{r\theta}$	-0.2726	0.0602
$\rho_{r\delta}$	-0.2330	0.0367
$\rho_{\theta\delta}$	0.4126	0.0443
μ	0.00052	0.00019
λ_r	-0.1234	0.0375
λ_θ	-0.0265	0.0326
λ_δ	0.0005	0.0037
ρ_u	0.9218	0.0054

instantaneous credit spread (λ_θ and λ_δ). The long-run mean of the risk-free rate under the risk-neutral measure is 8.78% per year. The risk-premia coefficients, $\lambda_r, \lambda_\theta$, are negative, implying that term premia are positive and increasing with maturity.[28] It appears that the level factor of the risk-free term structure has relatively low mean-reversion (10%) and volatility (1%) compared to the second-factor, the long run tendency, which appears to have high mean-reversion (50%) and high volatility (8%). This is in line with the results of Jegadeesh and Pennacchi (1996) who interpret the central tendency as a proxy for a long run interest rate target, thus reflecting expectation about future inflation rates. On the other hand, we find a significant negative correlation between the long-run tendency and the short-term rate.[29] The long-run mean of the instantaneous credit spread for top-rated corporate firms is 38 basis points under the risk-neutral measure. The instantaneous credit spread is quite volatile (1.3%) and has a strong level of mean-reversion (1.42). The credit-deterioration parameter μ is estimated around 5 bp. It is very significant both statistically and economically. Recall that $\mu \equiv \lambda\nu$ is the expected depreciation **rate** in credit quality.[30] Our findings thus imply that the spread between LIBOR par-bond yields and swap rates is consistent with top-quality LIBOR issuers experiencing, on average, a depreciation in credit quality of 5 basis points per year under the risk-neutral measure. This is economically significant and implies an increasing term structure of credit spreads for top-rated LIBOR issuers.

The correlation between movements in the instantaneous risk-free rate and credit spread is negative (-0.27) implying that the credit spread tends to decrease when the risk-free rate rises.[31] Interestingly, the correlation between the long-run tendency of the treasury term structure and the credit-risk process

[28] Term premia are defined as the expected return on a risk-free bond in excess of the instantaneous risk-free rate. They are equal to $-\lambda_r\kappa_r B_r(t,T) - \lambda_\theta\kappa_\theta C(t,T)$.

[29] Jegadeesh and Pennacchi are unable to precisely estimate that correlation, but they propose two possible interpretations depending on the sign of the correlation. We refer the reader to their discussion, p. 435-436.

[30] Unfortunately, as in Duffee (1999), we cannot disentangle the probability of downgrading from the jump size in the level of the instantaneous credit spread. In principle, if we had time-series data on individual credit-risky bond prices, our model would allow to estimate both parameters separately. Here, since for comparison with generic swap rates we use only generic LIBOR yields at contract initiation, we have no observation of actual credit-depreciation events. It would be interesting to analyze individual corporate-bond data, as in Duffee (1999) for example, using our model of corporate bonds.

[31] This is also consistent with the recent results in Duffee (1998) and Duffee (1999).

is positive. There are also macro-economic explanations for the correlation between interest rates and the credit spread. For example, the Treasury curve flattens in response to a slow-down in economic activity which should translate into higher spreads to compensate for credit risk. Part of the latter effect may actually be captured in our model by the correlation between the short rate and the credit spread.

It is interesting to assess the quality of the estimation by looking at the properties of the error terms for the various swap, LIBOR and Treasury rates, with maturities ranging from 0.5 to 10 years.

Table 2: Mean and standard deviation of the conditional errors (ϵ_i) in bp resulting from the Maximum Likelihood estimation described in section 4.2. Notice that the 1-year and 5-year Treasury and 1-year LIBOR are fitted perfectly because they are chosen for inversion.

Maturity	0.5 years	1 year	2 years	3 years	4 years	5 years	7 years	10 years	average
mean (Treasury yields)	N.A.	0	0.4	-0.0	0.2	0	0.3	0.3	-0.06
R.M.S.E	N.A.	0	4.7	3.9	2.6	0	3.3	4.3	3.8
mean (Swap rates)	N.A.	N.A.	-0.7	-0.6	-0.6	-0.5	-0.5	-0.3	-0.5
R.M.S.E	N.A.	N.A.	5.4	4.6	4.2	3.7	4.5	4.8	4.5
mean (LIBOR rates)	0.8	0	0.1	0.1	-0.0	-0.0	N.A.	N.A.	0.1
R.M.S.E	8.9	0	7.4	8.7	8.9	9.1	N.A.	N.A.	8.1

The error terms (u_i) are strongly autocorrelated[32] ($\rho_u = 0.92$), but the average and root mean square errors (R.M.S.E.) of the conditional error terms (ϵ_i) are quite low, as can be seen in Table 2. Depending on the maturity, the mean conditional error ranges from -0.7 bp (basis point) to +.5 bp across all maturities and all rates. The R.M.S.E is less than 9 bp for all rates and maturities. Notice that the R.M.S.E is less than 5 bp for swap and Treasury rates and slightly higher for LIBOR rates, i.e the model does better at capturing the dynamics of the swap and Treasury term structure. This may also indicate that the dynamics of the downgrading process chosen for this application is too simple and could be improved upon.[33]

5 Conclusion

In this paper we study the term structure of the spread between corporate bond yields and swap rates for top-quality counterparties. The swap term structure is widely used by bankers, investors and borrowers in lieu of the corporate term structure as the basic tool for pricing corporate assets and liabilities as well as all kinds of financial assets. This practice originates because swap rates are continuously quoted (and traded) for a wide range of maturities and therefore more readily updated than corporate yields. Yet, it is justified only if the swap term structure truly reflects the cost of financing for a top-rated corporate issuer for the various maturities. Indeed, we have shown that realistic modeling of default risk leads to a theoretical difference between the two curves (we call it the “LIBOR-swap spread”).[34]

Our model is consistent with the empirical fact that LIBOR-quality bond yields are in general higher than swap rates for similar maturities. Our two key assumptions are (1) swaps carry less credit risk than corporate bonds, (2) the credit quality of top-rated issuers may deteriorate over the life of the contract and in particular differ from that of a continuously updated “refreshed credit quality” index.

[32] Chen and Scott (1993) and Duffie and Singleton (1997) find similar results.

[33] In an earlier version we also looked at unconditional fitting errors and the volatility of the model implied spread. Results show a good fit of the model.

[34] Although none of the extant theories specifically study the spread between corporate bond yields and swap rates - they focus on the Treasury-Swap spread - they imply that these two term structures should be identical.

Interestingly, our results show that, the first assumption is not sufficient to explain a positive LIBOR-swap spread. LIBOR bond yields should be mostly below (not above) swap rates if swaps are free of default risk while LIBOR bonds carry default risk, and if all counterparties are sure to maintain their credit quality over the life of the contracts. Our second assumption is thus crucial to explain the observed positive LIBOR-swap spread. Because swap payments are indexed on the 6-month LIBOR rate, a continuously updated, “refreshed” credit quality rate, we argue the LIBOR-swap spread captures the expected credit-quality deterioration of a top-rated credit-quality issuer. We provide an explicit model of the difference between a refreshed credit-quality term structure and an actual top-rated credit-quality term structure that includes the possible jumps in credit quality and derive the swap rate in this framework.

Our empirical results show the existence of an economically and statistically significant expected credit-quality deterioration for top-rated LIBOR-bond issuers.

There are several ways in which our work could be extended,[35] but we believe that our analysis highlights an important dimension in swap pricing that has been neglected so far in the academic literature.

[35] For example, we could: use different processes for the state variables, introduce a stochastic intensity for credit deterioration, model the fact that swap contracts carry some default risk (although less than bonds), add a liquidity convenience yield or other factors in the Treasury market.

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A The formulas

This appendix gives the different formulas used in the text. All the derivations, proofs and further details about the empirical analysis can be found in an appendix available at http://www.cmu.edu/user/dufresne

- The risk-free discount bond price

$$\begin{aligned}
P^{\tau}(t) &= e^{A_r(t,T)-B_r(t,T)r(t)-C(t,T)\theta(t)} \quad (A.1)\\
A_r(t,T) &= -\int_t^T C(v,T)\kappa_{\theta}(v)\overline{\theta}(v)dv + \frac{1}{2}\int_t^T B_r(u,T)^2\sigma_r^2(u)du + \frac{1}{2}\int_t^T C(v,T)^2\sigma_{\theta}^2 dv + \\
&\quad \int_t^T C(u,T)B_r(u,T)\sigma_r(u)\sigma_{\theta}(u)\rho_{r\theta}du \quad (A.2)
\end{aligned}$$

$$B_r(t,T) = \int_t^T \gamma_r(t,s)ds \quad C(t,T) = \int_t^T c(t,s)ds \quad c(t,s) = \int_t^s \gamma_r(u,s)\gamma_\theta(t,u)\kappa_r(u)du$$

- The risky discount bond price

$$P_L^{\tau}(t) = \mathrm{E}_t^{\mathcal{Q}}\left[e^{-\int_t^T (r(s)+\delta(s))ds}\right]\mathrm{E}_t^{\mathcal{Q}}\left[e^{-\int_t^T\int_t^s(\nu_2+(\nu_1-\nu_2)\gamma_{\kappa_\delta}(u,s))dN^t(u)ds}\right] \quad (A.3)$$

where by standard arguments:

$$\mathrm{E}_t^{\mathcal{Q}}\left[e^{-\int_t^T (r(s)+\delta(s))ds}\right] = P^{\tau}(t)e^{A_\delta(t,T)-B_\delta(t,T)\delta(t)} \quad (A.4)$$

$$\begin{aligned}
A_\delta(t,T) &= -\int_t^T B_\delta(u,T)\left(\kappa_\delta(u)\overline{\delta} - \rho_{r\delta}\sigma_\delta(u)\sigma_r(u)B_r(u,T) - \rho_{r\theta}\sigma_\delta(u)\sigma_\theta(u)C(u,T)\right)du + \\
&\quad 1/2\int_t^T B_\delta(u,T)^2\sigma_\delta^2(u)du \\
B_\delta(t,T) &= \int_t^T \gamma_\delta(t,s)ds \qquad \gamma_\delta(t,s) = e^{\int_t^s -\kappa_\delta(u)du}
\end{aligned}$$

And as proven in an appendix available at the URL address quoted above, we have:

$$\mathrm{E}_t^{\mathcal{Q}}\left[e^{-\int_t^T\int_t^s(\nu_2(u)+(\nu_1(u)-\nu_2(u))\gamma_{\kappa_\delta}(u,s))dN^t(u)ds}\right] = \mathrm{E}_t^{\mathcal{Q}}\left[e^{-\int_t^T(1-e^{-\nu_2(u)(T-u)-(\nu_1(u)-\nu_2(u))B_\delta(u,T)})\lambda^t(u)du}\right] \quad (A.5)$$

Using (A.3),(A.4) and (A.5), we obtain the risky discount bond prices given in 8.

- The swap rate formula

Some calculations show that:

$$1 + Y_S^{\tau}(t) = \frac{\sum_{i=1}^{i=2n}\left[P^{.5(i-1)}(t)P_\delta^{.5(i-1)}(t)/P_\delta^{.5i}(t) * e^{\int_{t_{i-1}}^{t_i}\mu^{t_{i-1}}(s,t_i)ds} * \mathcal{C}(t,t_{i-1},t_i)\right]}{\sum_{i=1}^{i=2n}P^{.5i}(t)} \quad (A.6)$$

Where in the above, we have defined:

$$\begin{aligned}
\ln\mathcal{C}(t,t_{i-1},t_i) &= \int_t^{t_{i-1}} B_\delta(u,t_{i-1})\left(B_r(u,t_i)-B_r(u,t_{i-1})\right)\rho_{r\delta}\sigma_r(u)\sigma_\delta(u)du + \quad (A.7)\\
&\quad \int_t^{t_{i-1}} B_\delta(u,t_{i-1})\left(C(u,t_i)-C(u,t_{i-1})\right)\rho_{\theta\delta}\sigma_\theta(u)\sigma_\delta(u)du + \\
&\quad \int_t^{t_{i-1}} B_\delta(u,t_i)\left(B_\delta(u,t_i)-B_\delta(u,t_{i-1})\right)\sigma_\delta(u)^2du
\end{aligned}$$

After some rearranging and algebra, we obtain equation 14 in the text with:

$$\ln\mathcal{C}' = \int_{t_{i-1}}^{t_i}\mu^{t_{i-1}}(s,t_i)ds - \left(\int_t^{t_i}\mu^t(s,t_i)ds - \int_t^{t_{i-1}}\mu^t(s,t_{i-1})ds\right) \quad (A.8)$$