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Swap Pricing with Two-Sided Default Risk in a Rating-Based Model *

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1. Introduction

This paper analyzes the pricing of defaultable securities in rating based models where the default of more than one agent is involved. We extend the model of Duffie and Huang (1996) to a framework which explicitly takes into account the rating of each party. Although our method is by no means restricted to swap contracts we will use as our illustrative example a plain vanilla interest rate swap.[1] Our extension allows us to investigate the effects on swap spreads of early termination provisions, i.e., credit triggers, which are linked to the ratings of the contracting parties. Clearly, a credit trigger will make each counterparty look less risky, as illustrated for example in Wakeman (1996), simply because the trigger eliminates those defaults that occur after a sequence of downgrades. How much this affects swap spreads can be studied using the technique presented in this paper. We also consider the following questions:

- How does the degree of rating asymmetry affect swap spreads?
- How does the swap spread vary with rating when the two parties have the same rating?

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[1] For an introduction to swap valuation, see for example Sundaresan (1991). For discussions on reasons for the growth of swap contracts, see Smith, Smithson and Wakeman (1990), Litzenberger (1992) and Brown, Harlow and Smith (1994).

- How important is a stochastic specification of the transition intensities (as in Lando (1994, 1998))?

A second application of the numerical technique is to the study of default swaps. Here we study how the ratings of the reference security and the protection seller affect default swap premiums.

The inclusion of ratings is conveniently handled in an ‘intensity-based’ or ‘reduced form’ approach in which one focuses on modeling the default intensity of the parties directly.[2] In our setup, the rating category and the state variable (which is the short rate on the money market account) determine the default intensities. Introducing more state variables would complicate the numerical solution, but the reduced form approach can easily handle more firm-specific variables related to, for example, the asset value of the firm. Indeed, as shown by Duffie and Lando (1999), reduced form modeling is consistent with a full modeling of a firm’s assets and liabilities, if bondholders have incomplete information on issuers’ assets.

Rating-based models of default risk are popular for modeling defaultable bonds and credit derivatives since they use readily observable data which enable a financial institution to control credit risk without having to build detailed models for each counterparty. Using ratings alone may, however, result in a too crude approximation and approaches which allow stochastic variations in default intensities within each rating category are called for. To handle such extensions we will be working with the framework presented in Lando (1994, 1998) which extends the model presented in Jarrow, Lando and Turnbull (1997).[3] This framework allows for rating transition intensities to depend on random state variables, but requires a somewhat restrictive assumption on the stochastic transition intensities to obtain closed-form solutions for prices on defaultable bonds. And even if there may be analytical expressions for prices of corporate bonds many types of derivative prices, including swaps with settlement based on pre-default market value, may not have analytical solutions. Therefore, to allow for more complicated derivatives and to handle more realistic specifications of the stochastic transition intensities, a model based on ratings must have a numerical implementation to be of practical use. This paper provides such an implementation.

An elegant way of studying two-sided default risk in a reduced-form setting is presented in Duffie and Huang (1996). Here the authors show how to value swaps with a settlement payment depending on the pre-default market value of the contract – a problem which cannot be solved simply by analyzing the values of the floating-rate leg and the fixed-rate leg separately. Using a finite state space Markov

[2] Papers using this technique include Artzner and Delbaen (1995), Das and Tufano (1996), Duffie and Singleton (1999), Duffie, Schroder and Skiadas (1996), Jarrow, Lando and Turnbull (1997), Jarrow and Turnbull (1995), Lando (1994, 1998), Madan and Unal (1998), Schönbucher (1998). For a survey on the various approaches to modeling default risk, see for example Lando (1997).

[3] More recent contributions using rating-based modeling include Arvanitis, Gregory and Laurent (1999), Das and Tufano (1996), Kijima (1998), Kijima and Komoribayashi (1998), Li (1998) and Nakazato (1997).

chain as an additional factor in the default intensities, the solution equations become a system of quasi-linear PDEs. We pay special attention to an ADI-method which is well-suited for this problem which initially seems large due to the fact that both a spot rate and a two-dimensional rating process are involved. We also show in this paper a derivation of the valuation PDEs which does not build on recursive methods.

Earlier papers dealing with the pricing of one-sided default risk in swaps include Abken (1993), Artzner and Delbaen (1990), Cooper and Mello (1991), Rendleman (1992), Sundaresan (1991), Solnik (1990) and Turnbull (1987). Rendleman (1992) also considers two-sided risk in an analysis based on the asset values of two firms entering into the contract and Sorrensen and Bollier (1984) consider one- and two-sided default risk in general terms without stating an explicit term structure model or model for default risk. Hübner (1998) starts from different assumptions to obtain swap spreads analytically, and Jarrow and Turnbull (1997) present a discrete-time implementation which like the above mentioned approaches does not model ratings changes before default. For empirical evidence on swap spread behavior, see the papers by Brown, Harlow and Smith (1994), Dufresne and Solnik (1998), Duffie and Singleton (1997), Minton (1997), and Sun, Sundaresan and Wang (1993). In this paper, we do not estimate swap spreads from actual data, so it is difficult to compare the size of our spreads with the empirical findings of these papers. Our model agrees with Duffie and Huang (1996) in showing small swap spread sensitivities to changes in credit quality. This is consistent with the study of Sun, Sundaresan and Wang (1993) in which there is no significant difference in the mid-point prices (i.e., the average of bid and offer prices) for identical swap contracts bought and sold by a AAA-rated and an A-rated firm.

Since our paper is primarily concerned with implementation and comparing different specifications of the rating process, we have chosen very simple contracts to illustrate our methods. For example, the floating leg of the swap contract is tied to the default-free term structure, and not to a LIBOR rate.[4] If one viewed the AA category as being representative of LIBOR-rates, the swap spreads would increase by an amount roughly equal to the default adjusted spot rate for LIBOR. Working with a spread to treasuries allows us to focus on variations in the spread induced by variations in counterparty risk as opposed to variations induced by fluctuations in an underlying rate containing a default-risk adjustment.

We analyze one swap contract but our framework can easily handle a whole portfolio of contracts between two parties and effects of netting provisions may then be taken into account. Also, effects of using collateral could be handled in our framework but we have chosen not to include that into our analysis. Indeed, the spread results that we are getting can be used to compare the expenses of setting up a detailed system of collateral to the cost of offering a lower rated entity the same terms as, say, an AA-rated counterparty, or alternatively to say how much the

[4] For more on the impact on swap spreads of using LIBOR rates to determine the floating leg, see Dufresne and Solnik (1998).

spread should be increased in a contract with a lower rated counterparty posting no collateral.

The outline of the paper is as follows: In Section 2, we fix the notation and reiterate the model for recovery proposed by Duffie and Huang (1996) and set up the relevant system of quasi-linear PDEs which arises from our model formulation. We also propose an alternative derivation which does not build upon recursive methods. Section 3 sets up the ADI-method for computing prices in our framework. In Section 4 we address various problems in swap pricing with two-sided default risk: How sensitive are spreads to credit ratings if both parties have the same rating? How sensitive are swap spreads to differences in rating? The results we obtain in this section are similar to those obtained in Duffie and Huang (1996). In Section 5 we consider how varying the specification of the stochastic transition intensities changes the conclusions. In Section 6 we look at how the inclusion of a credit trigger affects the fair swap rate. Predictably, credit triggers reduce swap spreads so the contribution is to get a feel for the magnitude of the reduction. Section 7 considers an application of our numerical procedure to the valuation of default swaps. We investigate in this section the premium of a default swap as a function of the joint credit quality of the default protection seller and the underlying reference security. Finally, Section 8 concludes.

2. The Model

We consider a filtered probability space $(\Omega, \mathcal{F}, P, \mathbb{F})$ and assume the existence of an equivalent martingale measure Q which may or may not be uniquely defined. Jarrow, Lando and Turnbull (1997) propose one way of determining a martingale measure from an empirically observed transition matrix and an observed term structure of credit spreads.

The filtration will be defined as the natural filtration of the price and rating processes defined below. Given is a spot rate process

$$dr_t = \mu(r_t, t)dt + \sigma(r_t, t)dW_t$$

and from the associated money market account we define the discount factor

$$B_{t,s} = \exp\left(-\int_t^s r_u\, du\right)$$

such that under the martingale measure prices of zero coupon bonds can be computed as

$$P(t, T) = E^Q\left(B_{t,T}|\mathcal{F}_t\right)$$

We consider in this paper a two-dimensional stochastic process of ratings

$$\eta_t = (\eta_t^A, \eta_t^B)$$

where η is a continuous-time process with state space

$$\mathbb{L}^{K-1} = \{1, \dots, K-1\} \times \{1, \dots, K-1\}$$

describing the joint evolution of the rating of firms A and B. Here, η represents non-default states and the default intensities for firm i, when the joint rating of the companies is η, is given under Q by $\lambda^i(\eta)$. Later, we will also work with the formulation adopted in for example Jarrow, Lando and Turnbull (1997) and Lando (1998) where default is given by adding the absorbing state K to the state space. In both cases, the modeling of the simultaneous rating process η is convenient for notational purposes and, more importantly, for allowing correlation between the two rating processes. The joint rating process allows us to include cases in which, say, default of one party triggers the default of the other.

To specify the stochastic evolution of η we use a $(K-1)^2 \times (K-1)^2$ matrix of transition intensities which we refer to (with a slight abuse of language) as a generator matrix. In the case where Λ is constant it is simply the generator of a time-homogeneous continuous-time Markov chain. When Λ is time-dependent, but deterministic, it models a time-inhomogeneous Markov chain, as used for example in Jarrow, Lando and Turnbull (1997). When Λ depends on the stochastic short rate, we use the setup of Lando (1998) where, after conditioning on a sample path of the short rate process, Λ is the generator of a non-homogeneous Markov chain.

In the case where the two rating processes are independent, we may construct Λ from the generator matrices of the chains η_t^A and η_t^B by remembering that the chain may only jump in such a way that one of the ratings changes. We illustrate this in Appendix B. Two continuous-time chains which are independent do not jump simultaneously.

We are now interested in pricing an interest rate swap struck between the two parties, and to compute swap spreads in the usual fashion by setting the swap's price at initiation equal to zero.

A critical ingredient of the pricing of the swap is the rules for settlement in default. We work, as do Duffie and Huang (1996), with the following setup: In the event of default, if the contract has positive value to the non-defaulting party, the defaulting party pays a fraction of the pre-default market value of the swap to the non-defaulting party. If the contract has positive value to the defaulting party, the non-defaulting party will pay the full pre-default market value of the swap to the defaulting party.

Mathematically, this can be stated as follows: Assume that B defaults first. Assume also for simplicity that there are no lumpy payments planned on the swap at the default date. The payment received by A at the default date is given by

$$S(\tau-, r_\tau, \eta_{\tau-}) \left(1_{\{S(\tau-, r_\tau, \eta_{\tau-})<0\}} + \phi^B_{\tau-} 1_{\{S(\tau-, r_\tau, \eta_{\tau-})>0\}}\right) \tag{1}$$

where S is the price of the swap contract seen from counterparty A. Then $S(\tau-, r_\tau, \eta_{\tau-})$ is the price right before default[5] and $\phi^B_{\tau-}$ is the recovery paid by the defaulting

[5] Note that $r_{\tau-} = r_\tau$ due to the continuity of r.

party B. A similar expression can be written stipulating what happens if A defaults first.

In terms of arbitrage pricing technology, if we think of a security as a claim to a cumulative (actual) dividend process D, (which for defaultable securities may be different from the promised dividend process) then in a no arbitrage setting (see for example Duffie (1996), p. 118), the price of the security satisfies

$$S_t = E_t^Q \left(B_{t,T} S_T + \int_t^T B_{t,s} dD_s \right) \tag{2}$$

If the contract we are pricing has a maturity of T, then we are left with the expression

$$S_t = E_t^Q \int_t^T B_{t,s} dD_s \tag{3}$$

but note that by the definition of the swap contract the cumulative dividends depend through the settlement provisions on the future values of S and also on the random default time. A key result stated for two-sided default problems in various forms in Duffie and Huang (1996) and Duffie, Schroder and Skiadas (1996) tells us how to find S in terms of a pre-default process V and how to compute V as a function of *promised* cash flows, i.e. the cash flow paid if there is no default before expiration. Define

$$V(t, r_t, \eta_t) = E_t^Q \left[\int_t^T -R(s, V_s, \eta_s) V(s, r_s, \eta_s) ds + dD_s^p \right] \tag{4}$$

where the D_t^p is the *promised* cumulative dividend received by A up to time t and

$$\begin{aligned} R(t, v, k) &= r_t + s^A(t,k) 1_{\{v<0\}} + s^B(t,k) 1_{\{v>0\}} \\ s^i(t,k) &= (1-\phi^i)\lambda^i(k) \end{aligned} \tag{5}$$

Assume that

$$\Delta V(\tau, r_\tau, k) \equiv V(\tau, r_\tau, k) - V(\tau-, r_{\tau-}, k) = 0$$

almost surely for each category k.[6] Then the result of Duffie and Huang (1996) implies that the swap price is given as

$$S(t, r_t, k) = V(t, r_t, k) 1_{\{t<\tau\}}$$

[6] Note that this is a requirement that the default event is not occurring at the same time (with positive probability) as an event which would have caused a jump in the price of the security even if no change in rating had occurred. If, for example, we had a jump component in interest rates, then it would be easy to construct an example in which interest rates jump at the same time as a default. One could also imagine that transition intensities (including default intensities) are affected by the same jump process as the one triggering default, in which case the pre-default value would jump at the time of a default.

for each joint rating category k. Let $V_t = V(t, r_t, \eta_t)$ and $R_t = R(t, V_t, \eta_t)$ then the solution to (4) is

$$V_t = E_t^Q \left[\int_t^T e^{-\int_t^s R_u du} dD_s^p \right]. \tag{6}$$

Hence the equation which gives us the swap price as an expected discounted value of actual cash flows has been translated into an expression involving an expected, discounted value of promised cash flows, but with a more complicated discount factor.

We now derive two representations of the price process – one using a theorem in Duffie and Huang (1996) and one using the enlarged Markov chain which includes default categories (i.e., categories in which at least one of the parties has defaulted).

First, let $f(t, r_t, i)$ denote the (pre-default) price at time t, when the joint rating of the parties is given by i, of a contingent claim whose only promised payoff from B to A is $d(T, r_T, \eta_T)$ at time T. Let $\bar{f}$ be a $(K-1)^2$ dimensional vector of time t prices

$$\bar{f}(t, r_t) \equiv \begin{bmatrix} f(t, r_t, (1,1)) \\ \vdots \\ f(t, r_t, (K-1, K-1)) \end{bmatrix} \tag{7}$$

The following proposition gives a system of PDEs that this vector function must satisfy.

PROPOSITION 1 *Define $\tilde{\Lambda}$ as a $(K-1)^2 \times (K-1)^2$–matrix with elements (whose dependence on t, r_t is suppressed)*

$$\begin{aligned} \tilde{\lambda}_{ij} &= \lambda_{ij}, \qquad i, j \in \mathbb{L}^{K-1}, \quad i \neq j \\ \tilde{\lambda}_{ii} &= \lambda_{ii} - (1-\phi^A)\lambda^A(i) 1_{\{f(t,r_t,i)<0\}} - (1-\phi^B)\lambda^B(i) 1_{\{f(t,r_t,i)\geq 0\}} \end{aligned}$$

where the transition intensities λ_{ij} and the diagonal elements λ_{ii} may depend on time and the short rate process r. Assume that for t fixed the process

$$f(s, r_s, \eta_s) \exp\left(- \int_t^s R_u du \right) - f(t, r_t, \eta_t)$$

is a semimartingale whose local martingale part is a martingale.

Then $\bar{f}$ is the solution to

$$\frac{\partial}{\partial t}\bar{f} + \mu \frac{\partial}{\partial r}\bar{f} + \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial r^2}\bar{f} + \tilde{\Lambda}\bar{f} - r_t \bar{f} = 0 \tag{8}$$

with the boundary condition

$$f(T, r, i) = d(T, r, i) \quad \textit{for every } i \in \mathbb{L}^{K-1}.$$

The proof of the proposition is given in Appendix A.

Note that the system of PDEs is linked through the matrix $\tilde{\Lambda}$. Intuitively, the term $\tilde{\Lambda}\bar{f}$ takes into account the price changes that can occur due to rating changes and default whereas the remaining terms keep track of changing spot rates and time. To compute $f(t, r_t, i)$ for a claim with promised lumpy dividends at time points $T_1, \ldots, T_N = T$, we use the proposition above for $T_{N-1} \leq t < T_N$ and work our way backward in the usual way: For $T_{N-2} \leq t < T_{N-1}$ use the procedure above with $d(T, r, i)$ in the right hand side of the boundary condition replaced by

$$f\,(T_{N-1}, r, i) + d\,(T_{N-1}, r, i)$$

and so forth.

It is instructive and convenient for handling general settlement rules to see the derivation of the pricing formula starting with (2) but using a Markov chain *including* the default categories. Again, consider a two-dimensional Markov process of ratings, $\tilde{\eta}$, with a state space $\mathbb{L}^K$ which includes the default state represented by K. Define the default space

$$\mathbb{D} = \mathbb{L}^K \backslash \mathbb{L}^{K-1}$$

consisting of states where at least one party has defaulted. Let $t \in [T_{N-1}, T)$ and consider the discounted gain process for $t < u < T$

$$G_{t,u} = B_{t,u} S_u + \int_t^u B_{t,s} dD_s \tag{9}$$

where $\Delta D_s = \delta(s, r_s, \tilde{\eta}_{s-}, \tilde{\eta}_s)$ is a lump sum dividend payment in the event of a transition at time s from $\tilde{\eta}_{s-}$ to $\tilde{\eta}_s$. Note that this allows us to include payments made upon transitions between non-default states, which will be useful for contracts with credit triggers. The promised dividend at time T is $d(T, r_T, \tilde{\eta}_T)$. Holding t fixed we know from Duffie (1996) that the discounted gain process, $G_{t,u}$ is a martingale under Q and we have the pricing formula $S_t = E_t[G_{t,u}]$ for $t < u$. This time, let the function g be defined by

$$\begin{aligned} g(t, r_t, \tilde{\eta}_t) &= S_t, \quad t < T \\ g(T, r_T, \tilde{\eta}_T) &= d(T, r_T, \tilde{\eta}_T). \end{aligned}$$

Notice that for $\tilde{\eta}_t \in \mathbb{D}$, $g(t, r_t, \tilde{\eta}_t) = 0$ since we assume that there is a settlement payment when entering into the absorbing set of states $\mathbb{D}$, and after that no dividends are paid out. We will now find an expression for $dG_{t,u}$ and due to the martingale property of $G_{t,u}$, the drift must equal 0.

Again (see (25)) for $\tilde{\eta}_t \notin \mathbb{D}$ we find

$$dS_u = \left(\mathcal{D}g(u, r_u, \tilde{\eta}_u) + (\Lambda\bar{g}(u, r_u))_{\tilde{\eta}_u}\right) du + M_u \tag{10}$$

where M is a martingale, $\bar{g}$ is a K^2- dimensional vector obtained by stacking the functions $g(\cdot,\cdot,i),\quad i \in \mathbb{L}^K$ and $\mathcal{D}g$ is given as in (24). This will give us $B_{t,u}dS_u + S_u dB_{t,u}$ easily. Now, all we need is the second part of (9). For $t < u < T$ there are no dividends paid out unless a transition occurs so this contribution is

$$\begin{aligned}\int_t^u B_{t,s}dD_s &= \sum_{\{s:t<s\le u,\Delta\tilde{\eta}_s\neq 0\}} B_{t,s}\delta(s,r_s,\tilde{\eta}_{s-},\tilde{\eta}_s)\\ &= \hat{M}_u + \int_t^u \lambda_{\tilde{\eta}_s}\sum_{i\neq\tilde{\eta}_s}\frac{\lambda_{\tilde{\eta}_s,i}}{\lambda_{\tilde{\eta}_s}}\delta(s,r_s,\tilde{\eta}_s,i)B_{t,s}ds\\ &= \hat{M}_u + \int_t^u \sum_{i\neq\tilde{\eta}_s}\lambda_{\tilde{\eta}_s,i}\delta(s,r_s,\tilde{\eta}_s,i)B_{t,s}ds\end{aligned}$$

where $\hat{M}$ is a martingale.[7] This can be written with matrix notation

$$\hat{M}_u + \int_t^u \left[diag\left(\Lambda\Xi_s'\right)\right]_{\tilde{\eta}_s} B_{t,s}ds \tag{11}$$

where $diag$ is a vector of the diagonal elements and Ξ is a $K^2 \times K^2$ matrix whose element i, j represents the dividend payment in case of a transition from i to j.[8]

Now, insert (10) and (11) in (9) to see that a necessary condition for $G_{t,u}$ to be a martingale is

$$\mathcal{D}\bar{g}(s,r_s) + \Lambda\bar{g}(s,r_s) + diag(\Lambda\Xi_s') - r_s\bar{g}(s,r_s) = 0 \tag{12}$$

with boundary condition $g(T,r,i) = d(T,r,i),\ i \in \mathbb{L}^K$. If dividends associated with transition to default are given as in (1) and Ξ_{ij} is zero for $j \notin \mathbb{D}$, then the equations for g in the non-default categories are the same as in (8).

3. Numerical Solution

To justify our solution method, consider first using an ordinary implicit finite difference method to approximate the solution to (8). For each state of the Markov chain we would need a number of grid points to approximate the interest rate, say M. We would now have to solve MK^2 equations so for (say) $M = 50$ we would have 2450 equations for a Markov chain with 49 states. Using Gaussian elimination (the method could probably be improved by taking the special structure of the system into consideration) we would need in the neighborhood of $2450^3 \simeq 15$

[7] After the second equality sign we have replaced $\tilde{\eta}_{s-}$ by $\tilde{\eta}_s$ which does not change the integral with respect to Lebesgue measure.

[8] The diagonal of Ξ is zero since no transitions are associated with the diagonal. As an example, we could have $\Xi_{ij} = g(t,r_t,i) - g(t,r_t,j)$, which would capture a payment at each transition date compensating for the change in contract value due to a rating change.

billion multiplications and additions cf. Golub and Ortega (1991). On a moderate size system it takes about 13 minutes to solve such a system. In order to price a defaultable zero-coupon bond with maturity one year from today using a time step of $\Delta t = 0.1$ would take about 2 hours. Instead, we have chosen to approximate the solution of (8) using an *Alternating Direction Implicit Finite-Difference Method* (ADI-method) which seems to be more efficient when working with two state variables. The basic idea is to alternate which variable is implicit and which is explicit: In the first step the interest rate is implicit and the Markov chain is explicit and in the next step it is the other way around. This way we only have to solve two equation systems of dimension M and K, respectively. Think of this as first solving for the interest rate and then solving for the change in the Markov chain.

Define the approximating grid points as

$$S_n^{m,k} \simeq S(\Delta tn, \Delta rm + r_0, \Delta\eta k) = S(t, r, \eta)$$

where $n = 0, \dots, N \equiv \frac{T}{\Delta t}$, $m = -M, \dots, -1, 0, 1, \dots, M$, and $k = 1, \dots, L$, where L is the number of states in $\mathbb{L}^{K-1}$, i.e. 49 in this illustration. So for each time point we have an $(2M+1) \times L$ matrix with values of S. Strictly speaking, the values of S we compute in the following are pre-default values of the swap but this is equal to the value of the swap as long as there is no default.

Define the $(2M+1) \times L$ matrix S_n as the asset price at time point n in every grid point i.e.

$$S_n \equiv \begin{bmatrix} S_n^{M,1} & \dots & S_n^{M,L} \\ \vdots & & \vdots \\ S_n^{-M,1} & \dots & S_n^{-M,L} \end{bmatrix}$$

Notice, that the row numbers of S_n are opposite to the usual matrix notation.

Furthermore, define a discrete time version of $\bar{S}_t$ as

$$\bar{S}_n^m \equiv \begin{bmatrix} S_n^{m,1} \\ \vdots \\ S_n^{m,L} \end{bmatrix}$$

Instead of keeping track of which variable is implicit and which is explicit, we say that each step consists of two iterations. To do this we introduce half a step by, $t + \frac{1}{2}\Delta t$. Assume we know all the values at time point $n+1$ of S (that is for all values of m and k) then for r implicit and η explicit we approximate S at $n + \frac{1}{2}$ by using

$$\frac{\partial^2 \bar{S}_{\Delta t(n+\frac{1}{2})}}{\partial r^2} \simeq \frac{\bar{S}^{m+1}_{n+\frac{1}{2}} - 2\bar{S}^{m}_{n+\frac{1}{2}} + \bar{S}^{m-1}_{n+\frac{1}{2}}}{\Delta r^2} \tag{13}$$

$$\frac{\partial \bar{S}_{\Delta t(n+\frac{1}{2})}}{\partial r} \simeq \frac{\bar{S}^{m+1}_{n+\frac{1}{2}} - \bar{S}^{m-1}_{n+\frac{1}{2}}}{2\Delta r} \tag{14}$$

$$\tilde{\Lambda}\bar{S}_{\Delta t(n+\frac{1}{2})} \simeq \tilde{\Lambda}\bar{S}^{m}_{n+1} \tag{15}$$

$$\frac{\partial \bar{S}_{\Delta t(n+\frac{1}{2})}}{\partial t} \simeq \frac{\bar{S}^{m}_{n+1} - \bar{S}^{m}_{n+\frac{1}{2}}}{\frac{1}{2}\Delta t} \tag{16}$$

Using these approximations (8) leads to a system of equations

$$\left(I_{2M+1} + \tfrac{\Delta t}{2} D\right) S_{n+\frac{1}{2}} = S_{n+1}\left(I_{(K-1)^2} + \tfrac{\Delta t}{2}\tilde{\Lambda}'\right) \tag{17}$$

where D is a tridiagonal matrix and I_n denotes the $n-$dimensional identity matrix. The advantage of the ADI-method is now apparent: S is a $(2M+1) \times (K-1)^2$ matrix instead of a $(2M+1)(K-1)^2$ dimensional vector. Therefore, the equation system to be solved is much smaller.

For the next iteration we use the Markov chain as implicit and the interest rate as explicit. I.e. we change the approximations (15) and (16) to

$$\tilde{\Lambda}\bar{S}_{\Delta t\left(n+\frac{1}{2}\right)} \simeq \tilde{\Lambda}\bar{S}^{m}_{n} \tag{18}$$

$$\frac{\partial \bar{S}_{\Delta t\left(n+\frac{1}{2}\right)}}{\partial t} \simeq \frac{\bar{S}^{m}_{n+\frac{1}{2}} - \bar{S}^{m}_{n}}{\frac{1}{2}\Delta t} \tag{19}$$

Insert the approximations (13), (14), (18), and (19) into (8) which leads to a similar system of equations

$$\begin{aligned} \left(I_{(K-1)^2} - \tfrac{\Delta t}{2}\tilde{\Lambda}\right)(S_n)' &= \left(S_{n+\frac{1}{2}}\right)'\left(I_{2M+1} - \tfrac{\Delta t}{2}D\right)' \\ \Leftrightarrow \quad S_n\left(I_{(K-1)^2} - \tfrac{\Delta t}{2}\tilde{\Lambda}\right)' &= \left(I_{2M+1} - \tfrac{\Delta t}{2}D\right)S_{n+\frac{1}{2}} \end{aligned} \tag{20}$$

Solving (20) involves finding the inverse of $I_{(K-1)^2} - \frac{\Delta t}{2}\tilde{\Lambda}$ which causes a little problem since $\tilde{\Lambda}$ contains an indicator function. This can be done with the approximation

$$\begin{aligned} S_n\left(I_{(K-1)^2} - \frac{\Delta t}{2}\tilde{\Lambda}\right)' &= S_n\left(I_{(K-1)^2} - \frac{\Delta t}{2}\Lambda\right)' + S_n D^{\phi}_n \\ &\simeq S_n\left(I_{(K-1)^2} - \frac{\Delta t}{2}\Lambda\right)' + S_{n+\frac{1}{2}} D^{\phi}_{n+\frac{1}{2}} \end{aligned} \tag{21}$$

where D^{ϕ}_n is a diagonal $(K-1)^2 \times (K-1)^2$ matrix containing the indicator functions. Evaluated at time $n+\frac{1}{2}$ these indicator functions are known. This part can now be moved to the right hand side of (20). D^{ϕ} contains jumps so we have to make sure that this approximation is appropriate. $D_n^\phi \neq D_{n+\frac{1}{2}}^\phi$ only if the swap price changes sign in that time period. If the swap price changes sign over a very small period of time it must be close to zero, which justifies the approximation (21).[9]

To simplify notation we define

$$\begin{aligned} \mathtt{A}_1 &= \left(I + \frac{\Delta t}{2} D\right) \\ \mathtt{A}_2 &= \left(I - \frac{\Delta t}{2} \Lambda'\right) \\ \mathtt{B}_1 &= \left(I + \frac{\Delta t}{2} \tilde{\Lambda}'\right) \\ \mathtt{B}_2 &= \left(I - \frac{\Delta t}{2} D\right) \end{aligned}$$

Now it follows that the two systems needing to be solved are

$$\begin{aligned} \mathtt{A}_1 S^{n+\frac{1}{2}} &= S^{n+1}\mathtt{B}_1 \\ S^n \mathtt{A}_2 &= \mathtt{B}_2 S^{n+\frac{1}{2}} - S^{n+\frac{1}{2}} D^\phi \end{aligned}$$

In Appendix C we have included a description of the implementation and a discussion of the complications that arise when considering time-dependent generators of even stochastic (e.g. spot rate dependent) transition intensities.

4. Pricing Swaps with Default Risk

We have selected a 5 year "plain vanilla" swap with semiannual exchanges of fixed rate payments for floating rate payments. More precisely, consider 11 dates, $T_0, T_1, T_2, \ldots, T_{10}$, where $T_i - T_{i-1} = \frac{1}{2}$ for every $i = 1, \ldots, 10$. The payment at time T_i is defined as

$$X_i = F\left(\frac{1}{P(T_i, T_i + \frac{1}{2})} - 1 - \frac{c}{2}\right)$$

for $i = 1, \ldots, 10$ where c is the fixed rate and F is the constant notional amount. Without loss of generality we will assume that $F = 1$. Note, however, that we only study a version without payment in arrear. Also, note that the floating payment is based on the yield of a six-month treasury bond.

[9] Another way of eliminating the indicator functions is to use the method described at the end of Section 2. However, this is a slower solution due to the higher dimension of the intensity matrix. Our simulations indicate that the convergence properties of the two approaches are approximately the same. Therefore, for swap pricing we recommend using the $(K-1)^2 \times (K-1)^2$ intensity matrix.

For our interest rate model we have chosen the Vasicek model, but of course other models of the term structure could be used. Hence, under the risk neutral measure, the spot rate is described as

$$dr = \kappa(\mu - r)dt + \sigma dW_t$$

where W_t is a standard Brownian motion.

For our computations we have chosen typical values of parameters $\mu = 0.05$, $\kappa = 0.15$, and $\sigma = 0.015$. Furthermore, we have set $r_0 = 0.05$ and chosen a recovery rate for both counterparties equal to 0.4. The ratings of the two counterparties are assumed independent and follow Markov chains described by the constant generator matrix used in Jarrow, Lando and Turnbull (1997, Table 4, page 507).[10]

It is straightforward to calculate c such that the initial value of a default free swap is 0. Using closed form solutions for bond prices in the Vasicek model, we find that in a riskless swap the fixed side would have to pay a fixed rate of

$$c = 5,0125\%$$

to make the initial value of the swap zero. Swap spreads are computed with respect to this quantity.

We first consider the fair fixed rate to be paid if the initial ratings of the parties are the same and we consider what happens if this common rating varies. To be precise, the rating at time 0 is assumed to be the same for the two parties, but of course rating transitions can bring the two parties into different categories during the life of the contract. As can be seen from Figure 1 the spreads are very small (all smaller than 0.8 basis points) and do not vary a lot across ratings at an absolute level (but do show significant variation at a relative level). One may wonder why there is *any* spread in the case with symmetric ratings. It turns out that the asymmetry in payments (one pays floating, the other fixed) does indeed produce a small spread. Hence the small spreads we see are not due to numerical error. To see this, consider the case of identical, constant default intensities for two counterparties who can be in only one non-default category, and with a constant fractional recovery of $\phi = 0.4$. Using the same Vasicek model for riskless interest rates as above, we may compute analytically the value of the swap spread.[11] The size of the spread in the symmetric case as a function of the default intensity λ for different values of the initial short rate is shown in Figure 2.

Spreads change significantly, however, when we allow for rating asymmetries. Hence we next fix the (initial) rating of party A at AA and let the (initial) rating of

[10] It is straightforward to construct the generator matrix for the Markov chain of joint ratings, using the fact that the only non-zero entries off the diagonal are those corresponding to transition of one of the firms, see Appendix B.

[11] The analytic expression is easily obtained using Equation (6) and the fact that with symmetric default intensities we have $R(t) = r_t + (1 - \phi)\lambda$, hence the indicator functions vanish. The resulting expression for the spread is not zero.

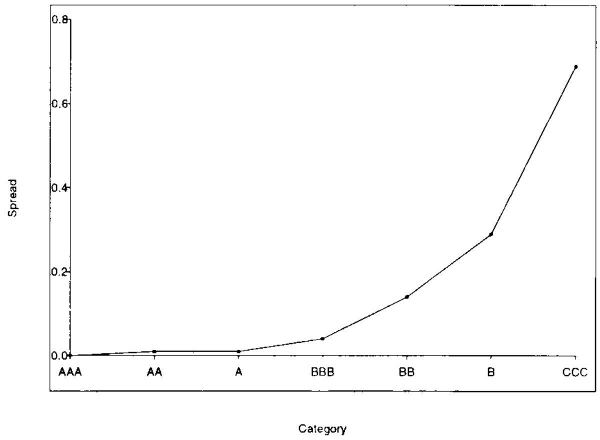

Figure 1. Swap spreads in basis points when the initial category is the same for both counterparties.

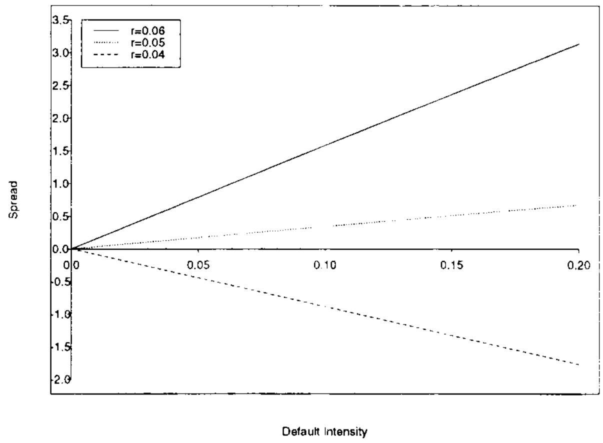

Figure 2. Swap spreads in the case of symmetric intensities as a function of the intensity. Recovery is fixed at 0.4 and the riskless rate follows a Vasicek model. Three different values of the initial short rate are used.

B vary. Both the case where A pays fixed and A pays floating are shown in Figure 3. It is worth noting that there is a high degree of symmetry in the spreads for the two cases. This symmetry is however sensitive to choice of initial spot rate level and spot rate model.

In Figure 4, we plot the spreads against spreads on defaultable bonds with same recovery and maturity as the swap and the graph seems to suggest an almost linear relationship between the two spreads. The same linear relationship is found in Duffie and Huang (1996). We find a bond yield spread of 100 basis points translates into a swap spread of approximately 1.7 basis points, which is very similar to Duffie and Huang (1996), who find a translation to 1 basis point.

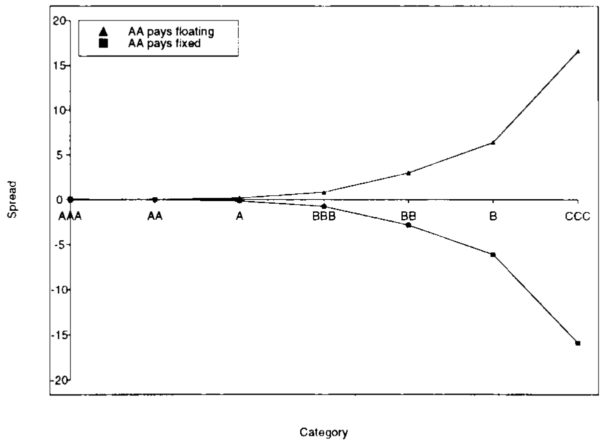

Figure 3. Swap spreads in basis points when the initial rating of one counterparty is fixed to AA and the rating of the other counterparty varies. Both the case where AA pays fixed and AA pays floating is considered. Notice the symmetry.

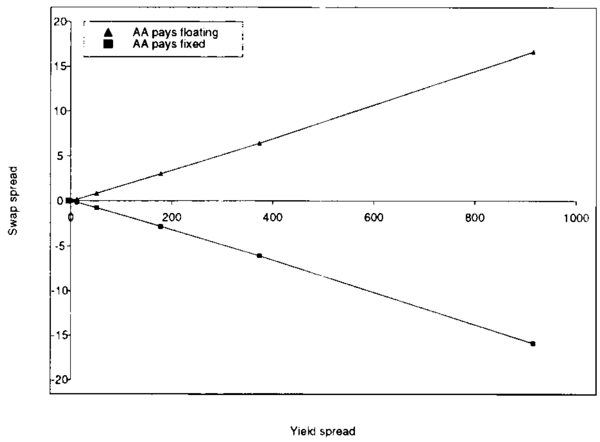

Figure 4. Swap spreads as a function of corporate bond yield spread for each category. Notice the almost linear relationship.

5. Time-Dependent and Stochastic Generators

The numerical method we have outlined can easily be adapted to the case where the generator of the Markov transition matrix is non-homogeneous and to the case where default intensities depend on the driving state variables also. The interest in non-homogeneous matrices arises primarily because of calibration issues: As shown in Jarrow, Lando and Turnbull (1997) one can use an empirical generator matrix and a time-dependent risk adjustment matrix to fit an initial structure of zero coupon bonds. The product of these two matrices gives a non-homogeneous generator. Another possibility designed to take into account changes in business cycles and correlation between rating migrations and interest rates is to let the generator matrix have elements which are functions of the interest rate (or, more generally, the state variables driving the interest rates). This approach is described in Lando (1994, 1998). In this section we will present some illustrations of the latter approach applied to swap pricing. We consider two cases of stochastic intensities. The first case is the 'affine' case studied in Lando (1998): The generator of the Markov chain which includes the default category is now given for each party as

$$A(r_s) = B\mu(r_s)B^{-1}$$

where B is a constant matrix of eigenvectors for the generator used in Jarrow, Lando and Turnbull (1997), and where $\mu(r_s)$ is a diagonal matrix whose elements are

$$\begin{aligned} \mu_i(r_s) &= \gamma_i + \kappa_i(r_s - r_0), \qquad i = 1, \dots, K-1 \\ \mu_K(r_s) &= 0. \end{aligned}$$

and where the coefficient vectors γ, κ are chosen to calibrate the level and the spot rate sensitivities of the initial corporate bond spreads to observed values.[12] Below, we will refer to this generator as the affine generator. In our example we have used the initial spreads (for zero recovery bonds)

$$\hat{s} = (16, 20, 27, 44, 89, 150, 255)'$$

and initial spread sensitivities

$$\hat{ds} = (-0.1, -0.15, -0.2, -0.25, -0.3, -0.5, -1.0)'$$

Now, the initial spreads and spot rate sensitivities are calibrated (see Lando (1998)) by defining

$$\begin{aligned} \gamma &= \beta^{-1}\hat{s} \\ \kappa &= \beta^{-1}\hat{ds} \end{aligned}$$

where β is a $K-1 \times K-1$ matrix with entries

$$\beta_{ij} = B_{ij}B^{-1}_{jK} \quad \text{for } i, j = 1, \dots, K-1.$$

This gives us

$$\begin{aligned} \gamma &= (-0.0057, -0.0161, -0.0200, -0.0227, -0.0272, -0.0328, -0.0345)' \\ \kappa &= (0.2674, 0.5600, 0.6320, 0.7069, 0.9013, 1.3590, 1.4002)' \end{aligned}$$

12 Since we work with intensities which are affine functions of the spot rate, there is a positive probability of having negative intensities in this framework. We solve this in our numerical implementation by setting the intensity equal to zero in such cases.

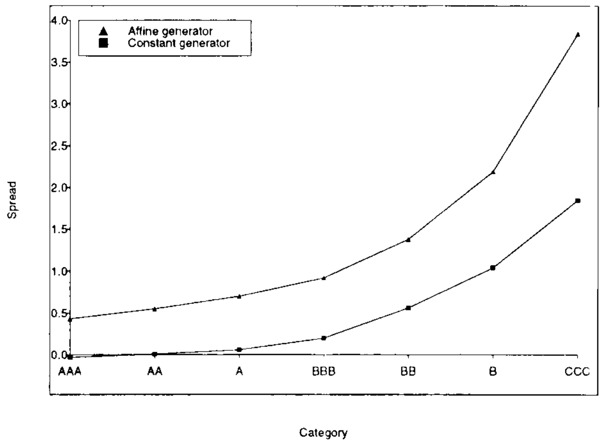

Figure 5. A comparison between swap spreads calculated with an affine generator and a constant generator. For both generators AA pays floating and fixed pay side varies. The constant generator is equal to the affine generator for $r_s = r_0$.

We study the effect of interest rate sensitivity in rating transitions and default intensities in this setting by comparing the affine generator with the 'base case' obtained by setting $\kappa = 0$. This base case corresponds to having zero interest rate sensitivity in the affine generator. Figure 5 shows that swap spreads change considerably when the interest rate sensitivity is taken into account. For example, the spread on a swap with an AA-rated floating payer and a B-rated counterparty paying fixed, increases from approximately 1 to 2.2 basis points. The intuition is the following: When interest rates go up, cash flows become less uncertain because the default intensities are calibrated to fall in this case and the opposite is true when rates go down. This means that the floating payer sees uncertain positive cash flows and less uncertain negative cash flows and consequently wants compensation for this through a higher spread. Also, the symmetry that we had in the non-stochastic case between spreads when AA paid fixed and AA paid floating vanishes. This is shown in Figure 6.

A problem with the affine generator specification is that it is hard to control the sign of all entries in the generator and it may take relatively small fluctuations in the riskless interest rate to produce negative entries in the off-diagonal elements of the generator.[13]

Therefore, we consider a second case in which rating migrations are held constant but default intensities are spot rate dependent. The default intensity from category i is assumed to be of a 'logit' form

$$\lambda_i(r_t) = \frac{a_i}{1 + \exp(b_i + c_i(r_t - r_0))} \qquad i = 1, \ldots, K - 1$$

[13] In our calibration, already at a spot rate level of 8% will some entries become negative.

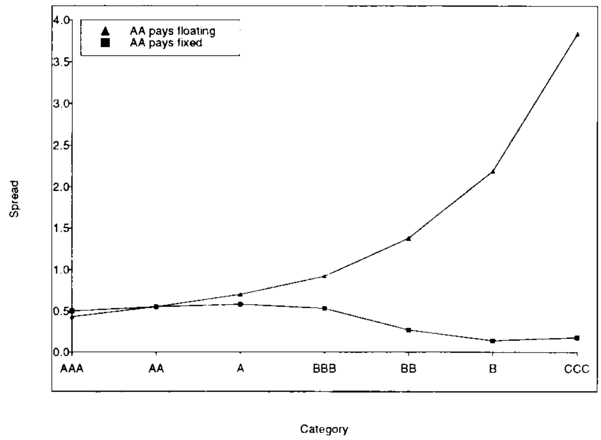

Figure 6. Swap spreads calculated with an affine generator. Notice that the spreads are no longer symmetric.

and negative sensitivity of intensities to changes in spot rates is obtained by having $c > 0$. This form is similar to the form proposed in Wilson (1997) and also used in Das and Sundaram (1998). We choose the parameter vectors a, c to match initial spreads and spread sensitivities[14] and we impose the condition $b = 0$ which gives symmetric dependence of intensities on spot rates around the initial level with our choice of a, c :

$$\begin{aligned} a_i &= 2\lambda_i(r_0) \\ c_i &= -\frac{2\lambda_i'(r_0)}{\lambda_i(r_0)} \end{aligned}$$

In Figure 7 we compare the results to those obtained using an affine generator. In both cases the short spreads and the sensitivity of changes in the short spreads to changes in the spot rate have been matched but the logit generator avoids negative intensities. The difference in spreads is not huge, suggesting that for some cases the computations using the affine generator are not invalidated by the positive probability of negative intensities.

6. Credit Triggers

As an example of how our methods can take into account provisions in contracts based on ratings, we will consider a credit trigger in the swap contract. The credit trigger is formulated as follows: If the triggering category is k then in the event that either of the counterparties drops to k or below, the contract terminates. The settlement payment in that case is defined to be equal to the value of a swap without

[14] Note that with zero recovery the initial spread for category i is $\lambda_i(r_0)$ and the sensitivity is the derivative $\lambda_i'(r_0)$.

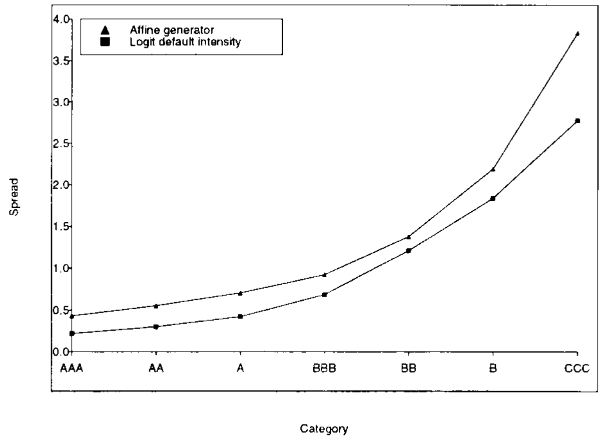

Figure 7. Swap spreads calculated with 'logit' default intensities compared with the results using an affine generator. Both models fit the initial spreads and spread sensitivities.

a credit trigger if the downgraded party has a rating one level above the triggering category, k. In the event that a counterparty defaults we use the partial settlement described in Section 2 where it is partial to the swap *with* a credit trigger. These conventions are for illustrative purposes only and it is easy to fit in other market conventions.

In Table I we have chosen the triggering category to be BB, (which often is said to mark the transition to 'speculative grade'). We compare the effect of a trigger in three different settings: The first uses the (empirical) constant generator used in Jarrow, Lando and Turnbull (1997). The second uses a stochastic generator with the affine specification which has been calibrated to match short spreads and spread sensitivities, and the third specification uses a stochastic generator where only the default intensities are changed using a logit functional form for calibration. In the constant case we see that introducing a credit trigger reduces the spreads to somewhere between one third and two thirds depending on ratings. For a B-trigger the spreads are reduced to between 20 and 40% of the original spreads, again depending on ratings. Similarly, the reduction for a BBB-trigger is between 60 and 80%.

In the second case (with the affine generator) a danger in using this calibration shows up: To compensate for the fact that the observed spreads are far from those generated by the empirical transition probabilities, there is a large risk adjustment, and this affects also the non-default transition probabilities. In this example, these take low values after calibration. The low dimension of the risk adjustment parameters only allows us to control for initial spreads and spread sensitivities but there is little control over non-default transition intensities. And in our example, the intensities of direct default become relatively much more important after calibration and this in turn renders the credit trigger much less efficient. This is true also in the final example with the logit generator.

Table I. A comparison of spreads on a swap with a BB credit trigger and regular swap spreads as studied above. The credit trigger has a significant effect in the case of a constant generator, little effect when using the affine and logit generator due to the calibration shifting probability mass from downgrade transitions to direct default transitions.

		AAA	AA	A	BBB
JLT	no trigger	−0.06	0.01	0.18	0.81
Generator	BB-trigger	−0.02	0.00	0.10	0.38
Affine	no trigger	0.43	0.55	0.70	0.92
Generator	BB-trigger	0.42	0.54	0.69	0.90
Logit	no trigger	0.22	0.30	0.42	0.68
Generator	BB-trigger	0.22	0.29	0.39	0.57

7. Default Swaps

As a second illustration of our numerical technique we consider default swaps – i.e., contracts in which one party (the default protection buyer) pays a fixed, periodic amount to the other party (the default protection seller) until whichever comes first: Default of an underlying reference security or maturity of the contract. If the reference security defaults before the maturity of the default swap, the protection seller pays an amount to the protection buyer which compensates (in a sense which varies among contracts) for the loss in value of the reference security. A contract specification with physical delivery would for example allow the protection buyer to exchange the defaulted reference security for its principal. For more on default swaps, see Duffie (1999).

We study the sensitivity of the price of a default swap to changes in the joint credit quality of the protection seller and the reference security. In particular, the effect of correlation can be studied more carefully. A natural intuition to have on default swaps is that the party buying default protection should worry about correlation in the credit quality between the underlying reference security and the default protection seller. We show that if the correlation is ‘weak’ in the sense that defaults do not occur simultaneously, then the effect of correlation is negligible.

For the illustration, we have selected a 5 year default swap with semiannual fixed rate payments where the reference security is a zero-coupon bond. Since our focus is on the correlation between the protection seller and the issuer of the reference security, we assume for simplicity that the protection buyer cannot default. It is possible to include default of the protection buyer, in which case we would need a three-dimensional stochastic process to represent the joint ratings movements.

Table II. Default swap premiums using the JLT generator for different combinations of credit quality of protection seller and reference security. Protection buyer is assumed default-free. Swap premiums are paid in arrear.

Ref \ Ps	AAA	AA	A	BBB	BB	B	CCC
AAA	2.66	2.65	2.65	2.62	2.54	2.43	2.12
AA	7.17	7.16	7.15	7.09	6.90	6.63	5.88
A	19.48	19.47	19.43	19.32	18.94	18.38	16.87
BBB	61.05	61.02	60.94	60.67	59.76	58.41	54.73
BB	202.35	202.30	202.17	201.69	200.09	197.68	190.93
B	425.73	425.69	425.58	425.21	423.93	422.00	416.49
CCC	1131.75	1131.72	1131.63	1131.33	1130.31	1128.77	1124.27

Again, consider for our illustration 11 dates, $T_0, T_1, \ldots, T_{10}$ where $T_i - T_{i-1} = \frac{1}{2}$ for every $i = 1, \ldots, 10$ and $T_0 = 0$. The payment at time T_i is defined as

$$(T_i - T_{i-1})c = \frac{c}{2}$$

In case the protection seller defaults the settlement value is defined in terms of the contract value just before default as in the case of a swap. The settlement in case the issuer of the zero-coupon bond defaults is as follows: Assume the default occurs at time $t \in (T_{i-1}, T_i)$ then the payment made to the protection buyer is

$$(1 - \phi)P(t, T) - (t - T_{i-1})c$$

where ϕ is the recovery rate of the zero-coupon bond and where we have subtracted from the payment the premium for the period between default and next premium payment date. Other payments could have been considered, e.g. a default payment of $1-\phi$, or the fixed payments could be paid in advance instead. This would change the level of the fixed payments but not alter our conclusions.

Now, c is found as a fair fixed rate such that the contract value is 0 at time 0. In Table II we have used the JLT generator for two independent firms. As can be seen from the table the initial category of the protection seller is not that important for the fair fixed rate. This is due to the fact that in a setup with no simultaneous defaults, there is only a loss to the protection buyer in the event that the protection seller defaults before the reference security, but when that happens the loss is a fractional loss on a default swap where the reference security has not yet defaulted.

Next, we have modeled correlation through the transition intensities' joint dependence on the interest rate. In Table III we have used an affine generator to model correlation and Table IV shows the results when using logit default intensities. Again, the credit quality of the protection seller is a 'second order' effect compared to the credit quality of the reference security. It is clear from this, that

Table III. Default swap premiums using the affine generator for different combinations of credit quality of protection seller and reference security. The upper table considers a 'base case' in which the generator is calibrated in order to match short spreads but has zero sensitivity to changes in the short rate. The lower table is also calibrated to match short spreads but has spot rate sensitive intensities as well. Protection buyer is assumed default-free. Swap premiums are paid in arrear.

Ref \ Ps	AAA	AA	A	BBB	BB	B	CCC
AAA	8.90	8.90	8.90	8.90	8.89	8.89	8.89
AA	11.13	11.13	11.13	11.13	11.13	11.13	11.12
A	15.00	15.00	15.00	15.00	15.00	14.99	14.99
BBB	24.18	24.18	24.18	24.18	24.18	24.17	24.16
BB	47.96	47.95	47.95	47.95	47.95	47.94	47.93
B	79.75	79.75	79.75	79.75	79.75	79.74	79.73
CCC	134.59	134.59	134.59	134.58	134.58	134.57	134.56

Ref \ Ps	AAA	AA	A	BBB	BB	B	CCC
AAA	9.66	9.65	9.65	9.63	9.61	9.56	9.47
AA	12.07	12.07	12.05	12.04	12.00	11.94	11.81
A	16.11	16.09	16.08	16.06	16.01	15.93	15.76
BBB	25.46	25.44	25.43	25.40	25.34	25.24	25.04
BB	49.71	49.69	49.67	49.63	49.57	49.45	49.20
B	81.85	81.82	81.78	81.73	81.63	81.45	81.08
CCC	136.69	136.64	136.57	136.48	136.28	135.94	135.24

to get significant effects of correlation we need to have 'strong correlation' in the sense of simultaneous defaults of protection seller and reference security. Finally, the effect of having interest rate sensitive default intensities is small: the relative change in default premiums never exceeds 10% in our example. The relative effects in the plain vanilla swaps could be much larger.

8. Conclusion

We have presented a method for computing swap spreads in models of default based on ratings. The results confirmed findings in Duffie and Huang (1996) that swap spreads are relatively insensitive to credit quality for interest rate swaps. Of course, for a book with thousands of swap contracts the small spreads will add up and there is still a need to price and control the credit risk of the entire portfolio. The framework presented may also be applied to foreign currency swaps. This would,

Table IV. Default swap premiums using the 'logit' generator for different combinations of credit quality of protection seller and reference security. The upper table considers a 'base case' in which the generator is calibrated in order to match short spreads but has zero sensitivity to changes in the short rate. The lower table is also calibrated to match short spreads but has spot rate sensitive intensities as well. Protection buyer is assumed default-free. Swap premiums are paid in arrear.

Ref \ Ps	AAA	AA	A	BBB	BB	B	CCC
AAA	9.80	9.80	9.80	9.80	9.80	9.81	9.81
AA	12.71	12.71	12.71	12.71	12.71	12.71	12.71
A	17.54	17.54	17.54	17.54	17.54	17.54	17.54
BBB	28.42	28.42	28.42	28.42	28.42	28.42	28.42
BB	52.42	52.42	52.42	52.43	52.45	52.47	52.51
B	79.22	79.23	79.24	79.27	79.33	79.39	79.48
CCC	115.22	115.24	115.27	115.34	115.48	115.65	115.88

Ref \ Ps	AAA	AA	A	BBB	BB	B	CCC
AAA	10.14	10.13	10.13	10.12	10.10	10.08	10.04
AA	13.15	13.14	13.13	13.12	13.10	13.07	13.01
A	18.11	18.11	18.10	18.08	18.05	18.01	17.93
BBB	29.23	29.22	29.21	29.19	29.14	29.08	28.98
BB	53.63	53.61	53.59	53.56	53.51	53.43	53.30
B	80.88	80.86	80.84	80.81	80.77	80.69	80.54
CCC	117.34	117.33	117.31	117.30	117.30	117.27	117.15

due to the exchange of principal, produce larger spreads but the computational issues involved would be similar.

Our computations have also shown that using stochastic generators has a large impact on the results. We used two specifications: The affine generator permits analytical expressions for corporate bonds but also allows negative intensities to occur, whereas the logit specification only allows positive intensities but also to our knowledge requires numerical solution of bond prices. Both specifications changed our results for swap prices but mostly in the case with no credit triggers.

We also showed that in our setting the effect of the quality of the protection seller in a default swap was relatively small. This has implications for correlation as well: If changing the rating of the protection seller does not drastically alter the contract value, then, if we hold the marginal distribution of default fixed, but change the correlation between protection seller and reference security this will not have a huge effect. This result depends critically on the fact that we did not allow simultaneous default as a possibility in our example. We applied a 'weak' type correlation in which the correlation between defaults is obtained through the default intensities' joint dependence on the state variables. A 'strong' type of correlation in which the defaults are correlated by allowing for simultaneous transitions in the rating process will be considered in future work. Allowing for simultaneous transitions may capture not only default 'contagion' effects but also the widening of credit spreads on non-defaulted bonds following the default of a particular bond – an issue of great concern to risk managers.

Appendix A

PROOF OF PROPOSITION 1

Consider a time point $t < T$, the date of the contract's final promised cash flow $d_T \equiv d(T, r_T, \eta_T)$. Then

$$V_t = E_t^Q \left[\exp\left(-\int_t^T R_u du \right) d_T \right]$$

In the case where the price processes are driven by Markovian state variables we can compute this expectation by solving an appropriate system of PDEs. Let

$$f(t, r_t, \eta_t) = V_t, \quad t < T \tag{22}$$

$$f(T, r_T, \eta_T) = d_T \tag{23}$$

Define

$$\mathcal{D} f = \frac{\partial}{\partial t} f + \mu \frac{\partial}{\partial r} f + \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial r^2} f \tag{24}$$

By Ito's lemma for processes with jumps (see for example Protter (1990)) we have

$$\begin{aligned} &f(t, r_t, \eta_t) - f(0, r_0, \eta_0) \\ &\quad = \int_0^t \mathcal{D} f(s, r_s, \eta_s) ds + \sum_{0<s\le t} \Big(f(s, r_s, \eta_s) - f(s, r_s, \eta_{s-}) \Big) + M_t \end{aligned}$$

where M_t is a local martingale. The sum can be evaluated as[15]

$$\sum_{0<s\le t} f(s, r_s, \eta_s) - f(s, r_s, \eta_{s-})$$

[15] We here use the representation of a pure jump process as the sum of a (local) martingale and the compensator of the jump process: For illustration, a counting process N with intensity λ, can be written as

$$N_t = N_t - \int_0^t \lambda_s \, ds + \int_0^t \lambda_s \, ds = M_t + \int_0^t \lambda_s \, ds$$

where M is a local martingale. Also, we use the common notation η_{t-} for the left limit of η at time t.

$$= \int_0^t \lambda_{\eta_s} \sum_{l \in \mathbb{L} \setminus \{\eta_s\}} (f(s, r_s, l) - f(s, r_s, \eta_s)) \, P(\eta_s, l) ds + M_t^*$$

where $P(\eta_s, l)$ is the conditional probability that $\eta_s = l$ given that the value of η_{s-} and that the Markov chain jumps at time s, λ_{η_s} is the intensity by which the chain leaves the state η_s, and M_t^* is a local martingale. Continuing on, we find

$$\begin{aligned} &= \int_0^t \left(\lambda_{\eta_s} \sum_{l \in \mathbb{L} \setminus \{\eta_s\}} \frac{\lambda_{\eta_s, l}}{\lambda_{\eta_s}} \Big(f(s, r_s, l) - f(s, r_s, \eta_s) \Big) \right) ds + M_t^* \\ &= \int_0^t \left(\sum_{l \in \mathbb{L} \setminus \{\eta_s\}} \lambda_{\eta_s, l} \Big(f(s, r_s, l) - f(s, r_s, \eta_s) \Big) \right) ds + M_t^* \\ &= \int_0^t \Big(\Lambda \bar{f}(s, r_s) \Big)_{\eta_s} ds + M_t^*. \end{aligned}$$

Here, $\Big(\Lambda \bar{f}(s, r_s) \Big)_{\eta_s}$ denotes the η_s'th element of the vector $\Lambda \bar{f}(s, r_s)$. Also, in the last equality we use the fact that a diagonal element of Λ is minus the sum of the off-diagonal elements in the same row. We can now write the differential of f as

$$df(t, r_t, \eta_t) = \Big(\mathcal{D} f(t, r_t, \eta_t) + \big(\Lambda \bar{f}(t, r_t) \big)_{\eta_t} \Big) dt + d\tilde{M}_t \tag{25}$$

where $\tilde{M}_t$ is a local martingale.

Next, consider the discounted pre-default process

$$Y_s = f(s, r_s, \eta_s) \beta_{t,s} \qquad s > t \tag{26}$$

where β is defined as

$$\beta_{t,s} = e^{-\int_t^s R_u du}$$

Now, using integration by parts

$$Y_T = f(t, r_t, \eta_t) + \int_t^T f(s, r_s, \eta_s) d\beta_{t,s} + \int_t^T \beta_{t,s} df(s, r_s, \eta_s) \tag{27}$$

Inserting (25) in (27) gives

$$Y_T - f(t, r_t, \eta_t) \tag{28}$$

$$= \int_t^T f(s, r_s, \eta_s) d\beta_{t,s} + \int_t^T \beta_{t,s} \Big(\mathcal{D} f(s, r_s, \eta_s) + \big(\Lambda \bar{f}(s, r_s) \big)_{\eta_s} \Big) ds + \hat{M}_t$$

$$= \int_t^T \beta_{t,s} \left(\mathcal{D} f(s, r_s, \eta_s) + \left(\Lambda \bar{f}(s, r_s)\right)_{\eta_s} - R_s f(s, r_s, \eta_s) \right) ds + \hat{M}_t$$

where $\hat{M}_t$ by assumption is a martingale. We also have from the definition of f and the expression for V that

$$f(t, r_t, \eta_t) = E_t^Q \left(\beta_{t,T} d_T\right) \quad \text{for } t < T \tag{29}$$

By taking conditional expectation on both sides of (28) we have a second expression for $f(t, r_t, \eta_t)$, and for the two expressions to be equal we must have for all $t < T$ that

$$E_t^Q \left[- \int_t^T \beta_{t,s} \left(\mathcal{D} f(s, r_s, \eta_s) + \left(\Lambda \bar{f}(s, r_s)\right)_{\eta_s} - R_s f(s, r_s, \eta_s) \right) ds \right] = 0 \tag{30}$$

Hence $f(s, r_s, \eta_s) = V_s$ is a solution to the partial differential equation

$$\mathcal{D} f(s, r_s, \eta_s) + \left(\Lambda \bar{f}(s, r_s)\right)_{\eta_s} - R_s f(s, r_s, \eta_s) = 0. \tag{31}$$

and from the lumpy dividend paid at time T we have the boundary condition

$$f(T, r, \eta) = d_T.$$

Using a little algebra and writing (31) with the same vector notation as in (7) gives the result.

■

Appendix B

In this appendix, we illustrate the construction of joint generators where independence between the two chains is maintained and we illustrate the distinction between generators with and without the default category. First, let the marginal generator matrices without the default category be given as

$$\Lambda^A = \Lambda^B = \begin{bmatrix} -\alpha & \alpha \\ \beta & -\beta \end{bmatrix}$$

and the default intensities as a 2-dimensional vector

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$$

Then the generator matrix including the default category, Λ^D is defined as:

$$\Lambda^D = \begin{bmatrix} -(\alpha + \lambda_1) & \alpha & \lambda_1 \\ \beta & -(\beta + \lambda_2) & \lambda_2 \\ 0 & 0 & 0 \end{bmatrix}$$

where the third category (default) is absorbing.

The combined generator matrix without the default category will be a 4×4 matrix with state space $\{(1, 1), (1, 2), (2, 1), (2, 2)\}$. Using this sequence of the states and assuming that counterparties A and B are independent (in that case there are no simultaneous jumps) we define the combined generator by

$$\Lambda = \begin{bmatrix} -2\alpha & \alpha & \alpha & 0 \\ \beta & -(\alpha+\beta) & 0 & \alpha \\ \beta & 0 & -(\alpha+\beta) & \alpha \\ 0 & \beta & \beta & -2\beta \end{bmatrix}$$

I.e., Λ_{12} is the intensity that counterparty B jumps from category 1 to 2 and A remains in category 1. Similarly, Λ_{13} is the intensity that counterparty A jumps from category 1 to 2 and B stays in category 1. The 4-dimensional vectors of default intensities are defined by

$$\lambda^A = \begin{bmatrix} \lambda_1 \\ \lambda_1 \\ \lambda_2 \\ \lambda_2 \end{bmatrix} \qquad \lambda^B = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_1 \\ \lambda_2 \end{bmatrix}$$

Appendix C

A program solving for the swap price would look something like

```
Initialization
find A_1^{-1}
find A_2^{-1}
for l = 1 to # of payments
      swap = swap + Payment(l)
      for i = 1 to n
            temp = swap*B_1
            swap = A_1^{-1}*temp
            temp = B_2*swap-swap*D^φ
            swap = temp*A_2^{-1}
```

where `n` is the number of time steps between payments and `temp` is the right hand side of the equations, updated for each iteration. The initialization involves reading the matrix Λ from a file and initializing the matrix D.

In the case of a time dependent intensity matrix we need to do the inversion of $(I - \frac{\Delta t}{2}\Lambda')$ in each time step. This means that the program will be slower.

Our simulations suggest that the time needed for calculating swap prices with a time dependent intensity matrix is approximately 10% higher than for the constant intensity matrix. A sketch of the program is outlined below.

```
Initialization
find A_1^{-1}
for l = 1 to # of payments
      swap = swap + Payment(l)
      for i = 1 to n
            temp = swap*B_1
            swap = A_1^{-1}*temp
            find A_2^{-1}
            temp = B_2*swap-swap*D^φ
            swap = temp*A_2^{-1}
```

Another complication is an intensity matrix with dependence on the interest rate. This complicates the program, since each row of `swap` is the swap price for different values of the interest rate. This means that we need to split up the matrix `swap` and we will call the `r`'th row for `swap(r)` where `r` is between `rmin` and `rmax`. This idea is outlined below.

```
Initialization
find A_1^{-1}
for r = rmin to rmax by Δr
      find A_2(r)^{-1}
for l = 1 to # of payments
      swap = swap + Payment(l)
      for i = 1 to n
            for r = rmin to rmax by Δr
                  temp(r) = swap(r)*B_1(r)
                  swap(r) = A_1^{-1}*temp(r)
                  temp(r) = B_2*swap(r)-swap(r)*D^φ(r)
                  swap(r) = temp(r)*A_2(r)^{-1}
            collect swap(r) in the matrix swap
```

This program is approximately 50% slower than the original program.

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