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The Valuation of Default Risk in Corporate Bonds and Interest Rate Swaps

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The Valuation of Default Risk in Corporate Bonds and Interest Rate Swaps [1]

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Abstract : This paper implements a model for the valuation of the default risk implicit in the prices of corporate bonds. The analytical approach considers the two essential ingredients in the valuation of corporate bonds: interest rate uncertainty and default risk. The former is modeled as a diffusion process. The latter is modeled as a spread following a diffusion process, with the magnitude of this spread impacting on the probability of a Poisson process governing the arrival of the default event. We apply two variants of this model to the valuation of fixed-for-floating swaps. In the first, the swap is default-free, and the spread represents the appropriate discounted expected value of the instantaneous TED spread; in the second, we allow the swap to incorporate default risk. We propose to test our models using the entire term structure of corporate bonds prices for different ratings and industry categories, as well as the term structure of fixed-for-floating swaps.

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The Valuation of Default Risk in Corporate Bonds and Interest Rate Swaps*

1 Introduction

The valuation of securities subject to default risk has been of interest in the academic and practitioner literature for some time. Beginning with Merton (1974), a substantial body of literature has modeled the default event as dependent on the value of the firm, F, with

$$\frac{dF}{F} = \mu\, dt + \sigma\, dz,$$

and default event triggered by $F \leq B$. B can be either the maturity value of debt, or a prespecified default boundary. For the latter, see, e.g., the recent work of Hull and White (1992) and Longstaff and Schwartz (1994). In contrast, the approach we offer below is silent on the causality of default, and hence does not require specification of the precise events which cause the firm to seek, or be forced into, a state of bankruptcy. Hence, this approach enjoys a greater generality relative to its predecessors. Following Jarrow and Turnbull (1992), Jarrow, Lando and Turnbull (1994), Lando (1994), Madan and Unal (1994) and Duffie and Singleton (1994), the current model abstracts from an explicit dependence on an unobservable firm value. In contrast, our model explicitly models the instantaneous default spread on risky bonds, correlates that spread to the riskfree rate of interest, and links the default intensity to its presumed market-efficient predictor, the prevailing instantaneous credit spread.

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The approach we suggest leads to directly testable results in the valuation of corporate bonds. We submit our model to rigorous econometric testing using a data base composed of the entire term structures of corporate bonds for different ratings and industry classifications.

Practitioners have of late become interested in models of default risk for several reasons. First, understanding the default event permits a more accurate modeling of issues subject to default in general, and corporate bonds in particular. Second, the modeling of the default event permits the construction and valuation of alternative securities whose payoff is triggered by the default event (or lack thereof), which go by the generic name "credit derivatives."[1]

The valuation of interest rate swaps has attracted significant attention. One strand of literature, including Evans and Bales (1991), Litzenberger (1992), and Chen and Selender (1994) have reported on the relative *insensitivity* of swap spreads to default measures, while Grinblatt (1995) has modeled the swap spread as a compensation for illiquidity relative to Treasuries. In contrast, the works of Cooper and Mello (1991) and Longstaff and Schwartz (1995) have attributed swap spreads to default risk, while Bhasin (1995) has recently reported on the increasing cross-sectional credit-quality variety of OTC-derivative

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[1]Examples of such credit derivatives include structures whose maturity date-t payoffs are

$$\max\{Y_t - T_t - K, 0\}$$

or

$$\max\{P_t(Y) - P_t(T+K), 0\},$$

where

Y_t = Yield-to-maturity on defaultable bond or bond-index on date t

T_t = On-the-run benchmark Treasury yield on date t

K = strike rate

$P_t(Y)$ = price of defaultable bond at time t (given yield Y)

$P_t(T+K)$ = time t value of defaultable bond at pre-determined spread K to time t-prevailing Treasury yield T

counterparties. Swap contracts are natural candidates for two-factor modeling, and in this paper we present two such models: in the first, the swap spread is simply the riskfree discounted expected present value of the TED (LIBOR — Treasury) spread, whereas the second model generalizes the first to admit default risk.

The paper is organized as follows. Section 2 proposes the two-factor corporate-bond model. Section 3 discusses the data and methodology, with Section 4 reporting on the model's empirical results. Section 5 presents default-free as well as risky two-factor swaps models, and their attendant empirical estimates. Section 6 concludes.

2 A Two-Factor Model of Default Risk

2.1 Model

The two-factor model of the default risk assumes the following process:

$$\begin{aligned} dr &= \mu_t\, r\, dt + \sigma_r\, r\, dz_1 \\ ds &= \sigma_s\, s\, dz_2 \end{aligned} \tag{1}$$

where

$r =$ is the instantaneously-riskless rate of interest

μ_t = the expected change in dr; μ_t is chosen to precisely match the observable term structure of interest rates

$\sigma_r =$ is the volatility of r

$s =$ instantaneous yield spread for risky corporate asset

$\sigma_s =$ is the volatility of s

and $\text{Corr}(dr, ds) = \rho\, dt$.[2]

[2] If we wished s to be lognormally distributed with zero drift, then the alternative specification to $ds = \sigma_s\, s\, dz_2$ in eq. (1) is $ds = \frac{1}{2}\sigma_s^2 s\, dt + \sigma_s\, s\, dz_2$.

As is well-known, there is a distinct relationship between the default intensity λ, the instantaneous default spread s and the after-default recovery rate D. Specifically, given our notation, consider the par value of an instantaneously-maturing risky debt issue promising a principal-cum-coupon redemption payment of $\exp\{-(r+s)\ dt\}$. Under risk-neutral valuation, this value is equal to the discounted expected value of the payoffs:

$$1 = \exp\{-r\,dt\}\left[D \cdot \lambda\,dt + \exp\{(r+s)\ dt\} \cdot (1 - \lambda\,dt)\right],$$

where λ is the default intensity. Multiplying through by $\exp\{r\,dt\}$ and making the substitution $\exp\{x\,dt\} \cong 1 + x\,dt$ results in

$$\lambda = \frac{s}{1-D}.$$

The intuition for this model incorporates the following aspects:

1. The corporation's bond prices are presumed to drift downward as default becomes increasingly likely (e.g., as they are downgraded).

2. Given the default event has occurred, the bond sells at a post-default price of D, some fraction of face value.[3]

3. We allow for a correlation in changes between r and s, and perform period-by-period discounting using riskless rates of interest.

4. Over the next interval, if the instantaneously-maturing security does not default, it will pay a return in excess of the riskfree rate; otherwise, it will sell at a price D.

2.2 Implementation

Denote by T the bond's maturity date. Since default can occur at any time $t \leq T$ — and equations (1) specify joint lognormal diffusions — we require a lattice implementation:

[3]In a more general model, the fraction D would not be a constant but presumably depend on the prevailing term structure of interest rates.

For constant σ's, step-size of Δ and $\lambda(s)\Delta < 1$, the quadrinomial lattice proposed for this process is:[4]

$$\begin{bmatrix} r(t+\Delta) \\ s(t+\Delta) \end{bmatrix} = \begin{cases} \begin{bmatrix} r(t)\exp\left\{\mu(t)\Delta + a_1\sqrt{\Delta}\right\} \\ s(t)\exp\left\{a_2\sqrt{\Delta}\right\} \end{bmatrix} & \text{with Prob.} = \lambda(s)\Delta \\ \begin{bmatrix} r(t)\exp\left\{\mu(t)\Delta + b_1\sqrt{\Delta}\right\} \\ s(t)\exp\left\{b_2\sqrt{\Delta}\right\} \end{bmatrix} & \text{with Prob.} = p - \alpha\lambda(s)\Delta \\ \begin{bmatrix} r(t)\exp\left\{\mu(t)\Delta + c_1\sqrt{\Delta}\right\} \\ s(t)\exp\left\{c_2\sqrt{\Delta}\right\} \end{bmatrix} & \text{with Prob.} = q - \beta\lambda(s)\Delta \\ \begin{bmatrix} r(t)\exp\left\{\mu(t)\Delta + d_1\sqrt{\Delta}\right\} \\ s(t)\exp\left\{d_2\sqrt{\Delta}\right\} \end{bmatrix} & \text{with Prob.} = 1 - p - q + (\alpha+\beta-1)\,\lambda(s)\Delta \end{cases} \tag{2}$$

The system (2) results in an efficient, recombining tree; see Appendix A for details.

The parameter set $\{a_i, b_i, c_i, d_i, p, q, \alpha, \beta;\ i = 1, 2\}$ is chosen to satisfy the mean, variance and covariance conditions implied in equations (1).[5] Specifically, we require that, under the risk-neutral probability measure,

$$E\begin{bmatrix} dr/r \\ ds/s \end{bmatrix} = \begin{bmatrix} \mu_t \\ 0 \end{bmatrix} dt \tag{3}$$

$$\text{Var}\begin{bmatrix} dr/r \\ ds/s \end{bmatrix} = \begin{bmatrix} \sigma_r^2 \\ \sigma_s^2 \end{bmatrix} dt \tag{4}$$

$$\text{Cov}\left(\frac{dr}{r}, \frac{ds}{s}\right) = \sigma_{12}\, dt \tag{5}$$

Appendix A demonstrates that these conditions give rise to 10 equations in 12 variables,

[4] The quadrinomial lattice contains the minimal number of nodes required to satisfy the first- and second-moment restrictions specified in eqs. (1).

[5] In addition to the constraint $\lambda(s)\Delta < 1$, we require that $p - \alpha\lambda(s)\Delta > 0$, $q - \beta\lambda(s)\Delta > 0$ and $1 - p - q + (\alpha+\beta-1)\,\lambda(s)\Delta < 1$.

and hence is underidentified. That appendix also demonstrates that the quadrinomial tree can be constructed with the following special values $\alpha = \beta = a_1 = d_1 = 0$ and $p = q = 1/4$. This choice of parameters satisfies the constraints that all probabilities be positive so long as $\lambda(s)\Delta \leq 1/2$.[6]

Note that the above tree is not a true quadrinomial, for whenever a default occurs, that branch of the tree terminates. In fact, for the postulated constant u's, we obtain a simple recombining tree.[7]

3 Data: Term Structures of Treasury Bonds, Corporate Bonds and Swaps

The data for our analysis consists of the term structures of interest rates for alternate-categories ratings Merrill Lynch & Co. bond indices, specified in Appendix B: these bond indices constitute examples wherein we observe an entire term structure of Treasury and corporate credit spreads through a 10-year maturity; with semi-annually separated intervals, this implies data on 20 par bonds with maturities ranging from one-half year to ten years. An interesting component of this data is the swap curve, which in turn permits the application of this methodology to the calculation of the default risk implicit in these important securities.

Our data includes observations on the par rates for Treasury and corporate bonds — including standard fixed-for-floating-LIBOR swap rates — at maturities of two, three, seven and ten years. Denote these par rates c_t^T and c_t^C for Treasury and corporate rates, respectively. From this data, we can extract the zero-coupon Treasury and corporate rates

[6] Clearly, when $\alpha = \beta = 0$ and $p = q = 1/4$, p, $q \in (0, 1)$ and

$$1 - p - q + (\alpha + \beta - 1)\,\lambda(s)\Delta = 1/2 - \lambda(s)\Delta,$$

which is always positive for finite s and $\lambda\Delta \leq 1/2$. We also tested the model using small values of A, which imply fewer restrictions on parameter values in order to assure positive state prices; the resulting empirical magnitudes did not change substantially.

[7] If the σ's were time-varying, recombination would require recalculating probabilities at each step of the tree.

through a "bootstrap" technique which complements the observable bonds by linearly interpolating the par rates for each semi-annually separated maturity from six months through ten years. The technique uses the postulated twenty par rates and sequentially solves for the zero-coupon rate that would give rise to the observable par rates. The analytical details of this technique are described in Appendix C.

Figure 1 displays an example of the Treasury and corporate par rate curve for several ratings categories, including the swap curve.

4 Default-Risk Parameter Estimation for Corporate Bonds

4.1 Methodology

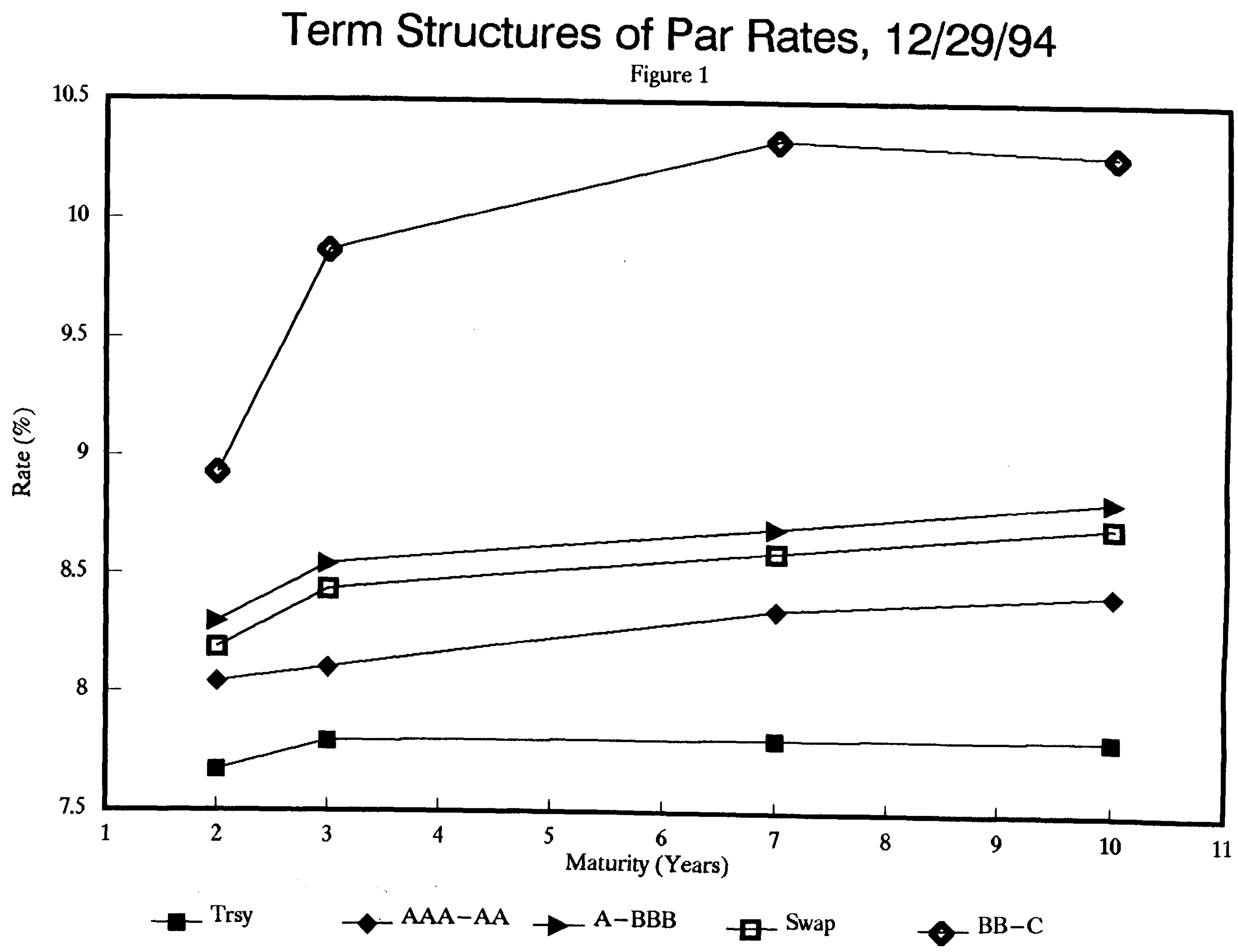
Using this data, our analysis determines the *implicit* values of the parameters which explain the cross-sectional term structures of interest rates for alternate ratings categories.

Specifically, assume that we observe the term structure of the riskless rate of interest, and the term structure of credit spreads for a given ratings category. This data is conveyed through the 20 par rates on the Treasury and corporate securities. Let P_t^T and P_t^C denote the market prices of the Treasury corporate securities, respectively, for maturity t, t = 0.5, 1, . . . , 10; by definition, these are par bonds so for a unit face value issue, $P_t^T = P_t^C = 1$ for all t. For a unit face value risky bond, the payoff on all dates $t \leq T$ on these bonds is

$$A = \begin{cases} c_T/2 & \text{if the bond has not defaulted and } t < T \\ 1 + c_T/2 & \text{if the bond has not defaulted and } t = T \\ D & \text{if the bond is in default} \end{cases}$$

With this definition of A, the V_t^C for all t are obtained as the present value of the expected payoffs under the risk-neutral expectation E.[8]

[8]The valuation of swaps is perforce different, since the widening/tightening of the swap spread does not necessarily imply a concomitant increase/decrease in the probability of a swap counterparty's default. Rather, the model relates the default intensity, λ, to the mark-to-market absolute value of the swap at a given point in time.


Term Structures of Par Rates, 12/29/94
Figure 1
Rate (%)
10.5
10
9.5
9
8.5
8
7.5
1
2
3
4
5
6
7
8
9
10
11
Maturity (Years)
Trsy
AAA−AA
A−BBB
Swap
BB−C

Recall that the time-dependent drift parameter μ_t is used to match the Treasury term structure exactly: for a unit face value, $P_t^T = V_t^T = 1$ for all t. Thus, any time-inhomogeneity is assumed captured in equation (3)'s μ_t term. For an initial value of s, the process for the default spread is then time-homogeneous and entirely determined by the fixed paramters σ_r, σ_s, ρ, D.

Ideally, one would implement an empirical procedure to estimate the value of the parameters σ_r, σ_s, ρ, D. The disadvantage in doing so, however, follows from the interaction between λ, s and D. Note that there exists a degenerate solution in which $D \to 1$: There is immediate default, but with full recovery. While this is a feasible empirical solution, it is clearly an unintuitive one. In order to preclude such a solution, we fix D at two arbitrary values, $D = 0$ and $D = .5$.

For fixed D, if the random error terms $e_t \equiv P_t^C - V_t^C$ are independently and identically distributed, then the estimation problem is

$$\min_{\{\sigma_r, \sigma_s, \rho\}} \sum_{t=1}^{20} \left[P_t^C - V_t^C(\sigma_r, \sigma_s, \rho)\right]^2 \tag{6}$$

subject to the obvious conditions $\sigma_r > 0$, $\sigma_s > 0$, $-1 \leq \rho \leq 1$. For each ratings category, we have 20 data points and three parameters to estimate.

In calibration estimations such as these, we infrequently encounter the possibility of estimating an implied correlation coefficient. In the current case, objective (6) permits the eliciting of this implied correlation parameter ρ.

The estimation procedure can be ratings category-specific or it can be market-wide: note that two parameters — σ_r and D — are in principle constant across different categories. Thus, the objective function (6) could be estimated with and without imposing the identical parameter values across different ratings categories.[9]

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[9] Of course, an estimate of σ_r and σ_s can also be obtained from a recent time series of these variables.

4.2 Statistical Properties of the Estimates

The above model is amenable to non-linear least squares estimation. If

$$\mathbf{y}(\boldsymbol{\beta}) = \mathbf{f}(X, \boldsymbol{\beta}) + \mathbf{e},$$

and $\mathbf{e} \sim (\mathbf{0}, \sigma^2 \mathbf{I})$ (independently and identically distributed), then the least-squares estimator is found by choosing $\boldsymbol{\beta}$ to minimize

$$\begin{aligned} S(\boldsymbol{\beta}) &\equiv [\mathbf{y} - \mathbf{f}(X, \boldsymbol{\beta})]' [\mathbf{y} - \mathbf{f}(X, \boldsymbol{\beta})] \\ &= \sum_{t=1}^{20} \left[P_t^C - V_t^C (\sigma_r, \sigma_s, \rho) \right]^2 \end{aligned}$$

where

$$\mathbf{y} = \left[P_{0.5}^C, P_1^C, \ldots, P_{10}^C \right]' = [1, 1, \ldots, 1]'$$

$$\mathbf{f} = \left[V_{0.5}^C, V_1^C, \ldots, V_{10}^C \right]'$$

$$\boldsymbol{\beta} = \{\sigma_r, \sigma_s, \rho\}'$$

and X constitutes the corporate and Treasury par rates for $t = 0.5, 1, \ldots, 10$.

If the resultant estimator for $\boldsymbol{\beta}$ is $\boldsymbol{\beta}^*$, then

$$\hat{\sigma}^2 = \frac{S(\boldsymbol{\beta}^*)}{T - K},$$

where $T = 20$, $K = 5$, and the variance-covariance matrix, $\hat{\boldsymbol{\Sigma}}$, is given by

$$\hat{\boldsymbol{\Sigma}} = \hat{\sigma}^2 \left[\left. \frac{\partial \mathbf{f}}{\partial \boldsymbol{\beta}} \right|_{\boldsymbol{\beta}^*} \left. \frac{\partial \mathbf{f}}{\partial \boldsymbol{\beta}'} \right|_{\boldsymbol{\beta}^*} \right]^{-1}.$$

4.3 Empirical Results

For 12/29/94, Table 1 reports the following set of parameter estimates:

Table 1 — Parameter Estimates for 12/29/94

Parameter	D = 0					
	AAA-AA	A-BBB	BB-C	All Three Categories		
				AAA-AA	A-BBB	BB-C
$\hat{\sigma}_r$	.0009 (7.19)	.0008 (17.7)	.376 (.129)		.188 (.084)	
$\hat{\sigma}_s$	.130 (1.23)	.115 (1.27)	.862 (.613)	.398 (6.67)	.446 (8.28)	.137 (.772)
$\hat{\rho}$	.973 (2.25)	.990 (2.07)	.255 (.050)	.235 (.606)	.394 (3.48)	1.0 (.432)

Parameter	D = 0.5					
	AAA-AA	A-BBB	BB-C	All Three Categories		
				AAA-AA	A-BBB	BB-C
$\hat{\sigma}_r$	.0005 (24.0)	.001 (19.1)	.408 (.319)		.259 (.088)	
$\hat{\sigma}_s$	.120 (3.49)	.116 (1.21)	.675 (3.95)	1.599 (151)	.605 (5.62)	.118 (.536)
$\hat{\rho}$	.991 (5.07)	.991 (2.02)	.256 (.644)	−.951 (29.7)	.108 (.774)	1.0 (.344)

Note: Numbers in parentheses represent standard errors.

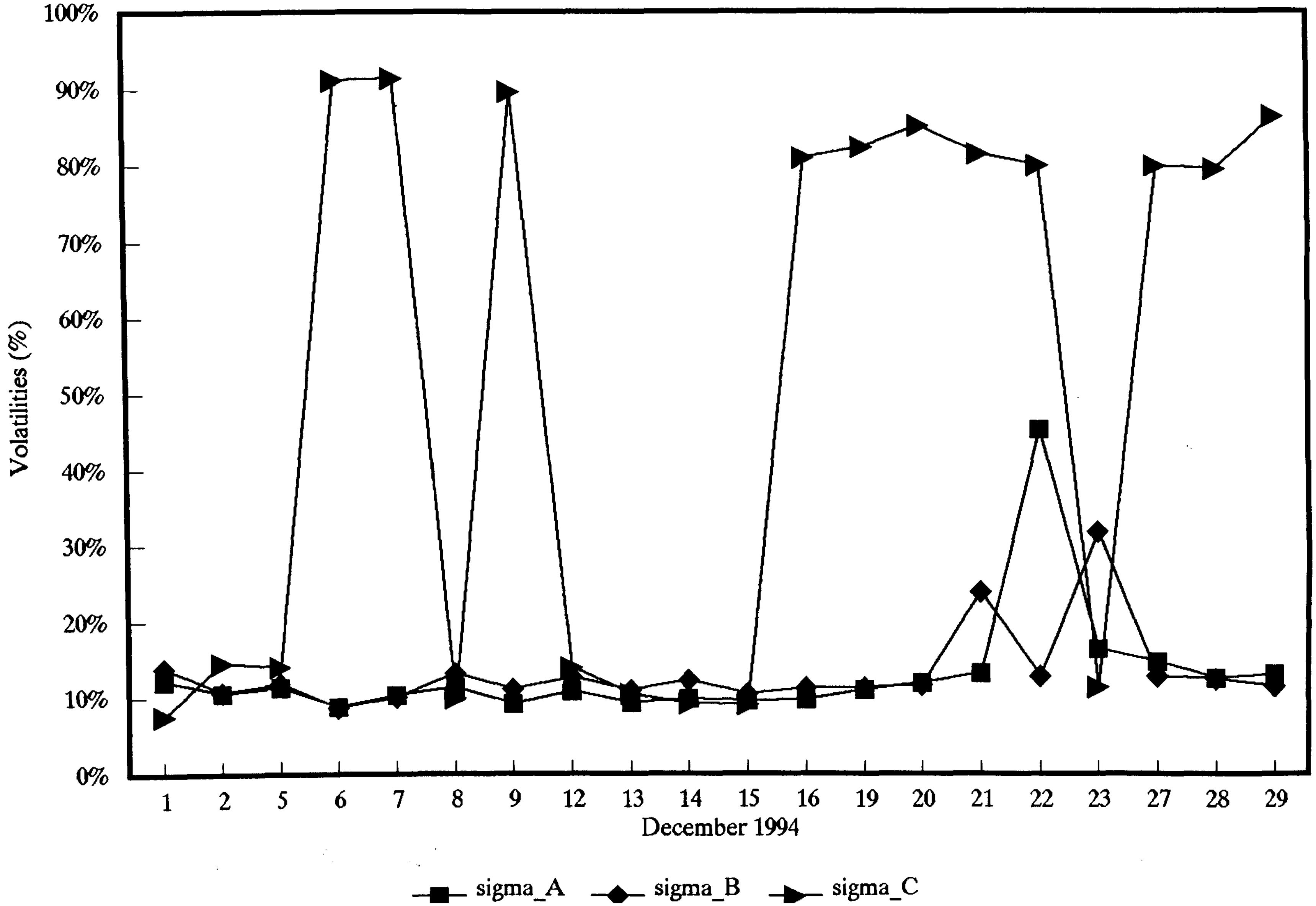
For D = 0, Panel A of Table 1 reports the separate-category estimates of implied volatilities for σ_s, s = { AAA-AA, A-BBB, BB-C } on 12/29/94 as equal to {.130, .115, .862}. Figure 2 contains a perspective on these values by presenting the time-series estimates of these parameters for the entire month of December 1994.

For D = 0.5, Figure 3 displays the time-series of implied volatility for the riskless rate of interest, i.e., σ_r. Other than two days in November, the estimated value of σ_r displays remarkable stability for the fourth quarter of 1994. Similar results also obtain for D = 0.

We performed sensitivity analyses on the parameter estimates and observed the follow-

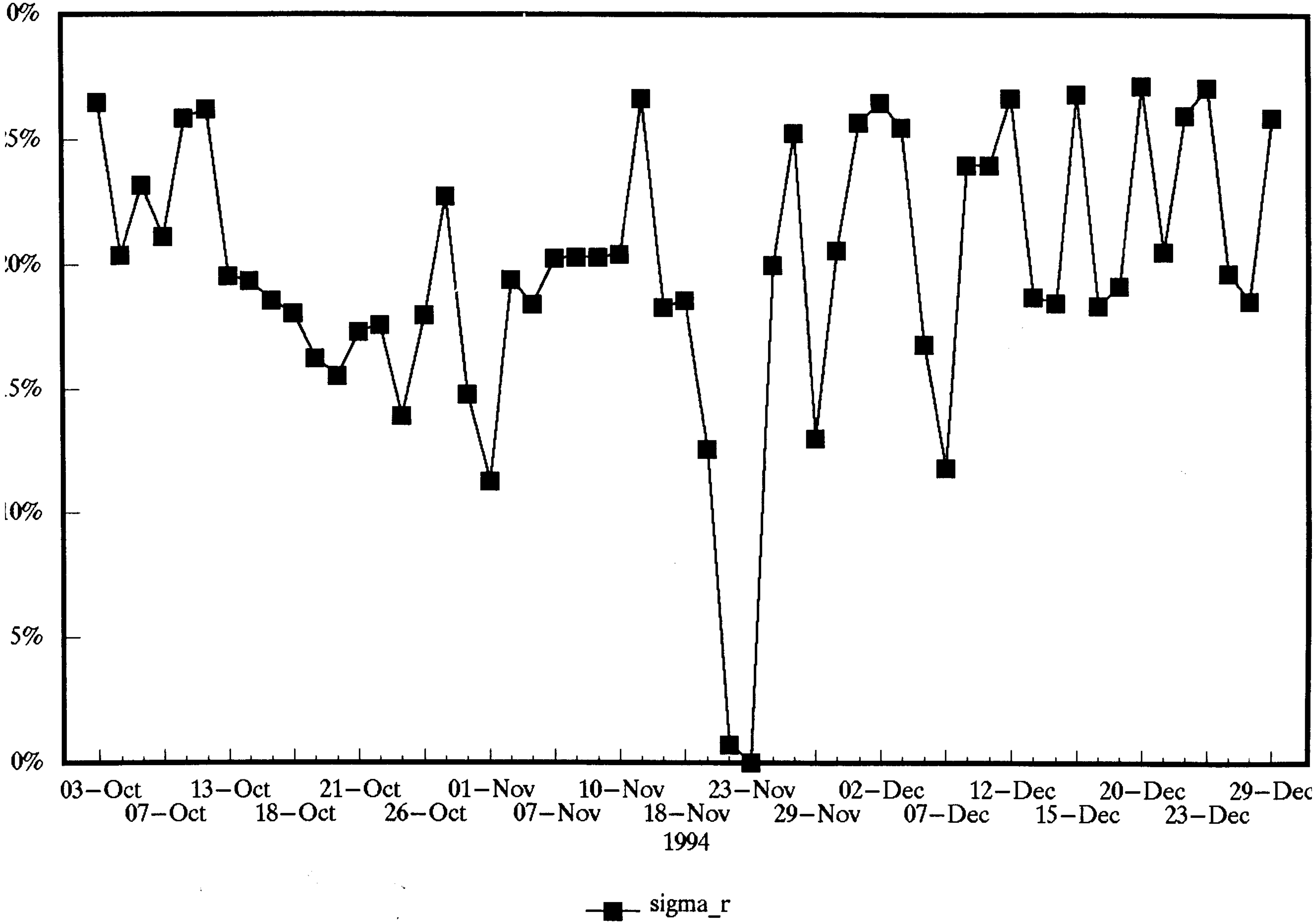
Implied Volatilities of Default Spreads (for D = 0)

Figure 2



Implied Volatility of Riskless Interest Rate (D = 0.5)

Figure 3



ing effects:

$$\frac{\partial V_t^C}{\partial \sigma_r} < 0, \quad \frac{\partial V_t^C}{\partial \sigma_s} < 0,$$
$$\frac{\partial V_t^C}{\partial \rho} < 0 \quad \text{(weak effect)}$$

Interestingly, the impact of the correlation effect was negative, but only weakly so. Our model's empirical result thus contrasts with the finding of Longstaff and Schwartz (1994), who find that "the correlation of a firm's assets with changes in the level of the interest rate can have significant effects on the value of risky fixed-income securities" (p. 23).

5 A Two-Factor Model for the Swap Curve

5.1 A Two-Factor Default-Free Model for the Swap Curve

As noted above, swap contracts are natural candidates for two-factor modeling. Thus, our first approach models the swap spread as the riskfree discounted expected present value of the TED spread.

Assume that a vanilla interest-rate swap is default-free. Let v_T^t be the time t mark-to-market value of an interest rate swap receiving fixed and paying floating LIBOR on a notional amount of \$1. The swap rate for maturity T, p_T, is then determined implicitly as follows. At time $T-1/2$, for $\Delta = 1/2$,

$$v_T^{T-1/2} = \tfrac{1}{2}\left(p_T - r_{T-1/2} - s_{T-1/2}\right)\exp\left\{-r_{T-1/2}/2\right\}$$

for all possible values of $v_T^{T-1/2}$, $r_{T-1/2}$ and $s_{T-1/2}$ at $T-1/2$. Further, for all values of v_T^t, r_t and s_t at time t, $0 < t < T-1/2$,

$$v_T^t = E_t\left[\tfrac{1}{2}\left(p_T - r_t - s_t\right) + v_T^{t+\Delta}\right]\exp\left\{-r_t/2\right\},$$

where $E_t(\cdot)$ is the time-t risk-neutral expectation operator. Now, setting $v_T^0 = 0$ (for $t = 0$)

yields the value of the T-maturity swap rate p_T.

Now consider the valuation of put swaptions. Let $P(t, T, K)$ be the price (per unit notional amount) of a put swaption with time to expiration t, swap-maturity T and exercise rate K. The value of the put swaption is the discounted expected value of the payoff, $E_0\left[\max\{-v_T^t, 0\}\int_0^t \exp\{-r_s\}\, ds\right]$. From the observable values of the swap rates p_T and put swaptions prices $P(t, T, K)$, we can obtain empirical estimates of the model's parameters σ_r, σ_s, ρ. Specifically, if

$V(\sigma_r, \sigma_s, \rho \mid t, T, K)$ = time 0 value of a put swaption with indicatives t, T, K, given distributional parameters σ_r, σ_s, ρ

$P(t, T, K)$ = market price of above put swaption

w_1, w_2 = weights

then the estimation procedure is

$$\min_{\{\sigma_r, \sigma_s, \rho\}} \; w_1 \sum_{t=1}^{20} \left(v_0^t\right)^2 + w_2 \sum_{\{t,T,K\}} \left[V(\sigma_r, \sigma_s, \rho \mid t, T, K) - P(t, T, K)\right]^2$$

subject to $\sigma_r > 0$, $\sigma_s > 0$, $-1 \leq \rho \leq 1$. In implementation, we choose $w_1 = 0$, $w_2 = 1$, in order to place emphasis on the higher-order volatility-dependence of swaption prices.

5.2 A Two-Factor Risk Model for the Swap Curve

A default-risk approach to the valuation of the swap curve requires a modification of the corporate bond-default model. Specifically, if the swap is the sole contract existing between the two parties, and the two parties currently have identical credit rating, then the probability of default increases with the absolute value of the mark-to-market on the swap. We require this assumption of default symmetry, since our empirical swap rates do not identify the counterparties or their ratings categories.[10] Let v_T^t be the time t mark-to-market value of an interest rate swap receiving fixed and paying floating LIBOR

[10]For a modeling of the swap contract with asymmetric default assumptions, see Duffie and Huang (1995).

on a notional amount of $1. The swap rate for maturity T, p_T, is then determined as the implicit solution to the following procedure. At time $T-1/2$, for $\Delta = 1/2$ and a partial-recovery-rate parameter $k < 1$,

$$\begin{aligned} v_T^{T-1/2} &= \left[1-\lambda\left(\left| v_T^{T-1/2} \right|\right)\Delta\right]\tfrac{1}{2}\left(p_T - r_{T-1/2} - s_{T-1/2}\right)\exp\left\{-r_{T-1/2}/2\right\} + \\ &\quad \tfrac{1}{2}k\lambda\left(\left| v_T^{T-1/2} \right|\right)\Delta\left(p_T - r_{T-1/2} - s_{T-1/2}\right)\exp\left\{-r_{T-1/2}/2\right\} \\ &= \tfrac{1}{2}\left[1-\lambda\left(\left| v_T^{T-1/2} \right|\right)\Delta + k\lambda\left(\left| v_T^{T-1/2} \right|\right)\Delta\right]\left(p_T - r_{T-1/2} - s_{T-1/2}\right)\exp\left\{-r_{T-1/2}/2\right\} \end{aligned}$$

for all possible values of $v_T^{T-1/2}$, $r_{T-1/2}$ and $s_{T-1/2}$ at $T-1/2$. Here, $|\cdot|$ denotes the absolute value operator and $\lambda(|x|)\Delta$ is the default probability as an increasing function of $|x|$; the use of $|v_T^t|$ embodies our modeling of the default intensity as symmetric about zero point. For all values of v_T^t, r_t and s_t at time t, $0 < t < T-1/2$,

$$\begin{aligned} v_T^t &= \left|1-\lambda\left(\left| v_T^t \right|\right)\Delta\right| E\left|\tfrac{1}{2}\left(p_T - r_t - s_t\right) + v_T^{t+\Delta} \mid \text{No default}\right| \exp\left\{-r_t/2\right\} + \\ &\quad k\lambda\left(\left| v_T^t \right|\right)\Delta\left|\tfrac{1}{2}\left(p_T - r_t - s_t\right) + v_T^{t+\Delta}\right| \exp\left\{-r_t/2\right\} \\ &\equiv y\left|1-\lambda\left(\left| v_T^t \right|\right)\Delta + ky\lambda\left(\left| v_T^t \right|\right)\Delta\right| \exp\left\{-r_t/2\right\} \\ &= y\left(k-1\right)\lambda\left(\left| v_T^t \right|\right)\Delta\exp\left\{-r_t/2\right\} + y\exp\left\{-r_t/2\right\} \\ &\equiv a\lambda\left(\left| v_T^t \right|\right)\left(k-1\right) + b \end{aligned} \tag{7}$$

where

$y \equiv E(\cdot \mid \text{No default})$ is the risk-neutral expectation operator, conditional on no default occurring over the next interval[11]

$a = y\Delta\exp\{-r_t/2\}$

$b = y\exp\{-r_t/2\}$

We can solve this equation by positing an increasing function $\lambda(|v_T^t|)$. For simplicity,

[11] In the implementation, $E(\cdot \mid \text{No default})$ uses the probabilities p, q and $1-p-q-\lambda(s)\Delta$ specified in the system (2).

assume that in the region of interest, we can write the linear relationship

$$\lambda\left(\mid v_T^t \mid\right) = c \cdot \mid v_T^t \mid .$$

In this case,

$$v_T^t = \begin{cases} \dfrac{b}{1 + ac\,(1-k)} & \text{for } y \geq 0 \\[2ex] \dfrac{b}{1 - ac\,(1-k)} & \text{for } y \leq 0 \end{cases}$$

Finally, with these new values of v_T^t, the values of the put swaptions are once again given by $E_0 \left|\max\left\{-v_T^t,\, 0\right\} \int_0^t \exp\left\{-r_s\right\}\, ds\right|$, since the short side of the put swaption has no incentive to default prior to the exercise of the put. However, the arguments of the value function $V\left(\cdot\right)$ are now incremented by $c\,(1-k)$. Due to the inability to distinguish between k and c in the valuation of put swaptions subject to default risk, we set $c' \equiv c\,(1-k)$ and solve for the new value function $V\left(\sigma_r,\, \sigma_s,\, \rho,\, c' \mid t,\, T,\, K\right)$.

5.3 Empirical Results

Using the above data, the estimation problem is now

$$\min_{\{\sigma_r,\, \sigma_s,\, \rho,\, c'\}} \sum_{\{t,T,K\}} \left[V\left(\sigma_r,\, \sigma_s,\, \rho,\, c' \mid t,\, T,\, K\right) - P\left(t,\, T,\, K\right)\right]^2$$

subject to $\sigma_r > 0$, $\sigma_s > 0$, $-1 \leq \rho \leq 1$ and $c' \geq 0$. This procedure permits the estimation of the model's coefficients, as well as the test of whether the incorporation of default risk significantly reduces the unexplained sum of squared residuals. Thus, letting SSR(u) be the unrestricted sum of squared residuals and SSR(r) be the restricted sum of squared residuals, the relevant test statistic for the nested test becomes

$$\frac{\left[\mathrm{SSR}(r) - \mathrm{SSR}(u)\right]/\,1}{\mathrm{SSR}(u)/\left(n-k\right)}$$

which is assumed to have the $F\left(1,\, n-k\right)$ distribution.

Table 2 reports the empirical tests based on vanilla fixed-for-floating swap rates out to a ten year maturity, and 17 put swaption prices for each of eight dates over the period Jan. 18, 1993 — May 3, 1994.

Table 2 — Estimation of Parameters for Defualt-Free and Risky Swap and Swaption Prices

Date	Default-Free Swap Model				Risky Swap Model					
	$\hat{\sigma}_r$	$\hat{\sigma}_s$	$\hat{\rho}$	$SSR(r) \cdot 10^3$	$\hat{\sigma}_r$	$\hat{\sigma}_s$	$\hat{\rho}$	$\hat{c}'$	$SSR(u) \cdot 10^3$	F-test
19940119	0.1665	0.1121	−1	0.2614	0.2104	0.09521	−0.756	0.7454	0.1722	*
19940228	0.1709	0.0624	−0.5772	0.1114	0.194	0.08651	−0.3142	0.3863	7.756e−02	*
19940302	0.1635	0.1508	−0.9859	7.075e−02	0.1763	0.1209	−0.9443	0.2461	5.728e−02	
19940330	0.1749	0.0781	−0.4844	0.181	0.2066	0.1838	0.151	0.4958	0.1199	*
19940503	0.1739	0.1469	0.1841	0.1538	0.2026	0.415	0.4521	0.5262	0.1042	*
All above dates jointly:										
	0.1707	0.1959	−0.6723	1.659	0.2034	0.4237	0.0397	0.5381	1.156	**

Notes:

1. $SSR(u)$ = unrestricted sum of squared residuals
   $SSR(r)$ = restricted sum of squared residuals
2. For each date, $n = 17$, $k = 4$
3. Nested F test examines whether the restriction $c' = 0$ (riskless swap) is statistically binding, using

$$F(1, n-k) = \frac{[SSR(r) - SSR(u)]/1}{SSR(u)/(n-k)}$$

   statistic
4. * : Statistically significant at 5% level
   ** : Statistically significant at 1% level

Several results from Table 2 are noteworthy:

1. At the 5% level, the sum of squared residuals on each observation date is significantly reduced through the addition of the possibility of default. If we posit the identical parameter values throughout the sample period, then the significance of default risk can be established at the 1% level.

2. Broadly speaking, the values of σ_r are within the reasonable range for the volatility of riskfree rates of interest.

6 Summary

This paper has derived a two-factor risk model for corporate bonds and interest rate swaps. The model is parsimonious in its assumptions, uses readily-observable inputs and is estimated empirically from the observable spreads of defaultable bonds and fixed-for-floating interest rate swaps. The empirical results suggest the importance of explicit modeling of the default event for both sets of securities. Further, we believe subsequent work should be focused on modeling the relevant stochastic processes so as to elicit the implied after-default values of corporate bonds and interest-rate swaps.

A Construction of the Quadrinomial Tree

Equation (2) in the text specifies the quadrinomial lattice to be used for corporate bond valuation. This appendix demonstrates the numerical validity of that specification in fulfilling the first and second moment requirements of the joint process eqs. (1).

Consider first the first-moment conditions (3). For $i = 1, 2$, we require that

$$\lambda\Delta a_i + (p - \alpha\lambda\Delta)b_i + (q - \beta\lambda\Delta)c_i + [1 - p - q + \lambda\Delta(\alpha + \beta - 1)]d_i = 0. \tag{8}$$

If, however, we wish to specify the parameters $\{a_i, b_i, c_i, d_i; i = 1, 2\}$ independent of the value of λ, then eq. (8) must be rearranged

$$\lambda\Delta[a_i - \alpha b_i - \beta c_i + (\alpha + \beta - 1)d_i] + pb_i + qc_i + (1 - p - q)d_i = 0 \tag{9}$$

to produce the following two conditions:

$$\begin{aligned} a_i - \alpha b_i - \beta c_i + (\alpha + \beta - 1)d_i &= 0 \\ pb_i + qc_i + (1 - p - q)d_i &= 0 \end{aligned}$$

For $i = 1, 2$ and $\sigma_1 \equiv \sigma_r$, $\sigma_2 \equiv \sigma_s$, the variance condition (4) is

$$\lambda\Delta a_i^2\Delta + (p - \alpha\lambda\Delta)b_i^2\Delta + (q - \beta\lambda\Delta)c_i^2\Delta + [1 - p - q + \lambda\Delta(\alpha + \beta - 1)]d_i^2\Delta = \sigma_i^2,$$

which implies

$$\begin{aligned} a_i^2 - \alpha b_i^2 - \beta c_i^2 + (\alpha + \beta - 1)d_i^2 &= 0 \\ pb_i^2 + qc_i^2 + (1 - p - q)d_i^2 &= \sigma_i^2 \end{aligned}$$

Finally, the covariance condition (5) is

$$\lambda\Delta a_1 a_2\Delta + (p - \alpha\lambda\Delta)b_1 b_2 + (q - \beta\lambda\Delta)c_1 c_2\Delta + [1 - p - q + \lambda\Delta(\alpha + \beta - 1)]d_1 d_2 = \sigma_{12}\Delta$$

which implies

$$
\begin{aligned}
a_1a_2 - \alpha b_1b_2 - \beta c_1c_2 + (\alpha + \beta - 1)d_1d_2 &= 0 \\
pb_1b_2 + qc_1c_2 + (1 - p - q)d_1d_2 &= \sigma_{12}
\end{aligned}
$$

We now demonstrate the tree can be constructed to satisfy all requirements under the special case when $\alpha = \beta = a_1 = d_1 = 0$ and $p = q = 1/4$. We obtain the following six equations in six unknowns:

$$a_2 + d_2 = 0 \tag{10}$$

$$b_1 + c_1 = 0 \tag{11}$$

$$b_1^2 + c_1^2 = 4\sigma_1^2 \tag{12}$$

$$b_2 + c_2 + 2d_2 = 0 \tag{13}$$

$$b_2^2 + c_2^2 + 2d_2^2 = 4\sigma_2^2 \tag{14}$$

$$b_1b_2 + c_1c_2 = 4\sigma_{12} \tag{15}$$

The solution to this system can be seen by inspection as follows. Equations (11) and (12) yield values for b_1 and c_1. Substituting theses into (15) yields b_2. Equations (13) and (14) jointly solve for c_2 and d_2. Finally, (10) provides the value of a_2. With $\rho \equiv \sigma_{12}/\left(\sigma_1\sigma_2\right)$, the solution to the system is

$$
\begin{aligned}
b_1 &= -\sqrt{2}\sigma_1 \\
c_1 &= \sqrt{2}\sigma_1 \\
a_2 &= \pm\sigma_2\sqrt{1-\rho^2} \\
b_2 &= \sigma_2\left(\sqrt{2}\rho \pm \sqrt{1-\rho^2}\right) \\
c_2 &= \sigma_2\left(\sqrt{2}\rho \mp \sqrt{1-\rho^2}\right) \\
d_2 &= \mp\sigma_2\sqrt{1-\rho^2}
\end{aligned}
$$

The non-uniqueness of this system can be resolved by appeal to the economics of the analysis. Specifically, note that a_2 is the proportional change in the spread in the default state; economic intuition would dictate that it be positive. The non-uniqueness of a_2 and d_2 is resolved:

$$\begin{aligned} a_2 &= \sigma_2\sqrt{1-\rho^2} \\ d_2 &= -\sigma_2\sqrt{1-\rho^2} \end{aligned}$$

The tree's computational efficiency may be observed as follows. At time $t + n\Delta$,

$$\begin{aligned} r\,(t+n\Delta) &= r(t)\exp\left\{\sum_{i=1}^{n}\mu\,(t+i\Delta)\,\Delta + (n_1 b_1 + n_2 c_1 + n_3 d_1)\,\sqrt{\Delta}\right\} \\ s\,(t+n\Delta) &= s(t)\exp\left\{(n_1 b_2 + n_2 c_2 + n_3 d_2)\,\sqrt{\Delta}\right\} \end{aligned}$$

The computational efficiency follows from the fact that the expressions for both $r(t+n\Delta)$ and $s(t+n\Delta)$ depend only on n_1, n_2 and $n_3 = n - n_1 - n_2$. Thus, the relevant nodes on the tree are fully identified by the triplet $(n,\, n_1,\, n_2)$.

B Data

The data consists of the term structures of interest rates for alternate ratings-categories Merrill Lynch & Co. bond indices.

Selected Indices and Swap Rates

No.	Index	Description	Spread to Treasury Security: Index	Spread to Treasury Security: Description
1.	C1B0	Corporate, 1 – 2.99 years, AAA-AA rated, all coupons	GA02	On-the-run two-year Note
2.	C1C0	Corporate, 1 – 2.99 years, A-BBB rated, all coupons		
3.	J1A1	High-yield, 1 – 2.99 years, BB-C rated, all coupons		
4.	$s_{\text{swap},2}$	Two-year swap rate		
5.	CVB0	Corporate, 1 – 4.99 years, AAA-AA rated, all coupons	GA03	On-the-run three-year Note
6.	CVC0	Corporate, 1 – 4.99 years, A-BBB rated, all coupons		
7.	J2A1	High-yield, 1 – 4.99 years, BB-C rated, all coupons		
8.	$s_{\text{swap},3}$	Three-year swap rate		
9.	C6B0	Corporate, 5 – 9.99 years, AAA-AA rated, all coupons	GA07	On-the-run seven-year Note
10.	C6C0	Corporate, 5 – 9.99 years, A-BBB rated, all coupons	GF07	Off-the-run seven-year Treasury*
11.	J4A1	High-yield, 5 – 9.99 years, BB-C rated, all coupons		
12.	$s_{\text{swap},7}$	Seven-year swap rate		
13.	C7B0	Corporate, 10 – 14.99 years, AAA-AA rated, all coupons	GA10	On-the-run ten-year Note
14.	C7C0	Corporate, 10 – 14.99 years, A-BBB rated, all coupons		
15.	J7A1	High-yield, 10 – 14.99 years, BB-C rated, all coupons		
16.	$s_{\text{swap},10}$	Ten-year swap rate		

* : The Treasury has discontinued the issuance of seven-year par bonds.

C "Bootstrapping" the Yield Curve from Par Rates

For a unit par bond, consider the relation

$$1 = \frac{c_T}{2} \sum_{t=0.5}^{T} \mathrm{PV}_t + \mathrm{PV}_T. \tag{16}$$

Write

$$\sum_{t=0.5}^{T} \mathrm{PV}_t = \sum_{t=0.5}^{T-0.5} \mathrm{PV}_t + \mathrm{PV}_T. \tag{17}$$

Now, substitute from eq. (17) into (16):

$$1 = \frac{c_T}{2} \sum_{t=0.5}^{T-0.5} \mathrm{PV}_t + \left(1 + \frac{c_T}{2}\right) \mathrm{PV}_T. \tag{18}$$

From eq. (18), solve for PV_T using the c_T's:

$$\mathrm{PV}_T = \frac{1 - \left(c_T \sum_{t=0.5}^{T-0.5} \mathrm{PV}_t\right) / 2}{1 + c_T/2}.$$

As noted in the text, the c_T's are interpolated linearly between observed values at the on-the-run points 2, 3, 7 and 10 yrs.

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