Pricing the Risks of default

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Abstract

We survey the developments in the finance literature with respect to pricing default. A broad description of the issues is follwed by a detailed summary of the main points and features of the models proposed. Both option theoretic and hazard rate models are presented and critically reviewed.

1 Introduction

Default may generally be defined as the failure of a counterparty to deliver in full the agreed upon quid pro quo as per contract. Examples cover a range of activities like the failure of timely loan repayments, airline bankruptcies and their related inability to honor their ticket sales, or baby sitters not showing up. Primarily the stronger party collects quid pro quo first and the weaker waits, so we buy the airline ticket or deliver labor on jobs and then wait for the ride or wages as the case may be. Sometimes this may be reversed as when banks provide individuals housing loans and then wait for their repayments for 30 years. The timing of the default is in many cases after the delivery of the quid pro quo, but it can be before as when the baby sitter doesn't show. The cost of the default, on the counterparty, goes beyond the loss of the quid pro quo and includes an assessment of the inconvenience of the absence of delivery that was being relied upon: For otherwise the baby sitter not showing up is no default as then he is not paid for services he did not render.

Associated with each default is the *default time* that we take to be the time at which it is announced that the quid pro quo would not be paid in full. But as there is in general some partial payment, we also have to assess the risk of the default magnitude. This magnitude can be difficult to assess as it amounts to ascertaining what it may cost the counterparty to restore the status quo of the contract in either dollar or utility terms. For example, one may get baby sitting services at short notice for a relatively high rate and the cost of the default is this higher value and not the value of the original fee. In other situations it is impossible to restore the status quo, as for example when you promise to deliver an original painting by Renoir and find it destroyed in transport, or an e-commerce site delivers a three year old their Christmas present two weeks too late. We then have to resort to mechanisms for compensating monetarily for a wide array of intangible welfare losses.

Our focus here will be restricted to financial contracts where all deliveries and payments are sums of money, albeit at different times and under different contingencies. The *default magnitude* is then the shortfall in the payment delivered relative to the promised contractual payments. Financial claims can be quite complicated even though they deal with just money transfers as they may involve multiple transfers among many parties under a variety of contingencies. One only has to consider the myriad of life insurance and retirement plans to appreciate the level of the underlying complexity. To this we may add convertible bonds that are callable and putable as well as swaps and options to swap or swaptions.

The success of modern finance in dealing with this maze of securities and associated eventualities lies in the reduction of these complicated claim structures to a portfolio of simpler claims. The simpler claims may be described as unit face state contingent bonds that pay off a dollar when a particular eventuality is realized. We shall concentrate our efforts on assessing these simpler securities, commenting in passing on how they are employed in practice to manage the structure of the more complex claims of interest. The events of interest to us here, are the default time and its magnitude.

Much of the literature and the models we shall survey are concerned with pricing these simpler securities. After a general discussion of the properties of these securities in section 2 we shall consider the various models that have been proposed to price them. Section 3 will consider option-theoretic models that have as their cornerstone the property of defining the precise default event while section 4 is devoted to what are now called hazard rate models that focus their attention directly on event probabilities and their sufficient statistics, leaving the precise event undefined. We shall cover in section 3, the models of Merton [14], Longstaff-Schwartz [9], and an application of the model by Madan, Carr and Chang [11]. Section 4 surveys the models by Duffie and Singleton [3], Madan and Unal 1 [12], Jarrow, Lando and Turnbull [8], and Madan and Unal 2 [13].

2 State Contingent Zero Coupon Bonds

For each name that is an individual or a legal entity empowered to write financial contracts in our economies with similar counterparties we are interested in the following events. The first is that name A defaults on obligations at some future time T and the second is that there is partial recovery in default to the extent of y, i.e. $(1-y)$ times the promised obligation is lost in default at T.

Events by themselves are not very interesting when they are potential future possibilities as they may happen or not and we may describe the outcomes but there is little more to be said. We may live or die: So what is the issue? What concerns us now about these future events is their probability conditional on current information. We denote by $\pi(t, A, T)$ the probability at time t that name A defaults on a dollar obligation at time T. This function assesses for us the real timing risk of default.

Furthermore, as there may be partial recovery in default, we are also interested in $f(y; t, A, T)$ the conditional probability that the recovery level is y in this time T default. The default magnitude is assessed by this conditional probability distribution.

We may associate with these two events, two securities that are zero coupon state contingent bonds paying out a dollar if the event of default occurs while the other pays out a dollar just if the default occurs and the recovery level is y. Just as the events have probabilities that we may determine today, the securities have prices that we may determine today. We denote by $p(t, A, T)$ the price at time t, of the security paying a dollar if name A defaults at time T. We also have the prices of the securities paying a dollar if there is default at T and the recovery level is y that we denote by $h(y; t, A, T)$. It is clear that

$$p(t, A, T) = \int_0^1 h(y; t, A, T) dy$$

and hence the counterpart in the world of prices to the conditional probability distribution $f(y; t.A.T)$ is

$$g(y; t, A, T) = \frac{h(y; t, A, T)}{p(t, A, T)}$$

which is the number of default timing bonds one must sell to finance the purchase of bonds at recovery level y. After all one may finance all the recovery level bonds for all recovery levels by one timing bond. The timing bond is the appropriate numeraire asset for the recovery level bonds.

We have associated with our events of default timing and magnitude, their conditional probabilities and current market prices. A natural question is the relation between these numbers. Should not the market price for a state contingent zero coupon bond just reflect its probability? More precisely, will the ratio of market prices equal the ratio of probabilities. It is one of the founding principles of modern finance to observe that the answer to this question is no. The difference is termed a risk premium. The rationale for this terminology may be briefly explained.

Consider holding the zero coupon state contingent bond, contingent on some event X for which the outcome will be known in a year. There are two payouts, 1 if the event happens and zero otherwise and hence the expected cash flow is just the probability of the event or $P(X)$. Consider now the two claims that pay a dollar if X occurs, while the other pays a dollar if X does not occur. The sum of their prices must be the value of a bond or say $\frac{1}{1+r}$ where r is the annual compounding spot interest rate for a one year term. The prices are also non-negative and when multiplied by $(1+r)$ they sum to unity and behave like probabilities. These are called risk neutral probabilities and for the event X occurring are denoted by $Q(X)$. The price, say $V(X)$, of the claim paying a dollar if X occurs is then

$$V(X) = \frac{Q(X)}{1+r}.$$

The gross rate of return on the claim, $1+R(X)$,is the ratio of the expected cash flow to the price and is

$$1+R(X) = \frac{P(X)}{Q(X)}(1+r)$$

whereby the excess return or risk premium is precisely the ratio of probability to risk neutral probability, $\frac{P(X)}{Q(X)}$.

Price and probability ratios are equal when events have no risk premia. Events for which probability exceeds risk neutral probability have a positive risk premium and vice versa for events with the higher risk neutral probability. If events have a positive risk premium then their complements have a negative risk premium. It is useful to consider some examples. Consider the term security that pays out in case of death, like life insurance. The probability of death is low relative to its price and the security has a negative risk premium, which is why it makes sense to short this security and sell life insurance. The complementary security that pays out on living has a positive risk premium with a high probability of living and a low a price for the claim that pays out in this event. Hence we wish to be long this security and this explains the absence of markets selling claims that pay out on living as opposed to death. In fact insurance companies are long life as they collect monies during life and short death by paying out in this eventuality. Individuals go long on the death security and short the life security with a view to extending the consequences of life in death.

With a view to appreciating these differences in the context of default we present a short table contrasting probabilities, prices and implied risk premia in basis points on the event of no default for n years by firms in the rating categories AAA and B. The probabilities are obtained from a study of S&P default rates reported by Lucas [10]. The prices are inferred from rates of return earned on managing portfolios of bonds by rating category as reported in Altman [1].

Probability, Prices and Risk Premia for No Default on Bonds by Rating and Maturity in Basis Points						
		Maturity				
Rating	Item	1	3	5	8	10
AAA	Probability	1	1	1	.9922	.9861
AAA	Price	.9955	.9835	.9656	.9228	.8755
AAA	Risk Premia	45	168	356	752	1263
B	Probability	.9430	.8306	.7680	.7165	.6802
B	Price	.9618	.8540	.7840	.5942	.5533
B	Risk Premia	-195	-274	-204	2058	2294

We observe that no default in n years by AAA is an event like life, one with a relatively high probability compared to its price. There is therefore a market in shorting the security that pays out on such defaults, as then we access a high price and low probability event. For the rating category B and a short maturity say 3 years the situation is reversed and we have a high price and low probability for the event of no default as opposed to default. Hence we might consider selling a security that pays out unit face if a B rated firm does not default in a short time span. For the longer maturity the risk premium is positive on no default. In general the risk premium rises with maturity and downgrading.

Understanding the rationale for the risk premia on various securities is an important exercise of modern finance theory. In many situations the cost of default is in excess of the dollars lost and hence the premium on price over probability. The welfare or replacement cost of a baby sitter not showing up is often in excess of the dollars involved in the price of the service and sellers can charge a premium price for the service if it is coupled with reliability of delivery. Markets pay more than probability for events that are of particular concern to them. In the context of our default table, we note that the market is not too concerned in the default of a B rated firm in 3 years but the failure of a AAA is a matter of concern.

The activity of managing portfolios of cash flow promises by a multitude of names requires us to understand the determinants of the probabilities $\pi(t, A, T)$, $f(y; t, A, T)$ and prices $p(t, A, T)$, $g(y; t, A, T)$ of events associated with default. For the purposes of trading it is the price that is the primary focus of attention, but if the portfolio is to be held to maturity then forecasting the cash flow experience requires an understanding of the probabilities. We anticipate that models for these prices and probabilities would include as explanatory variables the dynamic structure of the asset backing for the firm, the term structure of interest rates in the market, the priority structure of the debt, its rating, and other firm specific and macro economic variables that impact on the fortunes of the names in question. Our focus here will be primarily on pricing models as they have developed in the finance literature.

An important consideration in this regard is the use to which the model is to be put. If one is marking to market the default risk exposure then a particular model and its set of explanatory variables may be adequate, however, if one wishes to hedge the exposure as well then it is important to include among the explanatory variables the prices of traded securities that one may take risk management positions in. Alternatively one needs to relate the prices of other traded securities to the set of explanatory variables in the model.

3 Option Theoretic Models

This class of models defines default as a contingent claim by describing precisely when default occurs and then prices the default security using the methods of derivative security pricing. We shall consider three models in this class, the model by Merton [14], an application of the model of Madan, Carr and Chang [11] and the Longstaff-Schwartz [9] model. Models in this class are differentiated by the way in which the default event is defined and particular definitions may well be better suited to describing particular defaults. All these models relate default to the process for the firm's asset backing and define the default event in terms of boundary conditions on this process. A major deficiency of these models noted in Madan and Unal is that they treat the value of asset backing as a primary asset of the economy when in fact it may be a derivative asset in its own right with exposure to other more primitive state variables of the economy. Prices should be reduced to exogenous state variables of the economy and it is unclear that a firm's asset value is such a variable.

3.1 The Merton Model

Default in this model occurs only at maturity if it occurs at all. Hence the timing of default is a particularly simple random variable that is either T or ∞, where T is the maturity of the debt. Default occurs at T if at this time the firm value is below the face value of the debt at issue and the default magnitude is the extent of the shortfall. It is supposed that the initial firm value V evolves under risk neutral probabilities as a geometric Brownian motion with a drift equal to a constant interest rate of r per unit time and a constant volatility of σ per unit time. Firm value at time T, $V(T)$ then has a log normal distribution with $\log(V(T))$ normally distributed with a mean of $\left(r - \frac{\sigma^2}{2}\right)T$ and a variance of $\sigma^2 T$. The payoff to the defaultable bond is that of a treasury risk free bond less the value of a put option with a strike F equal to the face value of the debt. Defining leverage by $d = Fe^{-rT}/V$ one may write the credit spread, cs, for the price of the defaultable debt over the constant interest rate r as

$$
\begin{aligned}
cs(d, \sigma, T) &= -\frac{1}{T}\log\left(N(h_2) + \frac{1}{d}N(h_1)\right) \\
h_1 &= \frac{\log(d)}{\sigma\sqrt{T}} - \frac{\sigma\sqrt{T}}{2} \\
h_2 &= -\frac{\log(d)}{\sigma\sqrt{T}} - \frac{\sigma\sqrt{T}}{2} \\
N(x) &= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}}\exp(-u^2/2)du
\end{aligned}
$$

We may verify that credit spreads rise with leverage, and rise with maturity for low values of leverage. For values of leverage below unity credit spreads are hump shaped with respect to maturity, rising at first and then falling. For leverage levels above unity credit spreads fall with maturity. Increases in volatility raise the value of the short put option and thereby raise credit spreads.

A feature of the model that is countered in data is that short maturity credit spreads are near zero in the model. This is a consequence of the fact that a continuous process requires some passage of time before it may enter the defined region of default. Other deficiencies include inconsistencies in the treatment of debt with multiple maturities as in the model they default at different times, each at its own maturity, while in reality they default all together. Finally, the assumption of constant interest rates and a flat yield curve are contrary to realistic term structures of interest rates. Since defaultable bonds are bonds, they must have an embedded exposure to interest rates that is here modeled unrealistically in terms of movements in a flat yield curve.

Risk premia in the Merton model are also unrealistically large. For a 2 year debt on a 20% volatility with leverage (F/V) at 2/3 in a 5% interest rate market and an equity risk premium of 10% the statistical probability of default is given by $N(h_2)$ on replacing the interest rate in the expression for h_2 by the mean rate of return. For the data cited this is .009,while the corresponding risk neutral probability is .0476, or a risk premium on the no default security of $(1-.009)/(1-.0476)-1 = 405$ basis points. Realistic values would be below 100 basis points.

3.2 Madan, Carr and Chang Model

The deficiencies of the Black-Scholes-Merton model as an option pricing model are well known to participants in equity derivative markets. The log-normal distribution at any maturity fails to calibrate to the market skew and kurtosis, where the former is significantly negative in fact and zero for the log-normal and the latter is significantly above the log-normal value of 3. An option pricing model that calibrates well to risk neutral distributions at each maturity was proposed by Madan, Carr and Chang [11]. This model has two extra parameters over the asset volatility and calibrates to the market skew and kurtosis. The model is called the variance gamma model in which the logarithm of the stock price is a Brownian motion with drift θ and volatility σ evaluated at an independent random time given by an increasing process with independent increments that have a gamma distribution of mean 1 and variance ν for the increment over unit time. The parameter θ allows for control over skewness while ν calibrates the kurtosis. The resulting process is purely discontinuous and the jumps in value can take asset value into the default region immediately with positive probability. This solves the problem with short maturity credit spreads witnessed in the Merton model.

For leverage defined as in the Merton model the variance gamma credit spread, *csvg*, may be written as

$$\begin{aligned} csvg(d,\sigma,\nu,\theta,T) &= -\frac{1}{T}\log\left(1-\frac{1}{d}vgp(1,d,0,0,\sigma,\nu,\theta,T,0)\right) \\ vgp(p,k,r,q,\sigma,\nu,\theta,t,u) &= VG \text{ Option Pricing Function} \end{aligned}$$

The VG option pricing function has arguments of the spot price, p, the strike price, k, the interest rate, r, the dividend yield, q, the process parameters, σ, ν, θ, the maturity T, and a flag u that is 1 for calls and 0 for puts. Matlab code for this function is available upon request.

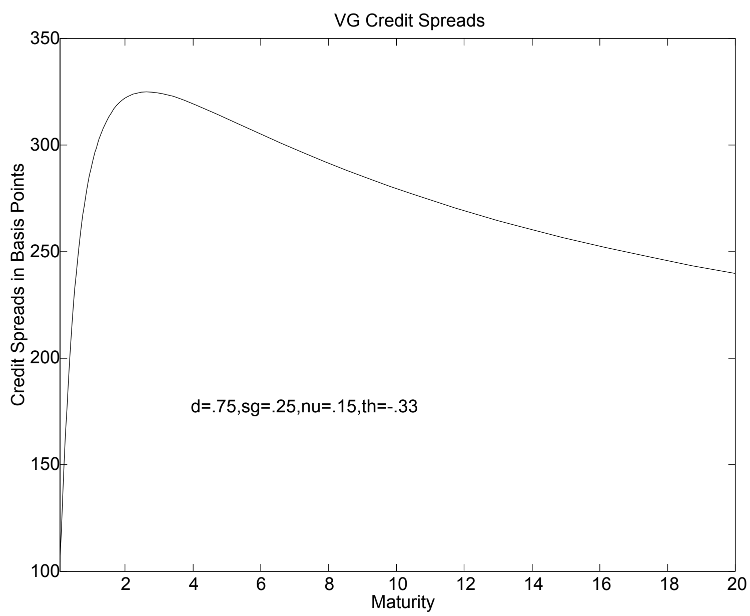

Figure 1:

We present a graph of VG credit spreads against maturity for leverage .75 and process parameters of $\sigma = .25$, $\nu = .15$, and $\theta = -.33$. The short maturity credit spread is seen to be substantial in contrast with what would be found in the Merton model. The humped shape for this leverage level is consistent with the Merton model.

3.3 Longstaff and Schwartz Model

This model attempts to address a number of shortcomings of the Merton model. By allowing for early default time consistency is attained in pricing multiple maturities. Default is defined as occurring at the first time that asset values reach a threshold level K and at this time all maturities still outstanding default simultaneously. The possibility of early default may also induce higher credit spreads and eliminate the problem of no short maturity credit spreads. Furthermore by allowing for stochastic interest rates one may better calibrate to existing term structures of interest rates. The model supposes that asset values $V(t)$, follow a geometric Brownian motion while interest rates $r(t)$, follow a Vasicek [15] process with movements that are correlated with the stock. Specifically it is supposed that

$$\begin{aligned} dV &= rVdt + \sigma V dW_V(t) \\ dr &= \kappa(\theta - r)\, dt + \eta dW_r(t) \\ dW_V dW_r &= \rho dt \end{aligned}$$

At the first passage time of V to K the default threshold, we have a default in which the recovery level is some constant write down of the face and creditors receive $(1-w)F$.

In pricing the defaultable claim one may account for correlation between interest rates and asset values by following Jamshidian [6] and Geman, El Karoui and Rochet [4] and expressing the dynamic evolution of asset values and interest rates using the price of the risk-free bond, $p(t,T)$, as the numeriare. Hence we let Q^T be the probability under which asset prices discounted by the price of the risk free bond price $p(t,T)$, are martingales. The price of the defaultable bond, $v(t,T)$, may then be written as

$$v(t,T) = p(t,T)\,[G(t,T) + (1-w)F(1-G(t,T)]$$

where $G(t,T)$ is the probability of no default in the interval (t,T) and in this case we receive the unit face, while under default, with probability $(1-G(t,T))$,we receive $(1-w)F$ at time T by definition. The claim is priced on determining the probability of no default $G(t,T)$ under the measure Q^T.

The explicit dynamics for asset values and interest rates under Q^T may be determined by an application of Girsanov's theorem as

$$\begin{aligned} d\ln V &= (r - \sigma^2/2 - \rho\sigma\eta M(T-t))dt + \sigma d\widetilde{W}_V, \\ dr &= (\theta - \kappa r - \eta^2 M(T-t))dt + \eta d\widetilde{W}_r, \\ M(t) &= \frac{1 - \exp(-\kappa(T-t))}{\kappa} \end{aligned}$$

where $\widetilde{W}_V, \widetilde{W}_r$ are Brwonian motions under Q^T.

To determine G, Longstaff and Schwartz present and solve an integral equation that equates the Gaussian probability of $\log(V/K)$ being negative at T to the integral over first passage times to zero of the first passage probability to zero times the Gaussian probability of then ending up below zero, starting now at zero at the first passage time. It has been observed by Robert Goldstein that this integral equation is incorrect as one must also integrate over the level of the random interest rate at the first passage time of $\log(V/K)$ to zero. It is simple to show using characteristic functions for example, that with two independent Brownian motions the distribution of the first Brownian motion at the first passge time of the second Brownian motion to zero is a Cauchy random variable with an infinite mean. The Longstaff and Schwartz model is, at this writing, as yet unsolved.

4 Hazard Rate Models of Default

This class of models focuses directly on describing the evolution of the probability of default in the next instant without defining the exact default event. Hence, at all times there is a probability that default will occur and a positive probability that it will not. We do not have a specification of variables with the property that if these variables are in a particular constellation then default happens with certainty immediately. The instantaneous likelihood of default is referred to as the hazard rate of default.

The motivation for this shift in focus was that default is often a complicated event and specifying the precise conditions under which it must occur are easily misspecified. The conditions one writes down may be too stringent so that it often occurs before these conditions are met, or the conditions are too weak and default fails to occur when all the requisite conditions have been met. A useful analogy can be made with the arrival of customers in a queue, one may hope to describe the evolution of the liklihood of the arrival rate of customers but not be in a position to pinpoint exactly when the next arrival will take place. Similarly one may describe variations in the likelihood of a patient suffering a heart attack without being able to write down precisely when it will occur. The exact event is by construction outside the span of ones observables.

In keeping with the notation of our earlier section, we denote by $p(t,T)$ the price at t of a risk free zero coupon bond maturing at T, while we let $v(t,T)$ be the price of a defaultable zero coupon bond. We also suppose the existence of a money market account paying a continuously compounded interest rate of $r(t)$ at time t. The time t value of the money market account is given by

$$B(t) = \exp\left(\int_0^t r(u)du\right).$$

The basic valuation equation of this class of models is that obtained on invoking the martingale principle whereby asset prices discounted by the money market account are martingales under a risk neutral measure denoted here by Q. We may then write that

$$v(t,T) = B(t)E^Q\left[\frac{X(T)}{B(T)}\right]$$

where $X(T)$ is the payout of the claim, in default or otherwise, as seen in T dollars.

The literature has formulated three distinct conventions for transfering early dollars in default to dollars at time T. Suppose in an early default creditors receive a dollars. Jarrow and Turnbull transfer this to payment in T dollars of

$$X(T) = \frac{a}{p(u,T)}$$

as the a dollars may be immediately invested in risk free bonds in the amount $X(T)$ for a time T receipt of $X(T)$ dollars. Madan and Unal place the a dollars in the money market account to receive at T

$$X(T) = a\frac{B(T)}{B(u)}.$$

Duffie and Singleton place the money in equivalent risky bonds to receive at T

$$X(T) = \frac{a}{v(u,T)}.$$

Since full payment amounts to having $X(T) = 1$ we have full payment in Jarrow and Turnbull on the delivery of a zero coupon Treasury bond of maturity T and unit face value. Full payment in Madan and Unal takes account of the actual money market experience in the interim while in Duffie and Singleton one only has to deliver a risky bond with the same face and maturity as the original bond pre-default. If we take the view that full payment is precisely the cost of replication of the original promise at time T then the definitions of Jarrow and Turnbull and Madan and Unal accomplish this objective in different ways while this objective is not attained in Duffie and Singleton as there may be another default. We further comment that as by Jensen's inequality

$$\begin{aligned} p(u,T) &= E^Q\left[\frac{B(u)}{B(T)}\right] \\ &\geq \frac{1}{E^Q\left[\frac{B(T)}{B(u)}\right]} \end{aligned}$$

the cost of full replication is lower on average using the Madan and Unal investment strategy than that of Jarrow and Turnbull.

If we now fix the recovery rate in T dollars at the level y then we may write the price of the defaultable bond as

$$v(t,T) = p(t,T)E^Q\left[F(t,T) + y(1 - F(t,T))\right]$$

assuming independence of the interest rate process and the default event to factor out the bond price. Alternatively, one may change numeraire to the treasury bond of maturity T and construct the forward adjusted probability measure Q^T and write

$$v(t,T) = p(t,T)E^{Q^T}\left[G(t,T) + y(1 - G(t,T))\right].$$

$F(t,T)$ is the probability of no default under Q while G is the probability of the same event under Q^T. Most of the earlier papers assumed independence and worked with F, while for example Madan and Unal [13] use a calculation under Q^T.

The problem is now reduced to determining the probability $F(t,T)$. We shall continue to discuss the situation for $F(t,T)$, noting that similar calculations hold for $G(t,T)$ after one has altered the dynamics to reflect the measure change.

4.1 Modeling the probability of no default

Jarrow and Turnbull consider the case of a constant hazard rate of default λ for which it follows from classical arguments for Poisson processes that

$$F(t,T) = \exp\left(-\lambda(T-t)\right).$$

Madan and Unal generalize this result to the case when the hazard rate is adapted to a Brownian filtration and is given by a continuous process $\lambda(t)$ for the hazard rate at time t, showing that in this case

$$F(t,T) = E^Q\left[\exp\left(-\int_t^T \lambda(u)du\right)\right].$$

Duffie and Singleton generalize this result and show that when recovery is defined as a proportion $\theta(u)$ of the predefault value of the defaultable bond then

$$v(t,T) = E^Q\left[\exp\left(-\int_t^T (r(u) + \lambda(u)(1-\theta(u))\, du\right)\right].$$

The importance of these results is that they teach us how to value defaultable bonds using the mathematics of risk free bonds as similar calculations have been done for risk free bonds with just the interest rate process constituting the instantaneous discount process. We now learn that once the discount rate has been adjusted to reflect the exposure to the hazard of default one may essentially treat defaultable bonds as if they were risk free and in particular value coupon bonds are portfolios of zeros. In the absence of such an adjustment in the discount rate, the coupons of coupon bonds had cumbersome linkages in default, as all coupons subsequent to the default time, defaulted together while a portfolio of zeros approach would give them independent default times.

To appreciate these results we consider two increasing and complete information filtrations $\mathcal{G}_t \subseteq \Im_t$ such that the default time is outside the span of $\mathcal{G} = \mathcal{G}_T$ but is adapted to $\Im_t$ while the hazard rate process is adapted to $\mathcal{G}_t$. We next define the default process by $\Delta(t) = \mathbf{1}_{\tau \leq t}$ where τ is the random default time. The process $\Delta(t)$ jumps to unity at the default time and is zero prior to this time. Clearly $\Delta(t)$ is an increasing process and it is not a martingale. The hazard rate process $\lambda(t)$ is the precise compensating process such that

$$\Delta(t) - \int_0^t (1-\Delta(u))\lambda(u)du$$

is a martingale under Q. Note that once Δ has reached unity and default has occurred, the compensation automatically stops on account of the term $(1 - \Delta(u))$.

An important result that plays a crucial role in eliminating the jump process $\Delta(t)$ from our calculations states that

$$E\left[1-\Delta(t)|\mathcal{G}_t\right] = \exp\left(-\int_0^t \lambda(u)du\right)$$

In deriving this result we shall use the result of a simple calculation using iterated expectations and the natural information inclusions that shows that if $M(t)$ is an $\Im_t$ adapted martingale then $\overline{M}(t) = E\left[M(t)|\mathcal{G}_t\right]$ is a $\mathcal{G}_t$ adapted martingale. We call this the drop down lemma as it lets us drop $\Im_t$ martingales down to $\mathcal{G}_t$ martingales.

We now note that by the definition of the hazard rate process

$$m(t) = -\Delta(t) + \int_0^t (1-\Delta(u))\lambda(u)du$$

is a $\Im_t$ adapted Q martingale. It follows that its Doleans-Dade exponential, $M(t)$, is also an $\Im_t$ adapted Q martingale. But we explicity write this exponential as

$$M(t) = \exp\left(\int_0^t (1-\Delta(u))\lambda(u)du\right)(1-\Delta(t))$$

We now observe that we may delete the jump process from the exponential as it is unity while $\Delta(u) = 0$ and once $\Delta(t)$ is 1 then $(1-\Delta(t)) = 0$ and the exponential is no longer relevant. Hence we may write

$$M(t) = \exp\left(\int_0^t \lambda(u)du\right)(1-\Delta(t))$$

and applying the drop down lemma we have that

$$\overline{M}(t) = \exp\left(\int_0^t \lambda(u)du\right)E\left[1-\Delta(t)|\mathcal{G}_t\right]$$

is a $\mathcal{G}_t$ adapted martingale. But $\overline{M}(t)$ is easily seen to be a predictable process of integrable variation and hence it must be constant and equal to its initial value of unity and the result follows.

We are now in a position to model the price of a defaultable claim paying a face value of F at T and a stochastic coupon $c(u)$, for $u \leq T$, through time till the default time when the payment is $y(u)$. Writing the price as expected discounted future claims under the measure Q we have

$$v(t,T) = E_t\left[\begin{array}{c} \int_t^T e^{-\int_t^u r(s)ds}(1-\Delta(u))c(u)du+ \\ e^{-\int_t^T r(s)ds}(1-\Delta(T))F+ \\ \int_t^T e^{-\int_t^u r(s)ds}(1-\Delta(u))y(u)d\Delta(u) \end{array}\right]$$

We note in this expression how the jump process $\Delta(t)$ accomplishes the accounting for coupon payments till the default time and the default payout precisely at the default time for it is then that $d\Delta(t) = 1$. Now using iterated expectations and conditioning on the filtration $\mathcal{G}_t$ we may write

$$v(t,T) = E_t\left[\begin{array}{c} \int_t^T e^{-\int_0^u r(s)ds}E[1-\Delta(u)|\mathcal{G}_u]c(u)du+ \\ e^{-\int_0^T r(s)ds}E[1-\Delta(T)|\mathcal{G}_T]F+ \\ \int_t^T e^{-\int_0^u r(s)ds}E[1-\Delta(u)|\mathcal{G}_u]y(u)\lambda(u)du \end{array}\right]$$

We now apply our main result replacing $E[1-\Delta(u)|\mathcal{G}_u]$ by $\exp\left(-\int_0^u \lambda(s)ds\right)$ to deduce a pricing equation that eliminates all reference to the jump process $\Delta(t)$ as

$$v(t,T) = E_t \left[\begin{array}{c} \int_t^T e^{-\int_t^u (r(s)+\lambda(s))ds} c(u)du + \\ e^{-\int_t^T (r(s)+\lambda(s))ds} F + \\ \int_t^T e^{-\int_t^u (r(s)+\lambda(s))ds} y(u)\lambda(u)du \end{array} \right]$$

This expression allows us to price defaultable coupon bonds now using the mathematics of default free bonds as the default jump process has been eliminated from the calculations.

In the further special case defining

$$y(u) = \theta(u)v(u,T)$$

the Duffie and Singleton recovery definition we have that

$$v(t,T) = E_t \left[\begin{array}{c} \int_t^T e^{-\int_t^u (r(s)+\lambda(s))ds} c(u)du + \\ e^{-\int_t^T (r(s)+\lambda(s))ds} F + \\ \int_t^T e^{-\int_t^u (r(s)+\lambda(s))ds} \theta(u)\lambda(u)v(u,T)du \end{array} \right]$$

The associated discounted gains martingale is

$$\begin{aligned} N(t) &= \int_0^t e^{-\int_0^u (r(s)+\lambda(s))ds} c(u)du + \\ &\quad \int_0^t e^{-\int_0^u (r(s)+\lambda(s))ds} \theta(u)\lambda(u)v(u,T)du + \\ &\quad e^{-\int_0^t (r(s)+\lambda(s))ds} v(t,T) \end{aligned}$$

It follows from an application of Ito's Lemma that

$$c(t) + \lambda(t)\theta(t)v(t,T) - (r(t) + \lambda(t))v(t,T) + E[dv] = 0$$

or equivalently that

$$c(t) - (r(t) + \lambda(t)(1 - \theta(t)))v(t,T) + E[dv] = 0$$

and hence by Ito's lemma that

$$\begin{aligned} L(t) &= \int_0^t e^{-\int_0^u (r(s)+\lambda(s)(1-\theta(t)))ds} c(u)du + \\ &\quad e^{-\int_0^t (r(s)+\lambda(s)(1-\theta(t)))ds} v(t,T) \end{aligned}$$

is a discounted gains martingale and so

$$v(t,T) = E_t \left[\begin{array}{c} \int_t^T e^{-\int_t^u (r(s)+\lambda(s)(1-\theta(t)))ds} c(u)du + \\ e^{-\int_t^T (r(s)+\lambda(s)(1-\theta(t)))ds} F \end{array} \right]$$

Having reduced default pricing in the hazard rate formulation to the mathematical calculations used for default free models, we now consider some specific models. The models we present are those proposed by Duffie and Singleton [3], Madan and Unal models 1 and 2 [12], [13], and the Jarrow, Lando and Turnbull [8] model.

4.2 Duffie Singleton Model

The literature on pricing default free bonds has witnessed the application of two sucessful classes of models, these are the exponential affine models and the Heath, Jarrow and Morton models. Duffie and Singleton consider the development of defaultable bond pricing models along these lines. We briefly present the results of this approach.

4.2.1 Exponential Affine Models

For this purpose we introduce a vector of state variables $y(t)$ that represent unobservables driving the term structure of defaultable bond prices. In practice these would be inferred from the existing level of credit spreads and the model seeks to explain the universe of credit spreads from a finite set of such spreads. The state variables are constructed to be a vector of positive random variables through time.

We suppose that these state variable follow a vector stochastic differential equation that generalizes the Cox, Ingersoll and Ross [2] model for the instantaneous spot rate to the vector case. Let K denote a diagonal matrix of speeds of mean reversion while Θ is a vector of long term levels to which the state variables are reverting. The matrix Σ is a covariance matrix between the state variables and $W(t)$ is standard vector Brownian motion of dimension equal to that of y. The stochastic differential equation governing the evolution is given by

$$dy(t) = K\left(\Theta - y(t)\right)dt + \Sigma diag(y(t))^{1/2} dW(t)$$

where $diag(y(t))$ is the diagonal matrix with diagonal given by the state variable vector $y(t)$.

Both the instantaneous spot rate and the exposure to default measured by the hazard rate times the loss rate are supposed linear in the state variables.

$$\begin{aligned} r(t) &= \delta_0 + \delta' y(t) \\ \lambda(t)(1-\theta(t)) &= \gamma_0 + \gamma' y(t). \end{aligned}$$

For such a formulation one may derive in closed form an exponential affine expression for the joint characteristic function of the integral of the process $y(t)$. Specifically,

$$E\left[\exp\left(i\xi' \int_0^t y(u)du\right)\right] = \exp\left(a(t,\xi) + b(t,\xi)' y(0)\right)$$

where the coefficient function $a(t, \xi)$ and $b(t, \xi)$ may be explicitly solved for. Since the price of a defaultable bond requires that we evaluate the expectation of an exponential of the integral of a linear function of the state variables, these prices may be obtained in closed form by evaluating the joint characteristic function at an appropriate point. The result is a closed form model for defaultable bonds that may be empirically evaluated.

4.2.2 Defaultable HJM models

The Heath Jarrow and Morton [5] models, for the purposes of pricing fixed income claims, require a specification of the dynamics of the forward rates under the risk neutral measure associated with the money market discounting process. For defaultable bonds we specify directly the dynamics of defaultable forward rates. Let $g(t, T)$ be the forward rate prevailing at time t for a defaultable loan at time $T > t$. Let $W(t)$ be a standard vector Brownian motion of dimension appropriate for describing the stochastic evolution of forward rates. In practice this would determined by a principal components analysis of the covariance matrix of defaultable forward rates. The function $g(0, T)$ is the initial forward rate curve. It is supposed that

$$g(t, T) = g(0, T) + \int_0^t \mu(u, T) du + \int_0^t \sigma(u, T) dW(u)$$

Given a model for the martingale component, that is a choice by the modeler for the functions $\sigma(u, T)$, one needs to determine the drift coefficients $\mu(u, T)$ that enforce the condition that discounted prices are martingales under the risk neutral measure. Using the definition of prices in terms of forward rates, and applying Ito's lemma one deduces that the martingale condition determines completely the forward rate drifts $\mu(u, T)$. When recovery in default is defined as a proportion of the predefault price of the bond then this drift restriction is seen to be identical to its counterpart in default free bond pricing and is

$$\mu(t, T) = \sigma(t, T) \left(\int_t^T \sigma(u, T) du \right)'.$$

When recovery is a proportion of the price of a Tresaury or default free bond then this restriction is slightly more involved. Letting $f(t, T)$ be the default free forward rates one obtains that

$$\mu(t, T) = \sigma(t, T) \left(\int_t^T \sigma(u, T) du \right)' + \theta(t)\lambda(t) \frac{v(t, T)}{p(t, T)} \left(g(t, T) - f(t, T) \right).$$

These drift restrictions may be imposed in simulations designed to determine claim valuations, or in the construction of risk neutral trees for the same purpose.

4.3 Madan and Unal 1

Madan and Unal take as a sufficient statistic on the well being of a firm its equity value measured in units of the money market account. The hazard rate of default is modeled as a decreasing function of this relative price. If $S(t)$ is the process for the stock price then $s(t) = S(t)/B(t)$ is the value of equity in units of the money market account. This discounted price is modeled as a martingale with

$$ds(t) = \sigma s(t) dW(t)$$

where $W(t)$ here is a standard one dimensional Brownian motion. Madan and Unal further suppose that

$$\begin{aligned} \lambda(t) &= \phi(s(t)) \\ &= \frac{c}{\left(\ln\left(\frac{s(t)}{\delta}\right)\right)^2}. \end{aligned}$$

If δ lies below the current value of $s(t)$ then λ is a decreasing function of the equity value with the hazard rate tending to infinity as $s(t)$ approaches δ. On the other hand is δ is above current equity values then a positive relation is possible. This is appropriate if increases in equity value are seen as signals of greater risk increasing the value of equity viewed as an option. The exact location of δ is left to be empirically determined.

The price defaultable debt is easily determined in terms of the probability of no default as explained earlier. For this probability, we now have a Markov setting in which the probability of no default in the interim t, T is of the form

$$F(t,T) = \psi(s,t).$$

It is easily observed that $\exp\left(-\int_0^t \lambda(u)du\right)\psi(s,t)$ is a risk neutral martingale and hence by Ito's lemma one must have that ψ satisfies the partial differential equation

$$\frac{\sigma^2 s^2}{2}\psi_{ss} + \psi_t = \phi(s)\psi$$

subject to the boundary condition

$$\psi(s,T) = 1.$$

The solution is given by

$$\begin{aligned} \psi(s,T) &= G_a\left(\frac{2}{d^2(s,T)}\right) \\ d &= \frac{\log(s/\delta)}{\sigma\sqrt{T}} - \frac{\sigma\sqrt{T}}{2} \\ a &= \frac{2c}{\sigma^2} \end{aligned}$$

and $G_a(y)$ satisfies the ordinary differential equation

$$y^2 G'' + \left(\frac{3y}{2} - 1\right) G' - aG = 0$$

subject to $G(0) = 1, G'(0) = -\, a$.

4.4 Jarrow, Lando and Turnbull

Jarrow, Lando and Turnbull consider a finite state discrete time model with states represented by rating categories, modeling default as a transition to the default state. They incorporate information on statistical rating transition matrices into default pricing. Suppose there are m states with state m being the default state and let Q_{ij} be the observed statistical probability of a transition from state i to state j in a single period. From what we have noted, assuming a default payout of a proportion y of the value of a Treasury bond we may write

$$\frac{v(t,T)}{p(t,T)} = 1 - (1 - F(t,T))(1 - y)$$

and hence the risk neutral probability of no default may be inferred from market prices as

$$1 - F(t,T) = \frac{1}{1-y}\left(1 - \frac{v(t,T)}{p(t,T)}\right).$$

This information is employed to build the risk neutral transition probability matrix that is central to risk neutral calculations of default. Theoretically, the risk neutral probability of default is given by the m^{th} column of the product of the single period risk neutral transition matrices in the interim, or

$$\frac{1}{1-y}\left(1 - \frac{v_i(0,T)}{p(0,T)}\right) = \left(\prod_{n=1}^{T} \widetilde{Q}(n)\right)_{im}$$

where the subscript i denotes the i^{th} rating category. It remains to determine the risk neutral transition matrices from market data.

For this identification it is supposed that the relative transition probability to different states are risk neutrally equal to their statistical counterparts and so we may write

$$\widetilde{Q}(n)_{ij} = \pi_i(n) Q_{ij} \text{ for } i \neq j$$

while

$$\begin{aligned}\widetilde{Q}(n)_{ii} &= 1 - \sum_{j \neq i} \widetilde{Q}_{ij}(n) \\ &= 1 - \pi_i(n) \sum_{j \neq i} Q_{ij} \\ &= 1 + \pi_i(n)(Q_{ii} - 1)\end{aligned}$$

and hence we have that

$$\widetilde{Q}(n) - I = diag(\pi(n))(Q - I)$$

One may employ the prices of zero coupon bonds by rating category by maturity to recursively identify the vector $\pi(n)$. For this we solve the following equation, linear in $\pi(T)$,

$$\left(\prod_{n=1}^{T-1} \widetilde{Q}(n)\right)^{-1} \frac{1}{1-y}\left(1 - \frac{v_i(0,T)}{p(0,T)}\right) = [I + diag(\pi(T))(Q - I)]e_m$$

where e_m is the m^{th} column of the m dimensional identity matrix.

4.5 Madan and Unal 2

Madan and Unal in their later model take a structural approach to default in hazard rate models and define default as occuring when a cash flow event triggers a short-fall of equity. This approach avoids the necessity of building a complicated jump component to the firm's asset value process with the resultant complexities of the associated first passage times that are typical of structural default models allowing for early default. They also allow for stochastic interest rates correlated with the value process and obtain closed forms for the prices of defaultable bonds.

Default is here viewed as arising from the occurence of a single unforeseen loss of magnitude L that triggers default if L dominates the existing continuously evolving equity. The loss arrival rate is a constant λ and the loss distribution, $F(L)$, is exponential with mean loss μ_L.

Equity on the other hand is composed of non-financial assets of value V and financial assets less liabilities in the amount $g(r)$ where r is the spot interest rate. This formulation importantly allows for duration considerations in assesing the impact of interest rate changes. In the classical models an increase in interest rates is always good news for the firm as it lowers the value of liabilities leaving assets untouched at V. Equity is then given by

$$E = V + g(r).$$

Taking a local linear approximation of equity in $\log(V)$ and r around an expansion point V_0, r_0 the hazard rate is derived as

$$\begin{aligned}
\lambda(t) &= a - b\ln(V(t)) + cr(t) \\
a &= \lambda(1 - F(E_0)) + b\ln(V_0) - cr_0 \\
b &= \frac{\lambda}{\mu_L}\exp\left(-\frac{E_0}{\mu_L}\right)V_0 \\
c &= \frac{b}{V_0}D
\end{aligned}$$

where D is the firm's net asset duration.

Modeling the firm's non-financial asset value as

$$dV = rV dt + \sigma V dW_V(t)$$

and assuming a Vasicek interest rate model

$$dr = \kappa(\theta - r)dt + \eta dW_r(t)$$

that is correlated with the value process with correlation ρ, supposing a recovery of y as proportion of Treasury value, simple closed forms are derived for the prices of defaultable coupon bonds. Calibration is illustrated on data from Bloomberg on credit spreads by rating category.

5 Conclusion

A variety of models have been proposed in the literature, ranging from direct option theoretic approaches linked to the dynamics of the firm's asset backing to the class of hazard rate models that relate credit spreads and their evolution to existing credit spreads, credit ratings, and asset values. Much of the literature has concentrated on zero coupon bonds while the majority of claims trading are coupon bonds. At this stage we clearly need a comprehensive empirical study evaluating the performance of this wide array of models in explaining the implied prices of traded coupon bonds. The results of such a study will point to the directions for future research in developing model improvements that may be necessary.

The focus of attention has been almost exclusively on the assessing the arrival time of default with little attention paid to recovery rates. Madan and Unal [12] present an attempt to assess recovery rates from market data, but clearly more needs to be done in this direction as well.

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