# Pricing Credit Risk as ParAsian Options with Stochastic Recovery Rate of Corporate Bonds

By LingZhi Yu*

August, 2003

**Abstract**

Recovery rates are mostly treated as exogenous and constant in structural models. However, this assumption generates a number of problems: default probability is disassociated from the recovery rate; recovery rate is uniform for all classes of bond; there is often a problem of discontinuity in payoff at expiration; and the possibility of a negative duration. In this paper, we adhere to the original Merton (1974) framework and treat the recovery rate as endogenously determined, thus avoiding the above problems. This is achieved by modelling the process of default loss as ParAsian options, whose features can capture all possible bankruptcy resolutions. Instantaneous interest rates follow the CIR model (1985). We produce term structures of credit spreads consistent with the literature. The "window" feature of ParAsian options is influential. We also observe different results from other models when analysing the volatility of interest rates and their correlation with firm value.

**JEL Code**: G12, G13, G32

*ORS Awards Scheme is gratefully acknowledged. Corresponding address: Manchester Business School, Booth Street West, Manchester, M15 6PB, England. Phone: +44-161 275 6312 Fax: +44-161 275 6596 Email: `lyu@man.mbs.ac.uk`

# 1 Introduction

A major point of concern for investors in fixed income products is risk versus reward, for which the benchmark is the Treasury yield curve. Thus, investors in a risky corporate bond can expect an incremental yield over the Treasury bond rate; the so-called yield spread. There are several important factors related to this spread. First, normally investors are obliged to pay multiple layers of taxation, whereas governments make their Treasury bonds exempt from local tax. Second, they might be forced to surrender their bonds, if these are callable and the market rates are low (this source of uncertainty is called prepayment risk). Third, although the Treasury bond market is not exclusive of liquidity risk, we believe the liquidity risk is much higher in the relatively thin corporate bond market. Finally, there is a risk that the issuer will default, giving rise to the credit spread which is the principal concern. The expected loss due to credit risk can be decomposed into the probability of default and the amount which cannot be recovered. The way to calculate these two parts determines the relative merits of different pricing models.

One of the motivations of this paper is correct modelling of the second part, the recovery rate. Most modellers simply assume that the recovery rate is exogenous and constant. However, this assumption brings about a list of pitfalls. First, the default probability is calculated isolated from the recovery rate, despite the fact that they are strongly dependent on each other (Wei and Guo, 1997, Izvorski, 1997). Consequently, there are not only correctness problems in calculating the default probability, but also a consistent problem between the amount recovered and the asset value at default. Second, recovery rate is the same for all classes of bond. Although, the absolute priority rule violation is common in distress reorganizations, it should not be uniform for all firms. Additionally, within a firm, the priority is still realized to some degree. Recovery rate is a function of seniority (Altman and Kishore, 1996, Fridson et al. 2000). Next, some models adopt two recovery rates to guarantee the amount recovered is less than the asset value. (Kim et al. 1993, Briys and de Varenne 1997). However, because these rates are still exogenous and constant, a problem of discontinuity in default payoff at expiration can not be avoided. Finally, constant recovery rates may also lead to a negative duration (Acharya and Carpenter, 2002).

To prevent these problems, we comply with Merton's original framework (1974) and treat the recovery rate as endogenously determined, by which the recovery rate is virtually stochastic over time. Additionally, we treat the interest rate as stochastic according to the Cox, Ingersoll and Ross (1985) model (the CIR model hereafter). In this contingent claim model, we use ParAsian options to price the expected losses of the risky bonds. The way ParAsian options evolve conforms to our definition of recovery rate. Most importantly, ParAsian options can capture all possible resolutions in bankruptcy proceedings. Estimates of parameters are adopted from Huang and Huang (2003). As a result, we are able to generate credible shapes and magnitudes for the term structures of credit spreads, and find that the window feature of ParAsian option is very influential in forming the curves. Our findings regarding the volatility of interest rate and its correlation with firm's asset differ from those in the earlier literature.

Mixed results are found in this line of study (structural models), mainly stressing the magnitude of yield spread. Jones, Mason and Rosefeld (1984) find the average observed yield spread is much higher than that predicted by Merton (1974). However, Eom, Helwege and Huang (2000) find that a model can overestimate the credit risk and contingent-claim models do not systematically underpredict spreads. Gemmill (2002) is wary of the verity of sample in previous studies and finds similar magnitude between the Merton (1974) and market spreads in his sample. However, we acknowledge that market also contribute other risk premiums to the yield spread, as we discussed earlier. One theoretical appeal of our model is that we lay emphasis on the economic-understanding of credit risk. The yield spreads generated herein are effectively credit spreads.

The rest of the paper is organised as follows. In section 2, we properly define the recovery rate, and discuss the mathematical reason behind the choice of exogenous recovery rates, as well as the financial consequences of its pitfalls. Section 3 describes the main features of ParAsian options and justifies their suitability in pricing risky corporate bonds. We apply these features and build a contingent claim valuation framework in section 4. In section 5, we show the result of numerical analysis in a series of comparative statics. Conclusions are drawn in section 6.

## 2 Recovery Rate in Structural Models

In the literature for the pricing risky corporate bonds, there are two main avenues – structural models and reduced-form models, plus some recently developed mixed-form models. The reduced-form model gets its name from the use of an exogenous variable for default probability, calibrated to market data, and ignoring the underlying economics of corporate default. A motivation is to model the term structure of credit spreads relative to that of the risk-free interest rate, such that the well-developed short-rate models can be employed. However, its calibration gives rise to controversy surrounding the relative virtues of the structural and the reduced-form models. According to the structural model, represented by the seminal work of Merton (1974), the pricing scheme should be based on the actual operation of an individual firm, rather than fitting to historical default data, the statistics of which may only have a limited relationship with the process for the ongoing firm. Whether the default probability is exogenous or not differentiates structural models from reduced-form ones. Within the structural models, we can also classify into two groups according to whether the recovery rate is exogenous or not.

In the original Merton framework (1974), the recovery rate is endogenously determined by the asset value:

$$B_0(V,T) = Fe^{-rT} - \text{European Put} = Fe^{-rT} - N(-d_2)\left[Fe^{-rT} - \frac{N(-d_1)}{N(-d_2)}V_0\right],$$

where $B_t$ is the market value of debt – a risky zero-coupon bond at time t, $V_t$ is the market value of asset at time t, $F$ is the face value of the bond, $r$ is a constant interest rate, $T$ the time to maturity, $N(\cdot)$ is the cumulative normal distribution function as defined in the Black-Scholes model. As $N(-d_2)$ is the probability of an in-the-money put, the ratio $\frac{N(-d_1)}{N(-d_2)}$ is the expected recovery rate (Crouhy and Galai, 1997 in Cossin and Pirotte, 2000).

Starting from the work of Black and Cox (1976), the literature is able to account for default before bond maturity by means of barrier options rather than European options. This idea still can be formulated as

$$B_t(V,T) = Fe^{-\int_t^T r_t dt} - \text{down-and-out put}, \tag{1}$$

where $r_t$ might be stochastic. Theoretically, if a safety covenant can protect all the creditors' claims, the second part on the right hand side is worthless. Expressed in another way, the barrier put is knocked out whenever it goes in-the-money, as the barrier level is equal to the exercise price – the face value of the bond. One solution to generating a positive barrier option is to define an upward barrier $Fe^{-\phi t}$, where $\phi$ is a positive drift playing the same role as discount rate (Black and Cox, 1976, Kim et al., 1993, Zhou, 2001 etc.). Another solution is to define an exogenous recovery rate $\omega < 1$, so that the barrier option can be knocked out with a positive value. In terms of debt value, the creditors are supposed to suffer some loss. Under this circumstance, the barrier level is allowed to be a constant (i.e., $\phi = 0$), which may or may not be less than the face value (Longstaff and Schwartz, 1995). The second solution uniformly prices the risky bond in the form of:

$$B_0(V,T) = P(t,T) - \omega P(t,T)\mathbf{Q}_t^T(\tau < T),$$

where $P(t,T)$ is the price of risk-free bond at time $t$ with terminal value at $T$ of one unit, $\mathbf{Q}$ is a risk-neutral probability of default happening at time $\tau$. The right-hand-side can be read as the risk-free bond less the expected loss at default.

By definition, recovery rate is the fraction of the face value of the debt that creditors redeem at default[1] , which should be time-dependent, interest-rate-dependent as well as asset-value-dependent. At first glance, the second

[1]Other definitions include "the fraction of the risk-free market value" and "the fraction of the pre-default market value". Guha (2002) empirically compare these concepts and conclude that "recovery of face value of bond" has most support from the data.

method seems superior, since it can accommodate violation of the absolute priority rule. However, as pointed by Wei and Guo (1997), the constant recovery rate implies a huge assumption that the default probability and recovery rate have zero correlation. This is decisively different from the treatment of recovery rate in Merton's model (1974). Empirically, we also observe that safer bonds have higher recovery rate and vice versa (Fridson et al., 2000, Izvorski, 1997). Default probability is negatively correlated with recovery rate, though the direction of causality can be either way (Altman and Kishore, 1996, Izvorski, 1997). Uncertainty in the recovery rate is related to uncertainties in the interest rate and asset value. Cutting this link and assuming a constant recovery rate will only lead to an inaccurate result of the forward risk-adjusted probability,$\mathbf{Q}_t^T(\tau < T)$.

Moreover, this formula can not guarantee that the recovered debt value at default is less than corresponding asset value. Briys and de Varenne (1997) aim to solve this problem. Two recovery rates are proposed for default boundaries when default occurs before and at maturity, respectively. However, it is again the exogeneity of recovery rates that conveys a discontinuity at maturity. A similar problem can be found in Kim et al. (1993).

Finally, Acharya and Carpenter (2002) have proven both quantitatively and qualitatively that the bond duration should always be nonnegative. However, due to the exogeneity of recovery rates, the duration is of "U-shaped function of firm value" with a distinct possibility of negative value. Later, in our numerical analysis, we can see that the negative duration can be avoided in our model.

A noteworthy exception in this line is the work of Zhou (2001). The write-down component at default is of the form $\omega = \omega_0 - \omega_1 X$. The recovery/write-down component is endogenous as it is related to $X$, the ratio of asset and default boundary. Higher yield spreads are achieved, but by means of incorporating "counterfactually" high default probabilities (Huang and Huang, (2003)). This is because in Zhou's model (2001), the asset value follows a jump-diffusion process. The jump process (Poisson process) utilises an intensity parameter, which is not taken from the traits of the firm but reliant on historical data in the market. We believe the additional premium is due to market risks other than the credit risk. To some extent, Zhou (2001) reconciles the structural and the reduced-form models. In addition, the results are rendered even less convincing by using a negative recovery rate throughout the paper.

Hence, in this paper we aim to extend the Merton framework, comprising both early default and stochastic recovery rate. The original option-pricing approach is adhered to, and the recovery rate is endogenously determined.

# 3 ParAsian Option

## 3.1 Bankruptcy

The definition of default is that the debtors fail to comply with the provisions of bond indentures, a basic clause of which is making timely payments of coupons and principal. A natural consequence assumed in most risky bond models is that the creditors will liquidate and share the firm piecemeal. However, default is a process rather than an abrupt event. There are several default resolutions: forgiveness, private workout, prepackage bankruptcy filing, composition (Chapter 11 of the Bankruptcy Reform Act of 1978) or liquidation (Chapter 7) (Hart, 1995).

Bankruptcy is not costless. In liquidation, payments to creditors are badly in arrears and, according to the absolute priority rule, they follow payments to auditors, lawyers, government, payrolls etc. During composition, interest stops accruing; secured debt cannot seize collateral; creditors cannot cancel contracts. Private negotiation becomes the best resort for the lenders[2]. On the other hand, the borrowers may also not want to pay extra money to any third party, which is the incremental cost to them of Chapter 11 over workouts. Franks and Torous (1994) in their sample find "about one-half of Chapter 11 cases follow the abandonment of publicly announced workout attempts". Helwege (1999) finds a similar percentage of Chapter

[2]Franks and Torous (1994) and Helewege (1999) both show that hangout is not a plague problem in this circumstance.

11 cases, whose petitions were preceded by over six months in default.

Additionally, it is conceivable that the bankruptcy we talked above is due to stock insolvency only. In other words, we assume that there will be forgiveness if the firm value is higher than the debt obligation at cash insolvency.

### 3.2 ParAsian features

Parisian options are a variant of barrier options. Their barrier feature is only triggered when the underlying asset price breaches the barrier for a pre-specified period (the "window"). This overcomes the undesirable effect of termination in barrier options, when the price spikes only very briefly. Parisian options can be classified further as consecutive (if time since the last crossing of the barrier is measured continuously) and cumulative (if total time beyond the barrier is measured regardless of any re-crossing of the barrier). Cumulative Parisians are also called ParAsian options due to this aggregation feature like Asian options [3](Haber et al.,1999).

ParAsian options appear highly suitable for capturing the nature of financial distress. This is indicated in two aspects. First, for a window length equal to zero, the ParAsian option reduces to a standard barrier option; when the window is extended to (and beyond) maturity, the ParAsian option reduces to a standard European option. However, applying a down-and-out ParAsian put to modelling the default event is not to reach a compromise between Merton's European put (1974) and Black and Cox' barrier option (1976). In the previous subsection, we discussed the most likely choice of claimholders. It is plausible that bankruptcy can be negotiated away if there is only a temporary fluctuation in firm value. A ParAsian option can grasp this negotiation process in a way that a barrier option, with its one touch feature, cannot. Meanwhile, European options can also be treated as

[3]Strictly speaking, the ParAsian option applied in this paper is also a variant – the payment at the end of the window period is not zero, but the difference between the face value of the debt and the firm value, which can be regarded as a rebate. This will be addressed again when the boundary conditions are considered.

a special case of ParAsian options, when their durations are less than the window. We will apply the latter feature to short-term bonds.

Second, firm value at default can be decomposed into three parts: the amounts recovered by bondholders, the direct and indirect costs of financial distress. The direct costs are related to the legal and administrative procedures of bankruptcy, which creditors and debtors are both trying to avoid. The indirect costs are mainly related to losing business opportunity and the agency problem (over-investment behaviour by management on behalf of debtors)[4]. We will not distinguish these two kinds of cost, as in some cases the direct cost might not be incurred, in the other cases those fees are beyond modelling. Collectively, we treat them as the deployable amount at default.

Most models are not able to explain why the asset value at default is a stochastic variable between zero and the default threshold, if the dynamics of asset value are continuous. Zhou (2001) uses the jump-diffusion process to justify this. However, we think the indirect costs during the exclusivity period are the root cause. The exclusivity period given by the creditors can be modelled as the window in a ParAsian option. Creditors have long memories. They will not have the same tolerance for a repeated mistake by debtors. The firm will not be able to remain a going-concern, if its value has been low-ball cumulatively for a certain period. At the end of this period, the down-and-out feature is triggered while the closing firm value is stochastic.

It is noteworthy that François and Morellec (2002) also use a Parisian option to account for the impact of the U.S. bankruptcy procedures. However, our work takes a different path from theirs to model the problem. First, although recovery rate is also endogenously determined in their model, this is achieved by means of Nash Equilibrium to maximize the debtors' wealth, rather than a contingent claim model. Second, they only consider the

[4]Indirect costs also include claim-dilution and deviation from the absolute priority rule, which are accrued on creditors only. However, compared with the agency cost, they are on a much smaller scale.

Parisian option which needs consecutively breaching of the default barrier rather than a cumulative case (the ParAsian option) in our model. Third, the interest rate is assumed constant.

# 4 The Model

First, we adopt the standard assumptions on the dynamics for the asset value, $V$, geometric Brownian motion:

$$dV = (\mu\ V - \delta(t - t_c)cF_{t_d^-} - \delta(t - t_d)dV_{t_d^-})dt + \sigma_V V dZ_V, \tag{2}$$

where $\mu$ is the instantaneous expected rate of return on the firm, $\delta(\cdot)$ is the Dirac delta function, $c$ is coupon rate, paid semi-annually at $t_c$, and $d$ is dividend yield, paid annually at $t_d$, $\sigma_V$ is the volatility of the asset, and $dZ_V$ is a standard Gauss-Wiener process.

Next, the instantaneous risk-free interest rate is stochastic, following the CIR model:

$$dr = \kappa(\theta - r)dt + \sigma_r \sqrt{r} dZ_r, \tag{3}$$

where $\kappa$ is the speed of mean-reverting process of short rate; $\theta$ is the long-run mean level of interest rate; $\sigma_r$ is the volatility parameter; $dZ_r$ is also a standard Gauss-Wiener process. The instantaneous correlation coefficient between these two Gauss-Wiener processes is $\rho$:

$$dZ_V dZ_r = \rho dt.$$

A closed-form formula for risk-free bond with initial interest rate $r$ is:

$$P(r, T) = H(T)e^{-G(T)r},$$

with

$$H(T) = [\frac{2\gamma\ e^{(\kappa+\lambda+\gamma)T/2}}{(\kappa + \lambda + \gamma)(e^{\gamma\tau} - 1) + 2\gamma}]^{2\kappa\theta/\sigma^2},$$

$$G(T) = \frac{2(e^{\gamma\tau} - 1)}{(\kappa + \lambda + \gamma)(e^{\gamma\tau} - 1) + 2\gamma},$$

and

$$\gamma = \sqrt{(\kappa + \lambda)^2 + 2\sigma^2},$$

where $\lambda$ is the market price of interest rate risk (Cox, Ingersoll and Ross, (1985)).

Third, we assume that the underlying firm issues semi-annual coupon bonds of the same class at par, with face value $F$ and coupon rate $c$. This assumption is easily extended to several classes of bond. For instance, any subordinate class is valued as the difference between its risk-free value and a bear spread, which is comprised of a long and a short ParAsian puts. The long one has exercise price equal to total claim senior to this class, and the short with a higher exercise price by its own claim. In this sense, one class of bond in our model is a special case, as the long put is worth zero. Therefore, the recovery rates are not necessarily the same for all classes, as assumed in most literature. Each recovery rate depends on the claims in its own class, as well as those in others, and also on the time to maturity.

Collin-Dufresne and Golstein (2001) argue that firms mostly stick to their initial "target" financial policy. The debt level and asset value should grow at an equivalent speed. The leverage ratio can be at the discretion of the firm and is mean-reverting to an optimal level. With a constant boundary as in Longstaff and Schwartz (1995) and Zhou (2001)[5], $X$ will decrease exponentially over time. Accordingly, we define that the drift of debt level is partly dependent on the long-term mean level of interest rate $\theta$ (in a risk-neutral environment), and partly dependent on the firm's payout policy.[6] The payout ratio by the firm, $\zeta$, can be summarized as $\zeta V = d \cdot \text{equity} + c \cdot \text{debt}$. We write the expected face value of the debt at time $t$,

[5] $\phi$ is equal to zero, even though Zhou chooses an exponential form of default boundary.

[6] We assume initial leverage levels for rated firms are optimal. This is reasonable because firms of different rankings tend to adopt different optimal levels according to their individual characteristics. We do not expect that firms in distress or default are able to issue more bonds. Their debt levels are assumed constant. Collin-Dufresne and Goldstein (2001) assume a mean reverting leverage level of 40%. As we will show later, this is higher than investment grade firms but lower than speculative grade ones. Incorporating a mean reverting process will increase the credit spreads of the former firms but lower those of the latter firms, which makes the result more close to the reality, according to the findings in Huang and Huang (2003).

$K_t$ as:

$$K_t = F\ e^{(\theta-\zeta)t}. \tag{4}$$

Equation (4) in effect denotes the default threshold commonly observed in a safety covenant, which aims to protect creditors from further depreciation of the asset value (Black and Cox, 1976). This proposition can avoid the problem posed in Briys and de Varenne (1997) that the recovered amount is not related to asset level. Here, bondholders receive 100% of the asset value reached at default. However, violation of the absolute priority rule may also be applicable, as debtors are still able to grasp some positive value in the form of indirect cost incurred during the exclusivity period. What is more, this proposition is consistent to the definition of recovery rate. The write-down part is equal to the difference between face value of debt and asset.

Creditors are likely to prefer renegotiation to triggering the barrier feature, until they feel there is no more chance for the firm to emerge from the insolvency. Even in Chapter 11, the court will grant an exclusivity period for the debt-in-possession. According to Altman (1993), "over one-half of publicly owned Chapter 11 debtors emerge out of reorganization as a continuing entity". Either the creditors' patience or Chapter 11's grace can be denoted by the total time the firm spends under the default threshold, $\tau$. The dynamics of $\tau$ is

$$d\tau = \begin{cases} 0 & \text{if } V_t \geq F_t \quad \forall t, \\ dt & \text{else.} \end{cases}$$

# 5 Numerical Solution

Pricing default loss $D(V, r, t, \tau)$ is a two-factor modelling problem. We construct a portfolio with one bond long and a number $(1 - \frac{D_r}{P_r})$ of risk-free bond short (to eliminate interest rate risk) and a number $(D_v)$ of the underlying asset long (to eliminate asset risk). Applying Itô's lemma to equation (2) and (3) and no-arbitrage theory to the portfolio value, we obtain a partial differential equation:

$$D_t + \frac{1}{2} D_{VV} \sigma_V^2 V^2 + \frac{1}{2} D_{rr} \sigma_r^2 r + D_{Vr} \sigma_V V \sigma_r \sqrt{r} \rho$$
$$+ D_r \Big( \kappa(\theta - r) - \lambda \sigma_r \sqrt{r} \Big) + r D_v V + D_\tau = rD, \quad (5)$$
$$V(t_i^+) = V(t_i^-) - i,$$

where subscripts on $D$ denote partial derivatives, $i$ indicates the coupon or dividend payment.

### 5.1 Boundary Conditions

The features of ParAsian options are realized in the boundary conditions, which we need to solve the partial differential equation above. Before listing these conditions, first we summarize the main parameters of our ParAsian down-and-out put, as follows:

- term to maturity: $t = T$;
- window length: $\tau = \bar{T}$;
- barrier level/exercise price: $K_t = Fe^{(\theta - \zeta)t}$.

**At maturity ( terminal condition 1)** As long as the bond reaches its expiration, the ParAsian put has the same payoff as that of the normal vanilla put:

$$D(V, r, \tau, T) = Max[Fe^{(\theta-\zeta)t} - V_T, 0], \qquad \text{with } \tau < \bar{T}.$$

**At the end of window (terminal condition 2)** Whenever the ParAsian option is knocked out, creditors seize the firm and realise a loss of

$$D(V, r, \bar{T}, t) = K_t - V_t.$$

**Asset value is zero** When the firm is worthless, creditors are left with nothing to claim:

$$D(0, r, \tau, t) = K_t, \qquad \text{with } r < \infty.$$

**Asset value is infinite** In this circumstance, creditors are guaranteed their claims without suffering loss:

$$D(\infty, r, \tau, t) = 0.$$

**Interest rate is zero** When the interest rate is zero, according to the CIR model, the interest rate at the next instant is $\kappa\theta dt$ with certainty ($\sigma_r\sqrt{r}dZ_r = 0$). The PDE[7] reduces to

$$D_t + \frac{1}{2}D_{VV}\sigma_V^2 V^2 + D_r\kappa\theta + D_\tau = 0.$$

**Interest rate is infinite** When the interest rate is infinite, the present value of any asset becomes zero, so does the bond:

$$D(V, \infty, \tau, t) = 0$$

### 5.2 Credit Spread

Huang and Huang (2003) try to explain the discrepancies found in most analytical structural models. Under "empirically reasonable parameter choices", they generate consistent magnitude of credit risk premiums for different models, and conclude that up to 70% of yield spreads are due to systematic risk premium, such as business cycle, tax and illiquidity. Our result is not comparable with their findings, as recovery rate and default boundary level are important parameter choices for them. [8] However, we employ their data to calibrate our model with the same economic assumptions: the initial market value of debt $F = 100$[9] ; the initial asset level and the volatility of asset value $\sigma$ depend on firm's specific feature (e.g., credit rating)

[7] d'Halluin et. al (2001) prove that for the CIR model this PDE does not require boundary condition at $r = 0$, so long as the condition that $2\kappa\theta/\sigma_r^2 \geq 1$ is satisfied. In the next subsection, we will see that the LHS is around 23.32, with parameter choices in this model.

[8] Nevertheless, our result happens to corroborate their findings, as the credit spreads generated in our model are in a similar scale as theirs.

[9] Huang and Huang (2003) assume that bonds are issued at par and initial leverage ratio is the market value of debt to the market value of asset.

and is listed in Table 1[10]; the coupon rate $c = 8.162\%$, to guarantee the semi-annual coupon bond at par; the asset payout ratio $\zeta = 6\%$[11]; current interest rate level $r_0 = 8\%$; mean reverting speed $\kappa = 0.226$; long-term mean level of interest rate $\theta = 0.113$; volatility parameter $\sigma_r = 4.68\%$; the market price of risk $\lambda = 0$[12]; the correlation coefficient between asset value and interest rate $\rho = -0.25$. One parameter, not applicable to their work, is the window length. Altman (1993) finds that the reorganization experience on average is 21 months. Wagner (1996) in a similar study finds that the duration of default has a mean time of 26 months. In our model, $\bar{T} = 2$ years. The following results are obtained by solution of equation (5) using a four-dimension finite difference method.

#### 5.2.1 Credit Spread and Firm Rating

Practically, corporate bonds are quoted in terms of their durations and yields (or spreads over corresponding risk-free bonds). The most direct determinant for credit spread is the firm's credit rating. Among all the financial figures, we choose the leverage ratio as the proxy for the rating. Other related proxies can be the quasi-leverage ratio in Merton (1974) and the reverse leverage ratio $X$. In Table 1, we reproduce the inverse relation between rating and leverage in Huang and Huang (2003).

[10] Huang and Huang (2003) provide implied asset volatility for one-year, four-year and ten-year maturities. Here we assume for individual firm the asset volatility is constant, which we adopt the four-year-maturity data. The credit ratings are under Moody's system. Rating agencies normally do not report financial ratios for bonds rated Caa or lower. As we also study the distress and default cases, we assume their leverage ratios are 1 and 1.25, respectively; asset volatility are the same as B rated firm.

[11] This is only appropriate for firms whose assets are greater their debts. The asset payout ratio is determined by the function: $\zeta = c \cdot \ l + d \cdot (1 - l)$, where $d$ and $l$ are the dividend yield and the leverage ratio, respectively. Otherwise, the asset payout ratio depends on the specific debt ratio. For instance, at distress, $l = 1$ and $\zeta = c = 8.162\%$ at continuous level.

[12] Under these choices, $r < \frac{2\kappa\theta}{\gamma+\kappa+\lambda}$, the term structure of risk-free interest rate is upward-sloping (Cox, Ingersoll and Ross, 1985); $\lambda = 0$ is a routine assumption, which implies that liquidity premium is zero.

| | Aaa | Aa | A | Baa | Ba | B |
|---|---|---|---|---|---|---|
| Leverage ratio (%) | 13.1 | 21.2 | 32.0 | 43.3 | 53.5 | 65.7 |
| Asset Volatility (%) | 36.6 | 34.8 | 30.0 | 29.1 | 34.3 | 39.3 |

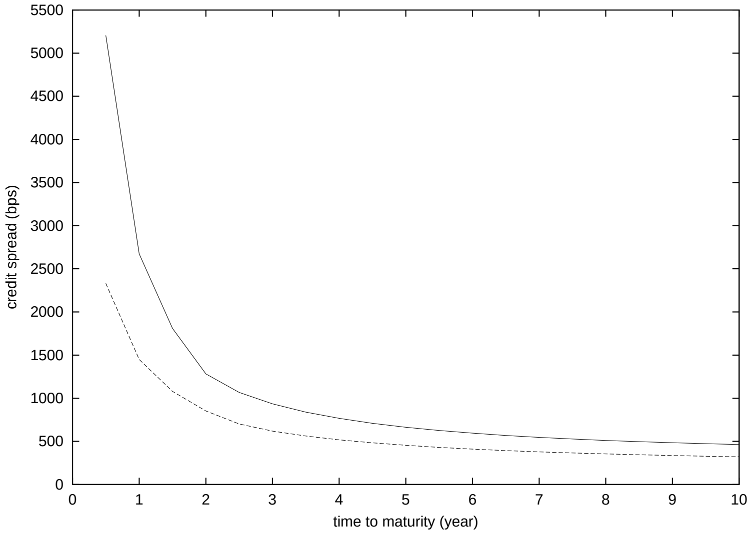
**Table 1: Leverage Ratio and Asset Volatility For Each Rating**

In general, our results are shaped consistently with Merton's (1974) and the empirical findings of Helwege and Turner (1999). That is , an inverted-shape term structure of credit spreads for a highly leveraged firm (debt ratio more than 1), humped-shaped for speculative grade bonds and normal shape for investment grade bonds. Figure 1 illustrates term structures of credit spreads for bonds whose leverage ratio is less than 1 and greater than or equal to 1.

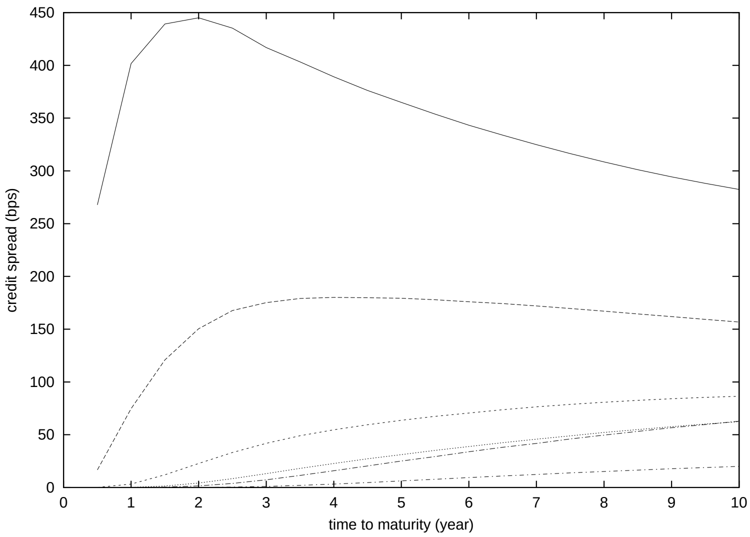
#### 5.2.2 Credit Spread and the Windows Period $\bar{\mathbf{T}}$

Figure 2 graphically depicts the relation between window length and credit spread. We can see that at the short end, the term structures are identical, even if their windows are different, so long as these windows are longer than maturity – bonds are effectively priced in European framework.

A second observation is that the longer a window period $\bar{T}$ is, the higher credit spreads are. During the window/exclusivity period, debtors keep convincing the creditors that the value of going concern is more than the liquidation value. This is also the purpose of the law to grant the debt-in-possession period. However, there is also a chance that debtors fail to do so, or even exacerbate the financial distress. Creditors *could have* downside protection in the form of the safety-covenant. In effect, they are giving up this protection for saving potential social welfare cost of liquidation (Hart, 1995). The longer the exclusivity period is, the more creditors forgo. Expecting this, they require higher yield to compensate at the beginning of the contract. If we call the positive relation between window length and credit spread the "window effect", we notice this effect is more apparent at the long end. This is readily appreciated in that for long-term bonds, the more creditors are tolerant, the higher chance that firm can emerge from

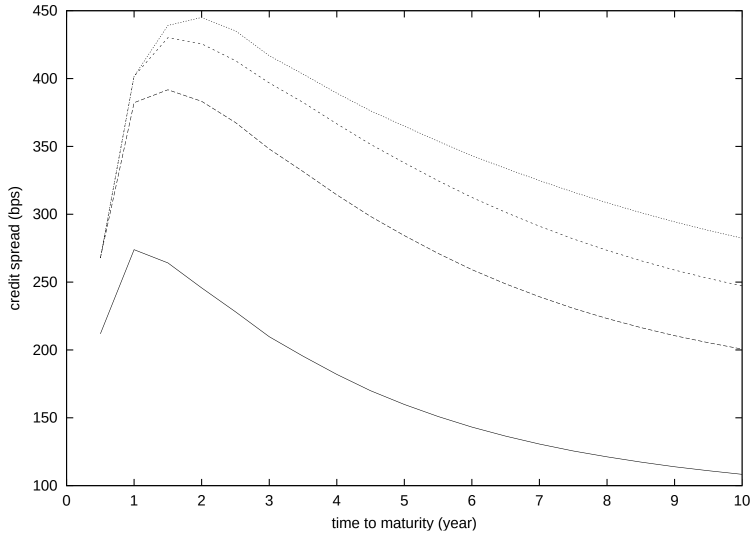



(a) **Leverage ratios greater than or equal to 1** From the top to the bottom are term structures of credit spreads for defaulted and distress bonds, respectively.

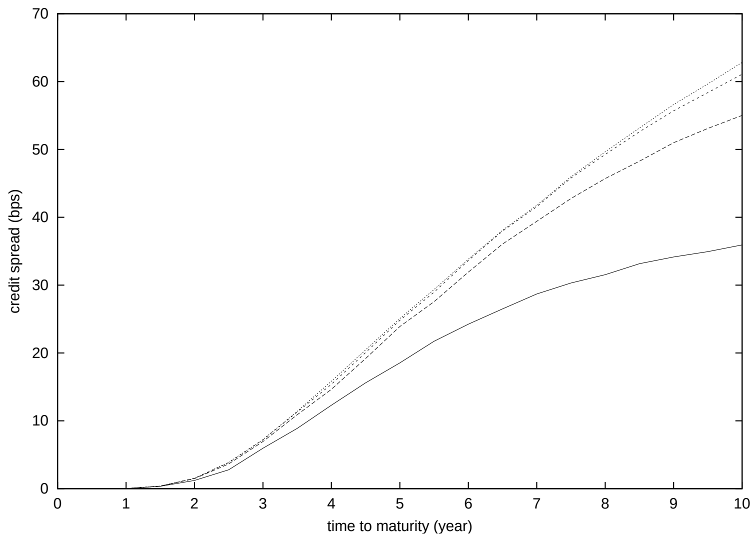



(b) **Leverage ratios less than 1** From the top to the bottom are term structures of credit spreads for B, Ba, Baa, A, Aa and Aaa rated bonds in turn.

Figure 1: **Term Structure of Credit Spreads**




(a) **a B rated bond** From the top to the bottom, the window length is equal to two years, one year, six months and two months respectively.




(b) **an Aa rated bond** From the top to the bottom, the window length is equal to two years, one year, six months and two months respectively.

Figure 2: **Term Structure of Credit Spreads and Length of Window**

reorganization.

Finally, we find the growth rate of credit spread with the window length is not linear. The window effect is less significant beyond a length of one year – the increment of credit spread between a one-year-window and a two-year-window is much less than that between a one-month-window and six-month-window. This is because however long the window is, all term structures of credit spreads are bounded by the one with the window equal to ten years, which is the longest maturity in our sample. To illustrate, in Figure 3, we plot the credit spreads for different window lengths but the same maturity (four years), for both Aa rated and B rated bonds. The starting points are virtually the standard barrier option case, since the window length is zero; the ending points are the European option case, since the window length is equal to maturity. The increment of credit spread presents a concave style. It is beyond the scope of this paper to decide how long the window should be before further increase has insignificant marginal effect. It is the characteristics of the firm that determine the window effect. For instance, for higher rated bonds, such as a double-A in the graph, when the window period exceeds one year, the underlying ParAsian differs little from a European, which means the default almost can be avoided completely. However, for a lower rated bond, the default is inevitable until the window length is much longer.

#### 5.2.3 Credit Spread and Interest Rate Model

In pricing a risky bond with stochastic interest rate, the Vasicek model (1974) is favoured over the CIR model (1985) in this line of literature. There are three main reasons. First, the square-root process for interest rate in the CIR model makes analytical solution very difficult, if not impossible. Second, in terms of numerical solution, Monte Carlo simulation might be the most intuitively appealing, but depends a lot on the distribution assumption. The Vasicek model implies a normal distribution for the instantaneous interest rate, while the CIR model implies a non-central chi-square distribution. Consequently, the latter requires more complicated discretisation

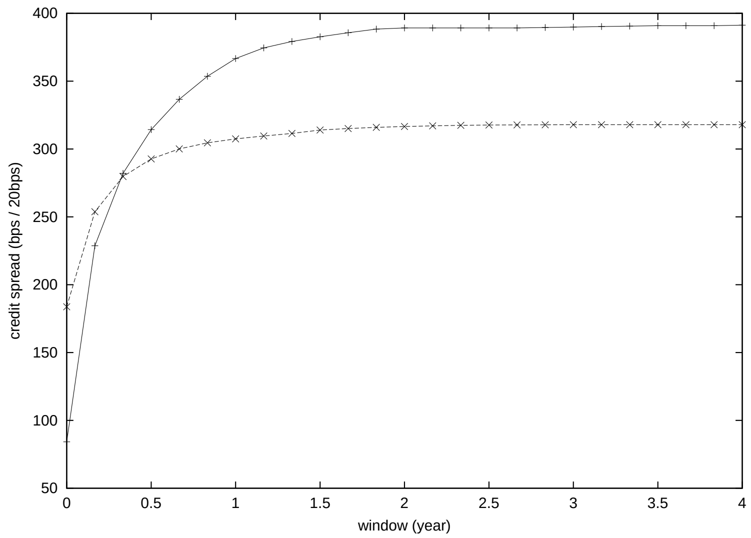



Figure 3: **Credit Spread and Window Length** An Aa rated bond is plot with x line points. Unit of credit spread is magnified by ten times for comparison sake. A B rated bond is plot with + line points. The increment of window length is two months.

method. Third, when adopting the Vasicek model, modellers can assert that the real possibility of negative interest rate is very low. However, when we choose the same parameters, we find the CIR model always generates higher yields for risk-free bonds. The difference widens with increasing time to maturity – under current choices, it can be up to 72 basis points. This result is illustrated in Figure 4. One possible explanation is the potential for negative results from the Vasicek model. Moreover, the corresponding partial differential equations are entirely different, with different interest rate models. What we emphasise here is that credit spreads calculated with one are not directly comparable with those from the other.

### 5.2.4 Credit Spread and CIR Model Parameters

*the Interest Rate Level* $r_0$

Duffee (1998) finds a negative relation between the yields of risk-free bonds and the yield spreads of risky bonds. This result is widely documented in most structural models, and is also observed in our model. Before dis-

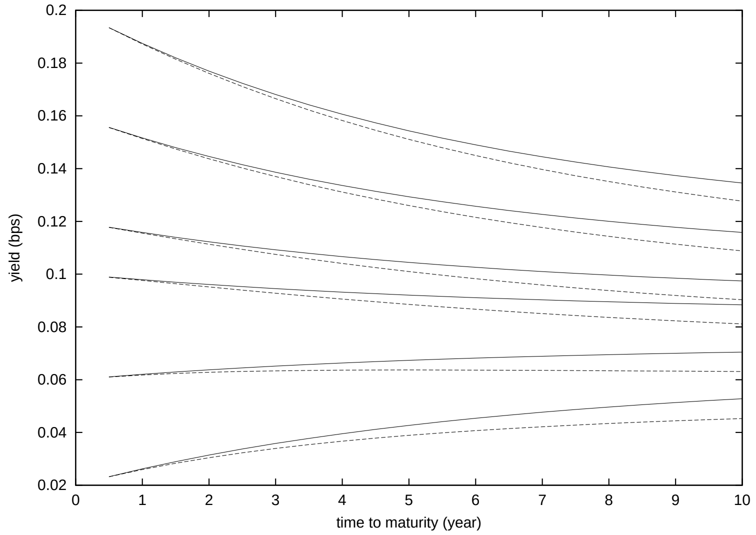
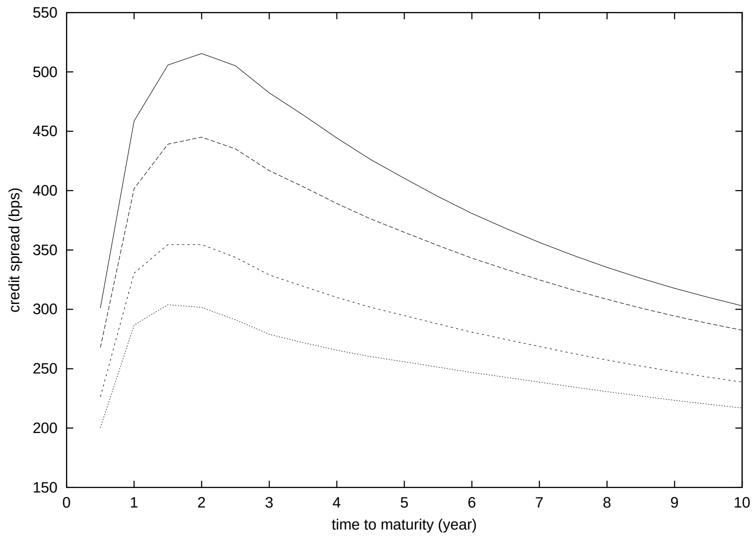
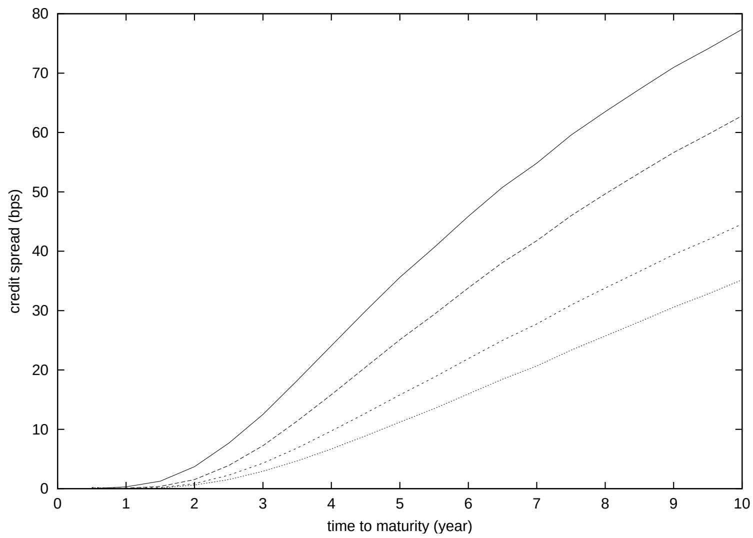



Figure 4: **the CIR Model vs. the Vasicek Model** Solid lines are the yield curves derived by the CIR model; Dash lines are the yield curves derived by the Vasicek Model. For each line style, interest rates are 0.02, 0.06, 0.1, 0.12, 0.16 and 0.2, from the bottom to the top.

cussing their difference – yield spread, we compare yield curves first. Yields of both risk-free bonds and risky bonds increase with the short rate $r_0$, though with different speeds – the short rate level increases the fastest, followed by its corresponding yield of the risk-free bond, with the yield of the risky bond ranked third. The reason is that, by definition, yield is the effective interest rate paid on the bond, which is averaged out over the life. Therefore, we expect the yield curve to have less steep slopes than for the current interest rate level, $r_0$. Moreover, from equation (1), we can tell that the short rate has an additional effect on the risky bond through the value of the put option. Debtors have a long put, which can protect them in financial distress. When the interest rate is higher, the drift of the firm's value will be higher in a risk-neutral world. There will be less chance for the put to be in-the-money. This offsetting effect is especially pronounced for junk bonds, and diminishes with advance in rating. Therefore, the yield of the risk-free bond (which is lower) increases at a higher speed than that of risky bond, (whilst at almost the same speed as a high-rated bond such as a triple-A).




(a) **a B Rated Bond Case** From the top to the bottom, interest rates are 0.04, 0.08, 0.12 and 0.16, respectively.




(b) **an Aa Rated Bond Case** From the top to the bottom, interest rates are 0.04, 0.08, 0.12 and 0.16, respectively.

Figure 5: **Term Structure of Credit Spreads and Interest Rate Level**

Accordingly, their difference, the yield spread, narrows. Comparative effects for junk bond and high grade bond are shown in Figure 5.

The reasoning above is also applied to the long-term mean level of interest rate, $\theta$.

*the Variance of Interest Rate* $\sigma_r^2 r$

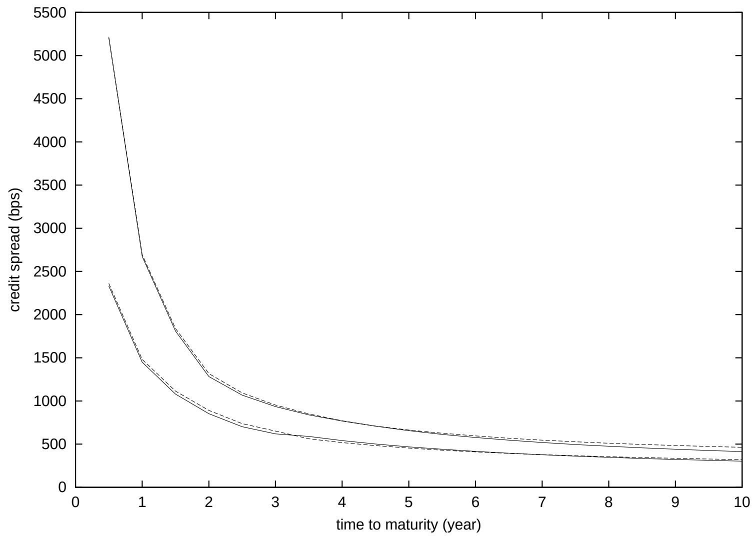
Kim, Ramaswamy & Sundaresan (1993) find that default risk is not sensitive to interest rate risk. This result is also confirmed in our model. However, we have two other different findings in respect with the volatility of interest rate. First, Kim et al. (1993) conclude that incorporating a stochastic term structure setting is able to explain a higher risk premium for risky bonds. In contrast, we find that a stochastic interest rate only generates lower credit spreads. This phenomenon is well explained in Huang and Huang (2003). When the interest rate is stochastic, the long-term mean level of interest rate is *always* higher under the risk-neutral measure than that under the real probability measure, as long as the pure expectation hypothesis holds. However, it is the same in both measures when interest rate is constant. We have seen in the previous subsection that a higher interest rate will decrease the credit spread, *ceteris paribus.* A stochastic interest rate process reduces the resulting credit spreads. In Figure 6, we contrast the results from one-factor and two-factor models, but find "conflict" results. When the leverage ratio is great than or equal to 1, one-factor models can explain more credit spread, as expected before, whilst for rated firms, the figure substantiates the remark of Kim et al.(1993). However, here the mechanism is different. In last section, we presented an upward barrier, which hinges on the long-term mean level of interest rate, $\theta = 11.3\%$. When the interest rate is constant, this drift reduces to 8%. Stochastic process of interest rate still produces lower credit spread, as we see in the constant barrier case (Figure 6(a)). Nevertheless, at the same time the speedy increase in the barrier level increases the credit spread, which more than offset the previous effect. Additionally, in the upper sub-figure, we can see that the differences are amplifying along the time. This is because in our model, the yield curve of the risk-free rate keeps its normal shape. As the variance of the changes in interest rates is proportional to the level of interest rate, it increases over the time. The interest rate is more volatile at the long end.

### 5.2.5 Credit Spread and Volatility of Asset Value $\sigma_{\mathbf{V}}$

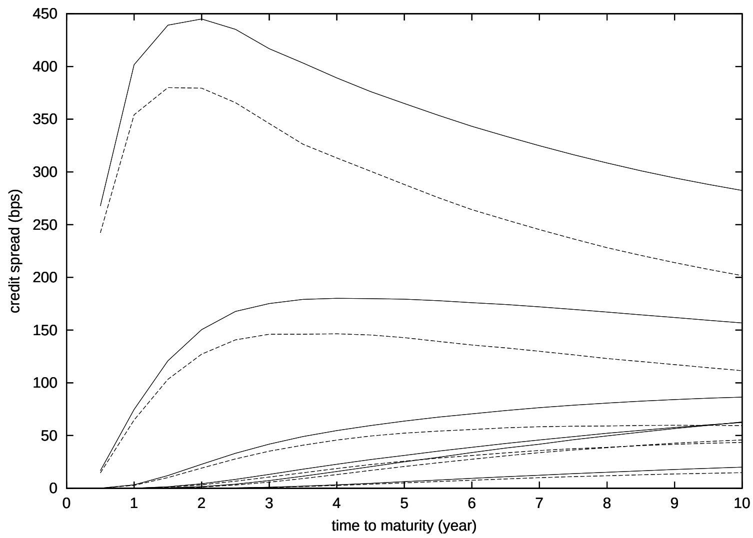
Higher rating firms may not be safer ones. As we can see from Table 1, the trend of implied volatility presents a "V" shape when rating hikes, rather than monotonically decreasing. However, we still believe that higher rated firms have lower credit spreads. This is a result of synergy, the volatility of asset value $\sigma_V$, the leverage ratio $l$ and others. However, a safer firm surely has a more secured bond. In Figure 7, we plot the positive relation between the term structure of credit spreads and asset volatility $\sigma_V$ by controlling the rating. As expected, the volatility of asset value is much more influential than the volatility of interest rate.

### 5.2.6 Credit Spread and the Correlation Coefficient $\rho$

Figure 8 plots the relation between credit spread and the correlation coefficient between interest rate and asset value. Consistent with the study of Longstaff and Schwartz (1995), we also find that the credit spread increases with the correlation coefficient $\rho$. However, they think that the effect of correlation coefficient on the credit spread is significant, because a negative correlation coefficient decreases total variance such that a lower probability of default will result. At this point, our conclusion is opposite. When the correlation coefficient is negative, the variances of interest rate and asset value, $\sigma_r^2 r$ and $\sigma_V^2$ offset each other. On the one hand, the firm is less risky and reduces the credit spread. On the other hand, the interest rate, $r$ is less stochastic, as we discussed before, and drives up the credit spread. In general, the first effect dominates the second, but also fairly balanced by the second. We find the effect of correlation coefficient is insignificant. Longstaff and Schwartz (1995) suggest that correlation can explain discrepancies among similarly rated bonds but in different industries. We believe that each industry has its own, unique features which affect the




(a) **Leverage ratios greater than or equal to 1** From the top to the bottom are term structures of credit spreads for defaulted and distress bonds.




(b) **Leverage ratios less than 1** From the top to the bottom are term structures of credit spreads for Aaa, Aa, A, Baa, Ba and B rated bonds in turn.

Figure 6: **Term Structure of Credit Spreads with Constant Interest Rates and Stochastic Interest Rates** Solid lines are results from one-factor ParAsian options; dash lines are results from two-factor ParAsian options and the CIR model.

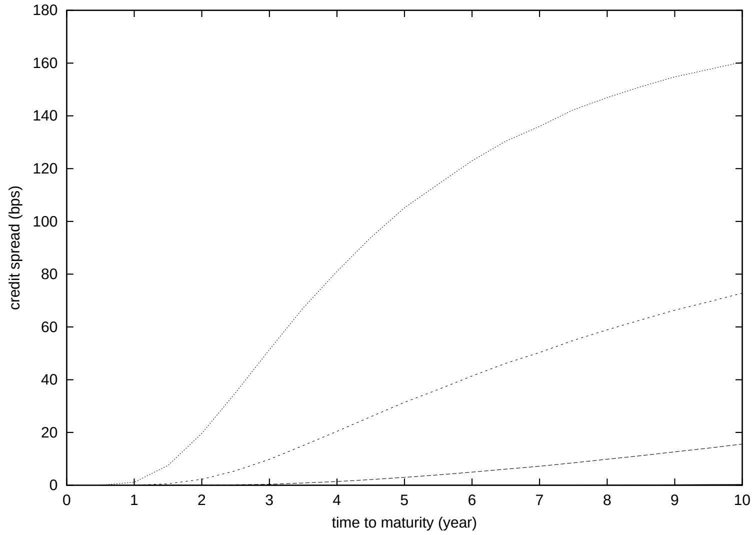



Figure 7: **Term Structure of Credit Spreads and Volatility of Asset Value, in an Aa Rated Bond Case** From the top to the bottom, the volatilities of asset value are 0.45, 0.36 and 0.27

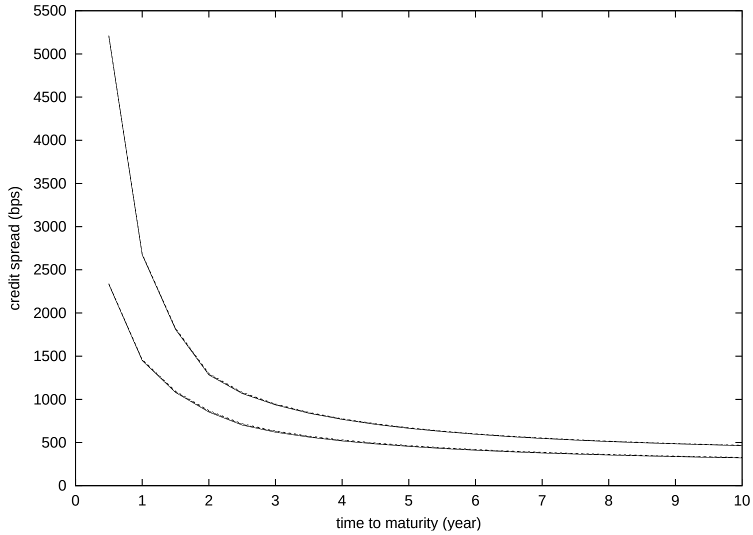
credit spreads. Other than the correlation with interest rate, those features include but are not limited to the leverage ratio, industry growth, industry concentration and physical asset obsolescence etc. (Izvorski,1997).

### 5.3 Duration

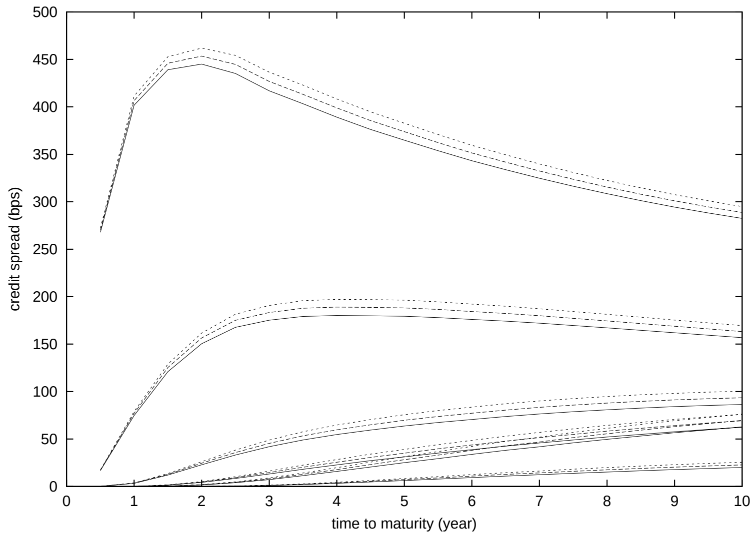
It is well documented that defaultable bonds have shorter durations[13] than otherwise risk free bonds (Chance, 1990, Longstaff and Schwartz, 1995). Figure 9 illustrates that the duration decreases with the increase of default risk.[14] According to Longstaff and Schwartz (1995), the reduced sensitivity of price to the change of interest rate is due to the offsetting effects of the interest rate on the bond, which we have explained in section 5.2.4. How-

[13]Here, we adopt the most common definition of duration, the negative percentage change in bond price to the change in its own yield. Another definition, for instance adopted in Acharya and Carpenter (2002), is a ratio to the change in a corresponding T-bond yield.

[14]This conclusion seems paradoxical, as duration is a proxy for interest rate risk. However, interest rate risk includes yield curve risk, reinvestment risk and others. Duration only represents the first one. Reinvestment risk is surely higher for low-ranked bonds, which is beyond the scope of this work.




(a) **Leverage ratios greater than or equal to 1** From the top to the bottom are term structures of credit spreads for defaulted and distress bonds. For each rating, correlation coefficient changes from 0.25 to 0 and then -0.25 downwards.




(b) **Leverage ratios less than 1** From the top to the bottom are term structures of credit spreads for B, Ba, Baa, A, Aa and Aaa rated bonds in turn. For each rating, correlation coefficient changes from 0.25 to 0 and then -0.25 downwards.

Figure 8: **Term Structure of Credit Spreads and Correlation Coefficient between Asset Value and Interest Rate.**

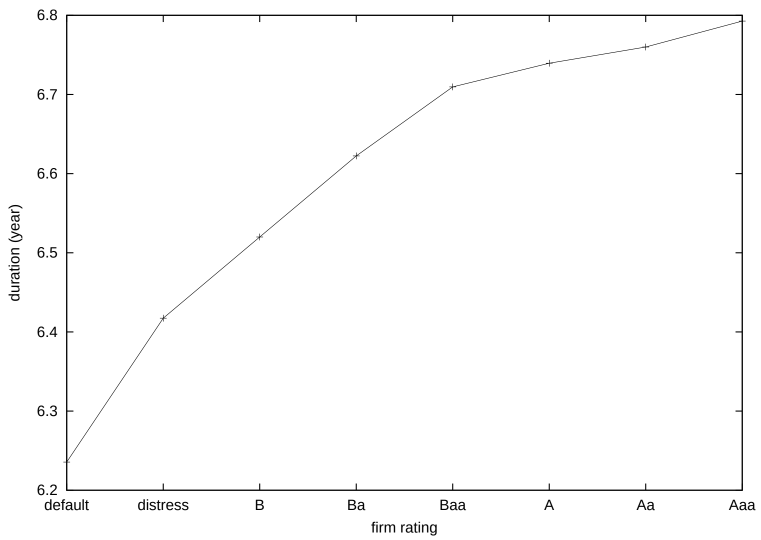



Figure 9: **Firm Ratings and Duration**

ever, we do not find any negative duration even for default bond, though observed in Longstaff and Schwartz (1995). According to Acharya and Carpenter (2002), an arbitrary low recovery rate will increase the value of very risky bond when default is effectively avoided. This situation sometimes is caused by an increase in interest rate. Consequently, a negative duration is observed. However, in our model, recovery rate is endogenously determined by the process of firm value. Creditors always lose money, as the firm value they get is always no more than default boundary. This is of a similar setting as that in Acharya and Carpenter's, though they model an endogenous bankruptcy process. Fooladi et al. (1997) study the duration for bonds with default risk. Our result is in comparable scale of theirs.

It is worth mentioning a scenario study in the work of Fooladi et al.(1997). They also acknowledge the possible delay between default and its settlement. They introduce "Doomsday" and "Phoenix" scenarios corresponding to a failure and a succuss in bankruptcy negotiation. The window length is two years, as well. They find durations shorten and lengthen in "Doomsday" and "Phoenix", respectively. However, our result is patterned that the duration decreases with the window length. Figure 10 plots this relation. In addition, we can see the duration of a safer bond decreases more slowly than

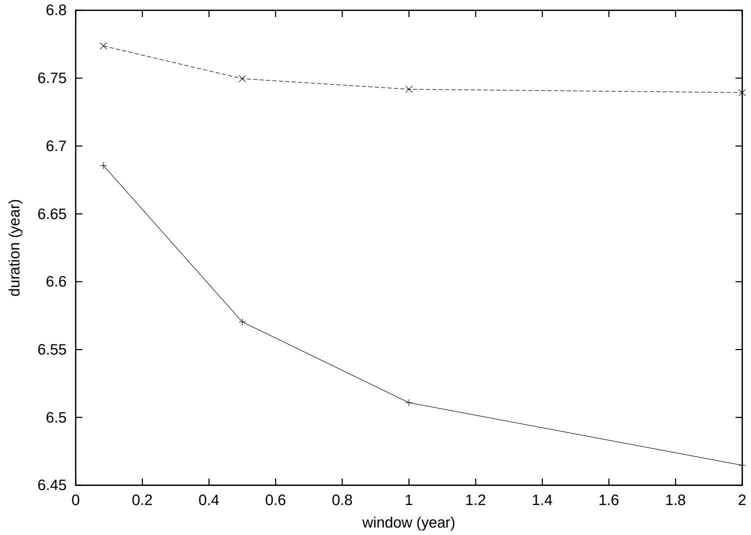



Figure 10: **Credit Spread and Window Length** An Aa rated bond is plot with x line points. A B rated bond is plot with + line points.

that of a riskier one, the fashion which is corresponding to that in Figure 3. This is consistent with the above finding that the duration decreases with risk, as the window length indicates degrees of risk undertaken by creditors. The discrepancy can be explained by the following two reasons. First, due to introducing a delay, Fooladi et al (1997) extend the bond maturity for an extra two-year. In effect, they are comparing two twelve-year bonds with a ten-year bond. We also carry out a comparison between a twelve-year bond and a ten-year bond, but only find shortened duration like the case in "Doomsday". Another reason is that they assume a Parisian case with a continuous window.

## 6 Conclusion

In this paper, we observe the Merton (1974) framework by pricing risky bonds with ParAsian options and the CIR interest rate model. Different bankruptcy procedures and possible (not assured) violation of absolute priority rule can be accounted for. The resulting term structures of credit spreads have shapes consistent with those in the literature. Additionally, we find that the window choice has enough of an influence to matter, in terms of both the level and shape of the term structure of credit spreads. Interest rates are assumed to follow a stochastic process, however, we find that the variance of the interest rate does not carry much weight in explaining more credit spread. Contrary to other structural models with stochastic interest rates, the level of credit spreads decreases with the volatility of interest rate, and so therefore, the change in the correlation coefficient between asset value and interest rate is not influential. The duration of risky bonds decrease not only with the default risk, but also with the window length.

One important property of our model is that the recovery rate is endogenously determined. A significant difference between structural models and reduced-form models is that the former are based on the characteristics of the underlying firm rather than on accounting data in the market. This is also the reason that the structural models can only generate part of the credit spreads observed in the market, other than the accounting noise and systematic risk premium. We believe that systematic (interest rate) and unsystematic (asset value) factors jointly determine the recovery rate, which is stochastic over time. Assuming a constant exogenous recovery rate, to some extent is a "reduced-form-kind" deviation from the original Merton (1974) framework. Although with a ParAsian option it is difficult to derive an analytical solution for the recovery rate, a possible method is to use Monte Carlo simulation to calculate the probability and then deduce the recovery rate from the put value.

Another possible line for further research is related to the barrier level. In our model, an upward barrier level is set up. It is deterministic, being derived from the (constant) long-term mean level of both interest and coupon rates. The instantaneous interest rate could take its place and generate a stochastic barrier which is widely believed to be more realistic in practice.

## References

[1] Acharya, Viral V. and Jennifer N. Carpenter, 2002 “Corporate Bond Vauation and Hedging with Stochastic Interest Rates and Endogenous Bankruptcy”, *Review of Financial Studies*, Vol. 15, 1355-1383.

[2] Altman, Edward I., 1993 “Evaluating the Chapter 11 Bankruptcy-Reorganization Process” in Edward I. Altman edited, 2002, *Bankruptcy, Credit Risk, and High Yield Junk Bonds*, Blackwells.

[3] Altman, Edward I. and Vellore M. Kishore, 1996, “Almost Everything You Wanted to Know About Recoveries on Defaulted Bonds”, *Financial Analysts Journal* November/December,57-64.

[4] Black, Fisher and John Cox, 1976, “On the Pricing of Corporate Debt: The Risk Structure of Interest Rates”, *Journal of Finance* 29, May, 449-70.

[5] Briys, Eric and François de Varenne, 1997, “Valuing Risky Fixed Rate Debt: An Extension”, *Journal of Financial & Quantitative Analysis* June, 239-49.

[6] Chance, Don M., 1990, “Default Risk and the Duration of Zero-coupon Bonds” *The Journal of Finance* 45, 265-274.

[7] Collin-Dufresne, Pierre and Robert S. Goldstein, 2001, “Do Credit Spreads Reflect Stationary Leverage Ratios?” *The Journal of Finance* October, 1929-1958.

[8] Cossin, Didier and Hugues Pirotte,2000, *Advanced Credit Risk Anaylsis* John Wiley & Sons, Ltd.

[9] Cox, John. C., Jonathan. E. Ingersoll and Stephen. A. Ross, 1985 “A Theory of the Term Structure of Interest Rates”, *Econometrica*53, 385-407.

[10] d'Halluin, Y, P.A. Forsyth, K.R. Vetzal, and G. Labahn, 2001, "A Numerical PDE Approach For Pricing Callable Bonds", Working Paper, University of Waterloo.

[11] Duffee, Gregory R., 1998, "The Relation Between Treasury Yields and Corporate Bond Yield Spreads",*Journal of Finance* December, 2225-41.

[12] Eom, Yong Ho, Jean Helwege and Jing-Zhi Huang, 2000, "Structural Models of Corporate Bond Pricing: An Empirical Analysis", Working Paper, Pennsylvania State University.

[13] Fooladi, Iraj J., Gordon S. Roberts and Frank Skinner, 1997, "Duration for Bonds with Default Risk", *Journal of Banking and Finance* Vol. 21, 1-16.

[14] François, Pascal and Erwan Morellec, 2002, "Capital Structure and Asset Prices: Some Effects of Bankruptcy Procedures", *Journal of Business* forthcoming.

[15] Franks, Julian R. and Walter N. Torous, 1994, "A Comparison of Financial Recontracting in Distressed Exchanges and Chapter 11 Reorganizations",*Journal of Financial Economics*, 35, 349-370.

[16] Fridson, Martin S., M. Christopher Garman and Kathryn Okashima, 2000 "Recovery Rates: The Search for Meaning", 13. March, Merrill Lynch & Co., Global Securities Research & Economics Group, High Yield Strategy.

[17] Gemmill, Gordon, 2002, "Testing Merton's Model for Credit Spreads on Zero-Coupon Bonds", Working Paper, City University Business School.

[18] Guha, Rajiv, 2002,"Recovery of Face Value at Default: Theory and Empirical Evidence", Working Paper, London Business School.

[19] Haber, Richard J., Philipp J. Schönbucher and Paul Wilmott, 1999, "Pricing Parisian Options", *Journal of Derivatives* Spring, 71-79.

[20] Hart, Oliver, 1995, *Firms Contracts and Financial Structure* Oxford University Press.

[21] Helwege, Jean, 1999, "How Long Do Junk Bonds Spend in Default?", *Journal of Finance*, February, 341-357.

[22] Helwege, Jean and M.Turner, 1999, "The Slope of the Credit Yield Curve for Speculative-grade Issuers", *Journal of Finance*, October,1869-1884.

[23] Huang, Jing-zhi and Ming Huang, 2003, "How Much of the Corporate - Treasury Yield Spread is Due to Credit Risk?", Working Paper, Stanford University.

[24] Izvorski, Ivailo, 1997 "Recovery Ratios and Survival Times for Corporate Bonds", International Monetary Fund Working Paper, Research Department, WP 97/84, July.

[25] Jones, E. Philip, Scott P. Mason and Eric Rosenfeld, 1984, "Contingent Claims Analysis of Corporate Capital Structures: An Empirical Investigation." *Journal of Finance*, July, 611-627.

[26] Kim, In Joon, Krishna Ramaswamy and Suresh Sundaresan, 1993, "Does Default Risk in Coupons Affect the Valuation of Corporate Bonds?: A Contingent Claims Model", *Financial Management* Autumn, 117-131.

[27] Longstaff, Francis A. and Eduardo S. Schwartz, 1995, "A Simple Approach to Valuing Risky Fixed and Floating Rate Debt", *Journal of Finance* July, 789-819.

[28] Merton, Robert C., 1974, "On the Pricing of Corporate Debt: The Risk Structure of Interest Rates", *Journal of Finance* 29, May, 449-70.

[29] Saá-Requejo, Jesús and Pedro Santa-Clara, 1999, "Bond Pricing with Default Risk", Working Paper, the Anderson Graduate School of Management at UCLA.

[30] Smith, Gordon D., 1978 *Numerical Solution of Partial Differential Equations*, 2nd edition, Oxford University Press.

[31] Vasicek, Oldrich, 1977, "An Equilibrium Characterization of the Term Structure", *Journal of Financial Economics* 5, 177-188.

[32] Wagner, Herbert S. III., 1996 "The Pricing of Bonds in Bankruptcy and Financial Restructuring", *the Journal of Fixed Income*, June, 40-47.

[33] Wei, David Guoming and Dajiang Guo, 1997, "Pricing Risky Debt: An Empirical Comparison of the Longstaff and Schwartz and Merton Models", *Journal of Fixed Income*, September, 9-28.

[34] Zhou, Chungsheng, 2001, "The Term Structure of Credit Spreads With Jump Risk", *Journal of Banking & Finance* 25, 2015-2040.