# A Series Expansion for the Bivariate Normal Integral

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KMV Corporation
1620 Montgomery Street, Suite 140
San Francisco, CA 94111 U.S.A.
Phone: +1 415-296-9669
FAX: +1 415-296-9458
email: support@kmv.com
website: http: // www.kmv.com

Authors:

Oldrich Alfons Vasicek

# Abstract

*An infinite series expansion is given for the bivariate normal cumulative distribution function. This expansion converges as a series of powers of* $\left(1-\rho^2\right)$, *where* $\rho$ *is the correlation coefficient, and thus represents a good alternative to the tetrachoric series when* $\rho$ *is large in absolute value.*



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# Introduction

The cumulative normal distribution function

$$N(x) = \int_{-\infty}^{x} n(u)\,du$$

with

$$n(u) = (2\pi)^{-1/2}\exp\left(-\tfrac{1}{2}u^2\right)$$

appears frequently in modern finance: Essentially all explicit equations of options pricing, starting with the Black/Scholes formula, involve the function in one form or another. Increasingly, however, there is also a need for the bivariate cumulative normal distribution function

$$N_2(x,y,\rho) = \int_{-\infty}^{x}\int_{-\infty}^{y} n_2(u,v,\rho)\,du\,dv \qquad (1)$$

where the bivariate normal density is given by

$$n_2(u,v,\rho) = (2\pi)^{-1}\left(1-\rho^2\right)^{-1/2}\exp\left(-\tfrac{1}{2}\frac{u^2-2\rho uv+v^2}{1-\rho^2}\right) \qquad (2)$$

This need arises in at least the following areas:

1. Pricing exotic options. Option with payout depending on the prices of two lognormally distributed assets, or two normally distributed factors, involve the bivariate normal distribution function in the pricing formula. Examples include the so-called rainbow options (such as calls on maximum or minimum of two assets), extendible options, spread and cross-country swaps, etc.

2. Correlation of derivatives. While the instantaneous correlation of two derivatives is the same as the correlation of the underlying assets, calculation of the correlation over non-infinitesimal intervals often requires the bivariate normal function.

3. Loan loss correlation. If a loan default occurs when the borrower's assets fall below a certain point, the covariance of defaults on two loans is given by a bivariate normal formula. This covariance is needed when evaluating the variance of loan portfolio losses.

A standard procedure for calculating the bivariate normal distribution function is the tetrachoric series,


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$$\mathrm{N}_2(x,y,\rho)=\mathrm{N}(x)\mathrm{N}(y)+\mathrm{n}(x)\mathrm{n}(y)\sum_{k=0}^{\infty}\frac{1}{(k+1)!}\mathrm{He}_k(x)\mathrm{He}_k(y)\rho^{k+1} \tag{3}$$

where

$$\mathrm{He}_k(x)=\sum_{i=0}^{[k/2]}\frac{k!}{i!(k-2i)!}(-1)^i 2^{-i}x^{k-2i}$$

are the Hermite polynomials. For a comprehensive review of the literature, see Gupta (1963).

The tetrachoric series (3) converges only slightly faster than a geometric series with quotient $\rho$, and it is therefore not very practical to use when $\rho$ is large in absolute value. In this note, we give an alternative series that converges approximately as a geometric series with quotient $\left(1-\rho^2\right)$.

# The Expansion

The starting point of this note is the formula

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\mathrm{N}_2(x,y,\rho)=\mathrm{n}_2(x,y,\rho) \tag{4}$$

proven in the appendix.

Because of the identity

$$\mathrm{N}_2(x,y,\rho)=\mathrm{N}_2\left(x,0,\frac{\rho x-y}{\sqrt{x^2-2\rho xy+y^2}}\mathrm{sign}x\right)+\mathrm{N}_2\left(y,0,\frac{\rho y-x}{\sqrt{x^2-2\rho xy+y^2}}\mathrm{sign}y\right)-\begin{bmatrix}0 & \text{if } xy>0\\ ½ & \text{if } xy<0\end{bmatrix}$$

for $xy\neq 0$, we can limit ourselves to calculation of $\mathrm{N}_2(x,0,\rho)$. Suppose first that $\rho>0$. Then by integrating equation (4) with $y=0$ from $\rho$ to 1 we get

$$\mathrm{N}_2(x,0,\rho)=\min(\mathrm{N}(x),½)-Q \tag{5}$$

where

$$\begin{aligned}Q&=\int_{\rho}^{1}\mathrm{n}_2(x,0,r)\mathrm{d}r\\ &=(2\pi)^{-1}\int_{\rho}^{1}\left(1-r^2\right)^{-½}\exp\left(-½\frac{x^2}{1-r^2}\right)\mathrm{d}r\end{aligned}$$

To evaluate the integral, substitute

$$r = \sqrt{1-s}$$

to obtain

$$Q = ½(2\pi)^{-1} \int_0^{1-\rho^2} (1-s)^{-½} s^{-½} \exp\left(-½\frac{x^2}{s}\right) \mathrm{d}s \tag{6}$$

Using the expansion

$$(1-s)^{-½} = \sum_{k=0}^{\infty} \frac{(2k)!}{(k!)^2} 2^{-2k} s^k$$

we get

$$Q = ½(2\pi)^{-1} \int_0^{1-\rho^2} \sum_{k=0}^{\infty} \frac{(2k)!}{(k!)^2} 2^{-2k} s^{k-½} \exp\left(-½\frac{x^2}{s}\right) \mathrm{d}s \tag{7}$$

Because

$$\frac{(2k)!}{(k!)^2} 2^{-2k} s^{k-½} \exp\left(-½\frac{x^2}{s}\right) \le s^{k-½} \le \left(1-\rho^2\right)^{k-½} \tag{8}$$

for $k > 0, 0 \le s \le 1-\rho^2$, the series in equation (7) converges uniformly in the interval $\left[0,1-\rho^2\right]$ and can be integrated term by term. It can be easily established that

$$\int_0^t s^{k-½} \exp\left(-½\frac{x^2}{s}\right) \mathrm{d}s = \frac{k!}{(2k+1)!} (-1)^k 2^{k+1} |x|^{2k+1}$$
$$\cdot\left[\exp\left(-½\frac{x^2}{t}\right) \sum_{i=0}^{k} \frac{(2i)!}{i!} (-1)^i 2^{-i} |x|^{-2i-1} t^{i+½} - (2\pi)^{½} \mathrm{N}\left(-\frac{|x|}{\sqrt{t}}\right)\right]$$

for $k \ge 0$. Substitution into (7) then gives

$$Q = \sum_{k=0}^{\infty} A_k \tag{9}$$

where

$$A_k = \frac{1}{k!}\frac{1}{2k+1}(-1)^k 2^{-k}|x|^{2k+1}$$
$$\left[(2\pi)^{-1}\exp\left(-½\frac{x^2}{1-\rho^2}\right)\sum_{i=0}^{k}\frac{(2i)!}{i!}(-1)^i 2^{-i}|x|^{-2i-1}\left(1-\rho^2\right)^{i+½} - (2\pi)^{-½}\mathrm{N}\left(-\frac{|x|}{\sqrt{1-\rho^2}}\right)\right] \quad (10)$$

Equations (5), (9) and (10) give an infinite series expansion for $\mathrm{N}_2(x,0,\rho)$, $\rho > 0$. When $\rho < 0$, integration of equation (4) from -1 to $\rho$ yields

$$\mathrm{N}_2(x,0,\rho) = \max\left(\mathrm{N}(x) - ½, 0\right) + Q \quad (11)$$

with $Q$ still given by (9) and (10).

A convenient procedure for computing the terms in the expansion (9) is using the recursive relationships

$$A_k = -\frac{2k-1}{2k(2k+1)}x^2 A_{k-1} + B_k$$
$$B_k = \frac{(2k-1)^2}{2k(2k+1)}\left(1-\rho^2\right)B_{k-1}$$

with

$$B_0 = (2\pi)^{-1}\left(1-\rho^2\right)^{½}\exp\left(-½\frac{x^2}{1-\rho^2}\right)$$
$$A_0 = -(2\pi)^{-½}|x|\mathrm{N}\left(-\frac{|x|}{\sqrt{1-\rho^2}}\right) + B_0$$

To determine the speed of convergence of (9), integrate the first half of inequality (8) from 0 to $1-\rho^2$. This results in the bound

$$0 < A_k \le ½(2\pi)^{-1}\frac{1}{k+½}\left(1-\rho^2\right)^{k+½}$$

for $k > 0$, and therefore the series (9) converges approximately as $\sum\left(1-\rho^2\right)^k / k$. As the tetrachoric series for $\mathrm{N}_2(x,0,\rho)$

$$\mathrm{N}_2(x,0,\rho) = ½\mathrm{N}(x) + (2\pi)^{-½}\mathrm{n}(x)\sum_{k=0}^{\infty}\frac{1}{k!}\frac{1}{2k+1}(-1)^k 2^{-k}\mathrm{He}_{2k}(x)\rho^{2k+1} \quad (12)$$

converges approximately as $\sum \rho^{2k}/k$, a reasonable method for calculating $N_2(x,0,\rho)$ is to use the tetrachoric series (12) when $\rho^2 \le ½$ and the expressions (5) and (11) with the series (9) when $\rho^2 > ½$.

The error in the calculation of $N_2(x,0,\rho)$ resulting from using $m$ terms in the expansion (9) is bound in absolute value by

$$\left|\sum_{k=m}^{\infty} A_k\right| < ½(2\pi)^{-1} \frac{1}{m+½} \frac{1}{\rho^2} \left(1-\rho^2\right)^{m+½}$$

# Numerical Results

A comparison of the convergence of the tetrachoric series (12) and the alternative calculation (5) or (11) with the series (9) in calculating $N_2(x,0,\rho)$ for high values of the correlation coefficient is given in the following tables:

**Table 1 Partial sums of the tetrachoric and alternative series**

| | $x=-1, \rho=.95$ | | $x=-1, \rho=.99$ | |
|---|---|---|---|---|
| Number of terms | Tetrachoric | Alternative | Tetrachoric | Alternative |
| 1 | .171033 | .158632 | .174894 | .158655 |
| 2 | .171033 | .158631 | .174894 | .158655 |
| 3 | .167298 | .158631 | .170304 | .158655 |
| 5 | .161764 | .158631 | .162651 | .158655 |
| 10 | .157961 | .158631 | .156068 | .158655 |
| 20 | .158466 | .158631 | .158068 | .158655 |
| 30 | .158660 | .158631 | .159374 | .158655 |
| 50 | .158632 | .158631 | .158599 | .158655 |
| 100 | .158631 | .158631 | .158711 | .158655 |
| 200 | .158631 | .158631 | .158657 | .158655 |
| 300 | .158631 | .158631 | .158654 | .158655 |
| Exact | .158631 | | .158655 | |


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**Table 2 Number of terms necessary for precision $10^{-4}$**

| $\rho =$ | .8 | .9 | .95 | .99 |
|---|---|---|---|---|
| Tetrachoric series | | | | |
| x=0 | 8 | 16 | 30 | 121 |
| x=±1 | 7 | 14 | 22 | 75 |
| x=±2 | 6 | 11 | 18 | 42 |
| Alternative series | | | | |
| x=0 | 4 | 3 | 2 | 1 |
| x=±1 | 3 | 1 | 1 | 1 |
| x=±2 | 1 | 1 | 1 | 1 |

# Appendix

We prove equation (4) by stating a slightly more general result. Let

$$\mathrm{n}_p(\boldsymbol{\xi},S)=(2\pi)^{-p/2}|S|^{-½}\exp\left(-½\boldsymbol{\xi}'S^{-1}\boldsymbol{\xi}\right)$$

$$\mathrm{N}_p(\boldsymbol{\xi},S)=\int_{-\infty}^{\boldsymbol{\xi}}\mathrm{n}_p(\boldsymbol{\upsilon},S)\mathrm{d}\boldsymbol{\upsilon}$$

be the $p$-variate normal density function and cumulative distribution function, respectively, where $\boldsymbol{\xi}=\left(x_1,x_2,\ldots,x_p\right)'$ is a $p\times 1$ vector and $S=\left\{\sigma_{11},\sigma_{12},\ldots,\sigma_{pp}\right\}$ is a $p\times p$ symmetric positive definite matrix. We prove the following lemma:

**Lemma.** *Let $p\geq 2$. Then for $i\neq j$,*

$$\frac{\partial}{\partial\sigma_{ij}}\mathrm{N}_p(\boldsymbol{\xi},S)=\frac{\partial^2}{\partial x_i\partial x_j}\mathrm{N}_p(\boldsymbol{\xi},S)$$
$$=\mathrm{n}_2\left(\boldsymbol{\xi}_{(1)},S_{(11)}\right)\mathrm{N}_{p-2}\left(\boldsymbol{\xi}_{(2)}-S_{(21)}S_{(11)}^{-1}\boldsymbol{\xi}_{(1)},S_{(22)}-S_{(21)}S_{(11)}^{-1}S_{(12)}\right)$$

*Here $\boldsymbol{\xi}_{(1)}=\left(x_i,x_j\right)'$ is the $2\times 1$ vector of $x_i,x_j$, $\boldsymbol{\xi}_{(2)}$ is the $(p-2)\times 1$ vector of the remaining components of $\boldsymbol{\xi}$, and $S_{(11)}$, $S_{(12)}$, $S_{(21)}$, $S_{(22)}$ is the $2\times 2$, $2\times(p-2)$, $(p-2)\times 2$, and $(p-2)\times(p-2)$ decomposition of $S$ into the $i$-th and $j$-th row and column and the remaining rows and columns.*

**Proof.** We have

$$\frac{\partial}{\partial\sigma_{ij}}\mathrm{n}_p(\boldsymbol{\xi},S)=\left(-½\mathrm{tr}\left(S^{-1}\frac{\partial S}{\partial\sigma_{ij}}\right)+½\boldsymbol{\xi}S^{-1}\frac{\partial S}{\partial\sigma_{ij}}S^{-1}\boldsymbol{\xi}\right)\mathrm{n}_p(\boldsymbol{\xi},S).$$

Define $\boldsymbol{\varsigma}=\left\{v_{11},v_{12},\ldots,v_{pp}\right\}$ by $\boldsymbol{\varsigma}=S^{-1}$ and put $\boldsymbol{\psi}=\left(y_1,y_2,\ldots,y_p\right)'=\boldsymbol{V}\boldsymbol{x}$. Since for $i\neq j$

$$\frac{\partial S}{\partial\sigma_{ij}}=\mathbf{E}_{ij}+\mathbf{E}_{ji}$$

where $\mathbf{E}_{ij}$ is the matrix having unity for the $ij$-th element and zeros elsewhere, we get on substitution

$$\frac{\partial}{\partial\sigma_{ij}}\mathrm{n}_p(\boldsymbol{\xi},S)=\left(-v_{ij}+y_iy_j\right)\mathrm{n}_p(\boldsymbol{\xi},S)$$

On the other hand,

$$\frac{\partial}{\partial x_j}\mathrm{n}_p(\boldsymbol{\xi},S) = -y_j\mathrm{n}_p(\boldsymbol{\xi},S)$$

$$\frac{\partial^2}{\partial x_i \partial x_j}\mathrm{n}_p(\boldsymbol{\xi},S) = \left(-v_{ij} + y_i y_j\right)\mathrm{n}_p(\boldsymbol{\xi},S)$$

and therefore

$$\frac{\partial}{\partial \sigma_{ij}}\mathrm{n}_p(\boldsymbol{\xi},S) = \frac{\partial^2}{\partial x_i \partial x_j}\mathrm{n}_p(\boldsymbol{\xi},S)$$

Integrating with respect to $\boldsymbol{\xi}$ and exchanging the order of integration and differentiation yields the first equality of the lemma. The second equality follows from the factorization

$$\mathrm{n}_p(\boldsymbol{\xi},S) = \mathrm{n}_2\left(\boldsymbol{\xi}_{(1)}, S_{(11)}\right)\mathrm{n}_{p-2}\left(\boldsymbol{\xi}_{(2)} - S_{(21)}S_{(11)}^{-1}x_{(1)}, S_{(22)} - S_{(21)}S_{(11)}^{-1}S_{(12)}\right)$$

# References

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