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Financial Markets Department – Leonia plc

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A Model for Estimating Recovery Rates and Collateral Haircuts for Bank Loans

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Abstract

We present a model of risky debt in which collateral value is correlated with the possibility of default. The model is then used to study: 1) the expected amount of debt recovered in the event of default as a function of collateral; and 2) the amount of collateral needed to mitigate the riskiness of a loan to a desired degree. The results obtained could prove useful for estimating recovery rates required by many popular models of credit risk and for determining collateral haircuts in debt transactions. The analysis also generates testable predictions of the behaviour of historical recovery rates of risky debt when collateral is involved. Regulators might benefit from the analysis in developing capital adequacy requirements and reviewing banks' lending standards relative to current collateral values.

JEL classification: G13, G21
Key words: credit risk, collateral, recovery rates, options theory

Malli luottoriskisen lainan vakuuksien asettamiseksi ja odotetun realisointiarvon laskemiseksi

Suomen Pankin keskustelualoitteita 2/2000

Esa Jokivuolle – Samu Peura
Rahoitusmarkkinaosasto – Leonia Oyj

Tiivistelmä

Työssä mallinnetaan luottoriskinen laina, jonka vakuuden arvo korreloi konkurssin todennäköisyyden kanssa. Mallin avulla voidaan arvioida 1) vakuuden odotettua realisointiarvoa konkurssitilanteessa sekä 2) alun perin tarvittavaa vakuuden määrää, jolla lainan riskipitoisuutta voidaan rajoittaa. Mallin tuloksia voitaneen hyödyntää luottoriskien hallinnassa sekä vaadittaessa riittäviä vakuuksia. Empiirisiä implikaatioita voidaan hyödyntää analysoitaessa luottotappioita. Työ saattaa myös tarjota hyödyllisiä näkökulmia kehitettäessä rahoituslaitosten vakavaraisuusvaatimuksia sekä valvottaessa luotonannon ehtoja.

Asiasanat: luottoriski, vakuus, luottotappiot, optioteoria

Contents

1 Introduction

Managing collateral provided by debt customers against their liabilities is important for many financial institutions; commercial and retail banks in particular. Collateral value in the event of default of a debtor can be a major determinant of the amount recovered by the creditor. Recovery rates in turn are one of the key parameters in the models of credit risk that are obtaining increasing attention both among practitioners and regulators. Further, it is a central problem both for practitioners and regulators to assess what is the sufficient amount of the so called *collateral haircut* in order to *ex ante* mitigate the risk of a certain loan contract to a desired degree[1]. Casual evidence from many countries in recent years suggest that collateral values can be notoriously volatile and, moreover, they tend to go down just when the number of defaults goes up in economic downturns (see also Schleifer and Vishny, 1992). Still, little quantitative analysis based on modern financial theory appears to exist to assist banks in setting sufficient collateral requirements, estimating recovery rates as well as pricing debt with stochastic collateral values. Instead banks often use rather crude rules of thumb when, e.g., accepting various types of collateral to back a certain amount of loan. Some of these rules, as will be shown later on in this paper, appear intuitively correct. Nevertheless, certain important features of the problem might be missed by them - let alone that any rule of thumb could potentially be much improved upon by proper quantitative analysis.

Why banks may in the first place put so much emphasis on risk mitigation of debt transactions rather than on careful pricing of loans according to their true risks, given any arbitrary amount of collateral? It is well known that in the presence of asymmetric information between the lender and the borrower of the borrower's true type credit rationing may emerge in equilibrium unless outside guarantees like collateral is used. Although direct evidence on credit rationing has been hard to catch, there is casual evidence that banks tend to apply rather uniform pricing schedules that are rather insensitive to customers' presumed riskiness, and that risks are levelled by requiring collateral. Indeed, the decision to grant a loan often depends on the availability of a sufficient amount of collateral. Even if banks do assess the creditworthiness of their customers by categorizing them into different risk classes, such classifications are often rather crude. Therefore collateral decisions potentially play an important role in levelling risks within each such category.

This paper is an attempt to provide a practical tool for estimating debt recovery rates as a function of collateral, and for quantifying collateral haircut decisions, which are both important aspects of bank loans. We use the option theoretic framework for modelling risky debt, pioneered by Black and Scholes (1973) and Merton (1974), including a collateral element whose value varies stochasticly. First we compute, and study the comparative statics of, *the expected recovery given default* (ERGD henceforth). This measure can be helpful

[1] A collateral haircut is equivalent to a limit on *the loan-to-value ratio*, i.e., the maximum amount of loan that can be granted against a given collateral in order to retain the risk of the loan at a desired level. Both recovery rates and collateral haircuts are dealt with in the proposals for reforming the regulatory minimum capital requirements (see Basel Committee on Banking Supervision, 1999, and the European Commission, 1999). European Commission already has detailed proposals regarding collateral haircuts when corporate net credit exposures are calculated for regulatory purposes.

in estimating the recovery rate of a debt transaction where collateral is involved. The analysis of the ERGD also implies empirically testable hypotheses concerning historical recovery rates, provided that data on collateral positions prior to default are available. Overall, we see our analysis of the expected debt recovery in the option theoretic framework, where recovery is endogenous, as a useful complement to the recent *reduced form* credit risk models (see, e.g., Jarrow, Lando, and Turnbull, 1997, and Duffie and Singleton, 1999)[2] where the recovery rate is typically taken as an exogenous constant parameter. The reduced form approach can also be seen implicit in much of the 1999 proposals of the Basel Committee on Banking Supervision and the European Commission on reforming the minimum capital adequacy standards. In particular, regulators might find our analysis useful when searching for ways to condition capital requirements on the type and the amount of collateral involved in a debt transaction. Secondly, we use our model to study the question of how much collateral should be required to begin with in order to mitigate the risk of a debt contract to a desired degree. As already noted above this question is a central practical concern to practitioners and regulators alike.

The model takes into account the stochastic properties of the collateral value; its volatility, drift, and correlation with the event of default. The borrowing firm's total asset value at the debt's maturity determines the event of default but does not affect the debt's payoff. Rather, we assume that collateral value is the only stochastic element determining recovery.[3] Because of this simplifying assumption the model can be implemented using an exogenous default probability estimate, so that the firm asset value parameters need not necessarily be estimated. Because the estimation of these parameters is generally found difficult, the use of the reduced form models with exogenous default probability parameters has become popular. In this respect our model, although cast in the traditional option theoretic setting, also contains features of the reduced form models. Moreover, we feel that allowing only the collateral value to affect recovery is a prudential first step in estimating expected recoveries, or assessing the ex ante sufficiency of the collateral. Additional elements like the firm's residual asset value apart from the collateral value could only increase the amount recovered in default.

We think this model structure also fits well with the data typically available for bank loans. First, loan customers are often private companies whose default probability estimates are usually based on various rating or scoring methods, i.e., they are exogenous to a credit risk model. Second, real estate is quite typically provided as collateral, so at least in this important case collateral value parameters could be readily estimated using publicly available real estate indexes. An appropriate stock market industry index could further be used as a proxy to estimate the correlation between the borrowing firm's asset value and collateral. The model might also be used for retail customers, as exogenous scoring based estimates of individuals' default probabilities are often available for banks and other credit institutions. In order to estimate the correlation parameter in that case the state variable driving default might well be proxied by

---

[2] Also the credit portfolio models of J.P. Morgan's CreditMetrics™, and Credit Suisse First Boston's CreditRisk+™ can largely be seen as reduced form models.

[3] The alternative interpretation of our approach is that we are modelling the credit risk exposure net of collateral.

the aggregate, industry or area-specific labor income, or by their combination, depending on how detailed data is available of individual attributes.

The analysis shows that the ERGD is a decreasing function of the collateral volatility and the correlation between collateral and the firm value driving default. What may be a less obvious result is that, other things being equal, the ERGD typically increases as the likelihood of default increases. We will provide an intuitive explanation to this result later on. Concerning collateral haircuts our analysis suggests the haircut should be increasing in the collateral volatility, the correlation between collateral and the firm value, and the probability of default of the debtor. Numerical examples are provided to demonstrate the size of the ERGD as well as the collateral haircut for various parameter values.

The paper is organised as follows. Section 2 presents the model, and analyzes the behaviour of the ERGD as a function of the model parameters both analytically and through numerical examples. The empirical implications are discussed. Section 3 analyses collateral haircuts. The final section concludes.

2 The model

Let the value of the firm's assets be denoted by A, and the value of the collateral associated with the firm's debt obligation be V. These state variables follow diffusion processes with stochastic differential representations

$$dA_t = \mu_A A_t dt + \sigma_A A_t dW_t \tag{1}$$

$$dV_t = \mu_V V_t dt + \sigma_V V_t dZ_t \tag{2}$$

where W_t and Z_t are standard Wiener processes with constant correlation parameter denoted by ρ.

We study a defaultable zero-coupon debt contract with a face value of F, and a maturity of T years, which is backed by collateral V. There may also be other debt in the company such that the total amount of debt is denoted by $\overline{A}$. However, the holder of F has a strict priority over V should the firm default. For simplicity we assume that default can only take place at maturity. The payoff to the holder of debt F at maturity is a function of the terminal values of A and V. The value of A determines whether the firm is in default at T or not. If the value of A at T exceeds $\overline{A}$, the total amount of debt, the debtholder receives the promised amount, F. When the value of A at maturity is below $\overline{A}$, the firm is in default. As was already discussed at some length in the introduction the recovery, conditional on default, is assumed to be determined only by the terminal value of the collateral, V_T. There is an upper boundary on the recovery, however, since the debtholder is not allowed to receive more of the liquidation value of the collateral than the total amount due to him, F. The payoff to the debtholder, conditional on default having taken place, is thus $Min(V_T, F)$.

So far we have just specified the correlation parameter between A and V but have not discussed their functional relationship. There are basically two alternatives. If the debtor owns the collateral, then V is part of A. However, it is also possible that the collateral is not owned by the debtor but is provided by a third party. A common example of this would be an entrepreneur whose privately held company is the debtor but who provides the collateral to the creditor as a private person. In this case it is possible in principal that the value of the collateral at the time of a default is greater than the firm's asset value, whereas in the first case it is not.

In the case where the debtor owns the collateral we could consider specifying $A = A' + V$, where A' and V would be correlated lognormal diffusion processes. The correlation between A and V would then be effectively derived within the model from the processes of A' and V, and the correlation between them. However, as the sum of two lognormal variables is not lognormal, by doing this we would lose the attractive analytic separation result, to be shown below, that the probability of default can be treated as an exogenous parameter to the model. Therefore we have chosen to keep V and A separate, which corresponds to the case where the debtor does not own the collateral. Nevertheless, we argue that in many cases such a specification can also serve as a good approximation to the case where collateral is actually part of the debtor's assets, although there is no guarantee that V would be less than F in the event of default. First, if F is small relative to $\overline{A}$, V could well exceed F even if a default occurs. Secondly, we may alternatively interpret the event of default in our model as a decision to liquidate the firm's assets. That is, if the actual business of the company becomes unprofitable, the remaining assets, e.g., buildings and the scrap value of machinery and equipment, might be sold to outside parties to repay debts. It may well be that V, as part of the liquidation value of assets, covers the part of total debt, F, it is pledged against. Of course, the probability of liquidation is generally different from the probability of default. Therefore, using the model in a case where the liquidation interpretation is more feasible, one would have to adjust the estimate for the probability of default accordingly.

2.1 Analysis of the expected debt recovery given default

We now turn to the derivation of the ERGD that is a natural candidate component when forming an estimate of the recovery rate of a debt contract involving a collateral agreement with a strict priority right. Let us denote by $I\left(A_T \leq \overline{A}\right)$ the indicator function for the event that the issuer defaults at time T. The payoff to the debtholder at maturity, P, can then be written as

$$P = FI\left(A_T > \overline{A}\right) + Min\left(V_T, F\right)I\left(A_T \leq \overline{A}\right) \tag{3}$$

Expected payoff at maturity, with respect to the true statistical probability measure, becomes

$$E[P] = F(1-p) + E\left[Min\left(V_T, F\right)I\left(A_T \leq \overline{A}\right)\right] \tag{4}$$

where we use p to denote the probability of default, i.e.,

$$p = E\left[I\left(A_T \leq \overline{A}\right)\right] \tag{5}$$

Expected payoff given default, i.e., the *ERGD*, obtains the form

$$\begin{aligned} ERGD &= E\left[P \middle| A_T \leq \overline{A}\right] \\ &= E\left[Min\left(V_T, F\right)I\left(A_T \leq \overline{A}\right)\right]/p \\ &= F - E\left[Max\left(0, F - V_T\right)I\left(A_T \leq \overline{A}\right)\right]/p \end{aligned} \tag{6}$$

which can be further rewritten as

$$ERGD = F - E\left[Max(0, F - V_T)\right] - Cov\left[Max(0, F - V_T), I(A_T \leq \overline{A})\right]/p \tag{7}$$

Although we ultimately have to resort to numerical analysis, (7) already provides some qualitative comparative statics results. The second term on the right-hand-side of (7) is equivalent to the terminal payoff function of a put option on the collateral value, where the face value of the debt, F, is the option's strike price. The third term involves the covariance of a function of the collateral value and a function of the firm asset value. If the firm asset value and the collateral value are uncorrelated, then the third term is zero. Suppose first that this is the case so that the ERGD equals F minus the put option part of (7). As the value of a put is an increasing function of the volatility of its underlying asset, ERGD is the lower the higher is the volatility of the collateral. In other words, the riskier the collateral the less we expect to recover in the event of default.

Next suppose the correlation between the firm asset value, A, and the collateral value, V, is positive, which should be the predominant case[4]. Functions $Max(0,F\text{-}V_T)$ and $I\left[A_T < \overline{A}\right]$ are both decreasing in V and A, respectively. This implies that the covariance between these functions is also positive, and increases as the correlation between V and A increases. This further implies two things. First, for positive correlation between V and A an increase in the collateral volatility decreases ERGD also via the third term in (7). Secondly, correlation between V and A in itself decreases ERGD. In other words, if collateral works as a good "hedge" in the case of default, the bank should have a higher recovery expectation, and vice versa. These conclusions are also confirmed via the numerical examples, displayed in figures 1 and 2, we will next turn to.

Regarding the qualitative effect of the probability of default on the ERGD - as well as obtaining quantitative results of changes in the ERGD as a result of changes in the model parameters in general - we will have to resort to numerical analysis. Note in particular that p affects both the numerator and the denominator of the third term in (7). Because of this we cannot use equation (7) - as we did above - to draw qualitative conclusions regarding the effect of changes in p on the ERGD. As shown in the appendix the term $E\left[Max\left(0, F - V_T\right)I\left(A_T \le \overline{A}\right)\right]$ on the right-hand-side of (6) obtains the form

$$
\begin{aligned}
& F \int_{-\infty}^{\overline{y}} N\left(\frac{\overline{z} - \rho y}{\sqrt{1-\rho^2}}\right) n(y)\,dy \\
& - V_0 \exp\left(h(T)\right) \int_{-\infty}^{\overline{y}} \exp\left(\rho \sigma_V \sqrt{T} y\right) N\left(\frac{\overline{z} - \rho y - (1-\rho^2)\sigma_V \sqrt{T}}{\sqrt{1-\rho^2}}\right) n(y)\,dy
\end{aligned}
\tag{8}
$$

where

$$h(T) = \left(\mu_V - \frac{1}{2}\sigma_V^2\right)T + \frac{1}{2}\left(1-\rho^2\right)\sigma_V^2 T,$$

$$\overline{y} = \frac{\ln \overline{A} - \ln A_0 - \mu_A T + 0.5\sigma_A^2 T}{\sigma_A \sqrt{T}},$$

$$\overline{z} = \frac{\ln F - \ln V_0 - \mu_V T + 0.5\sigma_V^2 T}{\sigma_V \sqrt{T}}.$$

$N(\ )$ is the standard normal distribution function, and $n(\ )$ is the standard normal density function. The solution is not quite an analytic formula, in that it involves a one-dimensional integral which has to be evaluated numerically. Such one-dimensional quadrature is not computationally intensive, though, and can be

---

[4] That is, as asset values in the economy in general are strongly driven by common market factors it is hard to find pairs of assets whose correlation would be negative.

performed very quickly using one of the many well-known numerical integration schemes (see, e.g., Press *et al*., 1992).

Using (6) and (8) we can readily compute the value of the ERGD. The ERGD is a function of the parameters of the collateral value process, which enter both formula (8) and the expression for $\bar{z}$. The parameters of the underlying asset value process, however, only enter through the expression for $\bar{y}$. Since there is a one-to-one mapping between the value of $\bar{y}$ and the default probability of the firm, we can select the value of $\bar{y}$ to match an exogenous estimate of the default probability *for a given horizon*. This is the *separation* result already discussed above that allows our model to be implemented in the similar way as the reduced form credit risk models. The parameters of the collateral value process as well as the correlation between the collateral and the firm asset value can be estimated using appropriate proxy variables such as stock market industry indexes (for the asset value) and real estate indexes (in the case of real estate collateral), as already discussed in the introduction.

Numerical examples of the behavior of the *ERGD*, expressed as a percentage of the face value of debt, are presented in Figures 1 and 2. It is assumed throughout the examples that the current value of collateral equals the face value of debt. The general result is that for positive correlation between the collateral and the firm asset value the *ERGD* is the higher, the higher is the likelihood of default. This might at first appear counterintuitive. The reason is, however, that a low probability of default implies that the firm asset value has to strongly decline in the future before default can occur. Positive correlation in turn implies that the collateral value is likely to be relatively low, too, in the case of default. For high probabilities of default the firm asset value does not have to decline equally substantially before default can occur. Hence the collateral value in default is on average also higher relative to its original value than in the case of low probability of default. Note from figure 1 that this relationship vanishes for zero correlation between the collateral and the firm value.

International rating agencies provide separate statistics on recovery rates of secured/unsecured and junior/senior defaulted bonds. Our analysis suggests that information on historical recovery rates should also be provided conditional on the rating, i.e., a proxy for the default probability, prior to default, and the ratio of the amount of loan to the value of collateral prior to default. According to the hypothesis, for a given loan-to-value ratio, the realised recovery rate should be a decreasing function of the company's creditworthiness prior to default[5].

[5] Care should be taken, though, when using public ratings as proxies for default probabilities as ratings may already include an assessment of the severity of loss in the event of default.

3 Collateral haircuts

The analysis of the ERGD in the previous section addresses the question of how much we should expect to recover from a debt contract in default *given* the amount and type of the collateral provided. We will now use our model to answer a question from a different angle: how much collateral should be required per the amount of loan in order to *ex ante* modify the risk of the debt contract to a desired degree, say, to make it effectively riskless. Alternatively, one can ask how much money can be lent againts a given amount of collateral to retain the risk of the loan at a desired level. Such decisions routinely made by credit institutions are called collateral haircuts or limits on loan-to-value ratios. For example, a bank may decide that a mortgage customer can be granted a loan which is maximum 80 per cent of the current value of the house or appartment to be purchased. That is, the loan-to-value ratio would be at most F/V = 80%. This is equivalent to saying that the loan equals the amount of collateral *after* a haircut of at least 20 per cent of the current value of the collateral has been made, i.e., $F = (1-x)V$, where $x \geq 0.20$ is the haircut percentage. Casual evidence suggests that banks have long been applying various rules of thumb when making these decisions. For example, a maximum of 80 per cent loan-to-value ratios may have typically been applied in the case of real estate or housing collateral, and a maximum of 60 per cent in the case of corporate shares. Ratios applied may also vary according to the presumed phase of the business cycle. Regulators are currently considering appropriate sizes of collateral haircuts to be applied in calculating capital requirements for credit transactions involving risk mitigation techniques such as collateral (see in particular European Comission, 1999).

On the whole we find the discussion on collateral haircuts and loan-to-value ratios up to date rather vague in many respects. First, the precise degree of risk mitigation pursued by taking collateral is typically not defined. That is, is perhaps the implicit goal of collateral haircuts to make the debt contract effectively riskless? Secondly, is the risk reduction, e.g., zero risk, defined via the payoff structure of the debt in all possible future states of the world, or via its fair value? Finally, haircut percentages applied thus far seem to have been based on practical experience rather than quantitative models, or even well defined statistics based on historical data.

In the following we suggest how a model like ours with a stochastic collateral element could be used to specify and calculate collateral haircuts. It is not clear, though, how much residual risk after taking collateral the creditor is exactly willing to bear but we think risklessness is a natural benchmark. Moreover, risklessness is the only benchmark that in principal could be achieved without knowledge of the probability of default of the credit customer. That is, a debt contract with a constant payoff structure regardless of the state of solvency, achieved through the use of collateral, is by definition riskless. Defining risklessness as a constant payoff structure of the debt contract is problematic, though, within the current model. The payoff would be *almost* certain only as the loan-to-value ratio approaches zero. Therefore we use a valuation based criterion for risklessness instead.[6] That is, we find the highest loan-to-value ratio that results in a yield spread over the riskless rate that is a maximum of 1 basis

[6] Some institutions define a collateral haircut as the reduction of the current value of the collateral such that the collateral value at default will be below the reduced value only in x% of the cases.

point (one hundredth of per cent)[7]. This requires an estimate of the probability of default. To account for the uncertainty regarding that parameter one could choose a loan-to-value ratio that is the most prudential one produced by a set of reasonable alternative values of the default probability.

In table 1 we have calculated loan-to-value ratios for various values of the collateral volatility, correlation between collateral and the firm value, and the (cumulative) probability of default for one and three-year horizons. Corresponding collateral haircut percentages are simply 100 minus the loan-to-value ratios. For example, the loan-to-value ratio for a three-year A-rated zero coupon loan with collateral volatility at 25%-points, and the correlation parameter at 40%, is 60%. In general, the numerical examples indicate that the loan-to-value ratio is a decreasing function of all three model parameters considered. Note, however, that for high levels of correlation, increasing the probability of default barely decreases the loan-to-value ratio. This is explained by the two counteracting forces at work, discussed earlier. As the probability of default increases the expected recovery given default also increases implying that the expected loss given default decreases. Hence the expected loss – the product of the probability of default and the loss given default - that is directly related to the value of the debt stays almost unchanged. Note also that loan-to-value ratios can be above 100 per cent (implying negative haircuts).

Some further care must be taken when interpreting our results. When computing the loan-to-value ratios in table 1 we have implicitly assumed that as we change the amount of debt the probability of default of the company remains the same. This would be consistent with requiring that the borrowing firm simultaneously increases the amount of its own funds so as to retain its debt equity ratio, and that the risk of the firm's assets stays the same. Although this may sound somewhat restrictive for practical purposes, allowing new debt to change the probability of default would require estimates of the firm asset value prior and after the change in the amount of debt. Alternatively, in many practical cases it is the amount of debt that is given, say, by an investment opportunity of fixed size, whereas the amount of collateral is variable. That is, to get the loan the company has to provide a sufficient amount of collateral either from its assets or from a third party. In the former case the correlation between the firm's assets pledged as collateral and the firm's total assets is in fact endogenous to the model although we have not taken that explicitly into account (see the discussion in section 2). We suggest that for practical reasons the effect on the correlation parameter should be assessed outside the model.

[7] For simplicity we assume a risk neutral investor who uses the true statistical probability measure to take expectations, and discounts risky payoffs with the riskless rate. As bank loans that are our primary practical focus are typically not traded in the secondary market, we do not wish to concentrate on pricing through market calibration. For these reasons our numerical results should only be taken as illustrative. In the more realistic case of a risk averse investor the loan-to-value ratios would be even lower, as the investor would effectively be using a higher discount rate.

4 Summary

In this paper we have presented a simple extension of the model of risky debt along the lines of Black and Scholes (1973) and Merton (1974) to include a stochastic collateral element. Instead of directly studying pricing issues, we have used the model to address two other practical problems particularly important to banks and their regulators.

The first of them is the estimation of bank loan recovery rates (as a function of collateral). This is central for many popular models of credit risk, used in banks' credit risk management and pricing, as well as for designing appropriate regulatory capital adequacy rules. The results of our analysis might be used as such to estimate recovery rates as a function of the fundamental parameters governing the stochastic value process of collateral and its correlation with the event of default. Moreover, our analytical results produce testable hypotheses regarding the behavior of historical recovery rates conditional on prior collateral positions of the creditors. Such information might be useful for arriving at more accurate estimates of recovery rates based on historical data. The second problem is the question of a sufficient collateral haircut – or, equivalently, limiting the loan-to-value ratio when granting loans - in order to *ex ante* mitigate the risk of a loan contract to a desired degree. Loan-to-value ratios are also often considered as indicators of banks' lending standards.

The two problems are closely related, but we believe it is worthwhile to deal with them separately as we have done. A bank that knows the default probability of its customer well may find the first approach useful for assessing expected recovery rates if collateral is involved in the debt contract. On the other hand, if there is much uncertainty concerning the default probability estimate of a customer the bank may prefer to demand enough of collateral so as to limit the risk of the loan to an acceptable level.

Our results show that if collateral is involved the recovery rate estimate should be a decreasing function of the collateral volatility and the correlation between the collateral and the firm value. Interestingly, in typical cases it should be an increasing function of the probability of default of the debtor. A collateral haircut in turn should be an increasing function of all the above factors, although often in highly nonlinear ways.

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Appendix 1. Derivation of a quasi-analytic expression for the ERGD

Here we derive the expression for the term $E\left[Max(0, F-V_T), I(A \leq \overline{A})\right]$ on the rigth-hand side of (6). Let us denote the joint density function of A_T and V_T as $f(A,V)$, the density function of V_T, conditional on A_T, as $f_{V|A}(A,V)$, and the marginal density function of A_T as $f_A(A)$. The expectation can then be evaluated as follows (we have dropped time-subscripts)

$$
\begin{aligned}
&E\left[Max(0,F-V)I(A \leq \overline{A})\right] \\
&= \int_0^{\infty}\left(\int_0^{\infty} Max(0,F-V)I(A \leq \overline{A})f(A,V)dV\right)dA \\
&= \int_0^{\overline{A}}\left(\int_0^{F}(F-V)f_{V|A}(A,V)dV\right)f_A(A)dA \\
&= F\int_0^{\overline{A}}\left(\int_0^{F} f_{V|A}(A,V)dV\right)f_A(A)dA - \int_0^{\overline{A}}\left(\int_0^{F} Vf_{V|A}(A,V)dV\right)f_A(A)dA \\
&= F\int_{-\infty}^{\overline{y}}\left(\int_{-\infty}^{\overline{z}} f_{z|y}(y,z)dz\right)f_y(y)dy - \\
&\quad \int_{-\infty}^{\overline{y}}\left(\int_{-\infty}^{\overline{z}} V_0 \exp\left(\left(\mu_V - \frac{1}{2}\sigma_V^2\right)T + \sigma_V\sqrt{T}z\right)f_{z|y}(y,z)dz\right)f_y(y)dy
\end{aligned}
\tag{A1}
$$

where y and z are jointly standard normally distributed random variables with correlation ρ, and

$$
\overline{y} = \frac{\ln \overline{A} - \ln A_0 - \mu \ T + 0.5\sigma^2 T}{\sigma \ \sqrt{T}},
$$

$$
\overline{z} = \frac{\ln F - \ln V_0 - \mu_V T + 0.5\sigma_V^2 T}{\sigma_V \sqrt{T}}.
$$

From the properties of the bivariate normal distribution, we know that conditional on y, z is distributed as

$$
z|y \sim N\left(\rho y, 1-\rho^2\right).
$$

Using this knowledge, the first term in the final expression in (A1) can be written as

$$
F\int_{-\infty}^{\overline{y}} N\left(\frac{\overline{z}-\rho y}{\sqrt{1-\rho^2}}\right)f_y(y)dy \tag{A2}
$$

The second term in the final expression in (A2) can be completed into a square prior to integrating. Let us denote $m_V = \mu_V - \frac{1}{2}\sigma_V^2$. The expression inside the inner integral becomes

$$
\begin{aligned}
& V_0 \exp\left(m_V T + \sigma_V \sqrt{T} z\right) f_{z|y}(y,z) \\
&= V_0 \exp\left(m_V T\right) \frac{1}{\sqrt{2\pi\left(1-\rho^2\right)}} \exp\left(-\frac{1}{2\left(1-\rho^2\right)}\left(z-\rho y\right)^2 + \sigma_V \sqrt{T} z\right) \\
&= V_0 \exp\left(m_V T\right) \frac{1}{\sqrt{2\pi\left(1-\rho^2\right)}} \exp\left(-\frac{1}{2\left(1-\rho^2\right)}\left(z^2 + \rho^2 y^2 - 2\rho z y - 2\left(1-\rho^2\right)\sigma_V \sqrt{T} z\right)\right) \\
&= V_0 \exp\left(m_V T\right) \frac{1}{\sqrt{2\pi\left(1-\rho^2\right)}} \exp\left(-\frac{1}{2\left(1-\rho^2\right)}\left(z^2 - 2\left(\rho y + \left(1-\rho^2\right)\sigma_V \sqrt{T}\right) z + \rho^2 y^2\right)\right) \\
&= V_0 \exp\left(m_V T\right) \frac{1}{\sqrt{2\pi\left(1-\rho^2\right)}} \\
&\quad \exp\left(-\frac{1}{2\left(1-\rho^2\right)}\left(\left(z - \left(\rho y + \left(1-\rho^2\right)\sigma_V \sqrt{T}\right)\right)^2 - \left(1-\rho^2\right)^2 \sigma_V^2 T - 2\rho\left(1-\rho^2\right)\sigma_V \sqrt{T} y\right)\right) \\
&= V_0 \exp\left(m_V T + \frac{1}{2}\left(1-\rho^2\right)\sigma_V^2 T + \rho\sigma_V \sqrt{T} y\right) \\
&\quad \frac{1}{\sqrt{2\pi\left(1-\rho^2\right)}} \exp\left(-\frac{1}{2\left(1-\rho^2\right)}\left(z - \left(\rho y + \left(1-\rho^2\right)\sigma_V \sqrt{T}\right)\right)^2\right)
\end{aligned}
\tag{A3}
$$

Substituting the final expression of (A3) into the last row in (A1) yields

$$
\begin{aligned}
& \int_{-\infty}^{\bar{y}} \left(\int_{-\infty}^{\bar{z}} V_0 \exp\left(m_V T + \sigma_V \sqrt{T} z\right) f_{z|y}(y,z) dz \right) f_y(y) dy \\
&= \int_{-\infty}^{\bar{y}} \left(V_0 \exp\left(h(T) + \rho\sigma_V \sqrt{T} y\right) N\left(\frac{\bar{z} - \rho y - \left(1-\rho^2\right)\sigma_V \sqrt{T}}{\sqrt{1-\rho^2}} \right) \right) f_y(y) dy \\
&= V_0 \exp\left(h(T)\right) \int_{-\infty}^{\bar{y}} \left(\exp\left(\rho\sigma_V \sqrt{T} y\right) N\left(\frac{\bar{z} - \rho y - \left(1-\rho^2\right)\sigma_V \sqrt{T}}{\sqrt{1-\rho^2}} \right) \right) f_y(y) dy
\end{aligned}
\tag{A4}
$$

where

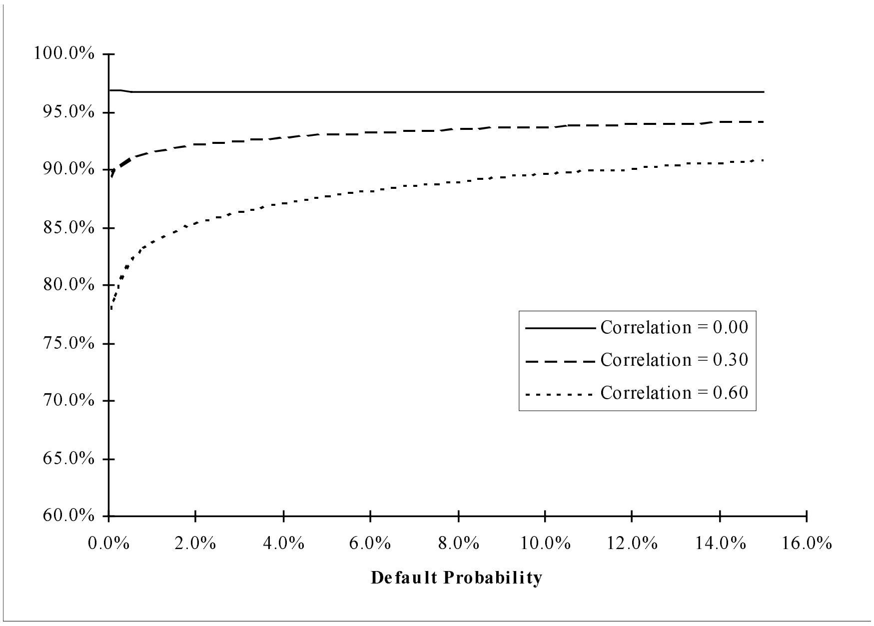
$$h(T) = \left(\mu_V - \frac{1}{2}\sigma_V^2\right) T + \frac{1}{2}\left(1-\rho^2\right)\sigma_V^2 T .$$

Combining (A1), (A2), and (A4) gives

$$
\begin{aligned}
E\left[Max\left(0, F - V\right) I\left(A \le \overline{A}\right)\right] &= F \int_{-\infty}^{\overline{y}} N\left(\frac{\bar{z} - \rho y}{\sqrt{1-\rho^2}} \right) n\left(y\right) dy \\
&- V_0 \exp\left(h(T)\right) \int_{-\infty}^{\overline{y}} \exp\left(\rho \sigma_V \sqrt{T} y\right) N\left(\frac{\bar{z} - \rho y - (1-\rho^2)\sigma_V \sqrt{T}}{\sqrt{1-\rho^2}} \right) n\left(y\right) dy
\end{aligned}
\qquad \text{(A5)}
$$

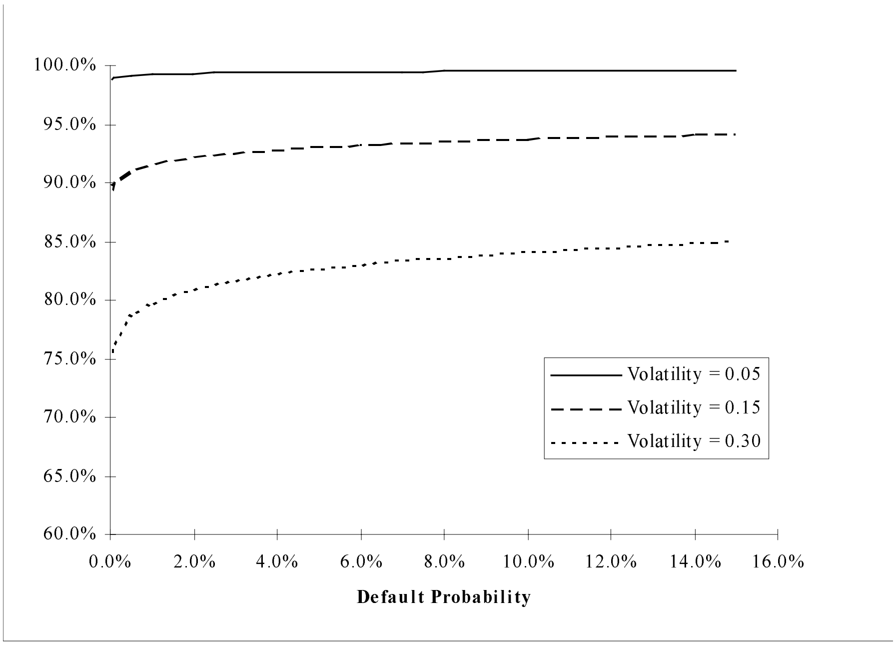
Appendix 2. Figures and tables

Figure 1. ERGD as a percentage of the face value of debt as a function of default probability for various levels of correlation between collateral and asset value



Other parameters: Face value of debt 100, maturity of debt 1 year, current collateral value 100, collateral value volatility 15.0% per annum, collateral value drift 7.0% per annum, risk free rate 5.0% per annum.

Figure 2. ERGD as a percentage of the face value of debt as a function of default probability for various levels of collateral value volatility



Other parameters: Face value of debt 100, maturity of debt 1 year, current collateral value 100, correlation between firm value and collateral value 30.0%, collateral value drift 7.0% per annum, risk free rate 5.0% per annum.

Table 1. Loan-to-value ratios,%

		σ_V=10%		σ_V=25%		σ_V=40%	
		1-yr.	3-yr.	1-yr.	3-yr.	1-yr.	3-yr.
ρ=0%	A	160	130	155	105	150	75
	BB	90	85	70	50	50	25
	B	85	80	60	40	40	20
ρ=40%	A	135	105	110	60	85	35
	BB	85	80	55	40	35	15
	B	80	75	50	35	30	15
ρ=80%	A	115	85	75	40	45	15
	BB	80	75	45	30	25	10
	B	75	70	45	30	25	10

Defined as the face value of the loan per the current collateral value, that makes the loan almost riskless. That is, the highest loan-to-value ratio for which the loan yield spread over the riskless rate is less than 1 basis point (< 0.01%), assuming risk neutrality. Expected collateral value change and the riskless rate equal 5% per annum. One and three-year bullet loans are considered. V is the annual collateral volatility, and is the correlation between collateral and the state variable driving default. The one and three-year (cumulative) probability of default estimates for A, BB, and B rated counterparties are 0.03%/0.22%, 1.32%/6.01%, and 5.58%/15.6%, respectively. They are obtained by assuming a time-homogenous transition matrix of annual rating transition probabilities, based on Standard and Poor's default statistics (see J.P. Morgan, 1997, table 6.11).