A traffic lights approach to PD validation

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Abstract

As a consequence of the dependence experienced in loan portfolios, the standard binomial test which is based on the assumption of independence does not appear appropriate for validating probabilities of default (PDs). The model underlying the new rules for minimum capital requirements (Basle II) is taken as a point of departure for deriving two parametric test procedures that incorporate dependence effects. The first one makes use of the so-called granularity adjustment approach while the the second one is based on moment matching.

The aim with this note is to present an approximate procedure for one-observation-based inference on the adequacy of probability of default (PD) forecasts. The PD forecast for a homogeneous portfolio of loans has to be compared to the realized default rate one year later. In case of independent default events, the natural procedure for this comparison would be the standard binomial test. However, the well-known fact that default events are correlated makes the binomial test appear unreliable.

Our approach here is to model the dependent default events in a Basle II-like style and to arrive this way at a means to compute critical values which respect correlations. However, in the Basle II-model the distribution of default rates cannot be calculated with elementary arithmetic procedures as they are available for instance in MSExcel. Therefore we suggest two approximation schemes which seem to work with reasonable precision.

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This note is organized as follows. In Section 1, we describe the general design of a traffic lights procedure for PD validation. Section 2 then specifies the stochastic model underlying the granularity adjustment and moment matching approximation procedures to be introduced in Sections 3 and 4 respectively. We conclude in Section 5 with a numerical illustration of the approaches.

1 Setting the colors

We fix two *confidence levels* α_{low} and α_{high}, e.g. $\alpha_{\text{low}} = 95\%$ and $\alpha_{\text{high}} = 99.9\%$. Assume that the forecast for the default rate is p. Under this assumption, we have to find *critical values* c_{low} and c_{high} such that the probabilities that the realized number of defaults exceeds c_{low} and c_{high} will equal $100\% - \alpha_{\text{low}}$ and $100\% - \alpha_{\text{high}}$ respectively.

The traffic light for the adequacy of the PD forecast will be set *green*, if the realized number of defaults is less than c_{low}. In this case there is no obvious contradiction between forecast and realized default rate.

The traffic light will be set *yellow*, if the realized number of defaults is equal to or greater than c_{low} but less than c_{high}. The yellow light indicates that the realized default rate is not compatible with the PD forecast. However, the difference of realized rate and forecast is still in the range of usual statistical fluctuations. As a consequence, the responsibility for the deviation of the forecast cannot without doubt assigned to the portfolio manager.

The traffic light will be set *red*, if the realized number of defaults is equal to or greater than c_{high}. In this case, the difference of forecast and realized default rate is so large that any disbelief in a wrong forecast would be unreasonable.

2 Specifying the stochastic model

The first step towards determining the critical values c_{low} and c_{high} is to fix a stochastic model that will enable us to carry out the necessary numerical calculations. We take the Vasicek one factor-model which underlies also the Basle II risk weight functions.

If n denotes the number of loans in the portfolio under consideration and D_n is the realized number of defaults in the observed period of time, we write D_n as

$$D_n = \sum_{i=1}^{n} \mathbf{1}_{\{\sqrt{\rho}\,X + \sqrt{1-\rho}\,\xi_i \le t\}}. \tag{1}$$

In (1), $\mathbf{1}_E$ is the indicator function assuming the value 1 on the event E and the value 0 on the complement of E. X and $\xi_1, \ldots, \xi_n$ are independent standard normal random variables. The threshold t has to be chosen in such a way that

$$\mathrm{E}[D_n] = n\,p. \tag{2a}$$

This will be achieved by setting

$$t = \Phi^{-1}(p), \tag{2b}$$

with Φ denoting the standard normal distribution function. The choice of the parameter ρ (sometimes called *asset correlation*) is not so obvious. It should not be chosen higher than 0.24 which is the highest correlation occurring in the Basle II rules. One way to handle this question would be to leave the choice of a value for ρ to the discretion of the national supervisors. For instance, $\rho = 0.05$ appears to be appropriate for Germany.

3 The granularity adjustment approach

In order to make work the traffic lights approach we have to find methods for calculating the critical values which have been introduced in Section 1. For instance, the critical value c_{low} is characterized by

$$c_{\text{low}} = \min\{k : \mathrm{P}[D_n \geq k] \leq 1 - \alpha_{\text{low}}\}. \tag{3a}$$

(3a) is equivalent to

$$c_{\text{low}} = q(\alpha_{\text{low}}, D_n) + 1, \tag{3b}$$

where $q(\alpha, D_n)$ denotes the usual α-quantile of D_n, i.e.

$$q(\alpha, D_n) = \min\{x : \mathrm{P}[D_n \leq x] \geq \alpha\}. \tag{3c}$$

Analogously, we have

$$c_{\text{high}} = q(\alpha_{\text{high}}, D_n) + 1. \tag{3d}$$

Write

$$R_n = D_n/n \tag{4a}$$

for the *default rate* corresponding to the number of defaults D_n. For computational reasons, it is often appropriate to consider R_n instead of D_n. But, of course, the quantiles of R_n and D_n are related by

$$n\,q(\alpha, R_n) = q(\alpha, D_n). \tag{4b}$$

Since the distribution of D_n cannot be calculated with elementary methods, Gordy (2002) suggested the *granularity adjustment* approach for approximating the quantiles $q(\alpha, R_n)$.

This approximation is based on a second order Taylor expansion of $q(\alpha, R_n)$ in the following sense

$$q(\alpha, R_n) = q(\alpha, R + h\,(R_n - R))\Big|_{h=1} \tag{5a}$$
$$\approx q(\alpha, R) + \frac{\partial}{\partial h} q(\alpha, R + h\,(R_n - R))\Big|_{h=0} + \frac{1}{2}\frac{\partial^2}{\partial h^2} q(\alpha, R + h\,(R_n - R))\Big|_{h=0}.$$

The random variable R in (5a) can be chosen as

$$R = \lim_{n\to\infty} R_n = \Phi\Big(\frac{t - \sqrt{\rho}\,X}{\sqrt{1-\rho}}\Big). \tag{5b}$$

The quantile $q(\alpha, R)$ turns out to be

$$q(\alpha, R) = \Phi\Big(\frac{\sqrt{\rho}\,\Phi^{-1}(\alpha) + t}{\sqrt{1-\rho}}\Big), \tag{5c}$$

and, as a consequence, can easily be calculated. Unfortunately, when defining R with (5b), the partial derivatives in (5a) may not exist because the distribution of D_n is purely discrete. However, Martin and Wilde (2002) noted that although (5a) was derived for *smooth* distributions its application may yield sensible results even in semi-smooth situations. Using the formulas for the derivatives by Martin and Wilde (2002) one arrives at (cf. Tasche, 2003)

$$q(\alpha, D_n) \approx n\,q(\alpha, R) + \frac{1}{2}\Bigg(2\,q(\alpha, R) - 1 \tag{6}$$
$$+ \frac{q(\alpha, R)\,(1 - q(\alpha, R))}{\phi\big(\frac{\sqrt{\rho}\,q(1-\alpha, X) - t}{\sqrt{1-\rho}}\big)}\Bigg(\frac{\sqrt{\rho}\,q(1-\alpha, X) - t}{\sqrt{1-\rho}} - \sqrt{\frac{1-\rho}{\rho}}\,q(1-\alpha, X)\Bigg)\Bigg),$$

where $\phi(x) = (2\,\pi)^{-1}\,e^{-x^2/2}$ denotes the standard normal density.

For given forecasted PD p and asset correlation ρ, by means of (3b), (3d), and (6) the critical values for the traffic lights approach respecting correlation can be calculated.

4 Fitting the default rate distribution with moment matching

A further approach to determine the critical values defined by (3b) and (3d) can be based on approximating the distribution of R_n as given by (1) and (4a) with a Beta-distribution. When proceeding this way, the parameters of the Beta-distribution are determined by matching the expectation and the variance of R_n (cf. Overbeck and Wagner, 2000).

Recall that the density of a $B(a,b)$-distributed random variable Z is defined by

$$\beta(a,b;x) = \frac{\Gamma(a+b)}{\Gamma(a)\,\Gamma(b)}\, x^{a-1}\,(1-x)^{b-1},\ 0 < x < 1, \tag{7a}$$

where Γ denotes the Gamma-function expanding the factorial function to the positive reals, and that the expectation and the variance respectively of Z are given by

$$\mathrm{E}[Z] = \frac{a}{a+b}, \tag{7b}$$

$$\mathrm{var}[Z] = \frac{a\,b}{(a+b)^2\,(a+b+1)}. \tag{7c}$$

Equating the right-hand sides of (7b) and (7c) with $\mathrm{E}[R_n]$ and $\mathrm{var}[R_n]$ respectively leads to

$$a = \frac{\mathrm{E}[R_n]}{\mathrm{var}[R_n]}\big(\mathrm{E}[R_n]\,(1-\mathrm{E}[R_n]) - \mathrm{var}[R_n]\big) \tag{8a}$$

and

$$b = \frac{1-\mathrm{E}[R_n]}{\mathrm{var}[R_n]}\big(\mathrm{E}[R_n]\,(1-\mathrm{E}[R_n]) - \mathrm{var}[R_n]\big). \tag{8b}$$

It is not hard to show that

$$\mathrm{E}[R_n] = p \tag{9a}$$

and

$$\mathrm{var}[R_n] = \frac{n-1}{n}\,\Phi_2(t,t,\rho) + \frac{p}{n} - p^2, \tag{9b}$$

with t defined by (2b) and Φ_2 denoting the bivariate standard normal distribution function. Since common tools like MSExcel have not got implemented algorithms for the calculation of Φ_2, the approximation

$$\Phi_2(t,t,\rho)\ \approx\ \Phi(t)^2 + \frac{e^{-t^2}}{2\,\pi}\,(\rho + 1/2\,\rho^2\,t^2) \tag{10}$$

can be used in (9b). (10) is based on a second order Taylor expansion of $\Phi_2(t,t,\rho)$ with respect to ρ (cf. Tong, 1990) and yields a fairly good approximation for moderate values of ρ and α. As will be shown in Section 5, the quality of approximation decreases for $\rho \geq 0.2$ and values of α close to one.

By means of (10), (9a), and (9b), the quantile $q(\alpha, D_n)$ can be calculated approximately via

$$q(\alpha, D_n)\ \approx\ n\,q(\alpha, Z), \tag{11}$$

where Z is a Beta-distributed random variable and the parameters a and b of its distributions are given by (8a) and (8b) respectively.

5 Numerical examples

For the purpose of illustration of the previous sections, we calculated[1] the lower and higher critical values of the traffic lights for various portfolio sizes n and two different asset correlation values. Table 1 shows the 95%-critical values for a test of PD ≤ 0.01 in case of low ($\rho = 0.05$) and high ($\rho = 0.2$) asset correlation. We compare the results obtained with the classical binomial test, the granularity adjustment of Section 3, and the moment matching of Section 4. Since the binomial test relies on the assumption of

n	50	250	1000	50	250	1000
Approach:	$\rho = 0.05$			$\rho = 0.2$		
Binomial	2	5	15	2	5	15
Granularity adjustment	3	7	24	3	11	39
Moment matching	4	8	25	4	12	42

Table 1: 95%-critical values for PD tests of hypothesis $p \leq 0.01$.

independent default events, there is no difference when switching the correlation regime from low to high. For the other approaches, this picture changes dramatically. Respecting the correlation makes grow a lot the critical values, with more than doubling in case of high correlation. At this quite moderate confidence level of 95%, the results of the granularity adjustment and the moment matching approach do not differ much, although differences appear to become larger with an increasing portfolio size and higher correlation.

As the numbers in Table 2 show, the effects described for the case of a moderate confidence level become much more pronounced when we consider a very high confidence level like 99.9%. In particular, in case of high correlation the differences between the results with

n	50	250	1000	50	250	1000
Approach:	$\rho = 0.05$			$\rho = 0.2$		
Binomial	4	9	21	4	9	21
Granularity adjustment	6	15	50	9	38	148
Moment matching	7	16	47	10	33	118

Table 2: 99.9%-critical values for PD tests of hypothesis $p \leq 0.01$.

the granularity approach and the moment matching approach respectively cannot be

[1] Upon request, an MSExcel-sheet with implementations of the algorithms can be obtained from the author.

any longer neglected. Since the granularity adjustment is based on an approximation procedure in the tail of the loss distribution, whereas the moment matching comes from an approximation in the center of the distribution, the granularity adjustment results will in general be the more reliable. But this observation should not be seen as a knock-out criterion against the moment matching since the use of high correlations like 0.2 might be considered being too conservative.

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