The Empirical Performance of Alternative Extreme Value Volatility Estimators *

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Abstract. This paper addresses the following issue: given a set of daily observations on an asset (historical opening, closing, high and low prices), how should one go about estimating the asset's volatility? We use high-frequency data on very liquid assets to construct daily realized volatility series, which enables us to treat volatility as observed rather than latent. We then compare the empirical performance of various estimators of asset return volatility against the realized volatility benchmark. This procedure makes it possible, for the first time, to study the bias and relative efficiency of the various estimators directly. The stock index results give overwhelming support to the use of extreme value volatility estimators, but the futures and currency results are less clear. We highlight a number of important instances in which extreme value volatility estimators are both less biased and more efficient than the traditional estimator.

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Since the events of October 1987, few would argue against the proposition that volatility is not constant. Engle (1982) introduced a class of parametric conditionally heteroskedastic models (ARCH models) that address this issue. While these models have proved very useful in uncovering a number of interesting facts about asset returns (Bollerslev, Chou and Kroner, 1992), a researcher may prefer not to impose a parametric econometric structure on the intertemporal behavior of the data. Highly efficient volatility estimators may prove useful in such a context: if a researcher has an estimator which is, say, ten times more efficient than the traditional, or close to close, estimator (i.e. its estimation variance is ten times lower), then he or she could use this estimator on data from a single day with the same confidence as if he or she were using the traditional estimator on two weeks of daily observations. Such a procedure may enable the researcher to estimate seemingly model-free daily volatilities when intra day data are unavailable or unreliable, see, e.g., Brown and Hartzell (2001) for a recent application along those lines.

Traditionally, volatility has simply been estimated by computing the sample standard deviation using daily close to close returns. However, following the work of Parkinson (1980), a number of volatility estimators based upon the daily high, low, opening and closing prices have been proposed in the literature. Extreme value variance estimators are theoretically 5 to 14 times more efficient than the traditional close to close estimator, yet their use in the literature is not widespread. It should be noted that these extreme value estimators are derived under rather strong assumptions about the data generating process, e.g., it is assumed that the asset price follows geometric Brownian motion, that trading is continuous and always monitored. In addition, one may suspect that high and low prices are more likely than closing prices to reflect certain liquidity-motivated trades, and thus may be less representative of the asset's fundamental value. [1]Therefore, it is possible that extreme value volatility estimators are misspecified: surely, the assumption of geometric Brownian motion cannot hold exactly; whether its accuracy, over the estimation window, is sufficient to warrant the use of extreme value estimators is therefore an empirical question which we address in this paper. The literature often implicitly assumes that the sample standard deviation estimate of volatility is unbiased. Indeed, this assumption underlies all previous research on the empirical performance of extreme value volatility estimators. However, as we discuss below, this assumption does not hold for general data generating processes even if the full (continuous) sample path were observable.

We use high-frequency data on very liquid and actively traded assets to construct measures of realized volatility. Andersen *et al.* (2000a, 2000b) show that, under very weak assumptions, such estimates may be considered model-free and largely free of measurement error. Thus, for our purposes, we can treat these volatilities as observed rather than latent. This, in turn, enables us to examine the empirical bias and efficiency of the various extreme value variance estimators directly: essentially, we have a large sample of realizations of extreme value volatility estimators, along with quasi error-free measures of the "true" parameters which these estimators are designed to estimate.

Discrete trading (in time or in price) can cause variance estimators to be biased. Marsh and Rosenfeld (1986) and Cho and Frees (1988) propose theoretical models to analyze the impact of discrete trading on the estimation of volatility. Marsh and Rosenfeld (1986) find that, in their model, discrete trading does not bias the close to close estimator (although it reduces its efficiency), while the extreme value estimator is biased downward and becomes far less efficient. Cho and Frees (1988) find that, in their model, discrete trading causes the traditional estimator to be upward biased. Garman and Klass (1980) propose a variance estimator based on high,

[1]We do not expect this to be a major issue in our empirical work, since we focus the analysis on very liquid assets.

low, opening and closing prices. They recognize that their estimator is biased downward in the presence of discrete trading (this is simply because the observed high is in general smaller than the true high, while the observed low is in general greater than the true low) and show that the close to close estimator is upward biased in this context, but only slightly so.

While such theoretical results are useful in showing the potential impact of discreteness on volatility estimation, they are again based upon specific assumptions about the data generating process. Ultimately, the performance of extreme value volatility estimators is an empirical issue. Beckers (1983) and Wiggins (1991, 1992) compared the empirical performance of the Parkinson (1980) and the close to close estimators. Their results suggest that the Parkinson (1980) estimator outperforms its close to close counterpart. Our approach differs from previous research in this area along three dimensions. First, our use of high frequency data allows us to observe volatility *directly*, thus we can compare the various volatility estimators without making any additional assumptions about the data generating process. In other words, the use of high frequency data gives us a benchmark along which the performance of the various estimators can be readily compared. Second, we explicitly recognize that the traditional close to close volatility estimator may not be unbiased. In contrast, previous empirical literature has assumed that the traditional close to close estimator provides unbiased (albeit very noisy) estimates and finds that the Parkinson (1980) extreme value estimator is downward biased *relative to* the close to close estimator. To wit, such a result could be consistent with the extreme value estimator being unbiased, and the close to close estimator upward biased! We show that the unbiasedness of the close to close estimator only holds under specific assumptions about the data generating process, even in the absence of any discrete trading-induced biases. Our methodology allows us to address the potential bias of the close to close estimator and the extreme value estimators separately, thereby resolving this issue. Third, we compare the performance of all six extreme value volatility estimators that have appeared in the literature, while previous research only considers Parkinson's estimator.

We believe that the use of high frequency data, along with the theoretical results of Andersen *et al.* (2000a, 2000b), provides a very clean way of examining the empirical performance of extreme value volatility estimators. The rest of this paper is organized as follows. Section 1 presents the various volatility estimators, discusses their properties, and describes our methodology and data. Section 2 presents the empirical tests, and section 3 concludes.

I. Volatility Estimation

We briefly introduce the various extreme value volatility estimators that have arisen in the literature, and show why it is generally erroneous to think of the close to close estimator as unbiased. We then describe our high frequency data and its use in constructing model- and error-free realized volatility series.

A. Extreme Value Estimators

The assumption is maintained that S_t, the price of the asset at time t, satisfies the following stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{1}$$

where μ and σ are nonnegative constants, and W_t is a standard Brownian motion. The solution to equation (1) is given by

$$S_t = S_0 \exp\{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t\}. \tag{2}$$

We will need the following notation. For the sake of concreteness, assume that we are dealing with n observations on daily data. Let O_t, C_t, H_t and L_t denote, respectively, the opening, closing, high and low prices on day t. To maintain consistency with the theoretical literature on extreme value estimators, the presentation that follows is in terms of variances, not volatilities. In the empirical section of the paper, however, we compare annualized volatilities, as the scale of these is more readily interpretable.

The traditional, or close to close, estimator of variance for a driftless[2] security is given by

$$\widehat{\sigma}^2_{cc} = \frac{1}{n}\sum_{t=1}^{n}(\ln\frac{C_t}{C_{t-1}})^2,\ n \geq 1. \tag{3}$$

A mean adjusted variant of this estimator is given by the sample standard deviation

$$\widehat{\sigma}^2_{acc} = \frac{1}{n-1}\sum_{t=1}^{n}(\ln\frac{C_t}{C_{t-1}})^2 - \frac{(\ln\frac{C_n}{C_0})^2}{n(n-1)},\ n > 1. \tag{4}$$

Parkinson (1980) introduces the following extreme value estimator for a driftless security

$$\widehat{\sigma}^2_{p} = \frac{1}{4n\ln 2}\sum_{t=1}^{n}(\ln\frac{H_t}{L_t})^2,\ n \geq 1. \tag{5}$$

We define the efficiency of an (unbiased) estimator $\widehat{\sigma}^2$ relative to the close to close estimator $\widehat{\sigma}^2_{cc}$ in the usual manner by the ratio $\mathrm{Var}(\widehat{\sigma}^2_{cc})/\mathrm{Var}(\widehat{\sigma}^2)$. It can be shown that if the stock price follows equation (1) with $\mu = \sigma^2/2$, and if trading is continuous and continuously monitored, then the Parkinson (1980) estimator $\widehat{\sigma}^2_p$ is about 5 times more efficient than the close to close estimator (i.e. its estimation variance is about five times lower).[3]

[2]By driftless, it is meant that the logarithmic price process is driftless, i.e. $\mu = \sigma^2/2$ (see equation (2))

[3]Kunitomo (1992) derives an extreme value estimator which is about two times more efficient than the Parkinson (1980) estimator. Kunitomo's estimator is based on the range of a Brownian bridge process constructed from the price process, which implies that the Kunitomo estimator cannot be computed from daily data. For this reason, we have decided not to analyze the Kunitomo estimator.

Under the assumptions of Parkinson (1980), Garman and Klass (1980) constructed a minimum variance unbiased estimator that simultaneously uses the opening, closing, high and low prices:

$$\begin{aligned}\widehat{\sigma}^2_{GK} \quad = \quad & \frac{1}{n}\sum_{t=1}^{n}(0.511(\ln\frac{H_t}{L_t})^2 - 0.019(\ln(\frac{C_t}{O_t})\ln(\frac{H_t L_t}{O_t^2}) - 2\ln(\frac{H_t}{O_t})\ln(\frac{L_t}{O_t})) \\ & -0.383(\ln\frac{C_t}{O_t})^2), \ n \geq 1. \end{aligned} \tag{6}$$

This estimator is theoretically 7.4 times more efficient than the traditional close to close estimator, but still maintains the assumption that $\mu = \sigma^2/2$. Rogers and Satchell (1991) relax this assumption and propose the following estimator

$$\widehat{\sigma}^2_{RS} = \frac{1}{n}\sum_{t=1}^{n}(\ln(\frac{H_t}{C_t})\ln(\frac{H_t}{O_t}) + \ln(\frac{L_t}{C_t})\ln(\frac{L_t}{O_t})), \ n \geq 1, \tag{7}$$

which has the desirable property that it is independent of the drift μ. They also propose an adjustment that is designed to take into account the fact that one may not be able to continuously monitor the stock price. Their adjusted estimator is the positive root of the following quadratic equation

$$\widehat{\sigma}^2_{ARS} = \frac{0.5594}{N^{obs}}\widehat{\sigma}^2_{ARS} + \frac{0.9072}{\sqrt{N^{obs}}}\ln(\frac{H_t}{L_t})\widehat{\sigma}_{ARS} + \widehat{\sigma}^2_{RS}, \tag{8}$$

where N^{obs} denotes the number of observations of the price during the trading day.[4] They propose a similar adjustment to the Garman and Klass (1980) estimator, the adjusted estimator is the positive root of the following equation

$$\begin{aligned}
\widehat{\sigma}^2_{AGK} &= 0.511[(\ln \frac{H_t}{L_t})^2 + \frac{0.9709}{N^{obs}}\widehat{\sigma}^2_{AGK} + \frac{1.8144}{\sqrt{N^{obs}}}(\ln \frac{H_t}{L_t})\widehat{\sigma}_{AGK}] \\
&\quad +0.038[\ln(\frac{H_t}{O_t})\ln(\frac{L_t}{O_t}) - \frac{0.2058}{N^{obs}}\widehat{\sigma}^2_{AGK} - \frac{0.4536}{\sqrt{N^{obs}}}(\ln \frac{H_t}{L_t})\widehat{\sigma}_{AGK}] \\
&\quad -0.019\ln(\frac{C_t}{O_t})\ln(\frac{H_tL_t}{O_t^2}) - 0.383(\ln \frac{C_t}{O_t})^2. \qquad (9)
\end{aligned}$$

Finally, Yang and Zhang (2000) propose a minimum variance unbiased estimator which is independent of the drift μ of the asset price process. Their practical estimator is given by

$$\widehat{\sigma}^2_{YZ} = \frac{1}{n-1}\sum_{t=1}^{n}(\ln(\frac{O_t}{C_{t-1}}) - \overline{o})^2 + k\frac{1}{n-1}\sum_{t=1}^{n}(\ln(\frac{C_t}{O_t}) - \overline{c})^2 + (1-k)\widehat{\sigma}^2_{RS},\ n > 1, \qquad (10)$$

where $\overline{o} = \frac{1}{n}\sum_{t=1}^{n}\ln(O_t/C_{t-1})$, $\overline{c} = \frac{1}{n}\sum_{t=1}^{n}\ln(C_t/O_t)$, and $k = 0.34/(1.34 + \frac{n+1}{n-1})$.

Except for the Yang and Zhang (2000) estimator $\widehat{\sigma}^2_{YZ}$, the extreme value variance estimators considered here do not incorporate an estimate of overnight (i.e. closed market) variance: they ignore the fact that O_t, the opening price on day t, is in general different from C_{t-1}, the previous closing price. Inspection of equation (10) reveals that $\widehat{\sigma}^2_{YZ}$ is simply the sum of the estimated overnight variance (the first term on the right hand side of equation (10)) and the estimated open market variance (which is a weighted average of the open market return sample variance and the Rogers and Satchell (1991) drift independent estimator, where the weights are chosen so as to minimize the variance of the estimator). The resulting estimator therefore explicitly incorporates a term for closed market variance; similar adjustments can be made to the other variance estimators, see Garman and Klass (1980) for details. In this paper, we have

[4] For a variance estimator over more than one day, simply take the arithmetic average of the $\widehat{\sigma}^2_{ARS}$'s over the interval of interest.

chosen not to estimate overnight variances, since obviously high frequency data do not allow the measurement of closed market realized variance. Therefore, in some applications, we work with open market equivalents of (4) and (10), e.g. we use open market sample variance

$$\widehat{\sigma}^2_{aoc} = \frac{1}{n-1}\sum_{t=1}^{n}(\ln(\frac{C_t}{O_t}) - \overline{c})^2 \tag{11}$$

in lieu of (4). This is not a restrictive assumption: any correction for close to open variance would be the same for the traditional estimator (equation 11), the various extreme value estimators, and our volatility measures derived from high frequency data. Therefore any correction for close to open variance would leave our empirical results unchanged.

B. The Biasedness of the Traditional Estimator

It is well known that extreme value estimators are derived under rather strong assumptions about the data generating process, but there is often an implicit assumption made in the literature that the traditional estimator provides unbiased variance estimates independently of the data generating process. This need not be true in general, however, as we now discuss. For simplicity, assume that the asset price process is a diffusion

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t, \tag{12}$$

where μ_t and σ_t may be stochastic. Then the logarithmic asset price process is given by

$$d\ln S_t = (\mu_t - \frac{1}{2}\sigma_t^2)dt + \sigma_t dW_t. \tag{13}$$

It then follows that $E[(d\ln S_t)^2] = \mathrm{Var}[d\ln S_t] = \sigma_t^2 dt$, since $E(dt \cdot dW_t) = E(dt^2) = 0$. In other words, squared infinitesimal returns provide unbiased variance estimates. Consider now the τ-period continuously compounded return $r_t(\tau) = \ln(S_{t+\tau}) - \ln(S_t)$:

$$r_t(\tau) = \int_t^{t+\tau} (\mu_s - \frac{1}{2}\sigma_s^2)ds + \int_t^{t+\tau} \sigma_s dW_s. \tag{14}$$

Clearly, in general, $\mathrm{Var}[r_t(\tau)] \neq \int_t^{t+\tau} \sigma_s^2 ds$. The equality would obtain, e.g., if $\int_t^{t+\tau}(\mu_s - \frac{1}{2}\sigma_s^2)ds$ were nonstochastic. In general, the relation between $\mathrm{Var}[r_t(\tau)]$ and $\int_t^{t+\tau} \sigma_s^2 ds$ depends upon the assumed data generating process. Let us look at a specific example in which volatility is actually constant: assume that returns are predictable and log prices follow the trending Ornstein-Uhlenbeck process

$$d\ln S_t = (-\gamma(\ln S_t - \mu t) + \mu)dt + \sigma dW_t, \tag{15}$$

where γ, μ and σ are positive constants. Log prices are the sum of a zero mean stationary autoregressive Gaussian process and a deterministic linear trend. This process has been studied by Lo and Wang (1995) in the context of option pricing. It can be shown that

$$\mathrm{Var}[r_t(\tau)] = \frac{\sigma^2}{\gamma}[1 - e^{-\gamma\tau}], \tag{16}$$

or equivalently

$$\sigma^2 = \frac{\mathrm{Var}[r_t(\tau)]}{\tau}[\frac{\gamma\tau}{1 - e^{-\gamma\tau}}], \tag{17}$$

which shows that under the trending Ornstein-Uhlenbeck specification, the sample variance of continuously compounded returns is not an unbiased estimator of σ^2. Note however that the adjustment factor on the right hand side of (17) vanishes in the continuous time limit:

$$\lim_{\tau \to 0} \frac{\gamma\tau}{1 - e^{-\gamma\tau}} = \lim_{\tau \to 0} \frac{\gamma}{\gamma e^{-\gamma\tau}} = 1, \tag{18}$$

which is consistent with our heuristic discussion of the unbiasedness of the sample variance estimator for infinitesimal returns. In other words sample variance is unbiased in the limit, as sampling frequency increases without bound; however, in general, sample variance need not be an unbiased estimator of (instantaneous) variance σ^2.

C. Measuring Realized Volatility

In order to directly examine the empirical bias and efficiency of the various estimators, we require that realized volatility be observable. We use high-frequency data on very liquid and actively traded assets to construct measures of realized volatility as in, e.g., Andersen *et al.* (2000a, 2000b), who show that, under very weak assumptions, such estimates may be considered model-free and largely free of measurement error. To be more precise, Andersen *et al.* (2000a) show that under the assumption that the logarithmic asset price process is a special semimartingale, the sum of squares of discretely sampled continuously compounded returns computed from equally spaced observations converges uniformly in probability to the quadratic variation of the process (which is interpretable as the realized cumulative instantaneous variability of the process) as the sampling frequency increases without bound. The generality of the special semimartingale assumption is worth emphasizing: it assumes little more than absence of arbitrage opportunities. It does not require that the process be Markov, nor does it rule out jumps, e.g. it encompasses Merton's (1976) jump diffusion model as well as pure diffusion models which are central to much stochastic volatility option pricing theory. For example, if the asset price process is a diffusion as in (12), then the sum of squares of discretely sampled returns over the time interval $[t_1, t_2]$ converges in probability to $\int_{t_1}^{t_2} \sigma_t^2 dt$ as the sampling frequency increases without bound. In short, with high frequency data, it is possible to construct volatility measures which are model- and error-free, the argument being essentially that observation of the full (i.e. continuous) sample path allows one to infer the true volatility process from the quadratic variation. Of course, in practice, the full sample path is not observable (one only has a discrete sequence of prices recorded at closely spaced times) nor is it desirable to sample the data at the highest available frequency, which would introduce a number of microstructure biases. Andersen *et al.* (2000a, 2000b) find that, in their data, 5-minute returns are largely free of microstructure bias yet are sampled frequently enough that the resulting volatility series is essentially free of measurement error, see e.g. Andersen and Bollerslev (1998) and Andersen *et al.* (1999) for more discussion of this issue. Our data also suggest that the 5-minute frequency is appropriate, therefore in the remainder of this paper we use the phrase realized daily volatility to mean the daily sum of squared 5-minute returns. To formally define our daily volatility measures, we denote the time series of five-minute returns by $r_{j,t} = \ln(s_{j,t}) - \ln(s_{j-1,t})$, where the s denotes the price of the asset, j denotes the intraday interval $(j = 1, 2, ..., J.)$, t denotes the day, and J denotes the number of periods in a day (for example, $J = 288$ in the currency markets, and $J = 79$ for the stock indices). We then form the corresponding five-minute squared return series $r_{j,t}^2$. The realized volatility for day t is then given by

$$\sigma_t = N \sqrt{\sum_{j=1}^{j=J} r_{j,t}^2}, \tag{19}$$

where N is an annualizing factor (we take $N = \sqrt{240}$).

D. Data

We use six sets of high frequency data: two stock indices, a stock index futures, and three exchange rates. These are all very actively quoted instruments.

The stock index series we consider are the S&P 100 and S&P 500 spot indices, and futures on the S&P 500 index. These were primarily chosen for the economic importance of obtaining precise estimates of their volatility. The most actively traded option contract in the United States is written on the S&P 100 index, which makes the estimation of S&P 100 volatility particularly relevant. Furthermore, S&P 100 index option traders are known to hedge their positions with S&P 500 index futures because of the high liquidity of that futures contract; a fact which leads us to consider the estimation of S&P 500 futures volatility. For comparison purposes, we also examine the S&P 500 spot index. The S&P 100 and S&P 500 index series consist of time-stamped levels of the indices recorded every 15 seconds, the data set covers the period January 3, 1989 - December 30, 1999. The S&P 500 index futures series consists of time-stamped tick by tick data from the CME,[5] the sample period is January 3, 1989 - December 30, 1999. The exchange rates we consider are the Deutsche Mark / US Dollar, Japanese Yen / US Dollar, and US Dollar / British Pound spot exchange rates, which are among the most actively traded and quoted currencies. The data are derived from all the bid - ask interbank quotes (not transaction prices) that have appeared on the Reuters screen over the sample period. Prices are obtained by averaging the log bid and log ask. The data were obtained from Olsen and Associates, Zurich and cover the period December 3, 1986 - December 30, 1998.

[5]These are prices recorded by exchange personnel who observe the pit and post the most recent transaction price. The observers record every change in price, but not successive trades at the same price.

Table 1 provides summary statistics for the unconditional distribution of the realized daily volatility series computed for these data. The mean values for the S&P 100 and S&P 500 are 8.99 and 10.34 percent, respectively. The mean value for the S&P 500 futures is 11.75 percent; we find that futures volatility is higher than spot volatility, which is consistent with previous research. The mean volatilities for the currencies range from 9.89 percent for the British Pound to 11.50 percent for the Japanese Yen. The standard deviations given in the second column show that realized volatility exhibits significant time series variation and that stock index volatility fluctuates somewhat more over time than currency volatility. The third and fourth columns indicate that the distributions of the realized volatility series are extremely right-skewed and leptokurtic. This may come as a surprise, as the realized daily volatilities are based on between 79 and 288 five-minute returns. Andersen, Bollerslev and Das (2001) show that standard central limit theorem arguments generally do not apply in a high frequency context, due to the strong dependence displayed by high frequency returns.

II. Empirical Results

A. Comparison Criteria

In order to analyze the empirical performance of the various estimators presented in Section 2, one must define finite sample criteria upon which meaningful comparisons may be based. While unbiasedness is a desirable attribute, it is rarely used by itself as an estimation criterion. Estimators are often compared on the basis of their mean squared error. Let σ_t denote the volatility realized during period t, the mean squared error of an estimator $\widehat{\sigma}_t$ is

$$MSE(\widehat{\sigma}_t) \quad = \quad E[(\widehat{\sigma}_t - \sigma_t)^2]$$

$$= \quad (E[\widehat{\sigma}_t - \sigma_t])^2 + \mathrm{Var}[\widehat{\sigma}_t - \sigma_t], \tag{20}$$

thus the mean square error equals the square of the bias (mean difference between the estimator and the parameter) plus the variance of the difference (between the estimator and the parameter). In a decision theoretic setting, minimizing the mean square error is equivalent to minimizing the expected loss associated with a loss function which is proportional to the square of the difference between the estimate and the true parameter. It should be noted however that the quantities in equation (13) are based on squares of variance estimators, so the fourth moments of the data are involved. To check that the presence of outliers is not driving our results, we also consider the mean absolute difference between the estimator and the true parameter:

$$MAD(\widehat{\sigma}_t) = E[|\widehat{\sigma}_t - \sigma_t|]. \tag{21}$$

To give some sense of the magnitude of the bias associated with a certain volatility estimator, we also report the relative bias:

$$\text{Prop. Bias} = E[\frac{\widehat{\sigma}_t - \sigma_t}{\sigma_t}]. \tag{22}$$

Finally, we also computed the following:

$$R^2 = 1 - \frac{MAD(\widehat{\sigma}_t)}{E[|\sigma_t - \overline{\sigma_t}|]}, \tag{23}$$

where $\overline{\sigma_t}$ is the mean realized volatility. This is simply a rescaled version of mean absolute deviation. Like the traditional R^2 from regression analysis, it has the interpretation of "percentage of variance explained", but unlike the traditional R^2 from a regression with a constant, nothing here prevents R^2 from being negative. This will occur if the mean absolute deviation for a given volatility estimator, $MAD(\widehat{\sigma}_t)$, is greater than the total variation in volatility, $E[|\sigma_t - \overline{\sigma_t}|]$, which can be interpreted as meaning that the noise component in the estimator is larger than the signal component, clearly an undesirable property.

Tables 2 through 5 report sample estimates of these criteria, along with standard errors in parentheses. These and other statistics (such as estimates of the variance-covariance matrix of the criteria) were obtained using standard bootstrap techniques, e.g. see Efron and Tibshirani (1986). The following sections present our main findings.

B. Stock Index Results

Table 2 compares the various estimators on S&P 500 index data, and table 3 contains the results for the S&P 100 index. Volatility estimators[6] are computed over non-overlapping estimation windows of one day, one week (5 trading days), and one month (24 trading days).[7]

The first two columns in the tables give the estimated mean proportional bias and the estimated bias, respectively. Surprisingly, we find that, in general, extreme value volatility estimators are not more biased than the traditional estimator. In fact, except at the daily frequency, *all* extreme value estimators are less biased than the traditional estimator. For example, consider the S&P 100 results: using a month of daily data, the traditional estimator has a mean bias of 2.75 percent, while the mean biases of extreme value estimators range from -0.12 percent for the adjusted Garman and Klass estimator, to 1.38 percent for the Parkinson estimator. A standard test cannot reject the null hypothesis that the adjusted Rogers and

[6] As discussed above, here we consider open market equivalents of the various estimators, e.g. we use (11) in lieu of (4).

[7] If the analysis were presented in terms of variances (not volatilities), then the average bias for each estimator would be independent of the estimation frequency (daily, weekly, or monthly).

Satchell and the adjusted Garman and Klass estimators are unbiased at the monthly frequency (at the 5 percent significance level); in terms of bias, these estimators seem to perform best.

Consistent with the previous research of Wiggins (1991), we find that the Parkinson estimator is downward biased *compared to* the traditional estimator, at the weekly and monthly frequencies. This is also true for the other extreme value volatility estimators. However, we find that this is not due to the fact that extreme value estimators are downward biased, but rather to the fact that the traditional estimator is upward biased. In fact, our results generally indicate that extreme value volatility estimators are upward, not downward, biased. Although theoretical research in this area has mostly argued that extreme value volatility estimators may be downward biased, we can think of reasons why extreme value estimators would be upward biased. For example, the Parkinson and Garman and Klass estimators explicitly assume that the asset price process has no drift; for a Brownian motion with drift the resulting estimator will be upward biased. To see why the bias will be upward, not downward, consider the case where the variance is very small. The behavior of the process will then be mostly determined by the drift, and the expected range will primarily reflect the expected rate of change. More generally, all extreme value estimators make strong assumptions about the distribution of asset prices, and the resulting bias may be upward or downward.

The third column in the tables gives the estimation variance. Consistent with previous research, we find that the use of extreme value volatility estimators results in significant efficiency gains. For example, the adjusted Garman and Klass estimator is about 9 times more efficient than the traditional estimator (on S&P 100 data). The fourth, fifth, and sixth columns give the estimated mean square error, mean absolute deviation, and our R^2 measure for the various volatility estimators. Consistent with our results regarding bias and efficiency, we find that extreme value volatility estimators significantly outperform the traditional estimator: extreme value estimators consistently produce lower MSE's and MAD's, and higher R^2 measures. For both indices, we find that the adjusted Garman and Klass estimator performs best according to all our comparison criteria (MSE, MAD and R^2). However, in all cases, we find that using any extreme value volatility estimator produces significantly better estimates than those obtained from the traditional estimator.

C. Index Futures Results

Table 4 compares the various estimators on S&P 500 index futures data. Looking at the first two columns, we find that the traditional estimator is significantly biased at the daily frequency, but that the bias essentially disappears (and is statistically insignificant) at the weekly and monthly frequencies. In contrast, we find that all extreme value volatility estimators are significantly downward biased at all frequencies. For example, at the monthly frequency, the traditional estimator has a bias of 0.23 percent (not significantly different from zero), while extreme value volatility estimators have mean biases that range from -2.45 percent (adjusted Garman and Klass) to -1.07 percent (Parkinson), all of which are significantly different from zero. The magnitude of the reported biases is consistent with previous research by Wiggins (1992). At the daily frequency, only the Parkinson and the Garman and Klass estimators produce lower estimated mean biases than the traditional estimator.

We find that extreme value volatility estimators are significantly more efficient than the traditional estimator. For example, at the monthly frequency, the Parkinson estimator is about 5 times more efficient than the traditional estimator. We also find that at the monthly frequency, the Parkinson estimator has a smaller MSE and MAD (and a larger R^2) than any other estimator. According to these criteria, the evidence in favor of using extreme value volatility estimators (at the monthly frequency) is not strong, however: the traditional estimator has an R^2 measure of 59.70 percent, which is higher than that of any extreme value estimator other than the Parkinson estimator, which has an R^2 measure of 66.82 percent.[8] At the daily and weekly frequencies, the evidence in favor of extreme value estimators is stronger; this is mainly due to the fact that they have much smaller estimation variances than the traditional estimator.

D. Foreign Exchange Results

Table 5 compares the various estimators on S&P 500 on foreign exchange data; Panel A gives the results for the Deutsche Mark / US Dollar exchange rate. Looking at the first two columns, we see that the close to close estimator is downward biased at all frequencies: while the bias is smaller at the monthly frequency, it remains significantly different from zero at all frequencies. There appears to be a significant difference between intraday volatility and day to day volatility in currency markets: it appears that intraday volatility is larger than day to day volatility. This is consistent with the results of Jorion (1995), who finds that implied volatilities derived from CME options on foreign currency futures are typically significantly larger than historical daily volatility; it may therefore be the case that currency option markets price intraday volatility, not day to day volatility. At the daily frequency, the traditional estimator is more biased than all the extreme value estimators: it has an estimated mean bias of -2.60 percent, while extreme value volatility estimators have estimated mean biases that range from -2.26 percent (adjusted Rogers and Satchell) to -1.59 percent (Parkinson). In contrast, at the weekly and monthly frequencies, the traditional estimator has a smaller bias than all the extreme value volatility estimators. For example, at the monthly frequency, the traditional estimator has an estimated mean bias of -0.60 percent, while extreme value volatility estimators have estimated

[8] The R^2 measure of the Parkinson estimator is significantly higher than that of the traditional estimator at the 5 percent level.

mean biases that range from -1.91 percent (adjusted Rogers and Satchell) to -1.12 percent (Parkinson).

The third column gives the estimation variance of the various estimators. Notwithstanding the large biases reported above, we find that extreme value volatility estimators are significantly more efficient than the traditional estimator. The greatest efficiency gains are achieved by the Garman and Klass, the adjusted Garman and Klass, and the Yang and Zhang estimators. For example, the adjusted Garman and Klass estimator is 7.65 times more efficient than the traditional estimator at the weekly frequency, and the Yang and Zhang estimator is 5.75 times more efficient than the traditional estimator at the monthly frequency. In terms of MSE, MAD or our R^2 measure, the Garman and Klass estimator is preferred at the daily frequency, while the Parkinson estimator is preferred at the weekly and monthly frequencies. If MSE or MAD is chosen as the appropriate criterion to compare the various variance estimators then our results suggest that the evidence in favor of the use of extreme value volatility estimators is strong at the daily and weekly frequencies, but not at the monthly frequency: at the monthly frequency, the traditional estimator does almost as well as the best extreme value volatility estimator (Parkinson), and strictly better than many (Garman and Klass, Rogers and Satchell, and their adjusted versions). We would like to point out, however, that MSE and MAD (and R^2) are rather arbitrary criteria. One may attach more weight to the bias of an estimator, and less to its variance. Furthermore, as a practical matter, one need not attach equal weight to similarly sized underestimations and overestimations of variance: e.g., an underestimation of variance will impart a downward bias to estimates of the price of call options, which may be of more concern to a seller than a buyer, while the reverse is true of overestimations of asset return variance.

Panel B gives the results for the Japanese Yen / US Dollar exchange rate. Our results are similar to those obtained for the Deutsche Mark. Briefly, we find that the traditional estimator is significantly downward biased at all frequencies. At the daily frequency, the traditional estimator is more biased than extreme value estimators, but at other frequencies, it is less biased than extreme value volatility estimators. Among all extreme value estimators, the Parkinson estimator has the smallest estimated mean biases, while the adjusted Garman and Klass generally achieves the largest efficiency gains. Overall, we find strong evidence in favor of the use of extreme value estimators at the daily and weekly frequencies (at which the Garman and Klass and Yang and Zhang estimators perform best in terms of MSE, MAD or R^2), but not at the monthly frequency: at the monthly frequency, the traditional estimator does almost as well as the best-performing extreme value volatility estimator (the Parkinson estimator).

Panel C gives the results for the British Pound / US Dollar exchange rate. Our results are similar to those obtained for the Japanese Yen and the Deutsche Mark, with one important difference: we find strong evidence in favor of the use of extreme value volatility estimators at all frequencies. For example, even at the monthly frequency, the MSE of the Parkinson estimator is less than half that of the traditional estimator.

III. Conclusion

Given a sample of daily observations on an asset, which includes the daily opening, closing, high and low prices, how should one go about estimating asset return volatility? Traditionally, volatility has been estimated by means of the sample standard deviation, but a plethora of alternative estimators is available in the literature. These have been shown to be theoretically several times more efficient than the traditional estimator, yet their use in the literature is rather limited. In this paper, we compare the bias and efficiency of a number of volatility estimators including the various extreme value volatility estimators that have appeared in the literature.

Using high frequency data, we construct measures of realized volatility for six sets of data (three currencies, two stock indices and a futures contract). Under very weak assumptions, such measures can be considered model-free and largely free of measurement error. In contrast, the traditional estimator derived from daily data is generally not unbiased: the unbiasedness of the traditional estimator only obtains under specific assumptions about the data generating process. Our methodology makes it possible, for the first time, to study the bias and relative efficiency of the various estimators directly. The stock index results give overwhelming support to the use of extreme value volatility estimators: we find that these estimators are both less biased and more efficient than the traditional estimator. The futures and currency results are less clear. While we confirm that extreme value volatility estimators afford significant efficiency gains, we also show that they are significantly biased. The choice of estimation procedure must then be one that strikes an optimal balance between bias and efficiency, taking into account the length of the estimation window.

Table 1

Descriptive Statistics for Realized Volatility

Entries are sample moments computed for daily realized volatility for the S&P500 Index Futures and exchange rates on Deutsche Mark, Japanese Yen and British Pound. The sample periods are from January 3, 1989 to December 30, 1999 and from December 3, 1986 to December 30, 1998 for the S&P500 Index Futures and exchange rates respectively. Skewness is the estimate of the skewness measure γ_1. Kurtosis is the estimate of the kurtosis measure γ_2. The skewness and kurtosis measures are defined, specifically, in terms of the central moments μ_j: $\gamma_1 = \mu_3/\mu_2^{3/2}$ and $\gamma_2 = \mu_4/\mu_2^2 - 3$. Both of them should be zero for normal random variables.

	Mean	Standard Deviation	Skewness	Kurtosis
Panel A. Equity Index				
S&P500	10.3444	5.0796	2.1832	7.9544
S&P100	8.9886	5.2542	4.8169	69.6890
Panel B. Futures				
S&P500 Futures	11.7475	5.7945	2.6320	13.5630
Panel C. Exchange Rates				
DM/US	10.6448	3.9179	1.7776	5.4022
JY/US	11.5016	4.8241	3.2642	30.9200
BP/US	9.8856	3.4380	1.8573	7.0636

Table 2

Comparison of Volatility Estimators on S&P 500 Index

This table details the proportional bias, bias, variance, mean square error (MSE), and mean absolute deviation (MAD) of various volatility estimators when volatility is estimated using daily data covering periods of 1 to 24 (trading) days. The sample period is January 1989 - December 1999. The traditional, close to close volatility estimator assuming no drift is denoted as $\widehat{\sigma}_{cc}$. $\widehat{\sigma}_{acc}$ denotes the close to close volatility estimator which adjusts for a possible drift. $\widehat{\sigma}_{p}$ denotes the Parkinson (1980) estimator. $\widehat{\sigma}_{GK}$ denotes the Garman and Klass (1980) estimator. $\widehat{\sigma}_{RS}$ denotes the Rogers and Satchell (1991) estimator. $\widehat{\sigma}_{ARS}$ and $\widehat{\sigma}_{AGK}$ are versions of, respectively, the Garman and Klass (1980) and Rogers and Satchell (1991) estimators which are adjusted for discrete trading, and $\widehat{\sigma}_{YZ}$ denotes the Yang and Zhang (2000) estimator.

		Prop. Bias	Bias	Variance	MSE	MAD	R^2
	1 day	0.1090	0.7905	67.7111	68.3094	5.9198	-0.6671
		(0.0170)	(0.1631)	(3.8542)	(3.9731)	(0.1149)	(0.0448)
$\widehat{\sigma}_{cc}$	5 days	0.3085	2.6461	16.1229	23.0932	3.6696	-0.1218
		(0.0186)	(0.1781)	(1.4783)	(1.9817)	(0.1384)	(0.0680)
	24 days	0.3500	3.1383	4.6114	14.4165	3.2223	0.0022
		(0.0195)	(0.2083)	(0.9916)	(1.9143)	(0.1954)	(0.1115)
$\widehat{\sigma}_{acc}$	5 days	0.2885	2.4827	17.7822	23.9112	3.6895	-0.1279
		(0.0192)	(0.1874)	(1.6734)	(2.1559)	(0.1429)	(0.0655)
	24 days	0.3465	3.1135	4.8285	14.4767	3.2079	0.0067
		(0.0199)	(0.2127)	(1.0192)	(1.9237)	(0.1986)	(0.1097)
	1 day	0.1561	1.2585	12.5807	14.1597	2.5020	0.2954
		(0.0072)	(0.0706)	(1.0649)	(1.0989)	(0.0559)	(0.0195)
$\widehat{\sigma}_{p}$	5 days	0.2004	1.6452	3.0022	5.7029	1.8480	0.4351
		(0.0076)	(0.0767)	(0.3603)	(0.4536)	(0.0670)	(0.0329)
	24 days	0.2078	1.7327	1.5854	4.5727	1.8602	0.4240
		(0.0103)	(0.1223)	(0.6858)	(0.5937)	(0.1020)	(0.0608)
	1 day	0.1135	0.9077	8.7322	9.5528	2.1071	0.4066
		(0.0056)	(0.0584)	(0.9223)	(0.9105)	(0.0449)	(0.0148)
$\widehat{\sigma}_{GK}$	5 days	0.1539	1.2217	2.6903	4.1776	1.5246	0.5339
		(0.0069)	(0.0727)	(0.6402)	(0.5943)	(0.0601)	(0.0255)
	24 days	0.1632	1.2856	1.4828	3.1213	1.4571	0.5488
		(0.0102)	(0.1181)	(0.7713)	(0.6043)	(0.0964)	(0.0487)

Table 2 (Continued)

		Prop. Bias	Bias	Variance	MSE	MAD	R^2
$\widehat{\sigma}_{RS}$	1 day	0.0366	0.2624	18.7799	18.8413	2.9650	0.1650
		(0.0052)	(0.0557)	(1.1116)	(1.1148)	(0.0417)	(0.0129)
	5 days	0.1269	0.9888	4.8286	5.7967	1.7057	0.4786
		(0.0065)	(0.0745)	(0.8980)	(0.8826)	(0.0571)	(0.0172)
	24 days	0.1485	1.1283	1.9417	3.1964	1.3969	0.5674
		(0.0095)	(0.1265)	(0.9368)	(0.8920)	(0.0993)	(0.0282)
$\widehat{\sigma}_{ARS}$	1 day	-0.0475	-0.4853	17.8929	18.1214	2.8500	0.1974
		(0.0084)	(0.0850)	(1.2429)	(1.2357)	(0.0628)	(0.0216)
	5 days	0.0427	0.2218	4.7826	4.8224	1.4649	0.5522
		(0.0092)	(0.0976)	(1.0059)	(0.9640)	(0.0749)	(0.0280)
	24 days	0.0641	0.3464	2.1122	2.2121	1.0046	0.6889
		(0.0125)	(0.1349)	(0.8551)	(0.7056)	(0.1083)	(0.0483)
$\widehat{\sigma}_{AGK}$	1 day	0.0012	-0.0857	8.0072	8.0114	1.8941	0.4666
		(0.0081)	(0.0830)	(1.3462)	(1.3718)	(0.0625)	(0.0208)
	5 days	0.0408	0.1923	2.8248	2.8562	1.0840	0.6686
		(0.0088)	(0.0972)	(1.1726)	(1.1546)	(0.0721)	(0.0227)
	24 days	0.0505	0.2361	1.7102	1.7496	0.8438	0.7387
		(0.0119)	(0.1405)	(0.9900)	(0.9253)	(0.1072)	(0.0332)
$\widehat{\sigma}_{YZ}$	5 days	0.1638	1.3316	3.2552	5.0219	1.6357	0.5000
		(0.0076)	(0.0801)	(0.6401)	(0.6315)	(0.0676)	(0.0281)
	24 days	0.1841	1.4842	1.6308	3.8181	1.6329	0.4944
		(0.0109)	(0.1238)	(0.7944)	(0.6354)	(0.1038)	(0.0528)

Table 3

Comparison of Estimators on S&P 100 Index

This table details the proportional bias, bias, variance, mean square error (MSE), and mean absolute deviation (MAD) of various volatility estimators when volatility is estimated using daily data covering periods of 1 to 24 (trading) days. The sample period is January 1989 - December 1999. The traditional, close to close volatility estimator assuming no drift is denoted as $\widehat{\sigma}_{cc}$. $\widehat{\sigma}_{acc}$ denotes the close to close volatility estimator which adjusts for a possible drift. $\widehat{\sigma}_p$ denotes the Parkinson (1980) estimator. $\widehat{\sigma}_{GK}$ denotes the Garman and Klass (1980) estimator. $\widehat{\sigma}_{RS}$ denotes the Rogers and Satchell (1991) estimator. $\widehat{\sigma}_{ARS}$ and $\widehat{\sigma}_{AGK}$ are versions of, respectively, the Garman and Klass (1980) and Rogers and Satchell (1991) estimators which are adjusted for discrete trading, and $\widehat{\sigma}_{YZ}$ denotes the Yang and Zhang (2000) estimator.

		Prop. Bias	Bias	Variance	MSE	MAD	R^2
	1 day	-0.0025	0.0525	80.1863	80.1926	6.4147	-0.7603
		(0.0138)	(0.1623)	(4.7369)	(4.7456)	(0.1148)	(0.0385)
$\widehat{\sigma}_{cc}$	5 days	0.2009	2.1753	22.2090	26.9041	3.5708	-0.0777
		(0.0159)	(0.1919)	(2.8974)	(3.4077)	(0.1523)	(0.0565)
	24 days	0.2518	2.7455	7.7007	15.1771	2.9350	0.0627
		(0.0187)	(0.2334)	(1.6096)	(2.5402)	(0.2142)	(0.0961)
$\widehat{\sigma}_{acc}$	5 days	0.1937	2.1259	26.6153	31.0910	3.7725	-0.1386
		(0.0170)	(0.2098)	(3.8288)	(4.3906)	(0.1665)	(0.0601)
	24 days	0.2501	2.7458	8.2372	1 5.7111	2.9447	0.0596
		(0.0193)	(0.2416)	(1.7227)	(2.6983)	(0.2221)	(0.0977)
	1 day	0.0743	0.7968	14.3386	14.9687	2.5730	0.2939
		(0.0059)	(0.0690)	(1.0400)	(1.0996)	(0.0532)	(0.0172)
$\widehat{\sigma}_{p}$	5 days	0.1209	1.2440	4.3076	5.8481	1.6372	0.5059
		(0.0067)	(0.0847)	(0.6912)	(0.8326)	(0.0720)	(0.0270)
	24 days	0.1321	1.3790	1.7049	3.5930	1.4233	0.5455
		(0.0091)	(0.1099)	(0.3479)	(0.5843)	(0.1055)	(0.0524)
	1 day	0.0498	0.5129	9.5596	9.8196	2.1676	0.4052
		(0.0049)	(0.0563)	(0.5911)	(0.6122)	(0.0413)	(0.0138)
$\widehat{\sigma}_{GK}$	5 days	0.0885	0.8688	2.5818	3.3323	1.3145	0.6033
		(0.0055)	(0.0653)	(0.2981)	(0.3607)	(0.0514)	(0.0203)
	24 days	0.0963	0.9584	0.9416	1.8526	1.0716	0.6578
		(0.0075)	(0.0844)	(0.1332)	(0.2461)	(0.0733)	(0.0397)

Table 3 (Continued)

		Prop. Bias	Bias	Variance	MSE	MAD	R^2
	1 day	-0.0109	-0.1335	19.8047	19.8160	3.0543	0.1619
		(0.0046)	(0.0527)	(0.4843)	(0.4802)	(0.0379)	(0.0129)
$\widehat{\sigma}_{RS}$	5 days	0.0628	0.5914	4.0184	4.3616	1.5429	0.5343
		(0.0051)	(0.0613)	(0.2254)	(0.2257)	(0.0426)	(0.0156)
	24 days	0.0774	0.7354	1.1753	1.7067	1.0366	0.6690
		(0.0068)	(0.0837)	(0.1140)	(0.1258)	(0.0553)	(0.0185)
	1 day	-0.0909	-0.9618	18.7756	19.6945	3.0305	0.1684
		(0.0070)	(0.0810)	(1.0108)	(1.0119)	(0.0593)	(0.0199)
$\widehat{\sigma}_{ARS}$	5 days	-0.0171	-0.2535	3.8270	3.8849	1.4243	0.5701
		(0.0072)	(0.0812)	(0.4049)	(0.4334)	(0.0573)	(0.0236)
	24 days	-0.0026	-0.1248	1.2125	1.2184	0.8221	0.7375
		(0.0084)	(0.0954)	(0.1583)	(0.2227)	(0.0710)	(0.0356)
	1 day	-0.0542	-0.5655	8.3763	8.6934	2.0978	0.4243
		(0.0068)	(0.0790)	(0.9502)	(0.9873)	(0.0594)	(0.0197)
$\widehat{\sigma}_{AGK}$	5 days	-0.0170	-0.2501	2.2882	2.3469	1.1130	0.6641
		(0.0068)	(0.0790)	(0.3772)	(0.3787)	(0.0551)	(0.0209)
	24 days	-0.0096	-0.1840	0.9086	0.9352	0.7372	0.7646
		(0.0080)	(0.0966)	(0.1720)	(0.1787)	(0.0646)	(0.0244)
$\widehat{\sigma}_{YZ}$	5 days	0.1008	1.0278	3.2512	4.3023	1.4570	0.5603
		(0.0062)	(0.0733)	(0.3886)	(0.4843)	(0.0601)	(0.0233)
	24 days	0.1163	1.1891	1.0736	2.4790	1.2407	0.6038
		(0.0077)	(0.0897)	(0.1534)	(0.3211)	(0.0848)	(0.0444)

Table 4

Comparison of Estimators on S&P 500 Futures

This table details the proportional bias, bias, variance, mean square error (MSE), and mean absolute deviation (MAD) of various volatility estimators when volatility is estimated using daily data covering periods of 1 to 24 (trading) days. The sample period is January 1989 - December 1999. The traditional, close to close volatility estimator assuming no drift is denoted as $\widehat{\sigma}_{cc}$. $\widehat{\sigma}_{acc}$ denotes the close to close volatility estimator which adjusts for a possible drift. $\widehat{\sigma}_{p}$ denotes the Parkinson (1980) estimator. $\widehat{\sigma}_{GK}$ denotes the Garman and Klass (1980) estimator. $\widehat{\sigma}_{RS}$ denotes the Rogers and Satchell (1991) estimator. $\widehat{\sigma}_{ARS}$ and $\widehat{\sigma}_{AGK}$ are versions of, respectively, the Garman and Klass (1980) and Rogers and Satchell (1991) estimators which are adjusted for discrete trading, and $\widehat{\sigma}_{YZ}$ denotes the Yang and Zhang (2000) estimator.

		Prop. Bias	Bias	Variance	MSE	MAD	R^2
	1 day	-0.2035	-2.2388	67.7634	72.7531	6.4036	-0.6004
		(0.0110)	(0.1495)	(4.0760)	(3.8458)	(0.1027)	(0.0340)
$\widehat{\sigma}_{cc}$	5 days	-0.0300	-0.3148	20.2550	20.3207	3.1209	0.1641
		(0.0128)	(0.1825)	(2.6930)	(2.6618)	(0.1325)	(0.0376)
	24 days	0.0131	0.2334	4.5343	4.5528	1.4280	0.5970
		(0.0127)	(0.1852)	(1.0590)	(1.0948)	(0.1366)	(0.0374)
$\widehat{\sigma}_{acc}$	5 days	-0.0342	-0.3267	23.8846	23.9520	3.3152	0.1120
		(0.0136)	(0.1980)	(3.3562)	(3.3082)	(0.1463)	(0.0397)
	24 days	0.0161	0.2809	4.8486	4.8890	1.5057	0.5751
		(0.0134)	(0.1911)	(1.1111)	(1.1569)	(0.1388)	(0.0380)
	1 day	-0.1439	-1.5808	12.6736	15.1682	2.9491	0.2629
		(0.0048)	(0.0647)	(0.7931)	(0.7557)	(0.0463)	(0.0159)
$\widehat{\sigma}_{p}$	5 days	-0.0967	-1.1621	3.4014	4.7464	1.6319	0.5629
		(0.0052)	(0.0749)	(0.4994)	(0.4877)	(0.0589)	(0.0198)
	24 days	-0.0858	-1.0652	0.9084	2.0359	1.1759	0.6682
		(0.0053)	(0.0835)	(0.1676)	(0.2761)	(0.0719)	(0.0252)
	1 day	-0.1764	-1.9903	8.6977	12.6559	2.6962	0.3262
		(0.0039)	(0.0538)	(0.4234)	(0.5273)	(0.0423)	(0.0144)
$\widehat{\sigma}_{GK}$	5 days	-0.1391	-1.7045	2.5055	5.4068	1.8127	0.5145
		(0.0041)	(0.0642)	(0.2892)	(0.4416)	(0.0590)	(0.0204)
	24 days	-0.1307	-1.6599	1.1453	3.8914	1.6693	0.5289
		(0.0046)	(0.0951)	(0.2637)	(0.5291)	(0.0937)	(0.0330)

Table 4 (Continued)

		Prop. Bias	Bias	Variance	MSE	MAD	R^2
	1 day	-0.2053	-2.3431	16.5261	22.0108	3.2929	0.1770
		(0.0037)	(0.0535)	(0.4804)	(0.6721)	(0.0470)	(0.0161)
$\widehat{\sigma}_{RS}$	5 days	-0.1480	-1.8232	4.2579	7.5751	2.0137	0.4606
		(0.0038)	(0.0703)	(0.3822)	(0.6648)	(0.0691)	(0.0266)
	24 days	-0.1340	-1.7259	1.8447	4.8089	1.7798	0.4977
		(0.0044)	(0.1155)	(0.3852)	(0.8887)	(0.1155)	(0.0473)
	1 day	-0.2621	-3.0154	16.0902	25.1774	3.6079	0.0983
		(0.0053)	(0.0741)	(1.2261)	(1.4171)	(0.0614)	(0.0188)
$\widehat{\sigma}_{ARS}$	5 days	-0.2068	-2.5290	4.6458	11.0340	2.5962	0.3046
		(0.0054)	(0.0836)	(0.6221)	(0.8075)	(0.0759)	(0.0244)
	24 days	-0.1930	-2.4462	2.2909	8.2567	2.4851	0.2987
		(0.0066)	(0.1209)	(0.5021)	(0.7972)	(0.1145)	(0.0366)
	1 day	-0.2431	-2.7820	8.5700	16.3068	3.1143	0.2217
		(0.0050)	(0.0733)	(1.2311)	(1.5021)	(0.0641)	(0.0200)
$\widehat{\sigma}_{AGK}$	5 days	-0.2083	-2.5384	3.0080	9.4464	2.5576	0.3150
		(0.0052)	(0.0874)	(0.7018)	(1.0080)	(0.0838)	(0.0291)
	24 days	-0.2004	-2.5141	1.6964	8.0035	2.5141	0.2905
		(0.0063)	(0.1342)	(0.5812)	(1.0921)	(0.1284)	(0.0476)
$\widehat{\sigma}_{YZ}$	5 days	-0.1237	-1.5024	2.5995	4.8524	1.6921	0.5468
		(0.0044)	(0.0654)	(0.3239)	(0.4125)	(0.0571)	(0.0201)
	24 days	-0.1090	-1.3895	1.0941	3.0162	1.4360	0.5948
		(0.0053)	(0.0931)	(0.2573)	(0.4282)	(0.0873)	(0.0298)

Table 5

Comparison of Volatility Estimators for Exchange Rates

This table details the proportional bias, bias, variance, mean square error (MSE), and mean absolute deviation (MAD) of various volatility estimators when volatility is estimated using daily data covering periods of 1 to 24 (trading) days. The sample period is December 1986 - December 1999. The traditional, close to close volatility estimator assuming no drift is denoted as $\widehat{\sigma}_{cc}$. $\widehat{\sigma}_{acc}$ denotes the close to close volatility estimator which adjusts for a possible drift. $\widehat{\sigma}_{p}$ denotes the Parkinson (1980) estimator. $\widehat{\sigma}_{GK}$ denotes the Garman and Klass (1980) estimator. $\widehat{\sigma}_{RS}$ denotes the Rogers and Satchell (1991) estimator. $\widehat{\sigma}_{ARS}$ and $\widehat{\sigma}_{AGK}$ are versions of, respectively, the Garman and Klass (1980) and Rogers and Satchell (1991) estimators which are adjusted for discrete trading, and $\widehat{\sigma}_{YZ}$ denotes the Yang and Zhang (2000) estimator.

Panel A. Deutsche Mark / Dollar Exchange Rate

		Prop. Bias	Bias	Variance	MSE	MAD	R^2
	1 day	-0.2632	-2.5987	40.7134	47.4525	5.5142	-0.9368
		(0.0101)	(0.1184)	(1.5032)	(1.4005)	(0.0770)	(0.0392)
$\widehat{\sigma}_{cc}$	5 days	-0.1030	-1.0180	10.0385	11.0575	2.6134	-0.0726
		(0.0118)	(0.1320)	(0.6751)	(0.7062)	(0.0857)	(0.0501)
	24 days	-0.0590	-0.6009	2.2469	2.5892	1.3219	0.3686
		(0.0122)	(0.1361)	(0.2950)	(0.3071)	(0.0834)	(0.0609)
$\widehat{\sigma}_{acc}$	5 days	-0.1091	-1.0742	12.6548	13.7868	2.8879	-0.1853
		(0.0129)	(0.1482)	(0.9260)	(0.9264)	(0.0977)	(0.0547)
	24 days	-0.0577	-0.5839	2.5060	2.8260	1.3787	0.3415
		(0.0127)	(0.1440)	(0.3356)	(0.3373)	(0.0876)	(0.0635)
	1 day	-0.1649	-1.5854	8.1745	10.6852	2.6650	0.0640
		(0.0046)	(0.0527)	(0.3382)	(0.3098)	(0.0350)	(0.0197)
$\widehat{\sigma}_{p}$	5 days	-0.1218	-1.2266	1.9125	3.4137	1.5224	0.3751
		(0.0054)	(0.0575)	(0.1380)	(0.1837)	(0.0436)	(0.0299)
	24 days	-0.1061	-1.1176	0.5075	1.7523	1.1685	0.4419
		(0.0059)	(0.0643)	(0.0732)	(0.1558)	(0.0563)	(0.0485)
	1 day	-0.1703	-1.6699	5.5061	8.2928	2.3124	0.1878
		(0.0037)	(0.0432)	(0.2308)	(0.2551)	(0.0318)	(0.0172)
$\widehat{\sigma}_{GK}$	5 days	-0.1405	-1.4340	1.3735	3.4274	1.5650	0.3577
		(0.0044)	(0.0488)	(0.1119)	(0.1809)	(0.0414)	(0.0296)
	24 days	-0.1279	-1.3568	0.4007	2.2381	1.3639	0.3485
		(0.0053)	(0.0574)	(0.0598)	(0.1727)	(0.0560)	(0.0530)

Panel A. Deutsche Mark / Dollar Exchange Rate (Continued)

		Prop. Bias	Bias	Variance	MSE	MAD	R^2
	1 day	-0.1885	-1.8813	7.2735	10.8101	2.5653	0.0990
		(0.0036)	(0.0417)	(0.2177)	(0.2746)	(0.0327)	(0.0184)
$\widehat{\sigma}_{RS}$	5 days	-0.1560	-1.6027	1.8518	4.4172	1.7763	0.2710
		(0.0043)	(0.0477)	(0.1107)	(0.2221)	(0.0436)	(0.0346)
	24 days	-0.1421	-1.5118	0.5103	2.7917	1.5138	0.2770
		(0.0051)	(0.0584)	(0.0696)	(0.2378)	(0.0584)	(0.0648)
	1 day	-0.2232	-2.2574	7.0734	1 2.1669	2.7591	0.0309
		(0.0042)	(0.0497)	(0.3855)	(0.4457)	(0.0382)	(0.0192)
$\widehat{\sigma}_{ARS}$	5 days	-0.1915	-1.9913	1.8206	5.7826	2.0874	0.1433
		(0.0050)	(0.0567)	(0.1609)	(0.2519)	(0.0470)	(0.0328)
	24 days	-0.1779	-1.9084	0.5298	4.1674	1.9084	0.0885
		(0.0061)	(0.0651)	(0.0687)	(0.2149)	(0.0647)	(0.0585)
	1 day	-0.2061	-2.0580	5.1332	9.3667	2.4991	0.1222
		(0.0041)	(0.0490)	(0.3902)	(0.4827)	(0.0396)	(0.0206)
$\widehat{\sigma}_{AGK}$	5 days	-0.1777	-1.8427	1.3129	4.7061	1.9018	0.2194
		(0.0048)	(0.0562)	(0.1612)	(0.2981)	(0.0498)	(0.0376)
	24 days	-0.1657	-1.7756	0.4117	3.5609	1.7756	0.1520
		(0.0059)	(0.0664)	(0.0770)	(0.2825)	(0.0664)	(0.0699)
$\widehat{\sigma}_{YZ}$	5 days	-0.1431	-1.4581	1.4226	3.5463	1.6081	0.3400
		(0.0045)	(0.0496)	(0.1112)	(0.1757)	(0.0410)	(0.0303)
	24 days	-0.1281	-1.3574	0.3908	2.2302	1.3624	0.3493
		(0.0054)	(0.0567)	(0.0525)	(0.1661)	(0.0557)	(0.0530)

Panel B. Japanese Yen / Dollar Exchange Rate

		Prop. Bias	Bias	Variance	MSE	MAD	R^2
	1 day	-0.2976	-3.1409	49.7602	59.6084	6.0022	-0.8104
		(0.0098)	(0.1318)	(3.0603)	(2.9549)	(0.0900)	(0.0378)
$\widehat{\sigma}_{cc}$	5 days	-0.1465	-1.5238	13.0811	15.3803	3.0662	-0.0544
		(0.0120)	(0.1497)	(1.0702)	(1.0175)	(0.1018)	(0.0550)
	24 days	-0.0928	-0.9761	2.9304	3.8587	1.6411	0.3529
		(0.0130)	(0.1546)	(0.4250)	(0.4007)	(0.0974)	(0.0784)
$\widehat{\sigma}_{acc}$	5 days	-0.1428	-1.4770	15.6324	17.7869	3.2949	-0.1331
		(0.0133)	(0.1642)	(1.2797)	(1.1883)	(0.1097)	(0.0611)
	24 days	-0.0916	-0.9742	3.1361	4.0590	1.6539	0.3479
		(0.0133)	(0.1598)	(0.4913)	(0.4676)	(0.1035)	(0.0780)
	1 day	-0.2089	-2.1178	10.5776	15.0588	3.1316	0.0554
		(0.0046)	(0.0608)	(0.7955)	(0.7143)	(0.0427)	(0.0215)
$\widehat{\sigma}_{p}$	5 days	-0.1623	-1.7124	3.1526	6.0795	2.0903	0.2812
		(0.0059)	(0.0741)	(0.4061)	(0.3652)	(0.0550)	(0.0352)
	24 days	-0.1385	-1.5181	0.7315	3.0299	1.5849	0.3751
		(0.0070)	(0.0774)	(0.1133)	(0.2070)	(0.0655)	(0.0717)
	1 day	-0.2183	-2.2417	7.6571	1 2.6798	2.8852	0.1298
		(0.0040)	(0.0519)	(0.5041)	(0.4842)	(0.0392)	(0.0196)
$\widehat{\sigma}_{GK}$	5 days	-0.1802	-1.9251	2.3358	6.0376	2.1169	0.2720
		(0.0049)	(0.0639)	(0.4153)	(0.3776)	(0.0525)	(0.0363)
	24 days	-0.1598	-1.7716	0.6222	3.7556	1.8196	0.2825
		(0.0063)	(0.0715)	(0.1513)	(0.2111)	(0.0605)	(0.0787)

Panel B. Japanese Yen / Dollar Exchange Rate (Continued)

		Prop. Bias	Bias	Variance	MSE	MAD	R^2
	1 day	-0.2361	-2.4617	10.1499	16.2065	3.1703	0.0438
		(0.0038)	(0.0497)	(0.4252)	(0.4514)	(0.0395)	(0.0214)
$\widehat{\sigma}_{RS}$	5 days	-0.1926	-2.0683	2.9355	7.2084	2.2787	0.2164
		(0.0047)	(0.0606)	(0.3171)	(0.3481)	(0.0529)	(0.0426)
	24 days	-0.1712	-1.9008	0.8201	4.4265	1.9587	0.2277
		(0.0060)	(0.0662)	(0.0958)	(0.2680)	(0.0615)	(0.0966)
	1 day	-0.2686	-2.8471	9.6574	1 7.7602	3.3791	-0.0192
		(0.0045)	(0.0598)	(0.6564)	(0.6713)	(0.0464)	(0.0216)
$\widehat{\sigma}_{ARS}$	5 days	-0.2261	-2.4704	2.7181	8.8163	2.6001	0.1058
		(0.0055)	(0.0714)	(0.5243)	(0.4924)	(0.0593)	(0.0395)
	24 days	-0.2053	-2.3151	0.7222	6.0759	2.3474	0.0745
		(0.0069)	(0.0821)	(0.2271)	(0.2687)	(0.0699)	(0.0846)
	1 day	-0.2522	-2.6437	7.0166	14.0031	3.0947	0.0665
		(0.0044)	(0.0583)	(0.5854)	(0.6532)	(0.0471)	(0.0234)
$\widehat{\sigma}_{AGK}$	5 days	-0.2157	-2.3514	2.0922	7.6175	2.4585	0.1546
		(0.0053)	(0.0688)	(0.4206)	(0.4591)	(0.0600)	(0.0454)
	24 days	-0.1962	-2.2131	0.5335	5.4267	2.2292	0.1210
		(0.0066)	(0.0772)	(0.1570)	(0.3241)	(0.0686)	(0.1008)
$\widehat{\sigma}_{YZ}$	5 days	-0.1787	-1.9047	2.4228	6.0465	2.1096	0.2745
		(0.0051)	(0.0651)	(0.4039)	(0.3738)	(0.0532)	(0.0368)
	24 days	-0.1579	-1.7452	0.6596	3.6999	1.7916	0.2936
		(0.0065)	(0.0737)	(0.1629)	(0.2195)	(0.0638)	(0.0791)

Panel C. British Pound / Dollar Exchange Rate

		Prop. Bias	Bias	Variance	MSE	MAD	R^2
	1 day	-0.2850	-2.4424	38.0600	44.0123	5.2972	-1.1189
		(0.0104)	(0.1151)	(1.6459)	(1.4346)	(0.0740)	(0.0427)
$\widehat{\sigma}_{cc}$	5 days	-0.1194	-0.9159	11.3410	12.1603	2.8333	-0.3041
		(0.0134)	(0.1401)	(0.8240)	(0.7348)	(0.0840)	(0.0588)
	24 days	-0.0598	-0.4220	3.4570	3.6063	1.5997	0.1530
		(0.0169)	(0.1693)	(0.4222)	(0.3728)	(0.0940)	(0.0782)
$\widehat{\sigma}_{acc}$	5 days	-0.1307	-1.0206	13.3283	14.3469	3.0238	-0.3918
		(0.0143)	(0.1519)	(1.0025)	(0.9037)	(0.0949)	(0.0650)
	24 days	-0.0577	-0.4020	3.6280	3.7593	1.6557	0.1233
		(0.0173)	(0.1733)	(0.4258)	(0.3724)	(0.0925)	(0.0793)
	1 day	-0.1727	-1.4406	8.2231	10.2955	2.6111	-0.0444
		(0.0050)	(0.0532)	(0.3714)	(0.3122)	(0.0348)	(0.0228)
$\widehat{\sigma}_{p}$	5 days	-0.1242	-1.0616	2.5753	3.6979	1.6389	0.2457
		(0.0068)	(0.0666)	(0.1790)	(0.1704)	(0.0419)	(0.0347)
	24 days	-0.1006	-0.9063	0.9346	1.7481	1.1248	0.4045
		(0.0095)	(0.0884)	(0.1157)	(0.1743)	(0.0638)	(0.0626)
	1 day	-0.1733	-1.5043	5.3224	7.5835	2.2635	0.0946
		(0.0041)	(0.0426)	(0.2055)	(0.1998)	(0.0291)	(0.0196)
$\widehat{\sigma}_{GK}$	5 days	-0.1377	-1.2396	1.5321	3.0662	1.4931	0.3128
		(0.0053)	(0.0513)	(0.1092)	(0.1441)	(0.0383)	(0.0325)
	24 days	-0.1202	-1.1342	0.5708	1.8525	1.1861	0.3720
		(0.0075)	(0.0693)	(0.0745)	(0.1709)	(0.0614)	(0.0629)

Panel C. British Pound / Dollar Exchange Rate (Continued)

		Prop. Bias	Bias	Variance	MSE	MAD	R^2
	1 day	-0.1891	-1.6977	6.7699	9.6498	2.4694	0.0123
		(0.0039)	(0.0406)	(0.1835)	(0.2041)	(0.0294)	(0.0207)
$\widehat{\sigma}_{RS}$	5 days	-0.1504	-1.3874	1.6949	3.6168	1.6136	0.2573
		(0.0051)	(0.0485)	(0.0971)	(0.1707)	(0.0404)	(0.0369)
	24 days	-0.1331	-1.2784	0.5499	2.1795	1.3048	0.3092
		(0.0071)	(0.0655)	(0.0666)	(0.2168)	(0.0650)	(0.0765)
	1 day	-0.2233	-2.0478	6.4693	1 0.6604	2.6326	-0.0531
		(0.0045)	(0.0480)	(0.3012)	(0.3266)	(0.0349)	(0.0209)
$\widehat{\sigma}_{ARS}$	5 days	-0.1858	-1.7506	1.5968	4.6587	1.8791	0.1351
		(0.0054)	(0.0540)	(0.1328)	(0.1800)	(0.0421)	(0.0338)
	24 days	-0.1692	-1.6500	0.5178	3.2362	1.6531	0.1247
		(0.0072)	(0.0681)	(0.0722)	(0.1865)	(0.0636)	(0.0669)
	1 day	-0.2088	-1.8661	4.8436	8.3241	2.4115	0.0354
		(0.0043)	(0.0470)	(0.2909)	(0.3461)	(0.0358)	(0.0221)
$\widehat{\sigma}_{AGK}$	5 days	-0.1748	-1.6211	1.3670	3.9927	1.7471	0.1959
		(0.0052)	(0.0524)	(0.1254)	(0.2138)	(0.0444)	(0.0390)
	24 days	-0.1582	-1.5263	0.5090	2.8344	1.5285	0.1907
		(0.0069)	(0.0661)	(0.0694)	(0.2374)	(0.0655)	(0.0792)
$\widehat{\sigma}_{YZ}$	5 days	-0.1409	-1.2675	1.5738	3.1776	1.5127	0.3037
		(0.0054)	(0.0521)	(0.1167)	(0.1527)	(0.0395)	(0.0326)
	24 days	-0.1202	-1.1308	0.5772	1.8512	1.1858	0.3721
		(0.0076)	(0.0699)	(0.0727)	(0.1663)	(0.0614)	(0.0642)

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