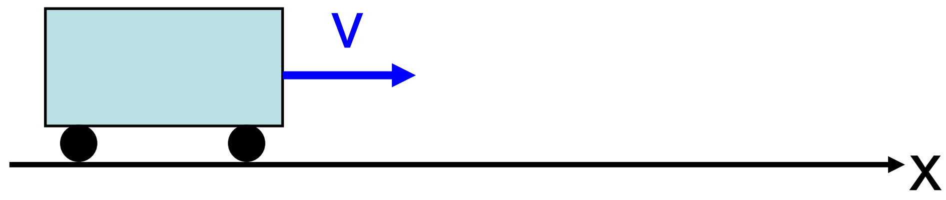
Spacetime diagrams

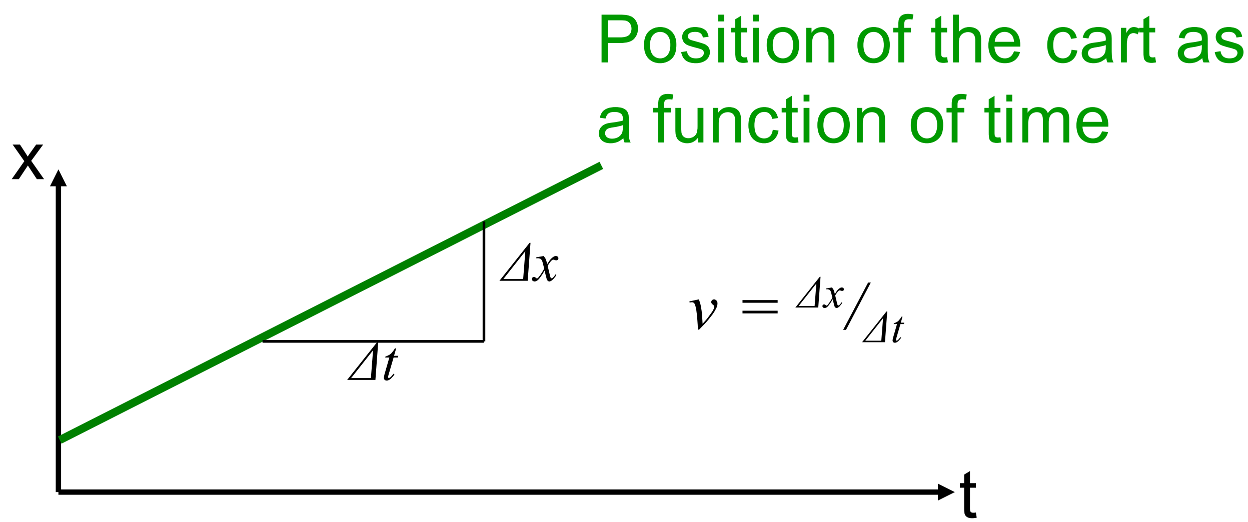
(also called “Minkowski diagrams”)

(very useful in SR!)

Spacetime Diagrams (1D in space)

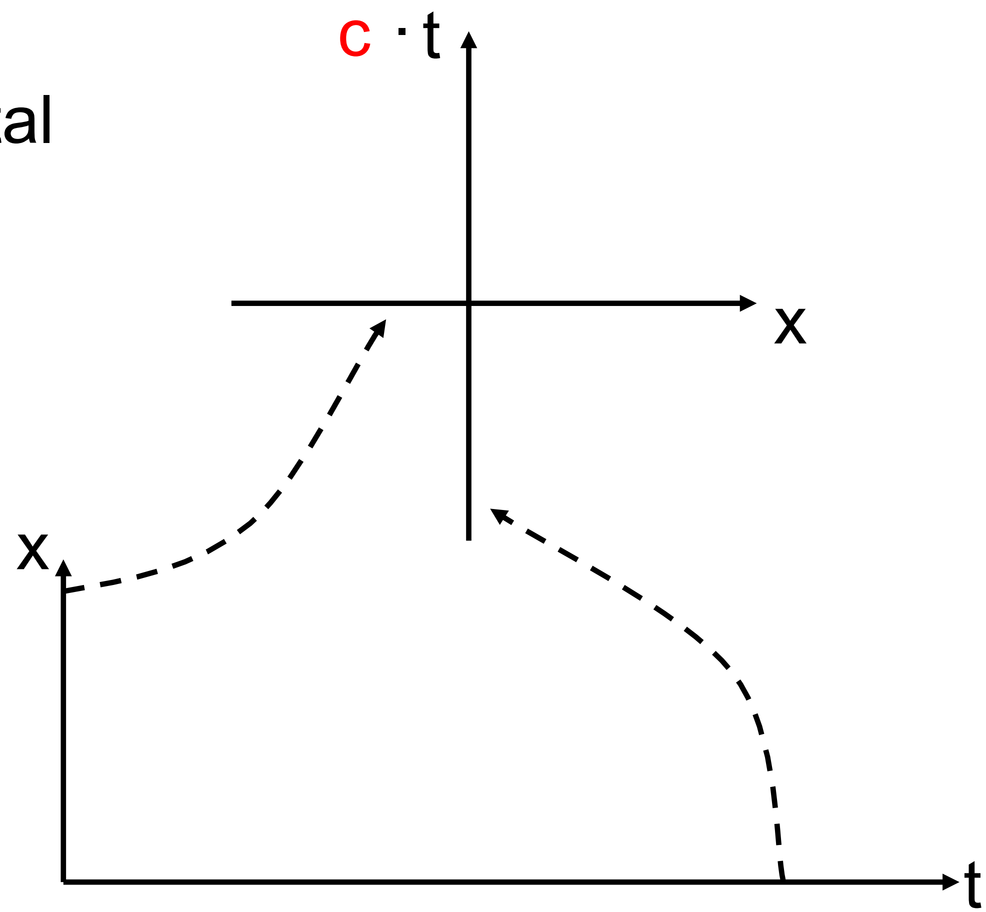
In Fundamental PHYS





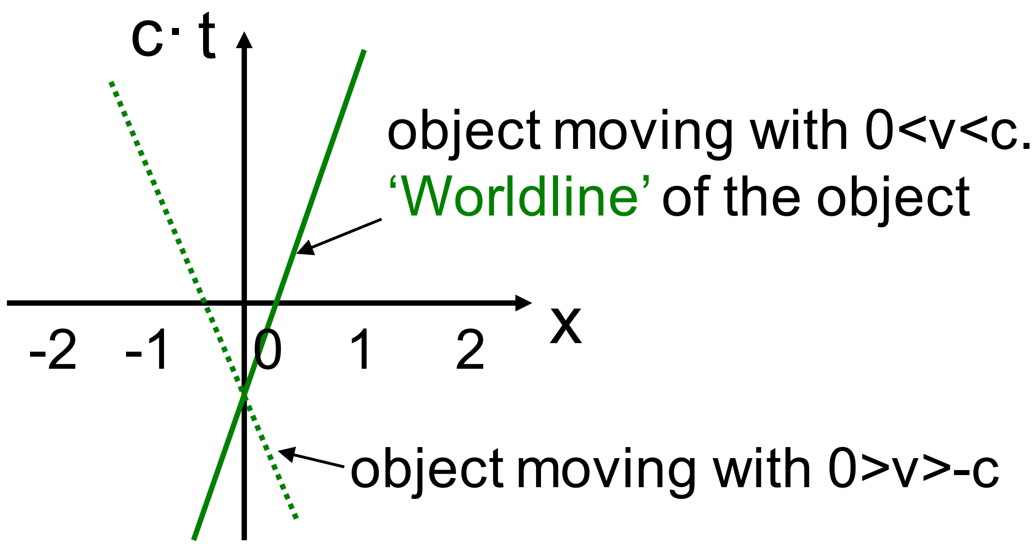
Spacetime Diagrams (1D in space)

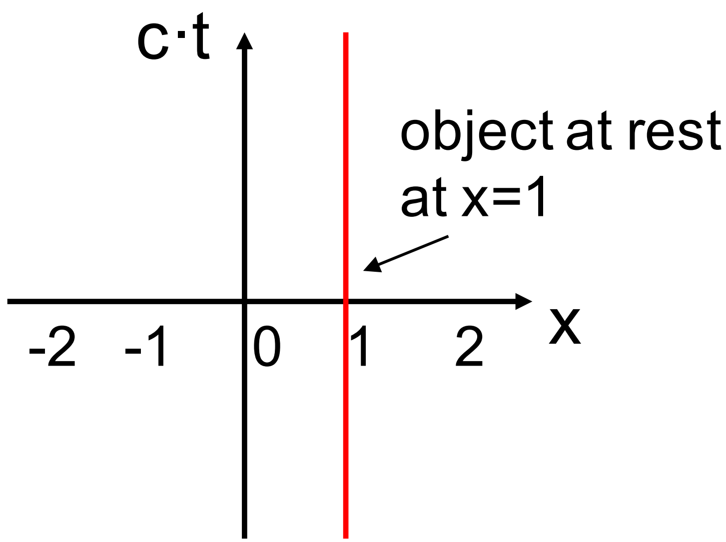
In PHYS fundamental physics:

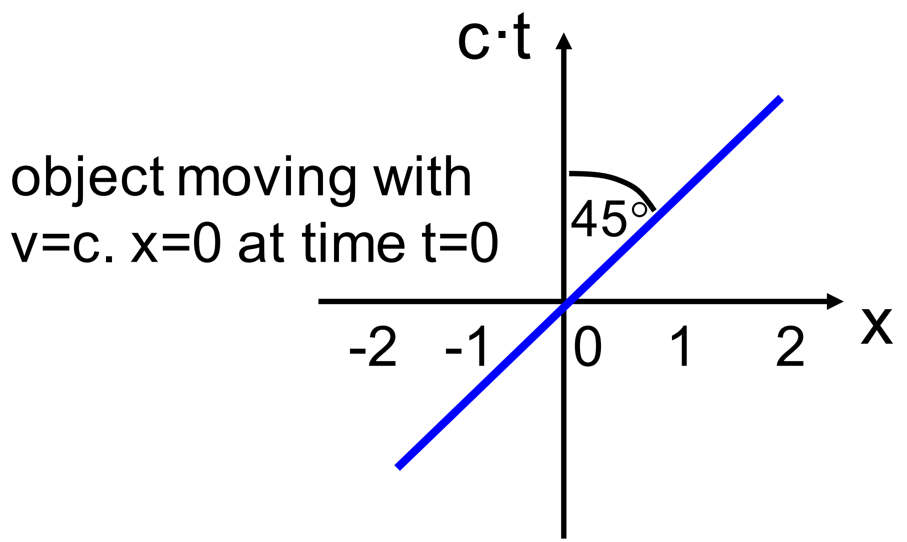


Spacetime Diagrams (1D in space)

In PHYS 2130:

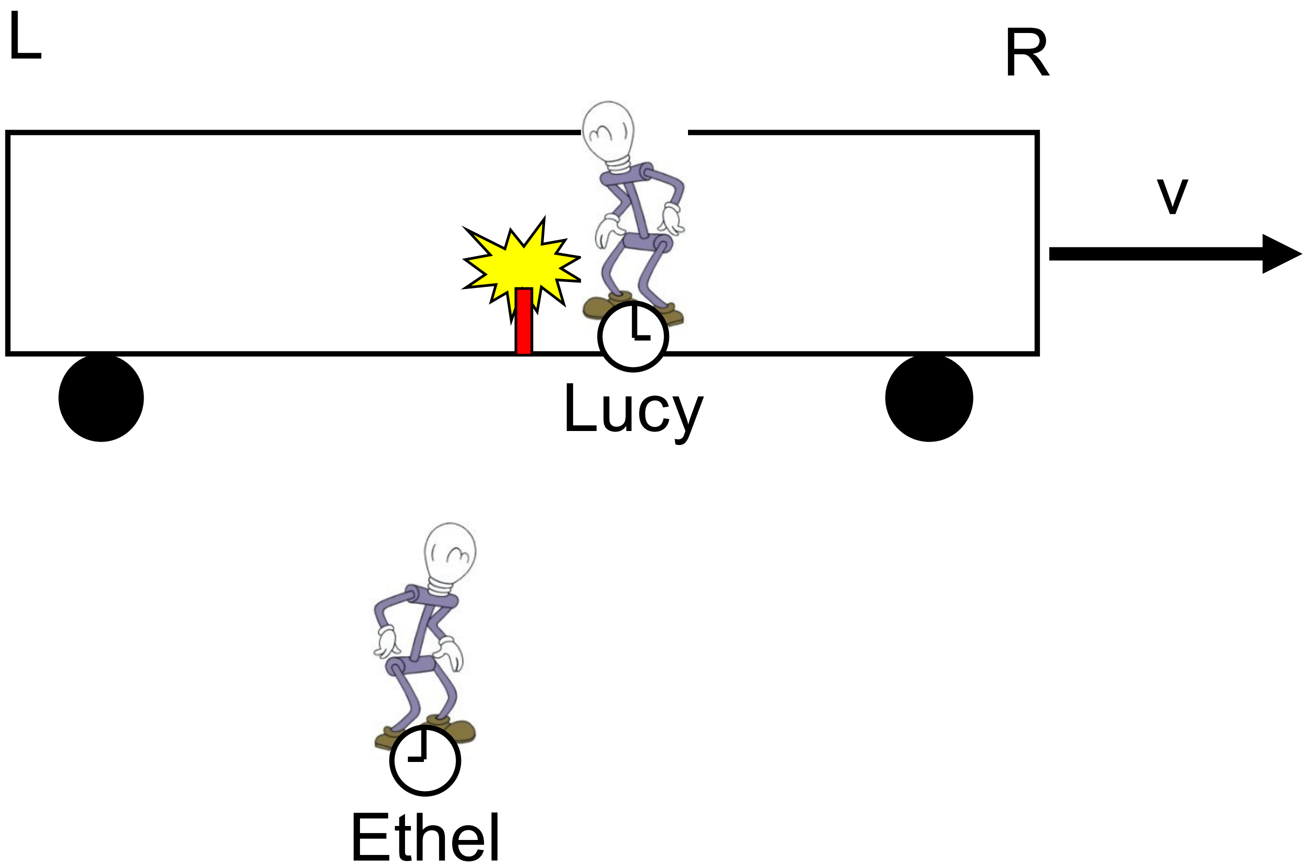




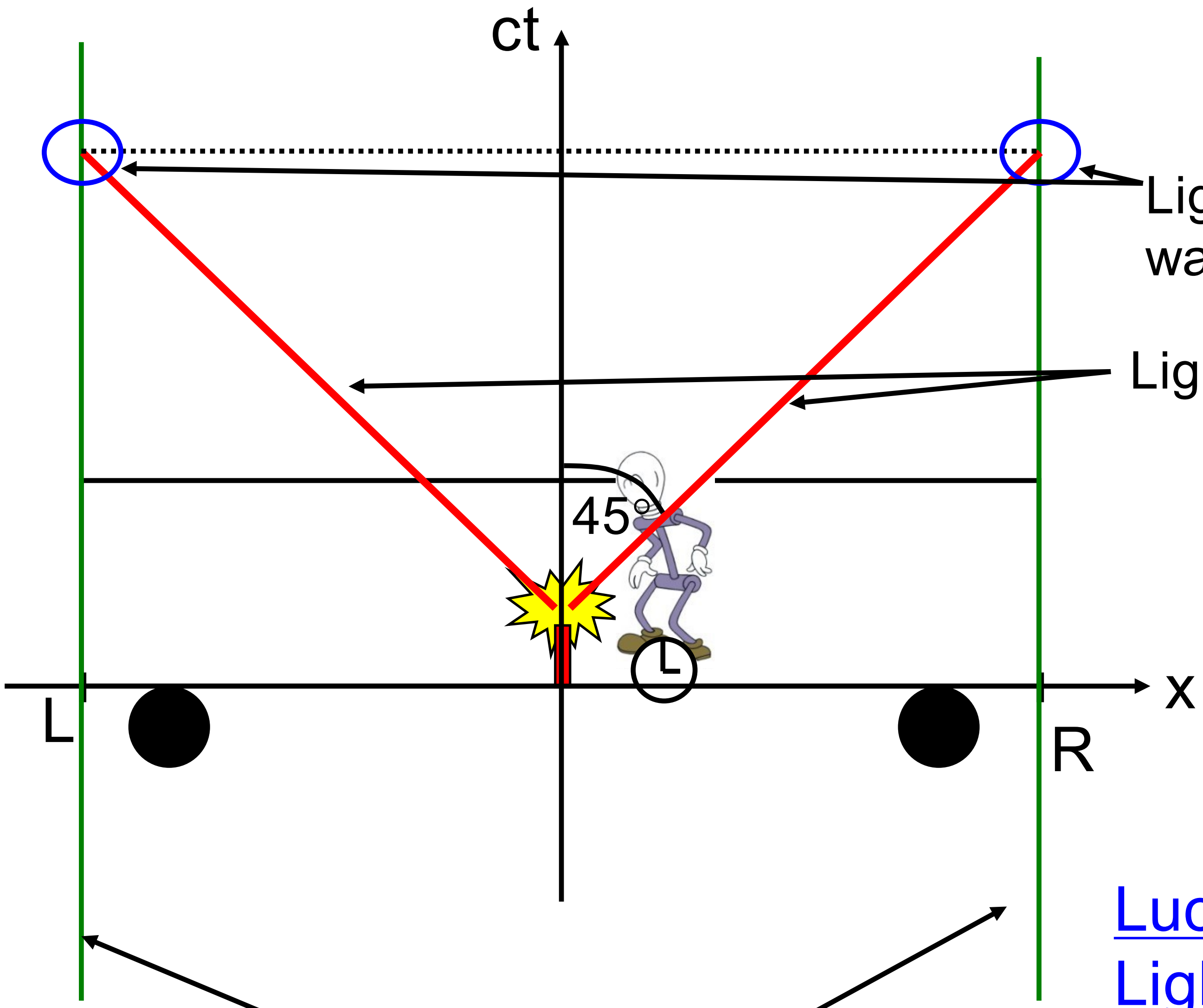


Recall:

Lucy plays with a fire cracker in the train.
Ethel watches the scene from the track.



Example: Lucy in the train



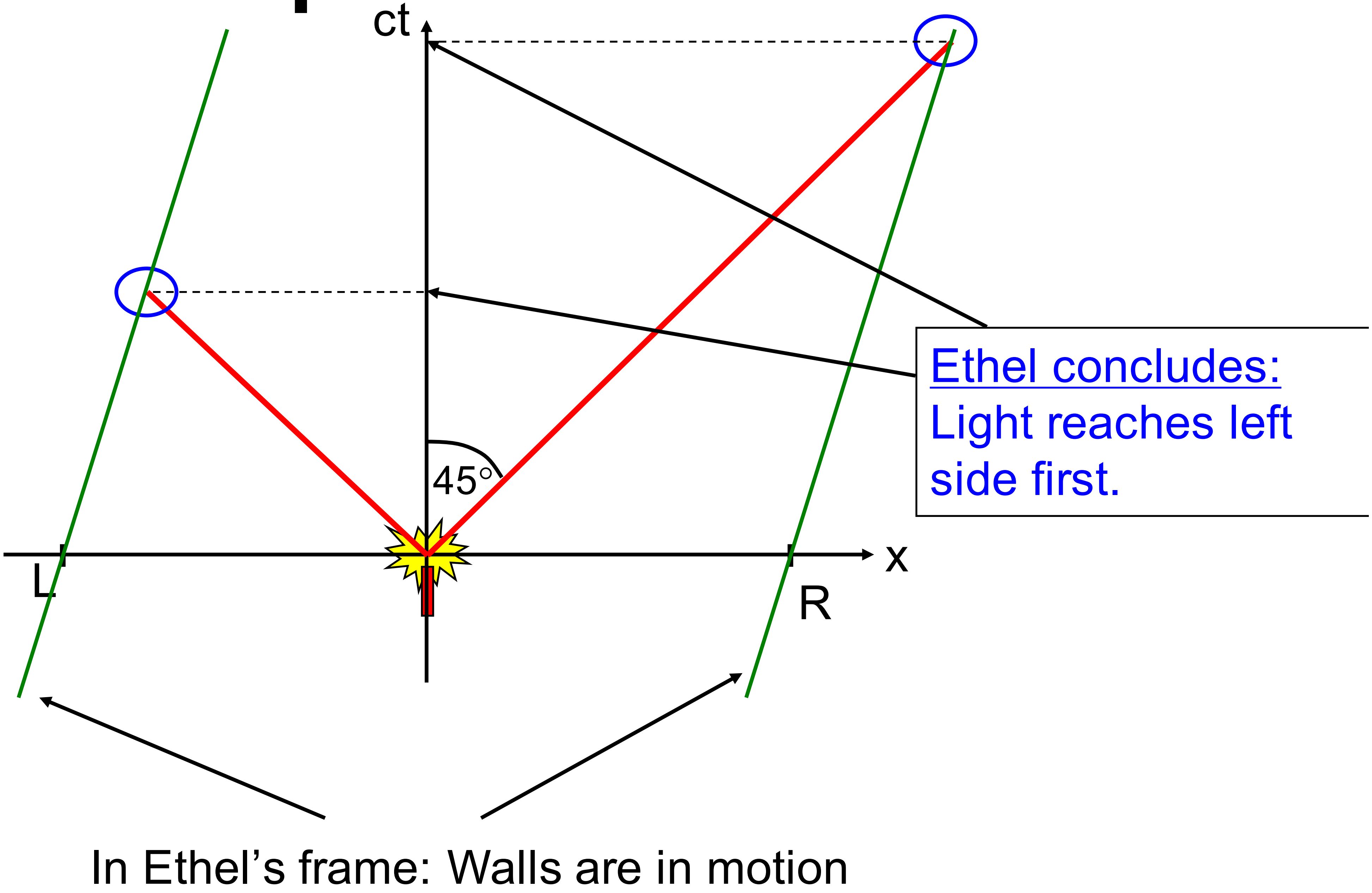
Light reaches both walls at the same time.

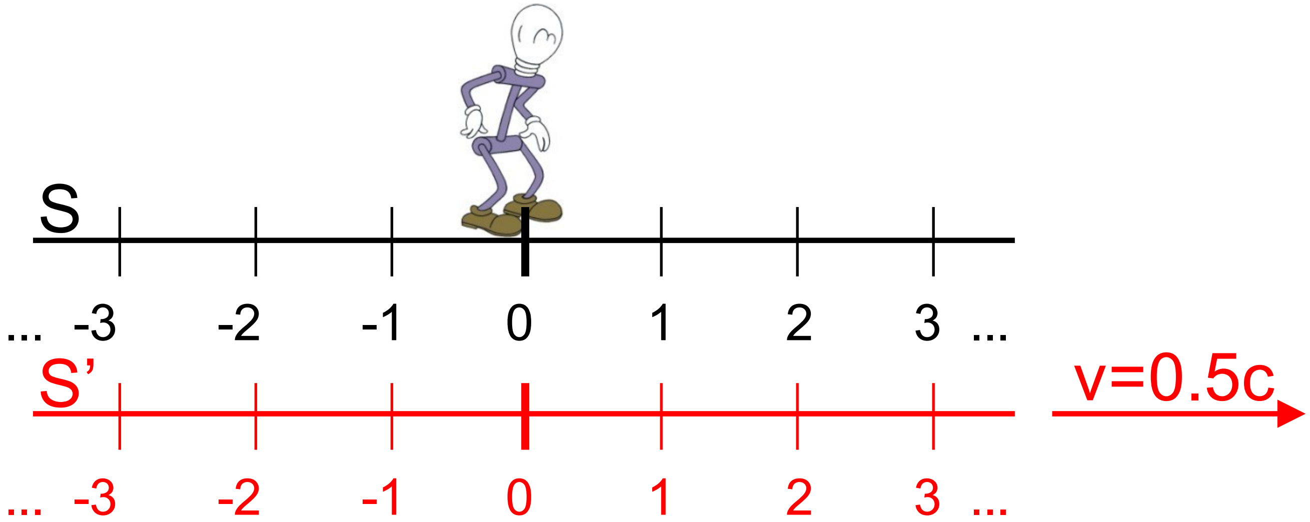
Light travels to both walls

<u>Lucy concludes:</u>
Light reaches both sides at the same time

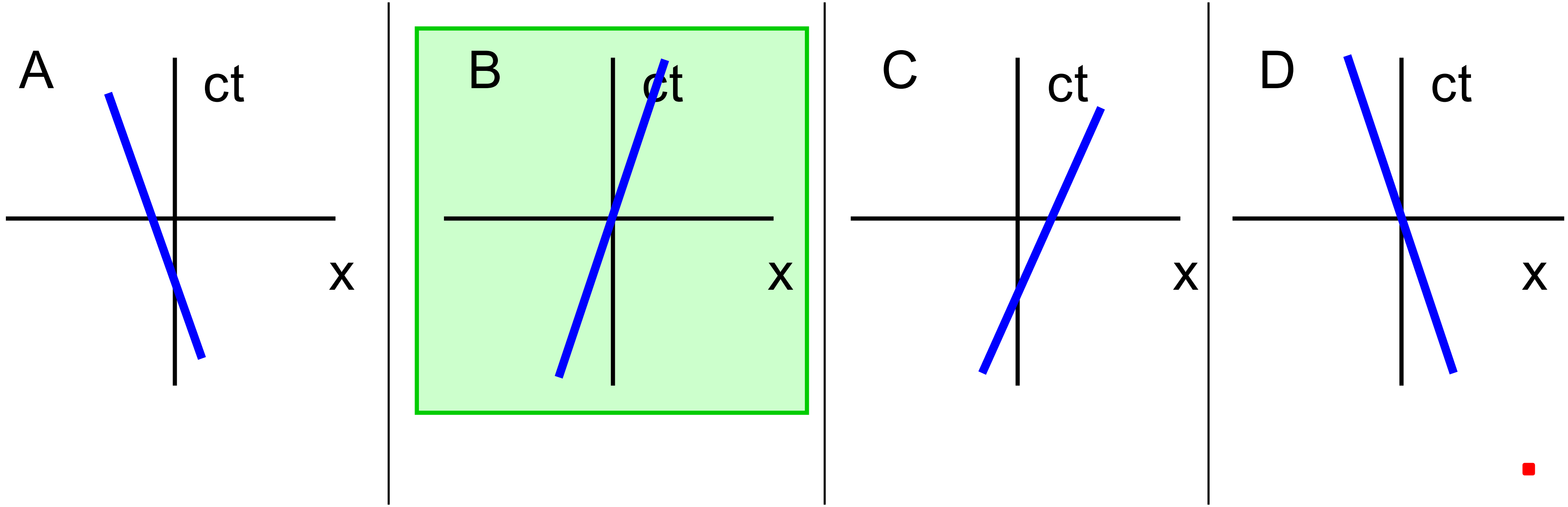
In Lucy's frame: Walls are at rest

Example: Ethel on the tracks

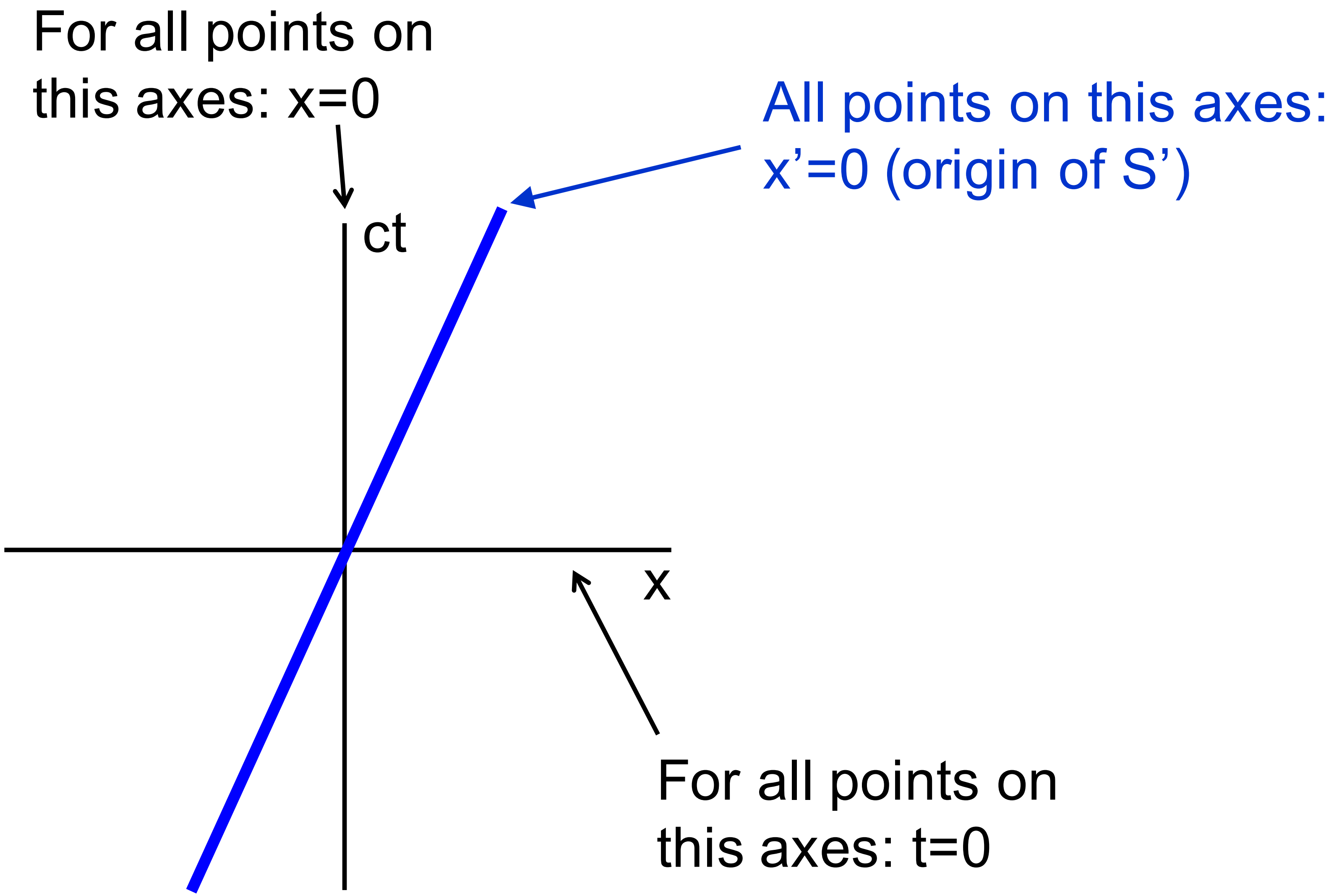




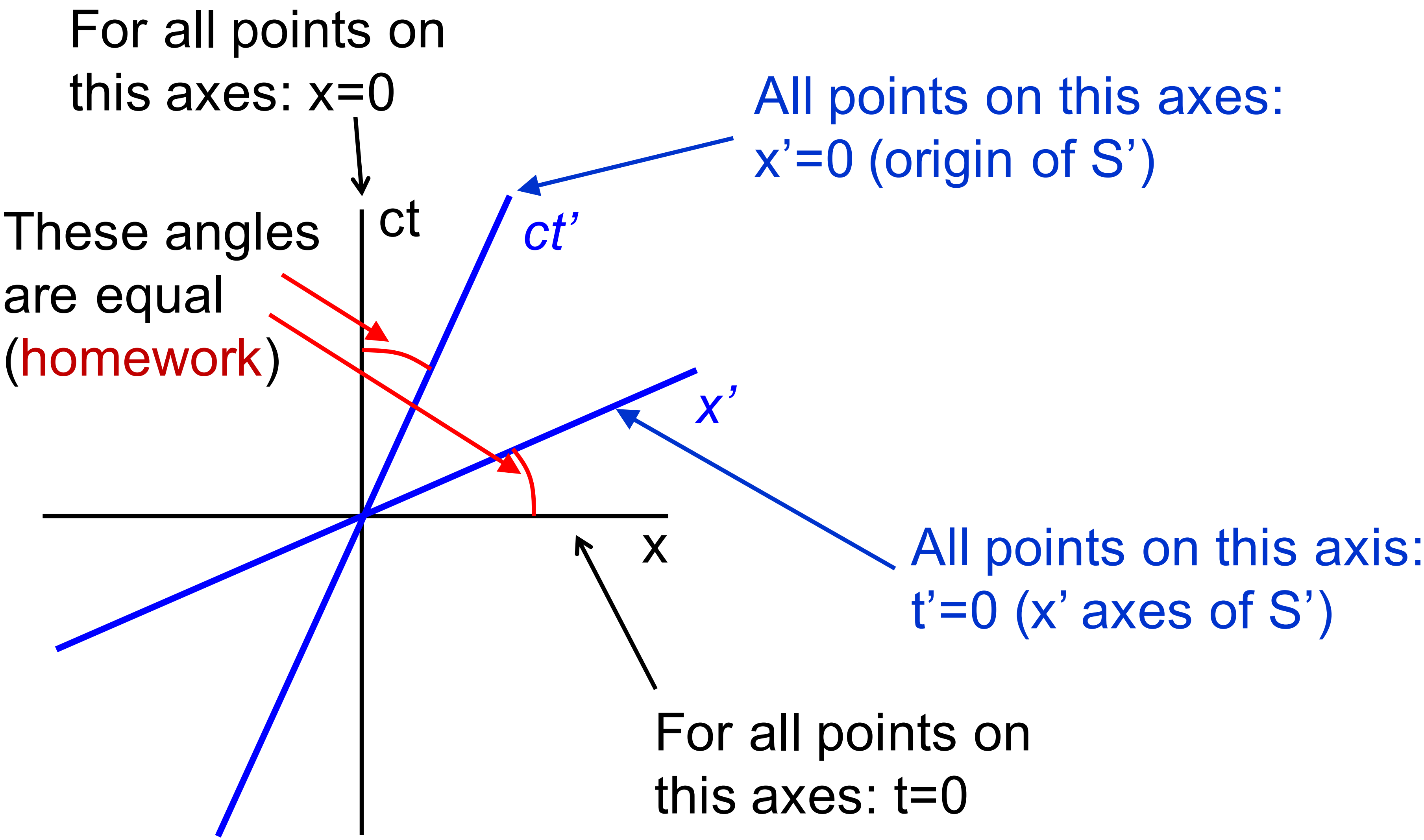
Frame S' is moving to the right at v = 0.5c. The origins of S and S' coincide at t=t'=0. Which shows the world line of the **origin of S' as viewed from S**?



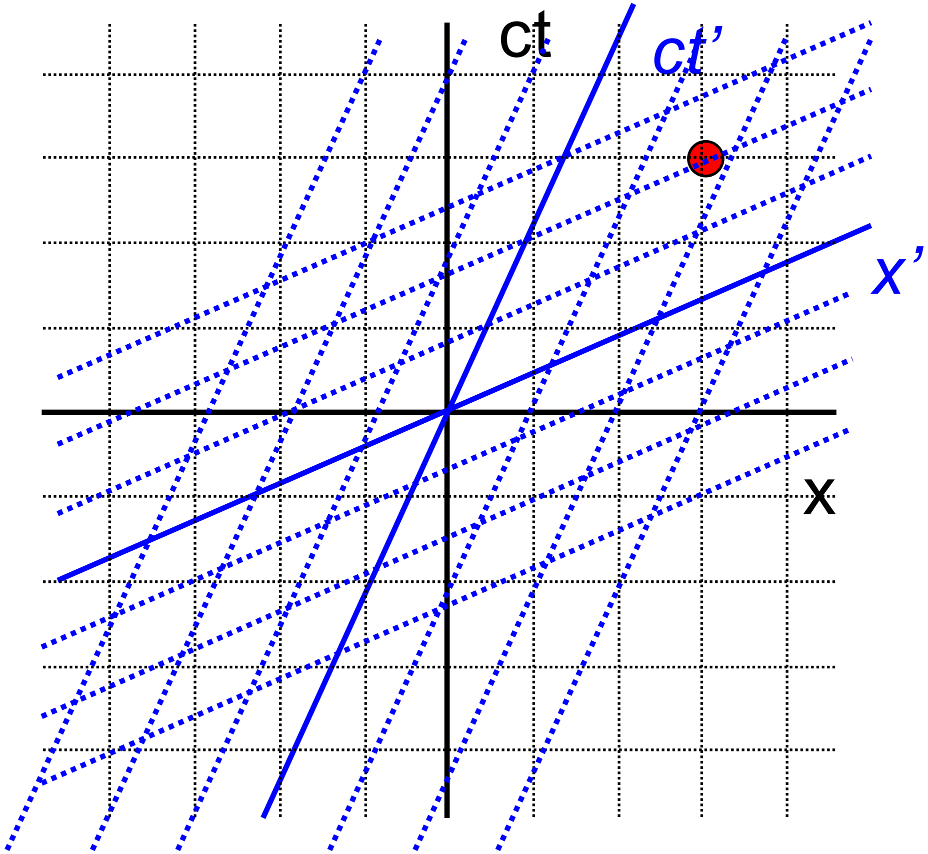
Origin of S’ viewed from S



Frame S’ as viewed from S



Both frames are adequate for describing events, but will generally give different spacetime coordinates. In S: (x,t), or in S': (x',t')



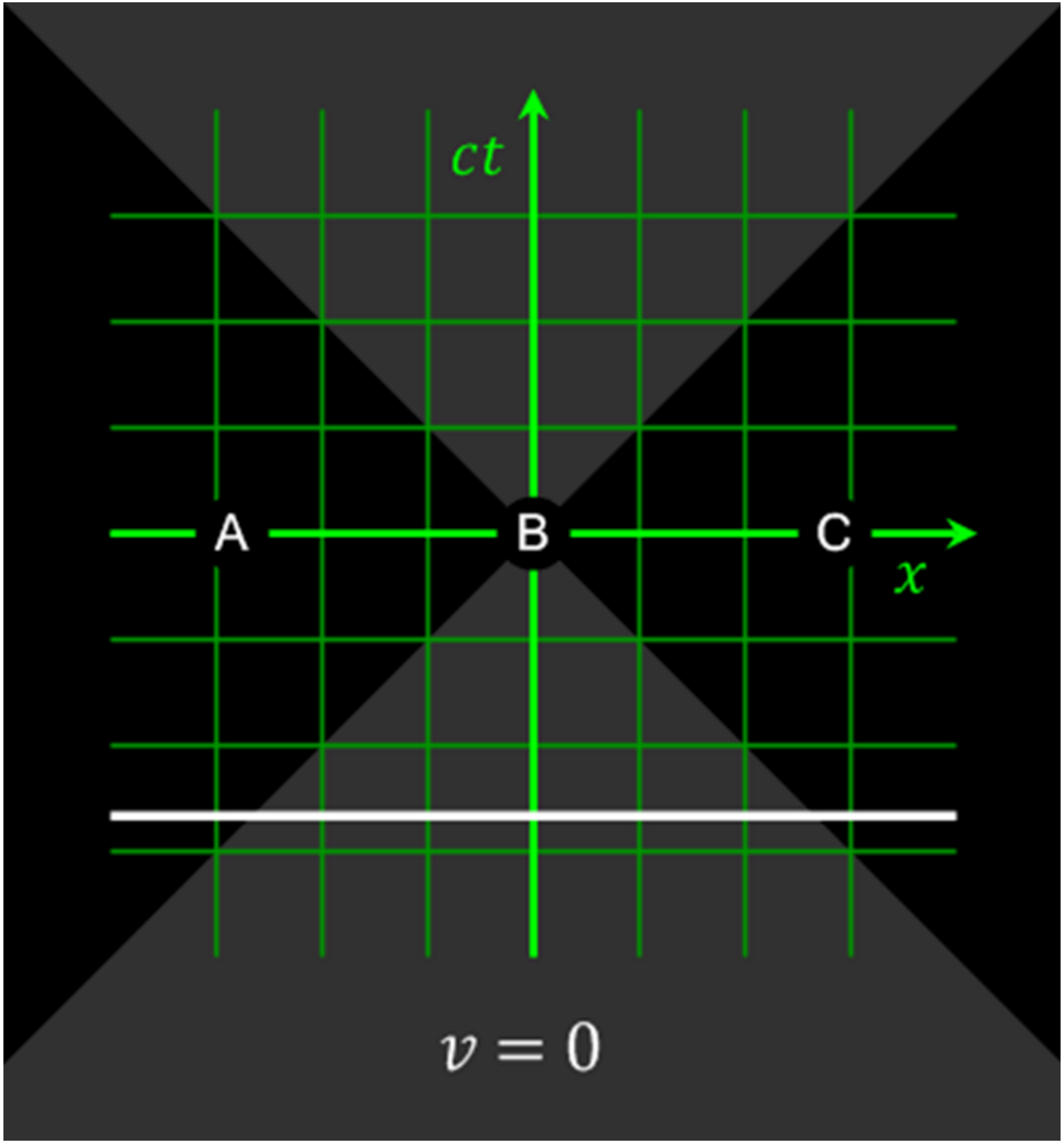
In S: (3,3)

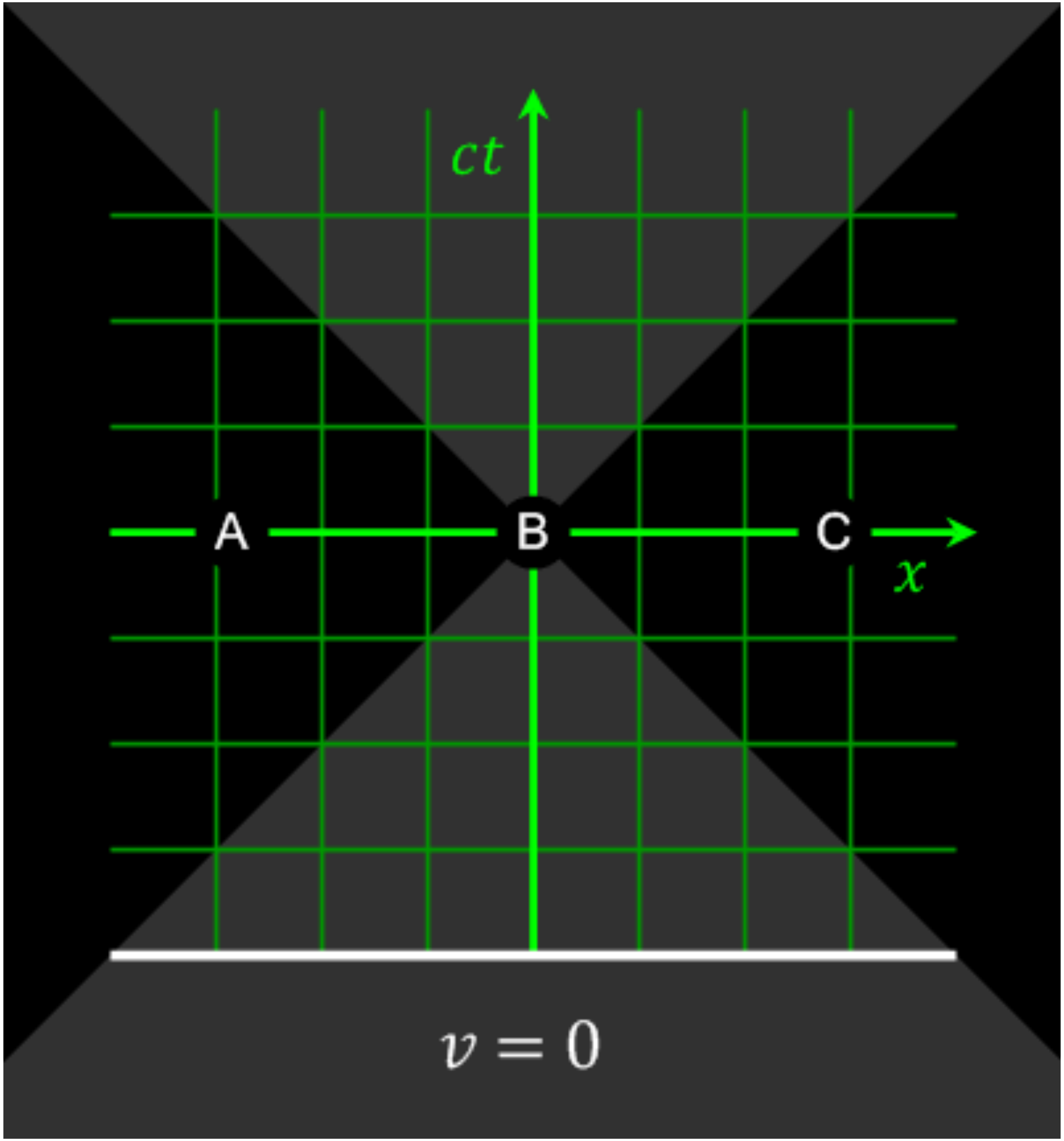
In S': (1.8,2)

In classical physics we had something similar: The Galileo transformations.

$$x' = x - v \cdot t$$
$$t' = t$$

(We know that they are no good for light. We'll fix them soon!)

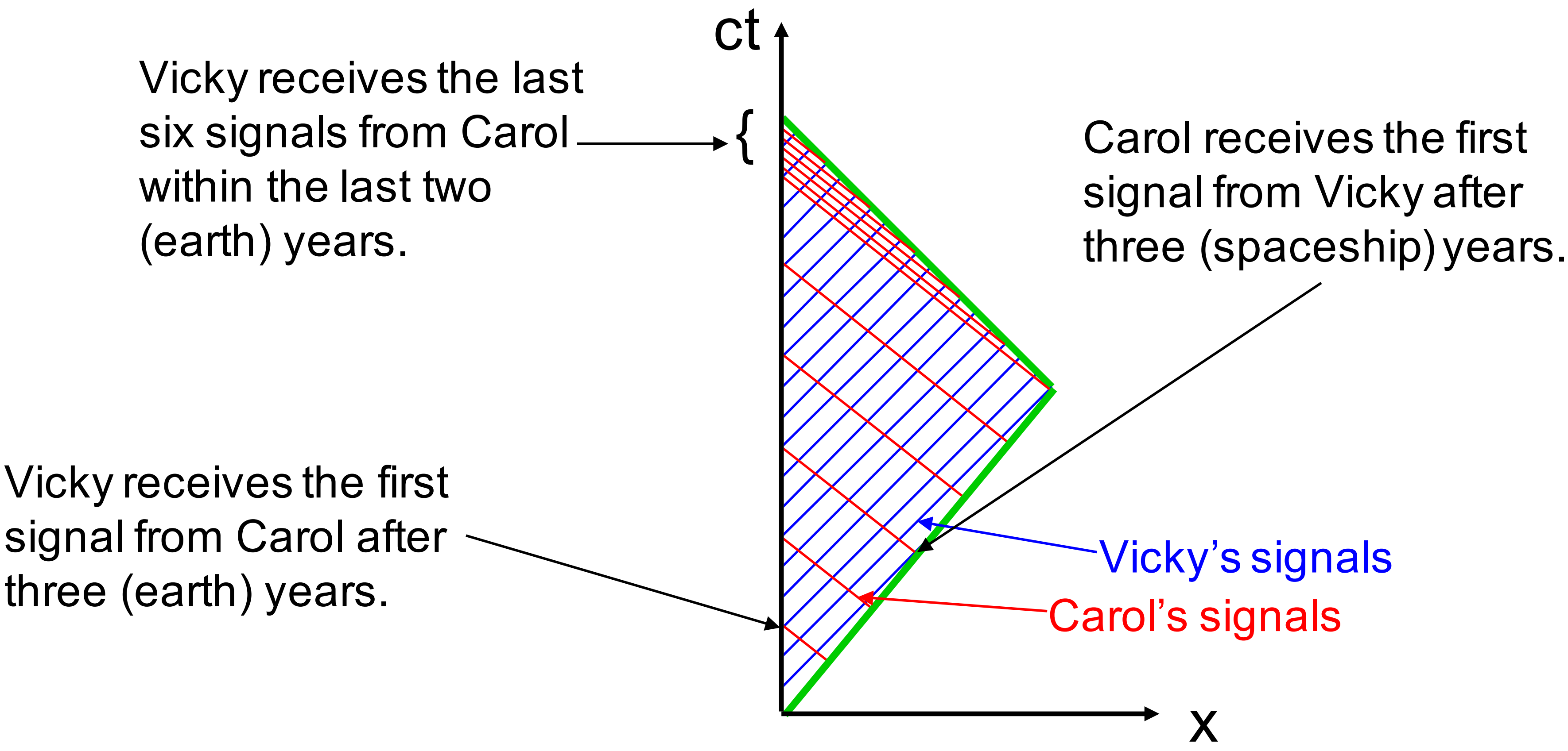

ct
A
B
C
x
$v = 0$



ct
A
B
C
x
$v = 0$

Example: Spacetime diagram for the twin paradox

Carol and Vicky send out radio signals at the beginning of every year (measured by their respective local clocks)

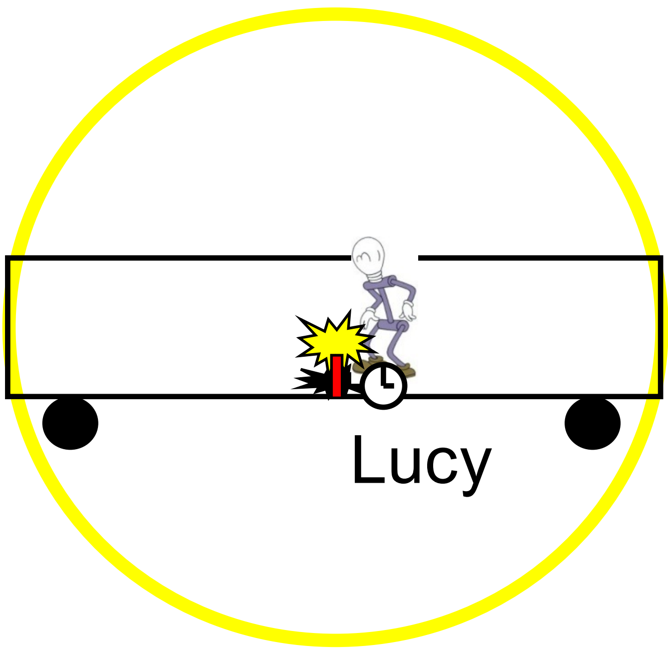


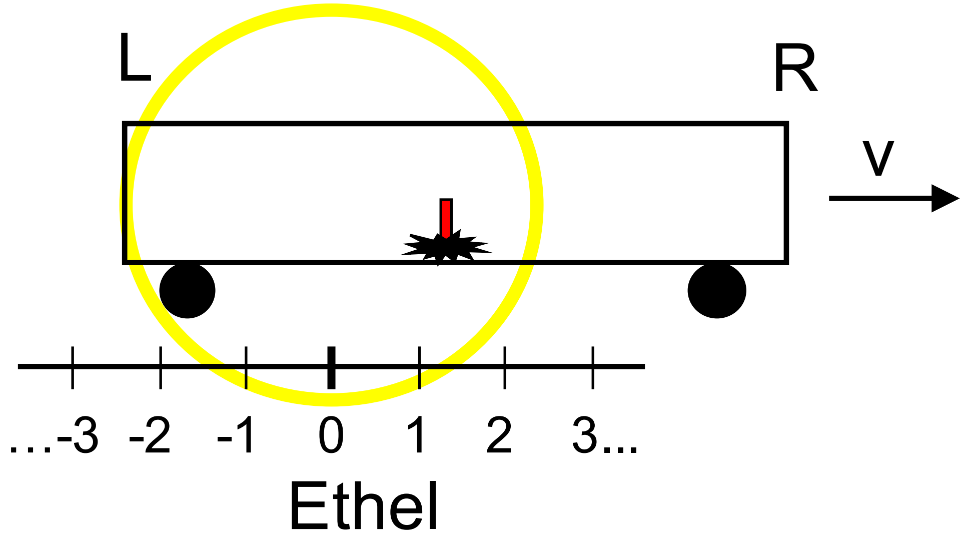
(End of Space time diagrams)

Now back to Time dilation!

What we found so far:

Simultaneity of two events depends on the choice of the reference frame



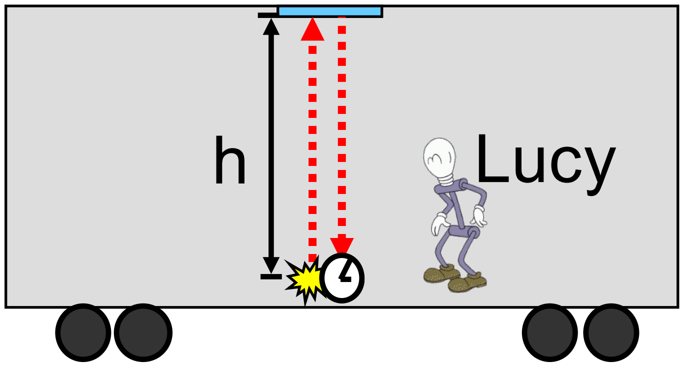


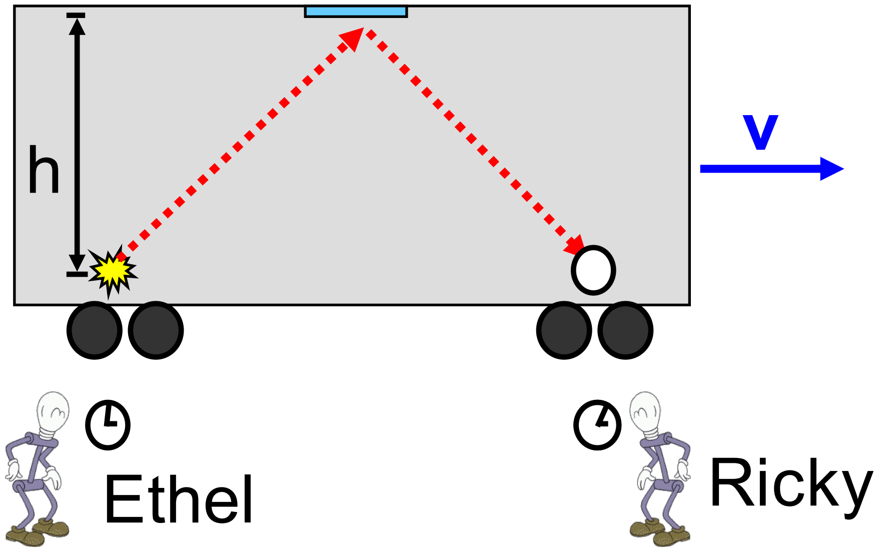
Lucy concludes:
Light hits both ends at the same time.

Ethel concludes:
Light hits left side first.

Time Dilation

Time Dilation: Two observers (moving relative to each other) can measure different durations between two events.





Lucy measures:

$$\Delta t' = {}^{2h}/_{c}$$

Here: $\Delta t'$ is the **proper time**

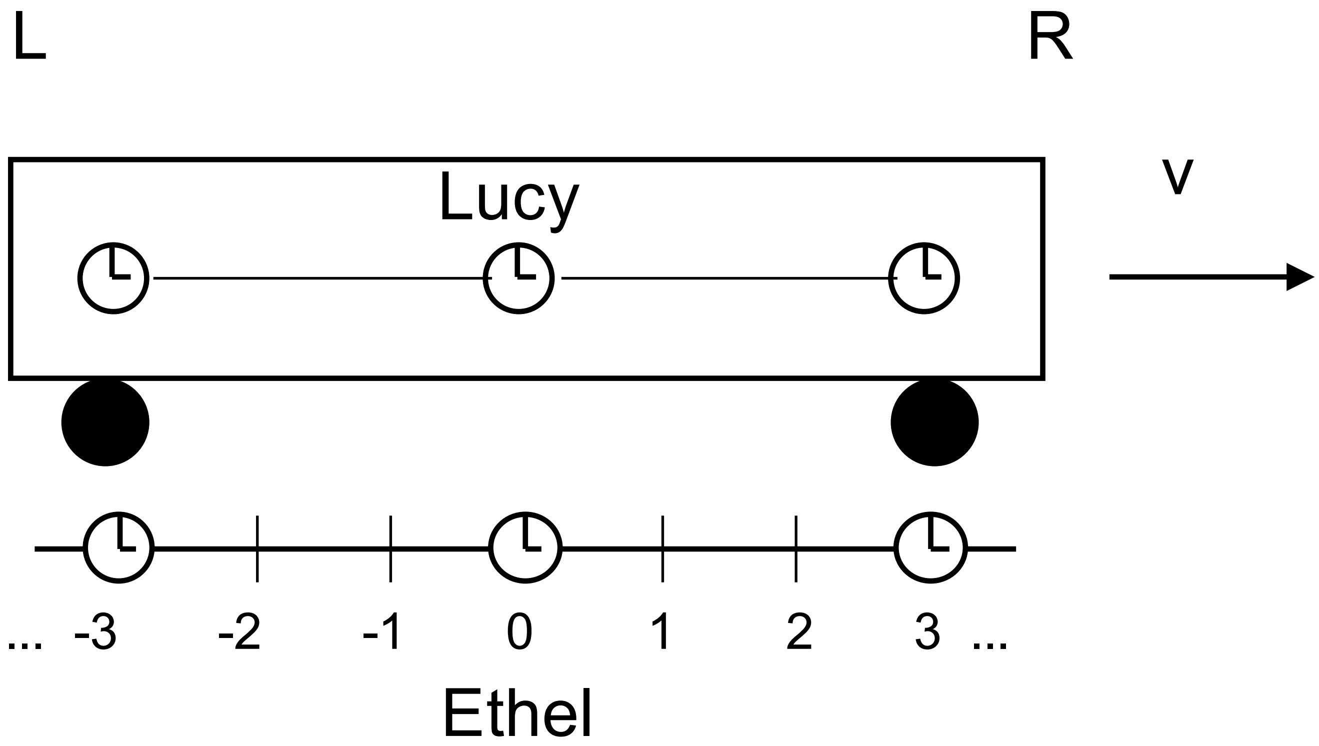
Ethel and Ricky:

$$\Delta t = \gamma \, {}^{2h}/_{c}, \text{ with } \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

More about time dilation:

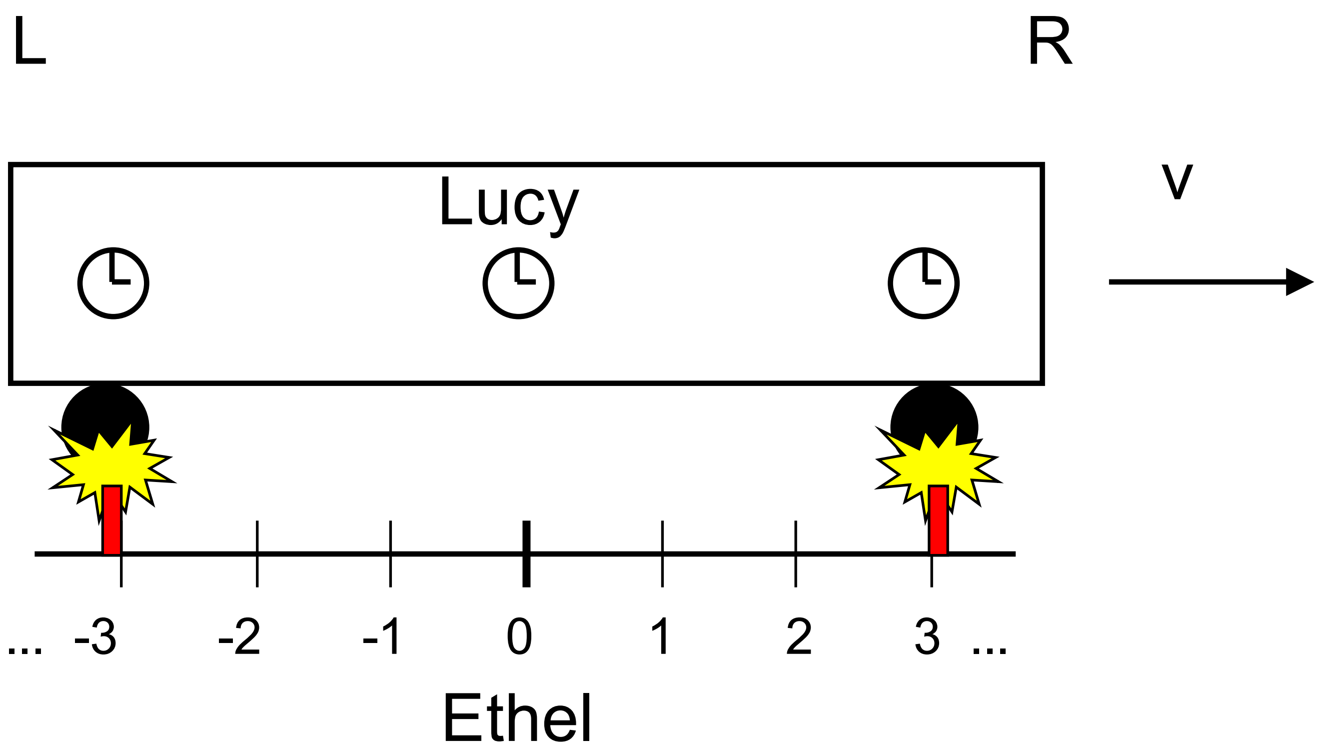
Are your clocks *really* synchronized?

They sure don't seem to be!…



Now Lucy and Ethel each have a set of clocks. Lucy's are synchronized in her frame (the train), while Ethel's are synchronized in her frame (the tracks).

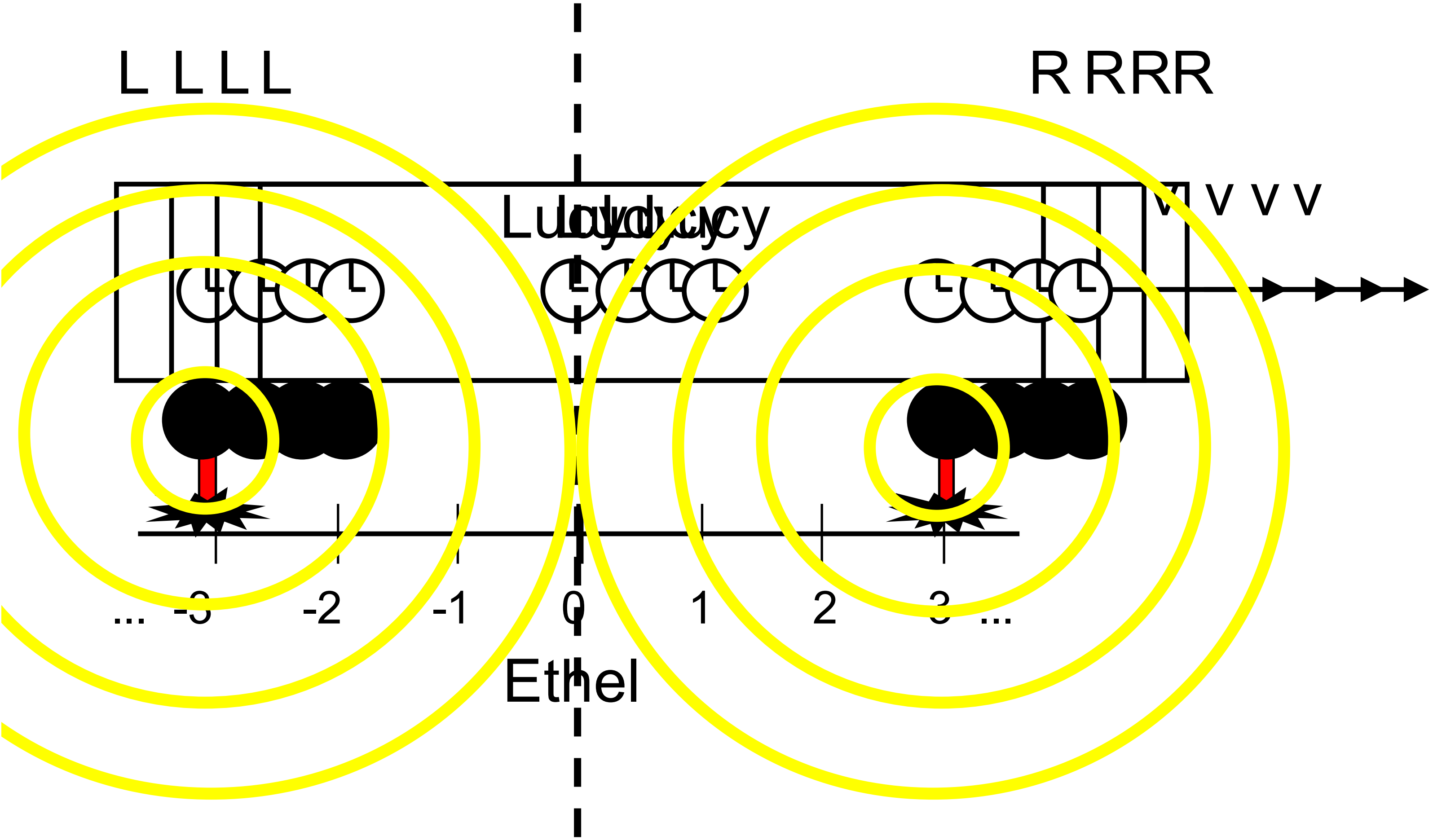
How do the clocks of one frame read in the other frame? (Pay attention, now!)



At 3 o'clock in Ethel's frame, two firecrackers go off to announce the time. It so happens that these firecrackers are at the left and right ends of the train, in Ethel's frame.

Event 1: firecracker 1 explodes at 3:00

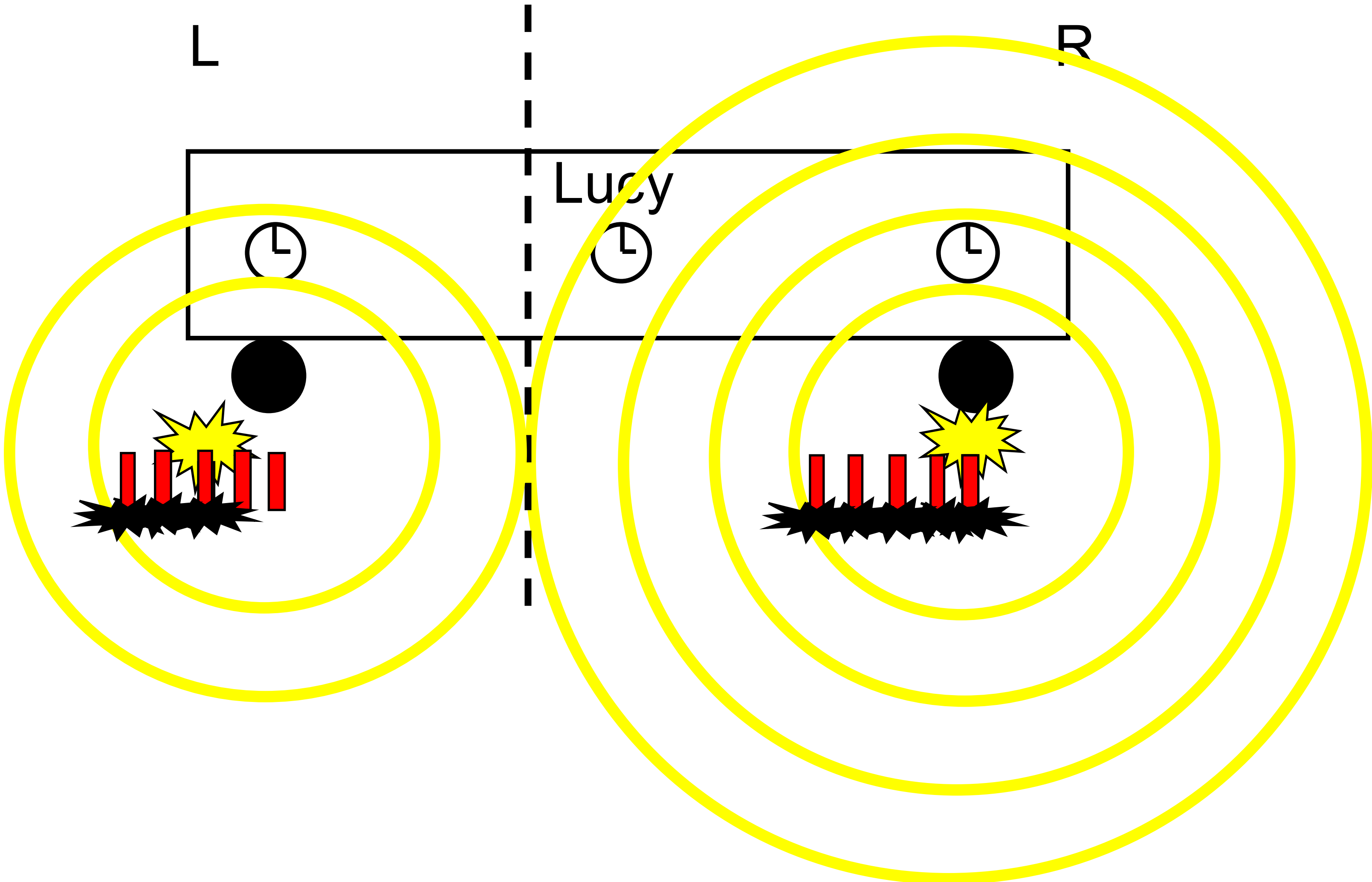
Event 2: firecracker 2 explodes at 3:00

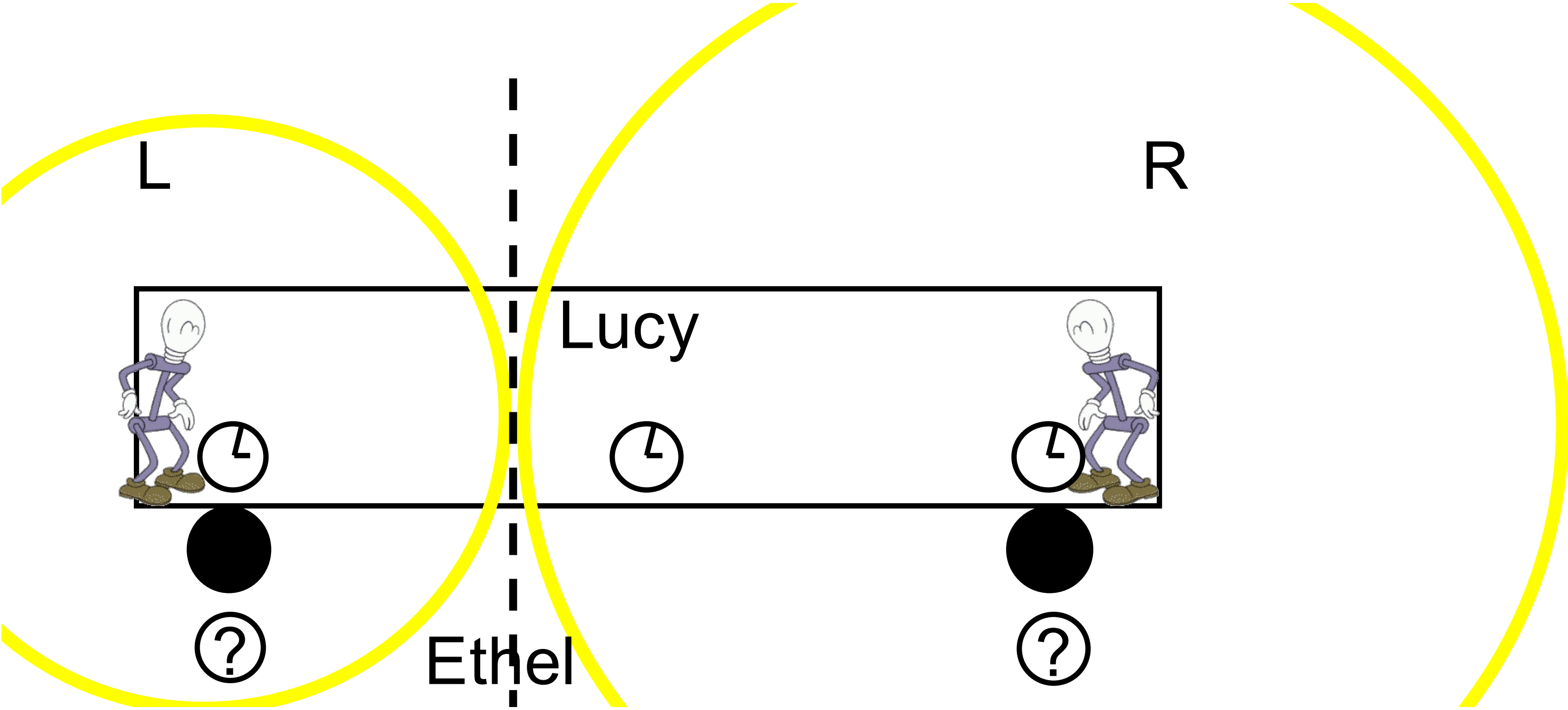


Sometime later, the wavefronts meet. The meeting point is halfway between the firecrackers in Ethel's frame, but is somewhere in the left of the train car, in Lucy's frame.

Event 3: two light pulses meet, shortly after 3:00.

The situation as seen by Lucy





In *Lucy's frame*, light left first from the right firecracker. But in Ethel's frame, both went off exactly at 3 o'clock!

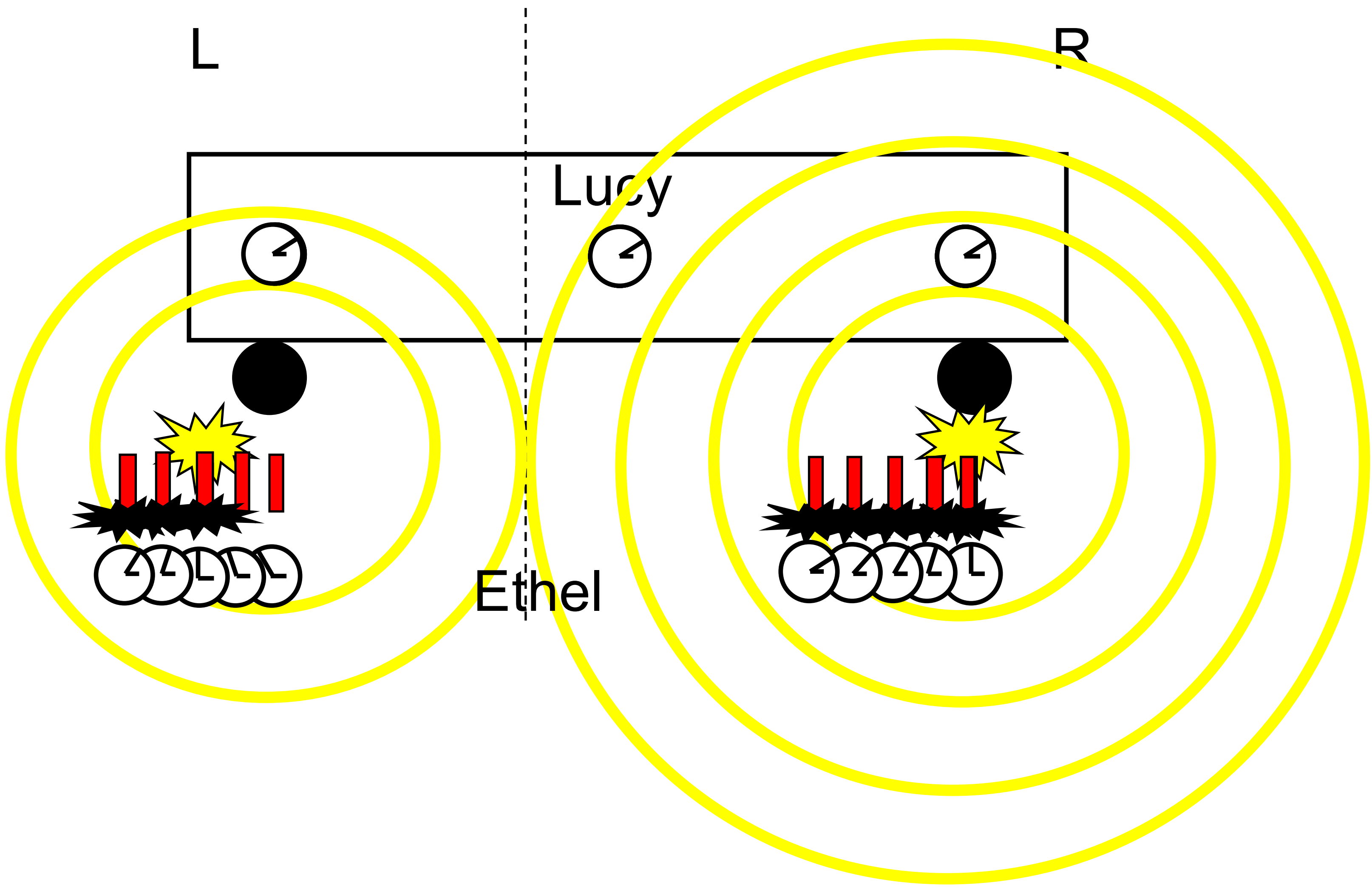
According to Lucy's reference frame, which of the following is true:

A) Ethel's clock on the left reads a later time than Ethel's clock on the right.

B) Ethel's clock on the right reads a later time than Ethel's clock on the left.

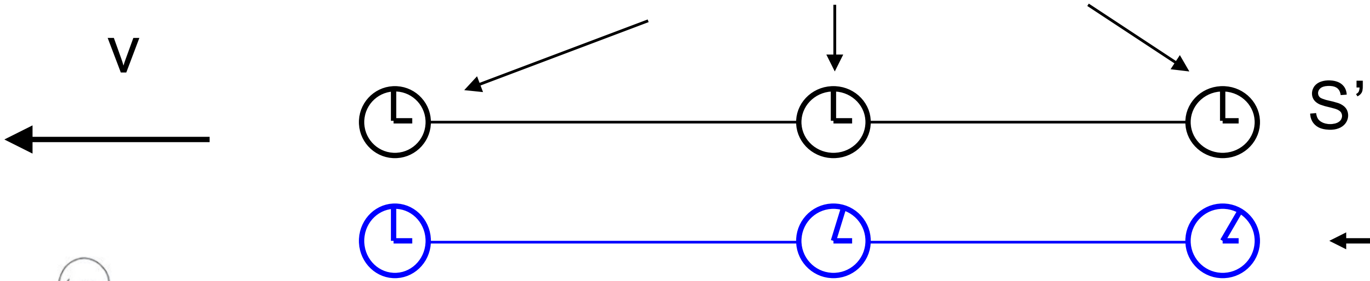
C) Both of Ethel's clocks read the same time.

In Lucy's frame:

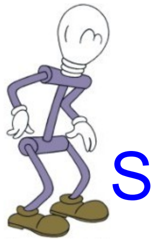


Important conclusion

Clocks in S' (synchronized in S') (Ethel) moving to the left with respect to S (Lucy)



Clocks in S' seen by observers in S



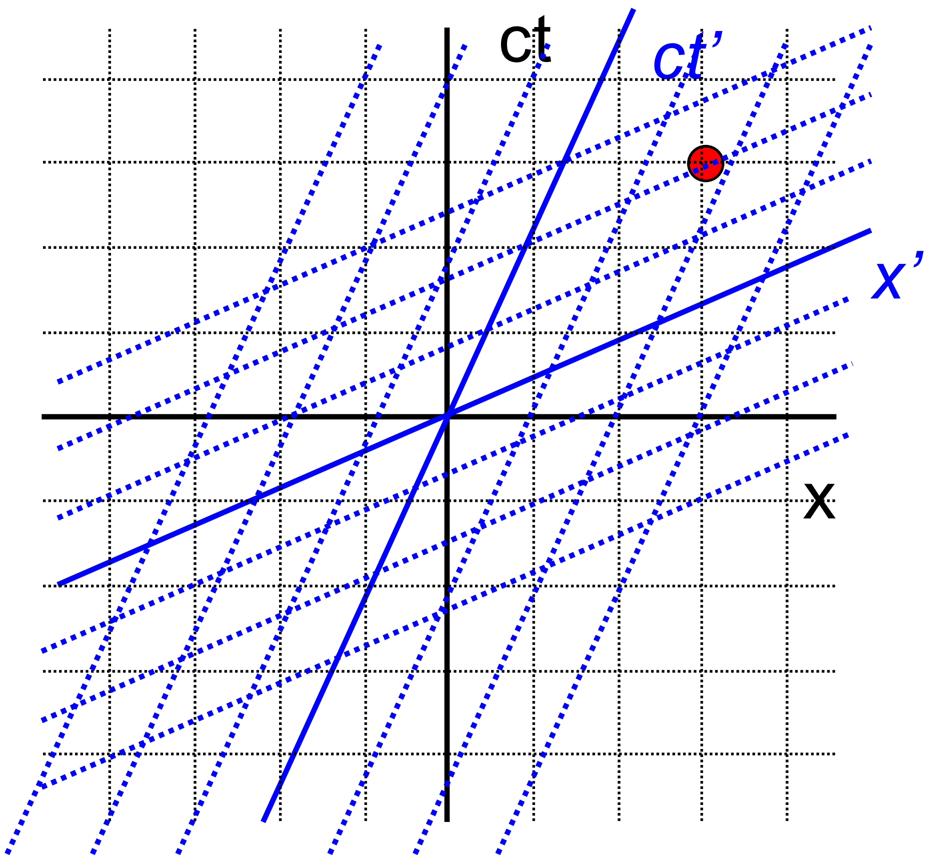
If this clock reads 3:00 in S, then:

This clock reads a little after 3:00 in S

This clock reads even a little later in S

Even though the clocks in S' are synchronized in S', observers in S find that they each show a different time!!

Both frames are adequate for describing events, but will generally give different spacetime coordinates. In S: (x,t), or in S': (x',t')



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In S': (1.8,2)

In classical physics we had something similar: The Galileo transformations.

$$x' = x - v \cdot t$$
$$t' = t$$

(We know that they are no good for light. We'll fix them soon!)

Length contraction

(Consequence of time dilation and vice versa)

Quiz on the reading

Proper length of an object is the length of the object measured…

A – …in its rest frame.

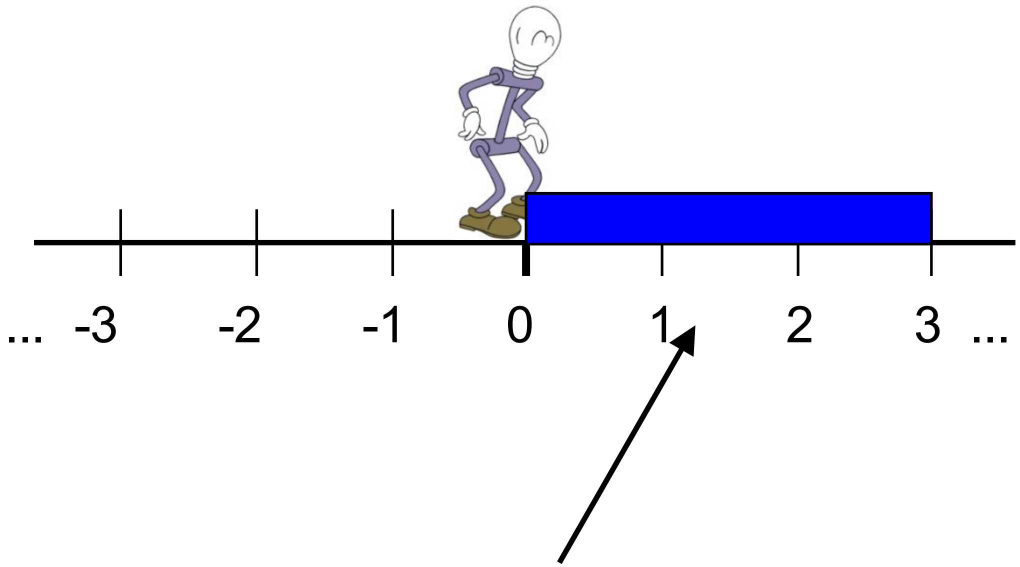
B – …in any inertial frame.

C – …in the inertial frame in which both ends of the object have the same event coordinates.

D – …in the frame in which the object is not rotating.

E – …by the speed of light.

Length of an object



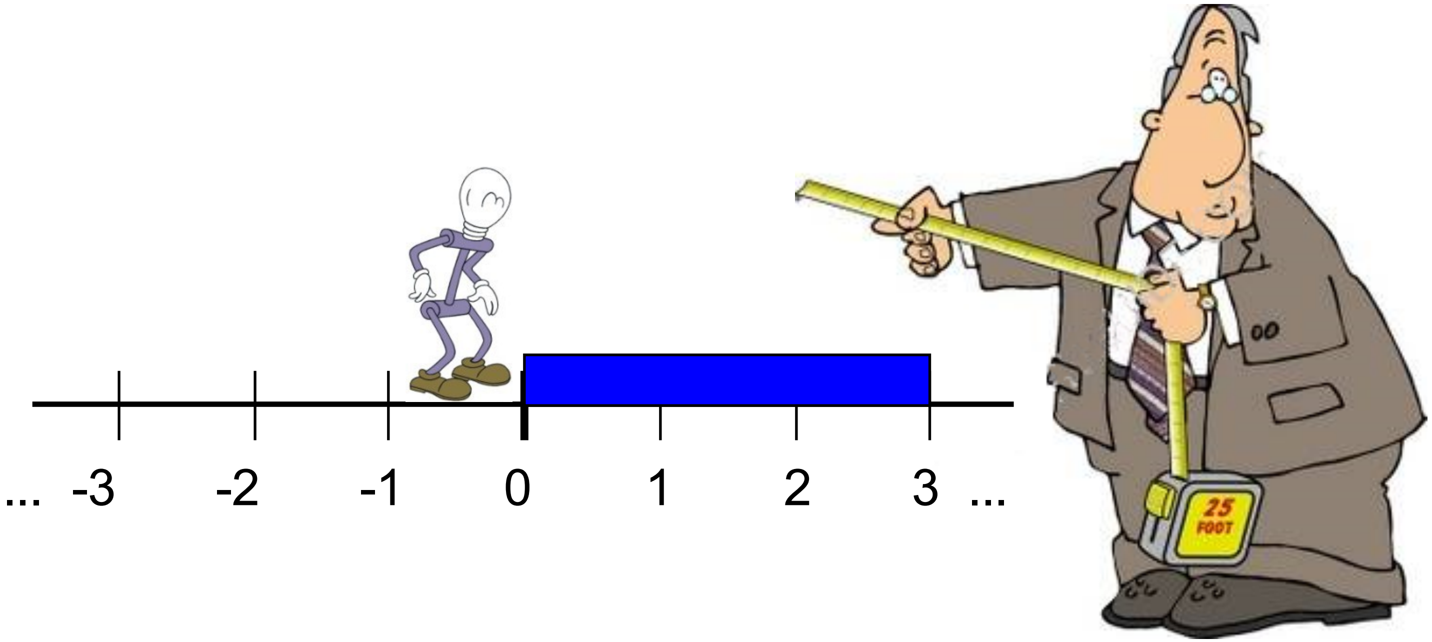
This stick is 3m long. I measure both ends at *the same time* in my frame of reference.

"Same time" or not doesn't actually matter here, because the stick isn't going anywhere.

This length, measured in the stick's rest frame, is its proper length.

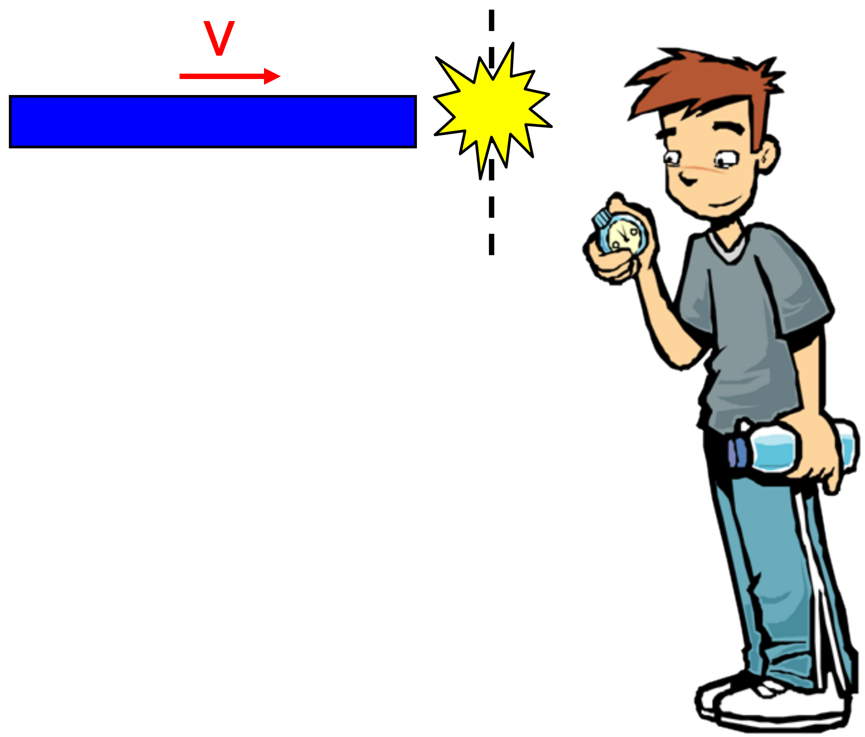
‘Proper length’

Proper length: Length of object measured at rest / object measured in the frame where it is at rest (use a ruler)

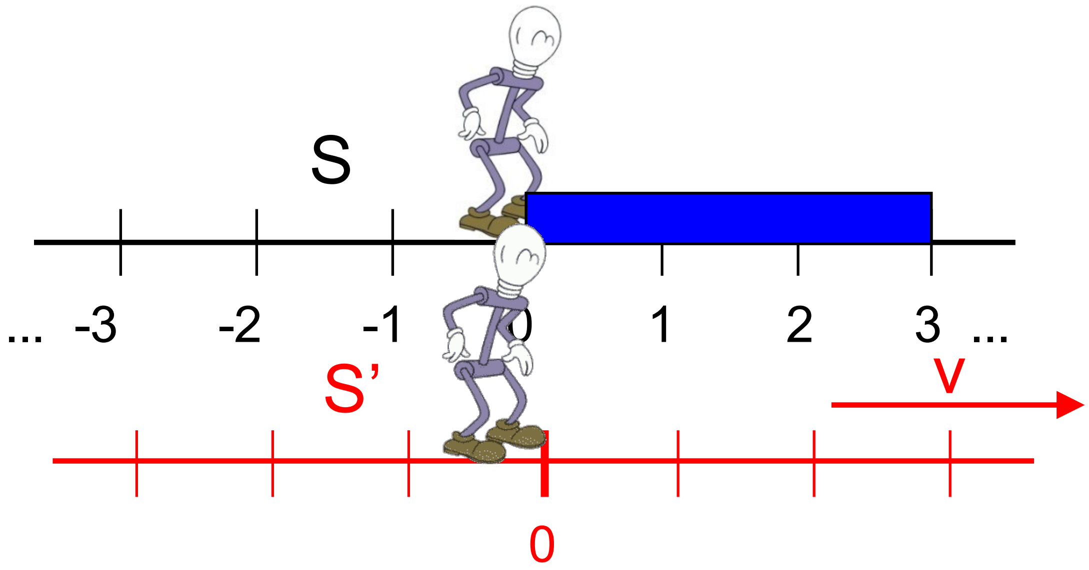


Remember 'proper time'

Proper time: Time interval $\Delta t = t_2 - t_1$ between two events measured in the frame, in which the two events occur at the same spatial coordinate, i.e. time interval that can be **measured with one clock**.



Length of an object

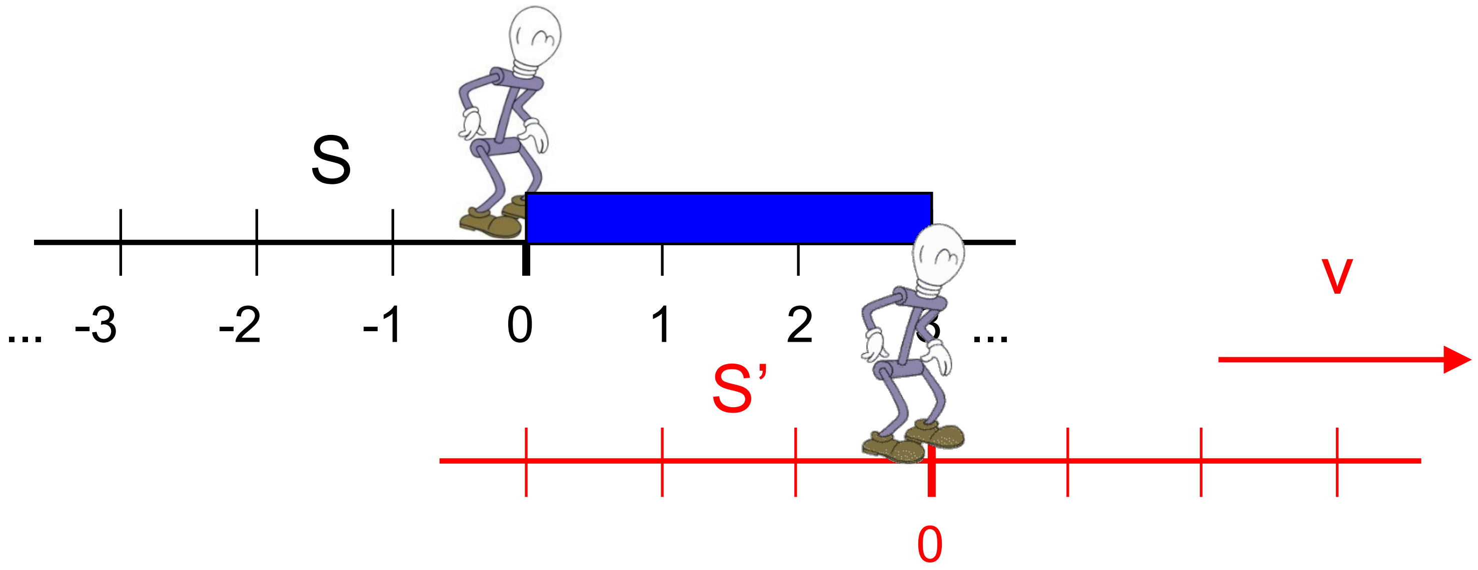


Observer in S measures the proper length L of the blue object.

Another observer comes whizzing by at speed v. This observer measures the length of the stick, *and keeps track of time*.

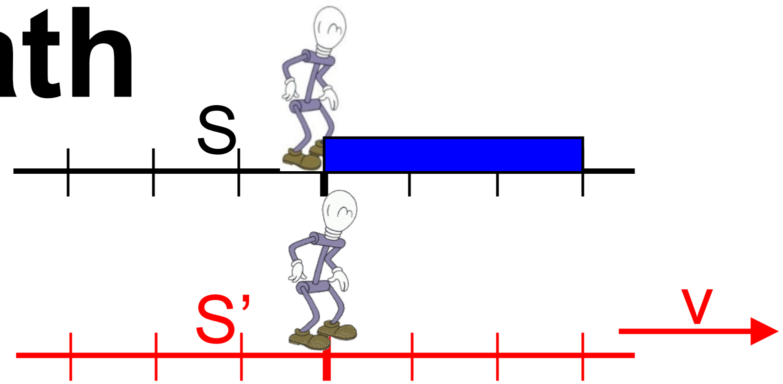
Event 1 – Origin of S' passes left end of stick.

Length of an object



Event 1 – Origin of S' passes left end of stick.
Event 2 – Origin of S' passes right end of stick.

A little math



In frame S: (rest frame of the stick)

length of stick = L (this is the proper length)

time between events = Δt

speed of frame S' is $v = L/\Delta t$

In frame S':

length of stick = L' (this is what we're looking for)

time between events = $\Delta t'$

speed of frame S is $v = L'/\Delta t'$

Follow the proper time!

Q: a) $\Delta t = \Delta t'$ b) $\Delta t = \gamma \Delta t'$ c) $\Delta t' = \gamma \Delta t$

A little math

Speeds are the same (both refer to the relative speed).

And so

$$|v| = \frac{L'}{\Delta t'} = \frac{L}{\Delta t} = \frac{L}{\gamma \Delta t'}$$

$$L' = \frac{L}{\gamma}$$

Length in moving frame

Length in stick's rest frame (proper length)

Length contraction is a consequence of time dilation (and vice-versa).

The Twin Paradox revisited

Quiz on proper time/length

Vicki

Carol

v →

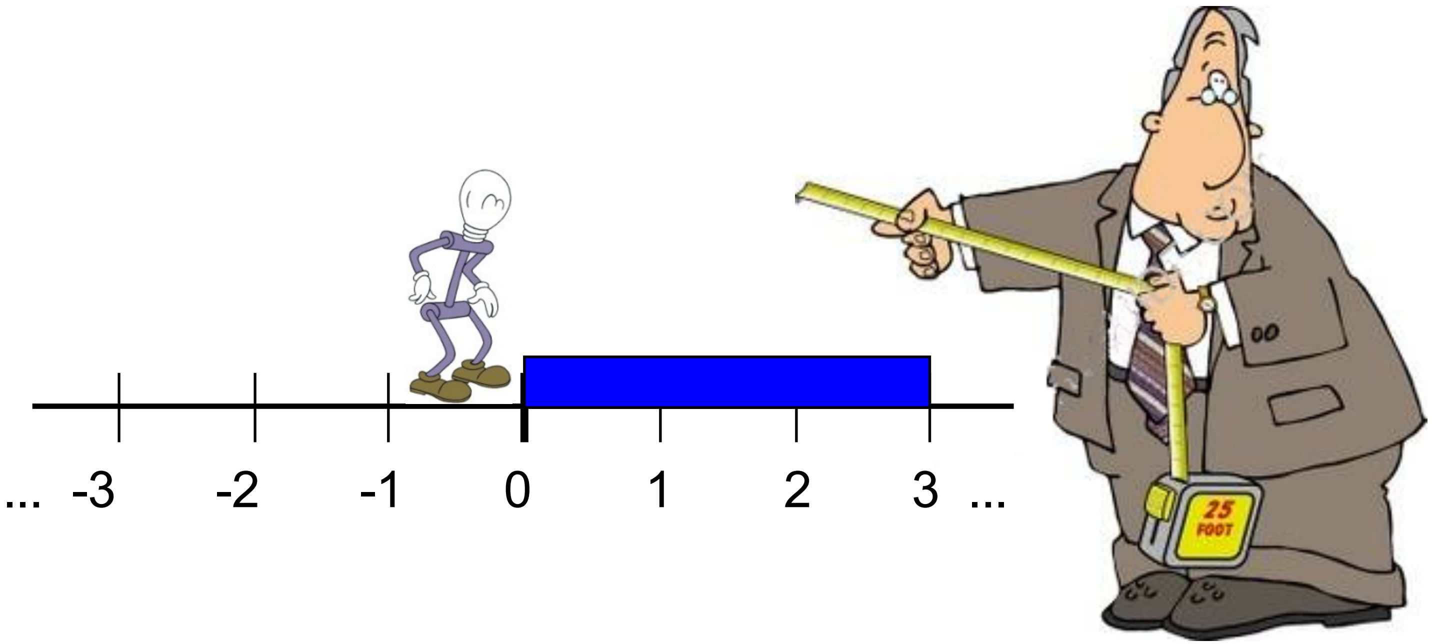
Sirius

Carol travels from the Earth to Sirius. Which of the following statements is correct? (Assume that Earth and Sirius are not moving relative to each other)

A – Vicky measures proper time and proper length of the journey.
B – Carol measures proper time and proper length of the journey.
C – Vicky measures proper time and Carol measures proper length of the journey.
D – Carol measures proper time and Vicky measures proper length of the journey.
E – none of the above

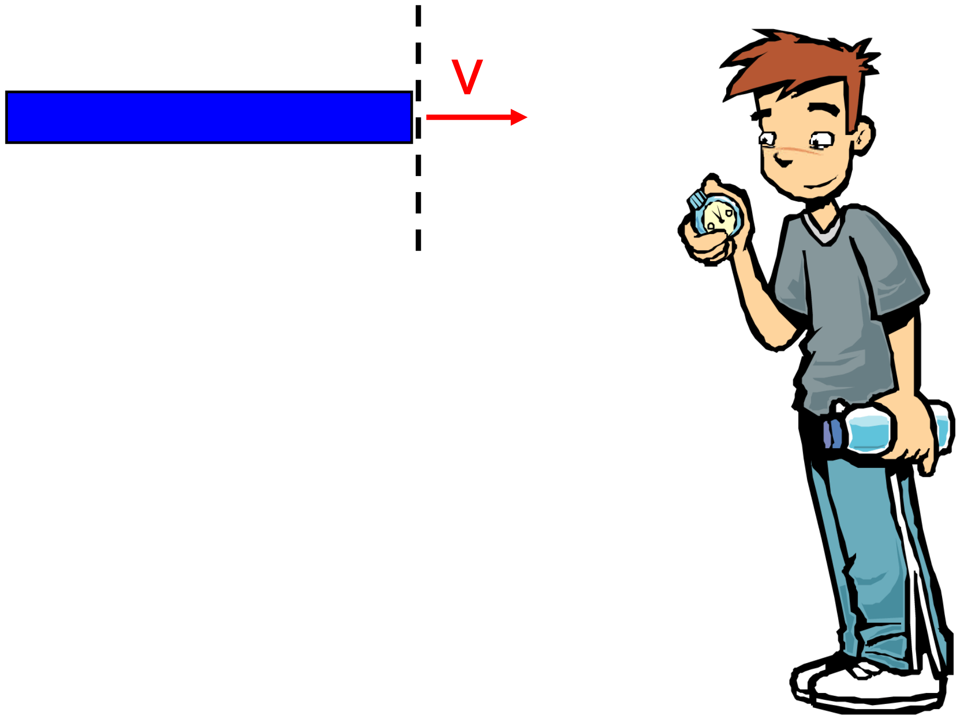
Review: Proper length

Proper length: Length of object measured in the frame, where it is at rest (use a ruler)



Review: Proper length

Proper time: Time interval $\Delta t = t_2 - t_1$ between two events measured in the frame, in which the two events occur at the same spatial coordinate, i.e. time interval that can be **measured with one clock**.



Proper time & proper length

Vicki

v →

Carol

Sirius

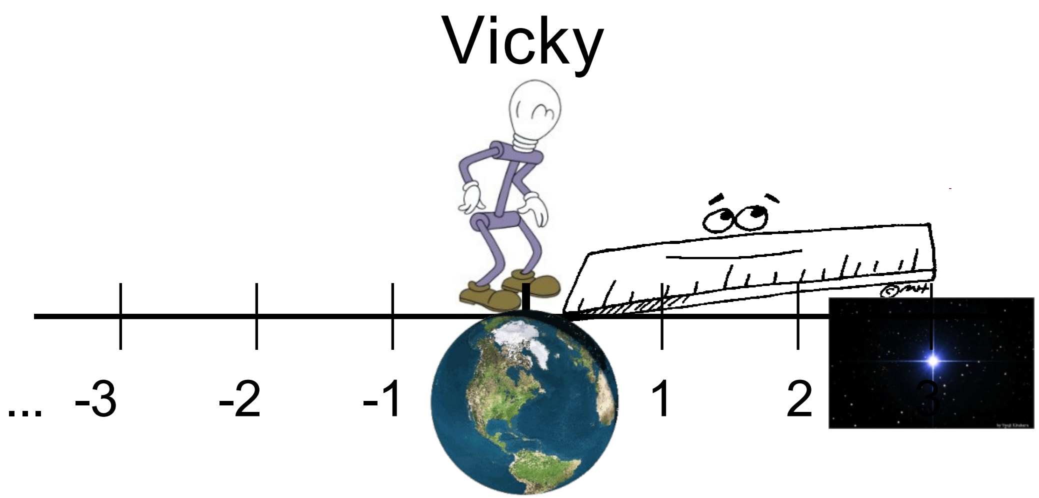
Now we know the following about this journey:

- Vicki measures the proper length: 8 light-years.
- Carol measures the proper time: 6 years.
- Both agree that Carol travels at a speed of v=0.8c relative to the earth.

From Carol's perspective:
Carol finds that she traveled only 6y*0.8c = 4.8 ly. But why does she find herself at Sirius after 6 years??

→ Length contraction!!

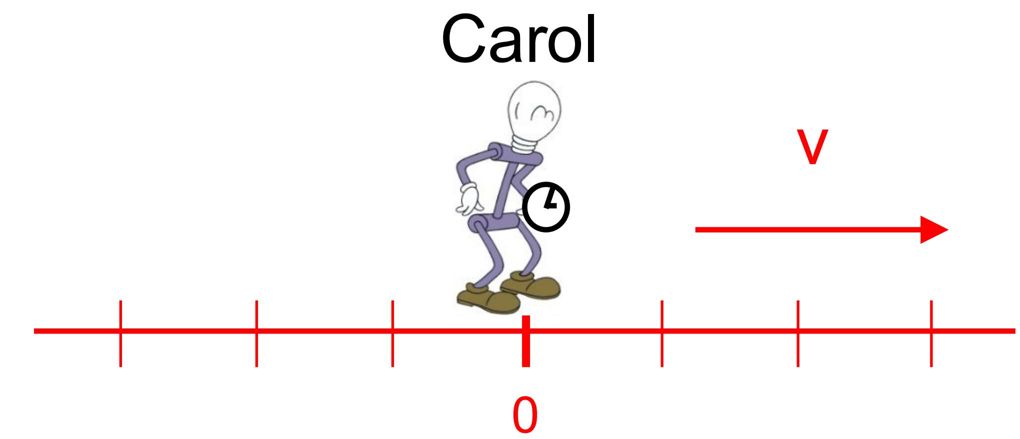
Length contraction vs. time dilation



Vicky measures:

Proper length: $L_{Vicky} = 8$ ly

Time: $\Delta t_{Vicky} = \gamma\, \Delta t_{Carol} = 10$y



Carol measures:

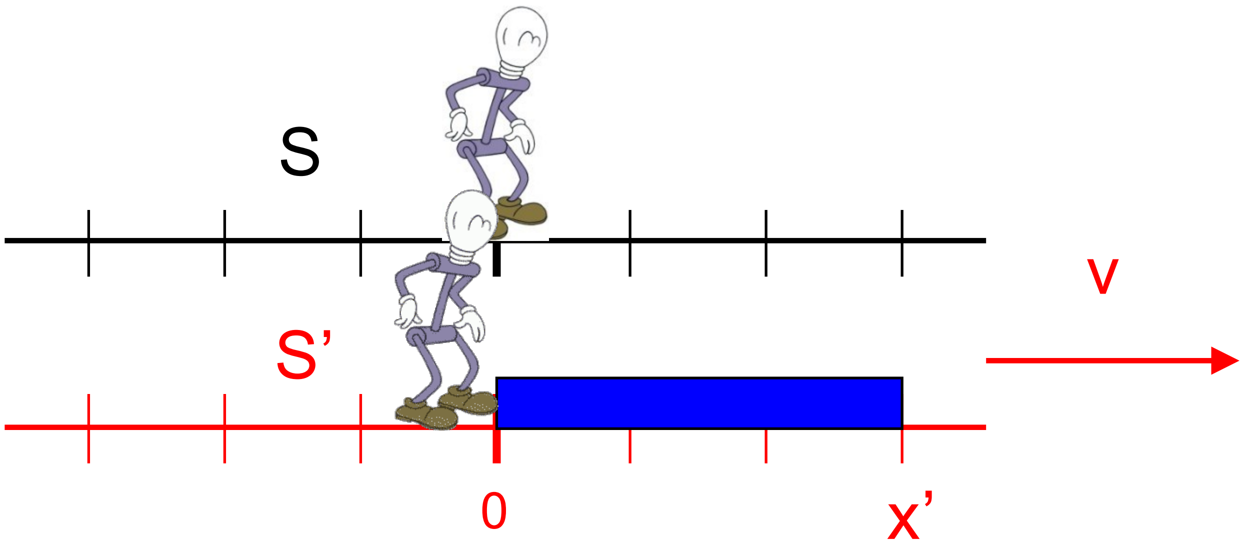
Length: $L_{Carol} = L_{Vicky}/\gamma = 4.8$ ly

Proper time: $\Delta t_{Carol} = 6$ y

Lorentz transformation

(Relativistic version of Galileo transformation)

The Lorentz transformation

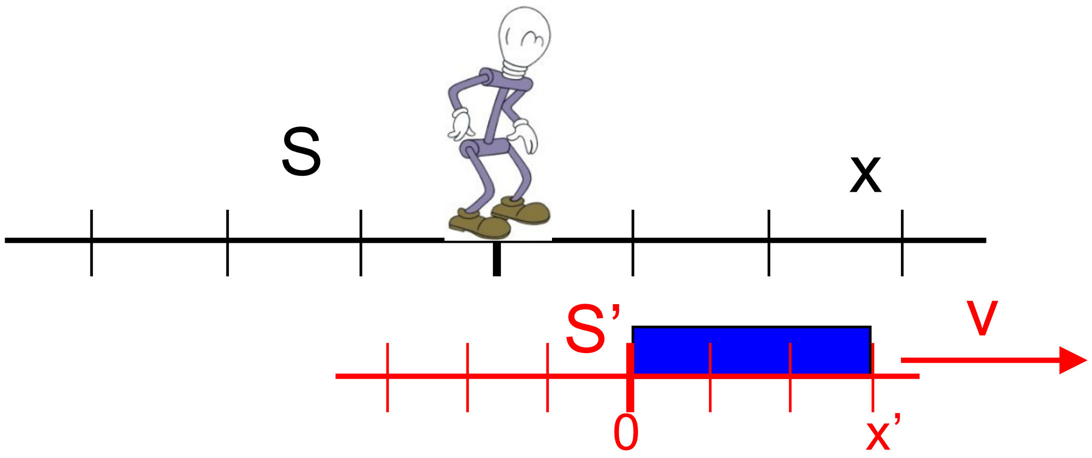


A stick is at rest in S'. Its endpoints are the events (x,ct) = (0,0) and (x',0) in S'. S' is moving to the right with respect to frame S.

Event 1 – left of stick passes origin of S. Its coordinates are (0,0) in S and (0,0) in S'.

Lorentz transformation

An observer at rest in frame S sees a stick flying past him with velocity v:

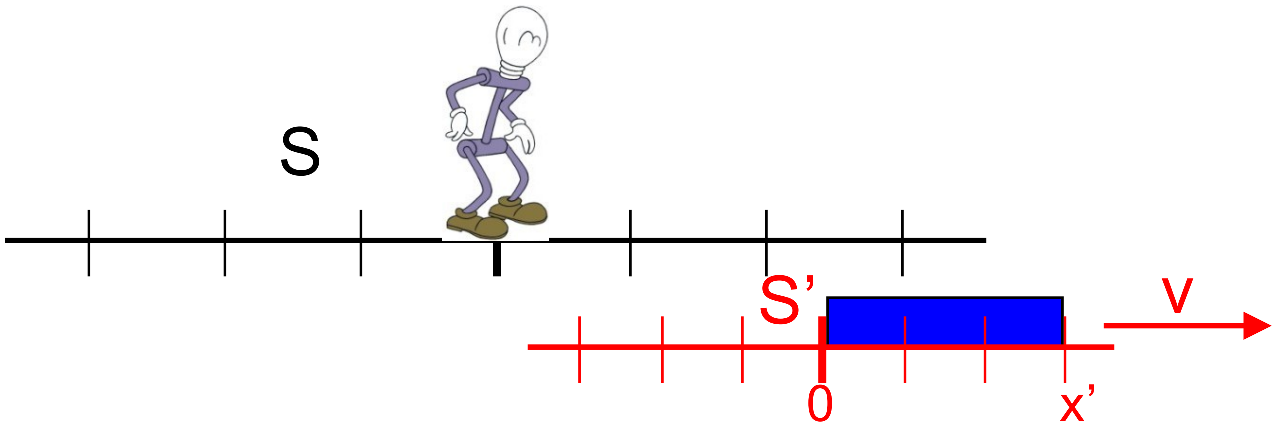


As viewed from S, the stick's length is x'/γ. Time t passes. According to S, where is the *right* end of the stick? (Assume the *left* end of the stick was at the origin of S at time t=0.)

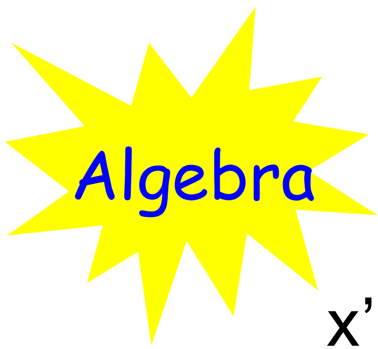
a) $x = \gamma vt$ b) $x = vt + x'/\gamma$ c) $x = -vt + x'/\gamma$

d) $x = vt - x'/\gamma$ e) something else

The Lorentz transformation



x = vt + x'/γ. This relates the spatial coordinates of an event in one frame to its coordinates in the other.



$x' = \gamma(x - vt)$

Transformations

If S' is moving with speed v in the positive x direction relative to S, then the coordinates of the same event in the two frames are related by:

Galilean transformation (classical)

$$x' = x - vt$$

$$y' = y$$

$$z' = z$$

$$t' = t$$

Lorentz transformation (relativistic)

$$x' = \gamma(x - vt)$$

$$y' = y$$

$$z' = z$$

$$t' = \gamma(t - \frac{v}{c^2}x)$$

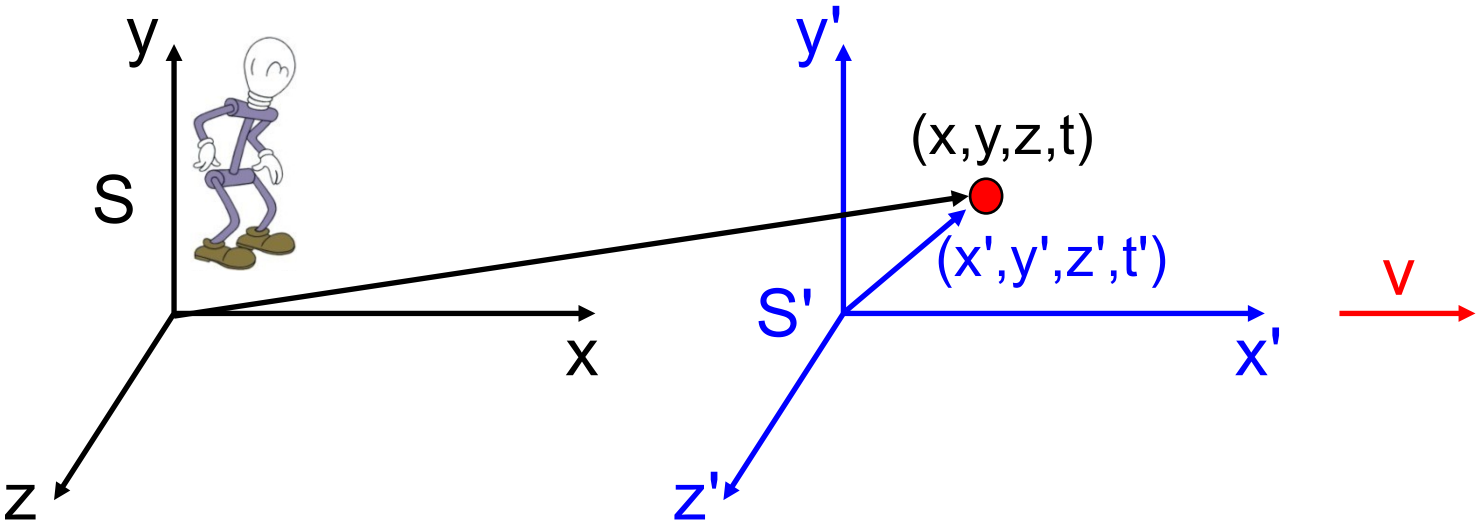
See homework #3

Note: This assumes (0,0,0,0) is the same event in both frames.

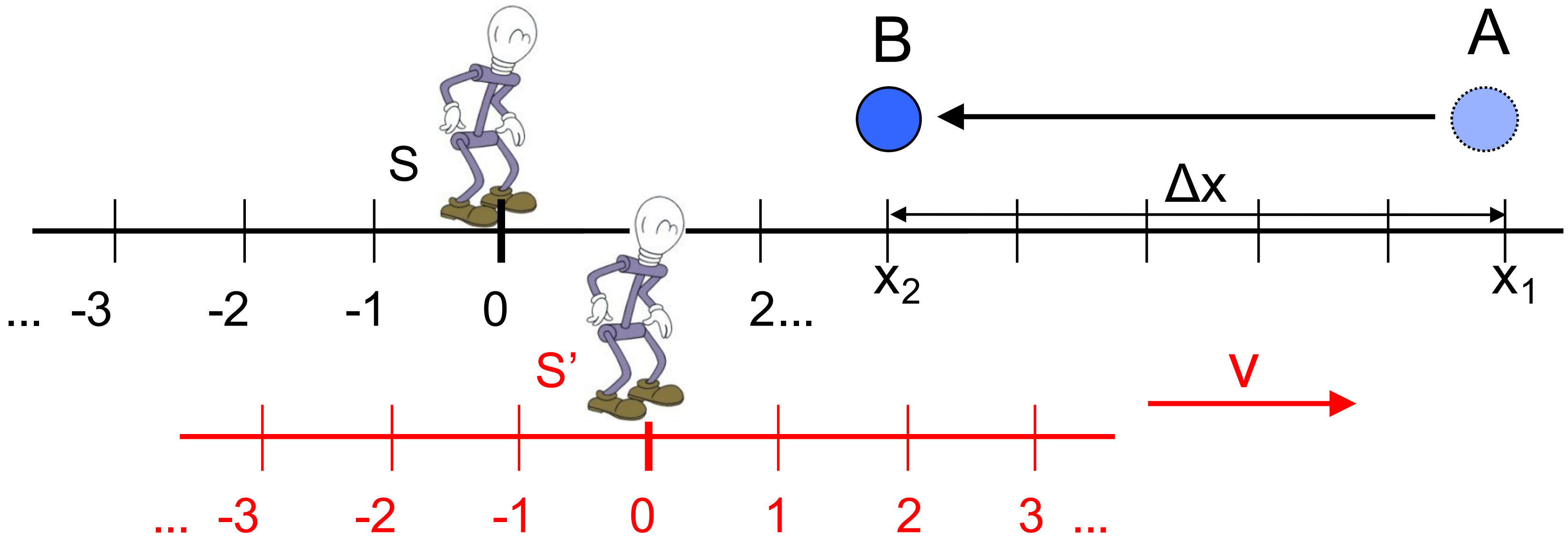
A note of caution:

The way the Lorentz and Galileo transformations are presented here (and in the textbook) assumes the following:

An observer in S would like to express an event (x,y,z,t) (in his frame S) with the coordinates of the frame S', i.e. he wants to find the corresponding event (x',y',z',t') in S'. The frame S' is moving along the x-axes of the frame S with the velocity v (measured relative to S) and we assume that the origins of both frames overlap at the time t=0.



Velocity transformation (1D)



An object moves from event A=(x_1,t_1) to event B =(x_2,t_2).

As seen from S, its speed is $u = \frac{\Delta x}{\Delta t}$ with: $\Delta x = x_1 - x_2$, $\Delta t = t_1 - t_2$

As seen from S', its speed is $u' = \frac{\Delta x'}{\Delta t'}$ with: $\Delta x' = x'_1 - x'_2$, $\Delta t' = t'_1 - t'_2$

Velocity transformation (1D)

$$u = \frac{\Delta x}{\Delta t}\ , \qquad u' = \frac{\Delta x'}{\Delta t'}$$

, where $\Delta x = x_1 - x_2$, $\Delta x' = x'_1 - x'_2$

Use Lorentz: x' = γ(x-vt)
:
:
:

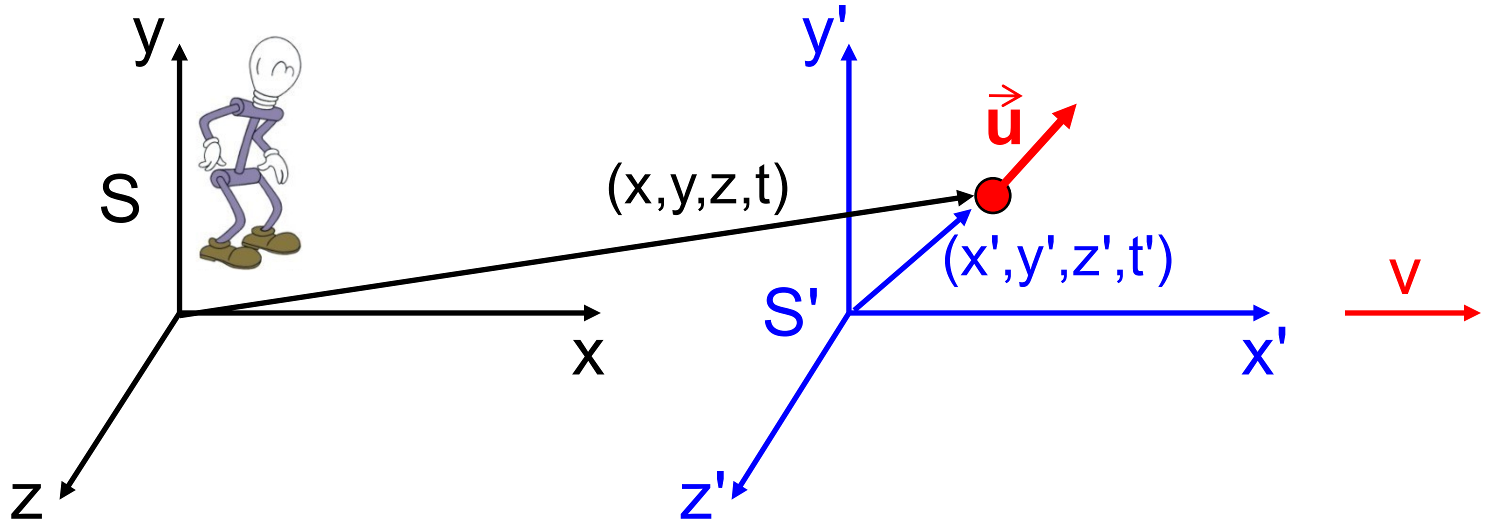
$$u' = \frac{\Delta x'}{\Delta t'} = \frac{\gamma(\Delta x - v\Delta t)}{\gamma(\Delta t - (v/c^2)\Delta x)}$$

$$u' = \frac{u - v}{1 - uv/c^2}$$

Galilean result

New in special relativity

Velocity transformation in 3D



In a more general case we want to transform a velocity $\vec{\mathbf{u}}$ (measured in frame S) to $\vec{\mathbf{u}}'$ in frame S'. Note that $\vec{\mathbf{u}}$ can point in any arbitrary direction, but <u>v still points along the x-axes</u>.

Velocity transformation (3D)

The velocity $\vec{\mathbf{u}}=(u_x, u_y, u_z)$ measured in S is given by:

$u_x = {}^{\Delta x}/_{\Delta t}$, $u_y = {}^{\Delta y}/_{\Delta t}$, $u_z = {}^{\Delta z}/_{\Delta t}$, where $\Delta x = x_1 - x_2$ …

To find the corresponding velocity components u'_x, u'_y, u'_z in the frame S', which is moving along the x-axes in S with the velocity v, we use again the Lorentz transformation:

$x'_1 = \gamma(x_1 - vt_1),$ and so on…

$t'_1 = \gamma(t_1 - vx_1/c^2),$ and so on…

Algebra

Velocity transformation (3D)

(aka. "Velocity-Addition formula")

$$u'_x = \frac{u_x - v}{1 - u_x v / c^2}$$

$$u'_y = \frac{u_y}{\gamma\left(1 - u_x v / c^2\right)}$$

$$u'_z = \frac{u_z}{\gamma\left(1 - u_x v / c^2\right)}$$

Some examples

Relativistic transformations

$$x' = \gamma(x - vt)$$

$$t' = \gamma(t - \frac{v}{c^2}x)$$

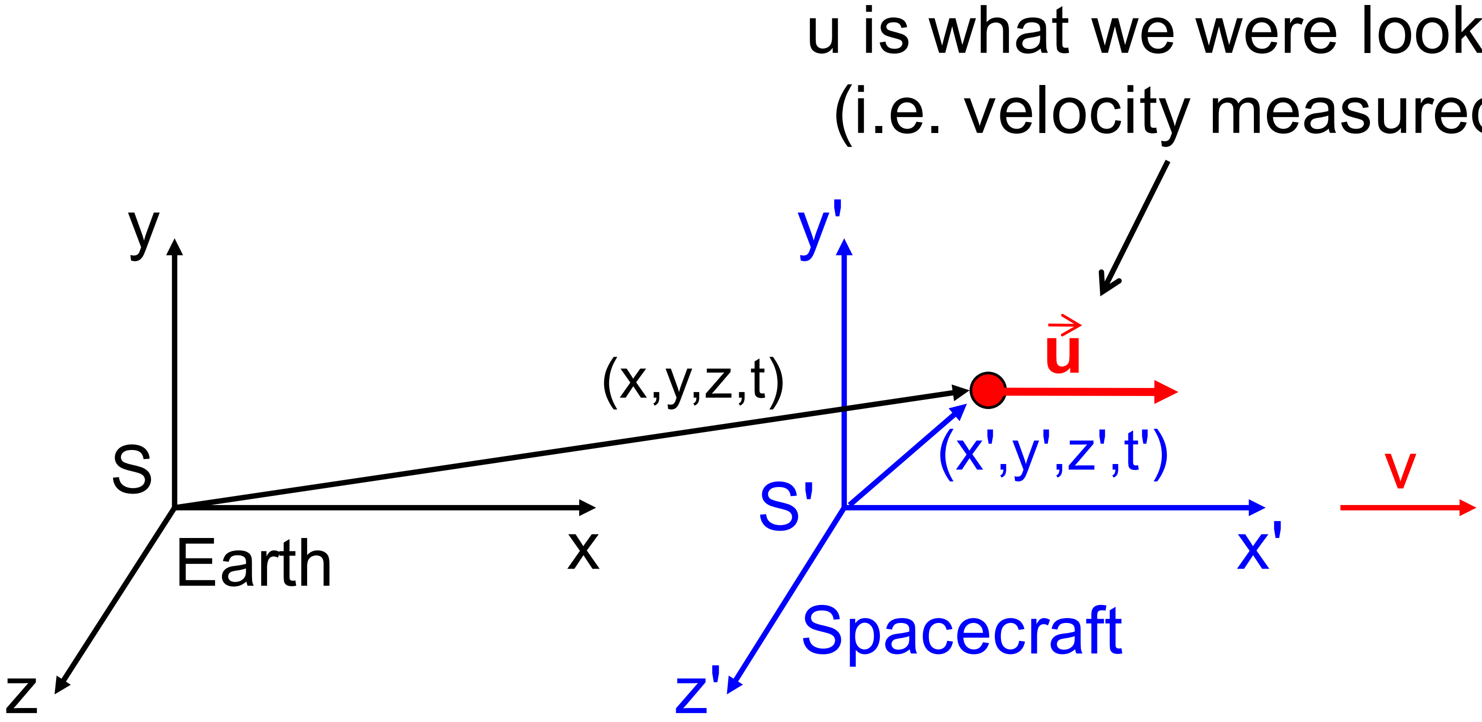
$$u' = \frac{u - v}{1 - uv/c^2}$$

$$u = \frac{u' + v}{1 + u'v/c^2}$$

Suppose a spacecraft travels at speed v=0.5c relative to the Earth. It launches a missile at speed 0.5c relative to the spacecraft in its direction of motion. How fast is the missile moving relative to Earth? (Hint: Remember which coordinates are the primed ones. And: Does your answer make sense?)

a) 0.8 c b) 0.5 c c) c d) 0.25 c e) 0

Velocity transformation: Which coordinates are primed?

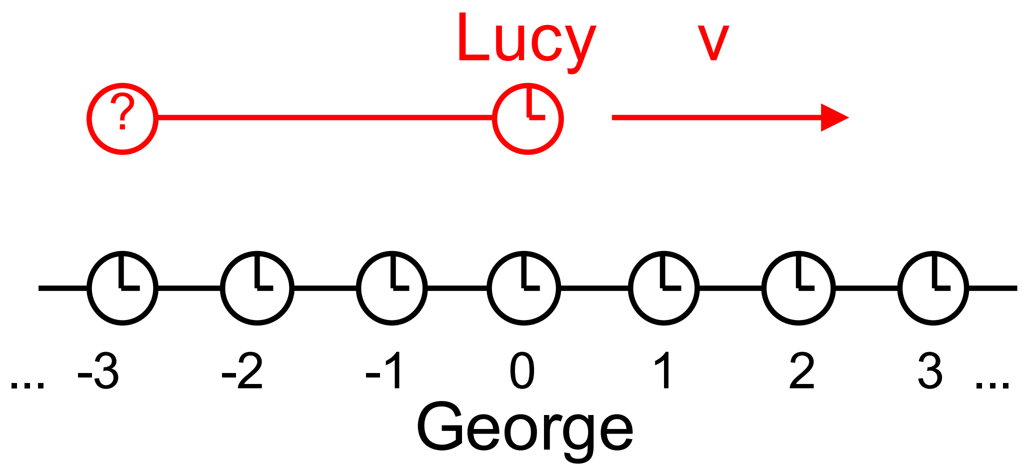


$$u = \frac{u' + v}{1 + u'v / c^2} \qquad u' = \frac{u - v}{1 - uv / c^2}$$

The "object" could be light, too!

Suppose a spacecraft travels at speed v=0.5c relative to the Earth. It shoots a beam of light out in its direction of motion. How fast is the light moving relative to the Earth? (Get your answer using the formula).

a) 1.5c b) 0.5 c c) c d) d e) e

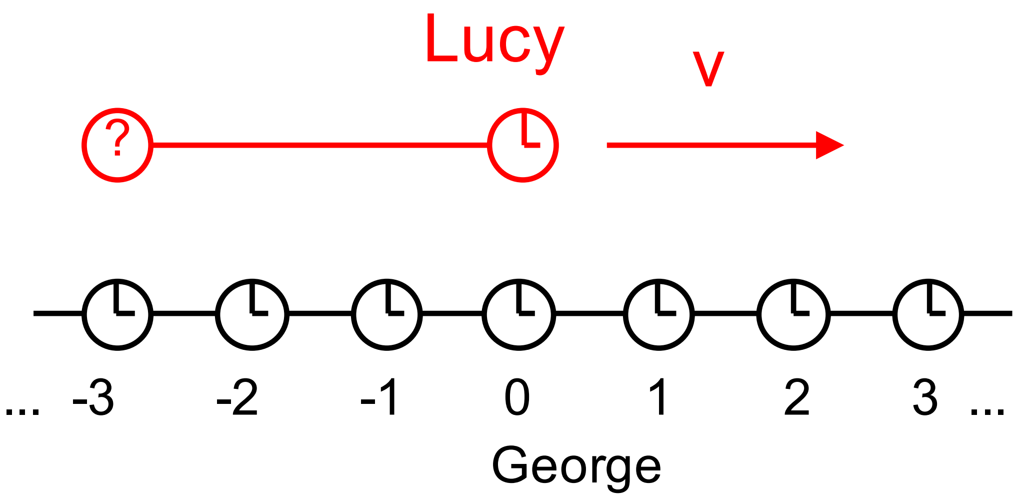


$$x' = \gamma(x - vt)$$

$$t' = \gamma(t - \frac{v}{c^2}x)$$

George has a set of synchronized clocks in reference frame S, as shown. Lucy is moving to the right past George, and has (naturally) her own set of synchronized clocks. Lucy passes George at the event (0,0) in both frames. An observer in George's frame checks the clock marked '?'. Compared to George's clocks, this one reads

A) a slightly earlier time B) a slightly later time C) same time



$$x' = \gamma(x - vt)$$

$$t' = \gamma(t - \frac{v}{c^2}x)$$

The event has coordinates (x = -3, t = 0) for George.

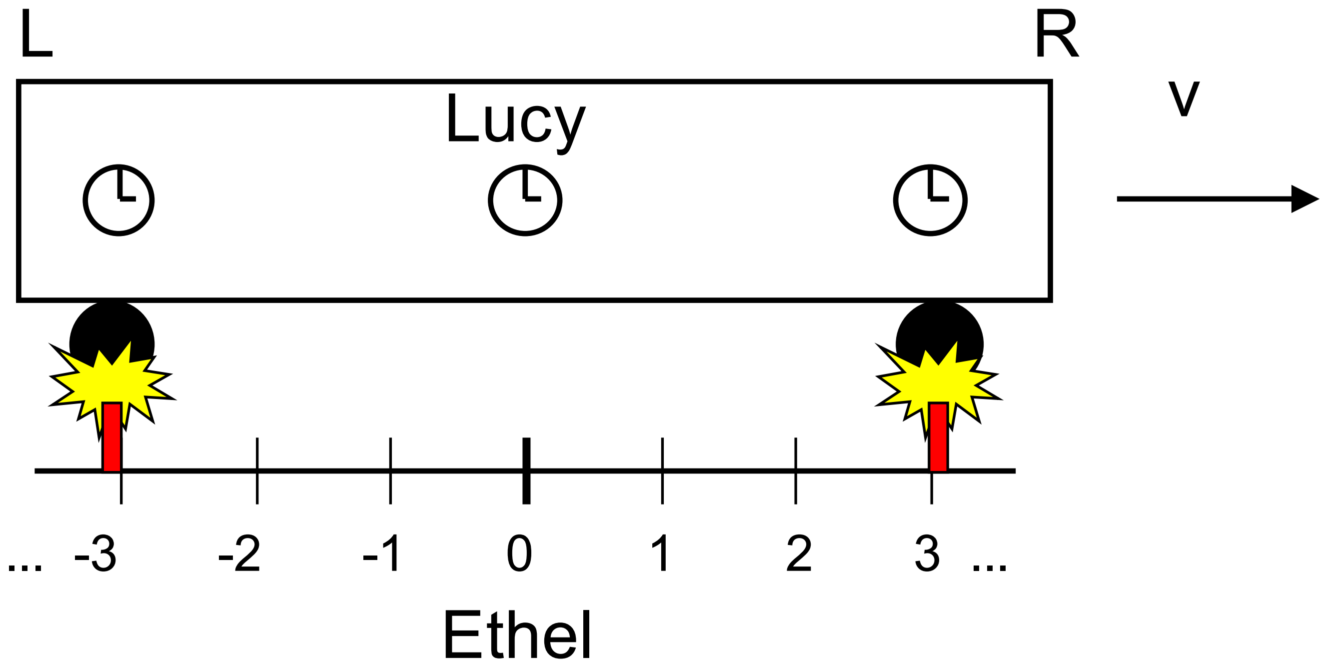
In Lucy's frame, where the ? clock is, the time t' is

$$t' = \gamma(0 - \frac{v}{c^2}(-3)) = \frac{3\gamma v}{c^2}$$

, a positive quantity.

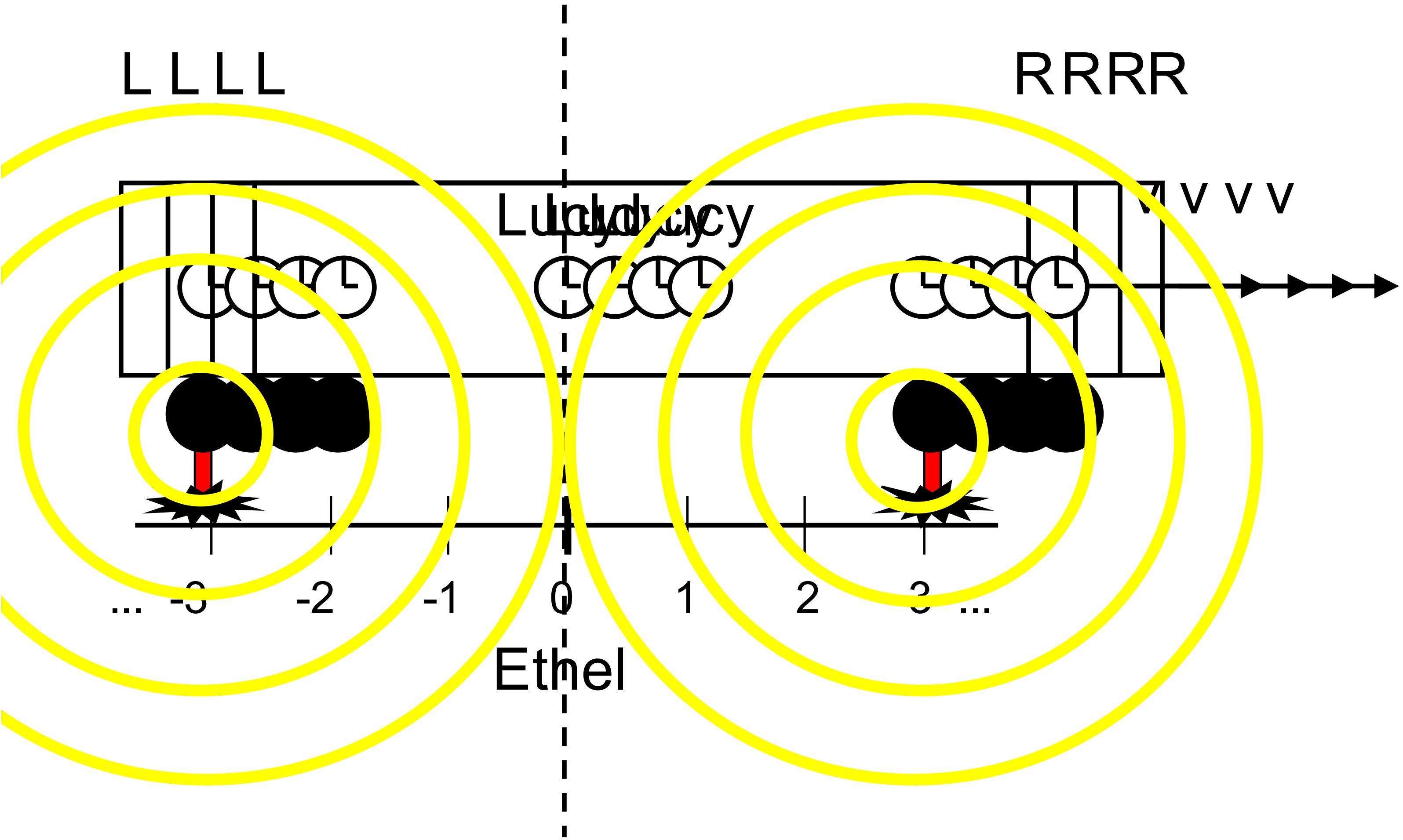
'?' = slightly later time

Remember Ethel and Lucy?



At 3 o'clock in Ethel's frame, two firecrackers go off to announce the time. It so happens that these firecrackers are at the left and right ends of the train, in Ethel's frame.

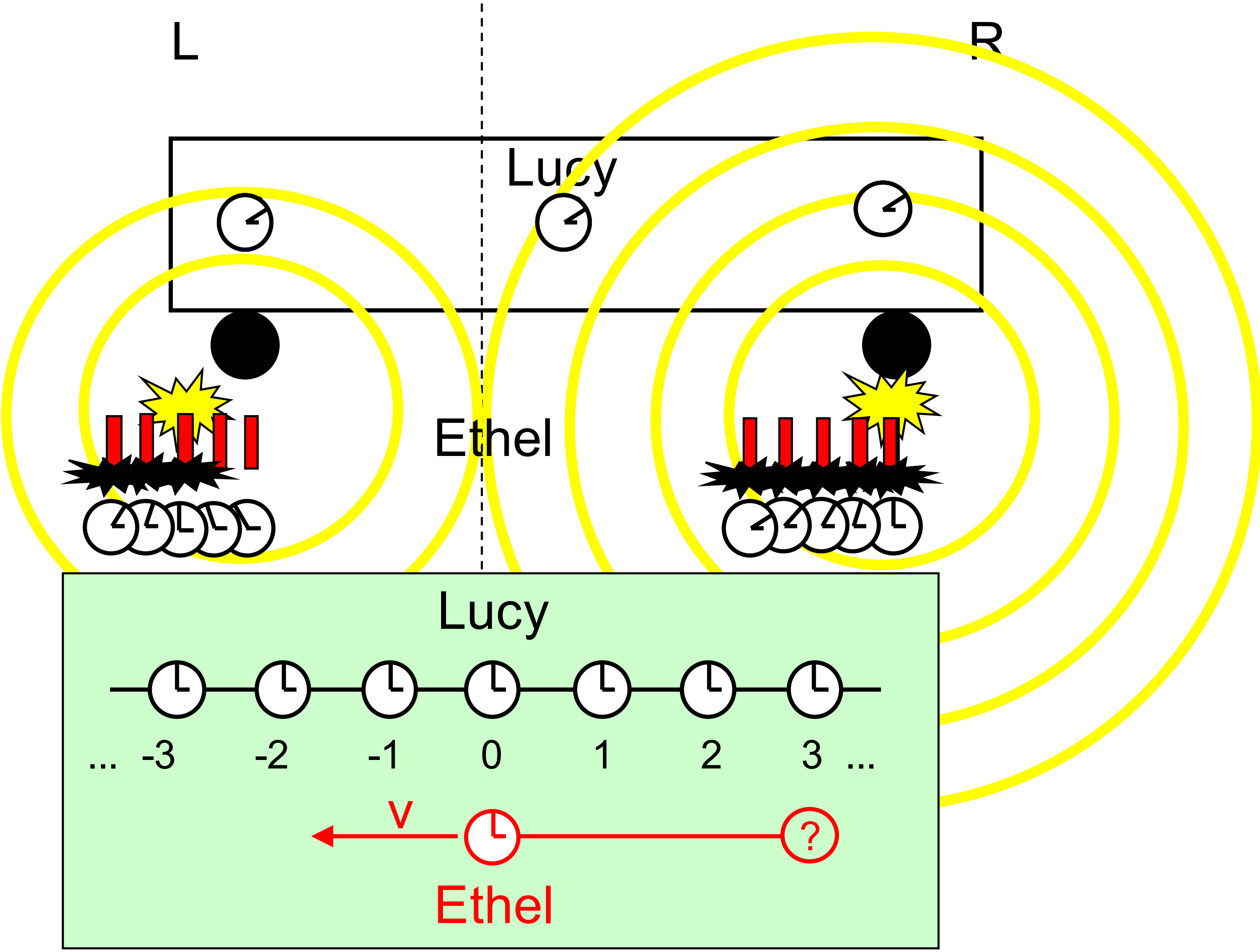
Event 1: firecracker 1 explodes at 3:00
Event 2: firecracker 2 explodes at 3:00

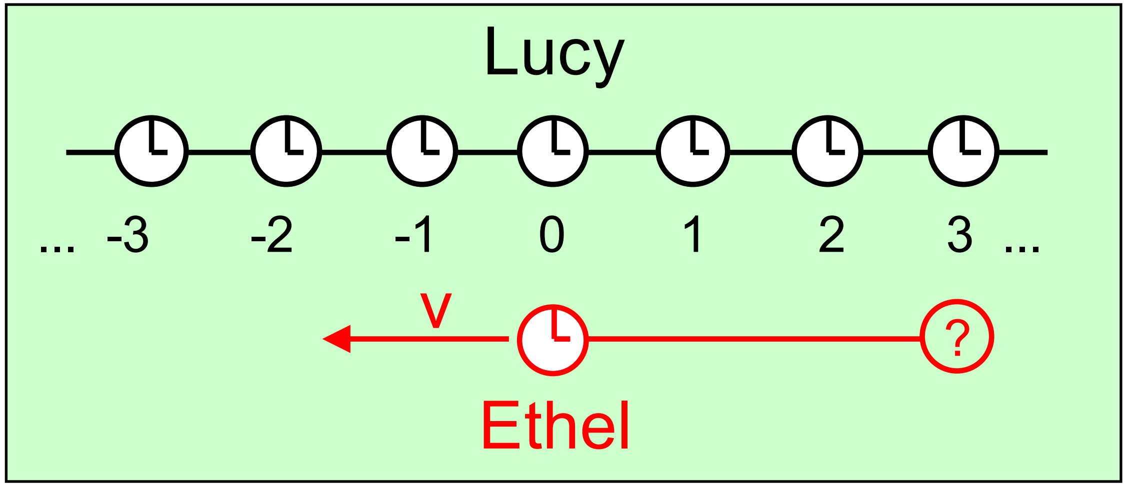


Sometime later, the wavefronts meet. The meeting point is halfway between the firecrackers in Ethel's frame, but is somewhere in the left of the train car, in Lucy's frame.

Event 3: two light pulses meet, shortly after 3:00.

Do you remember Lucy?





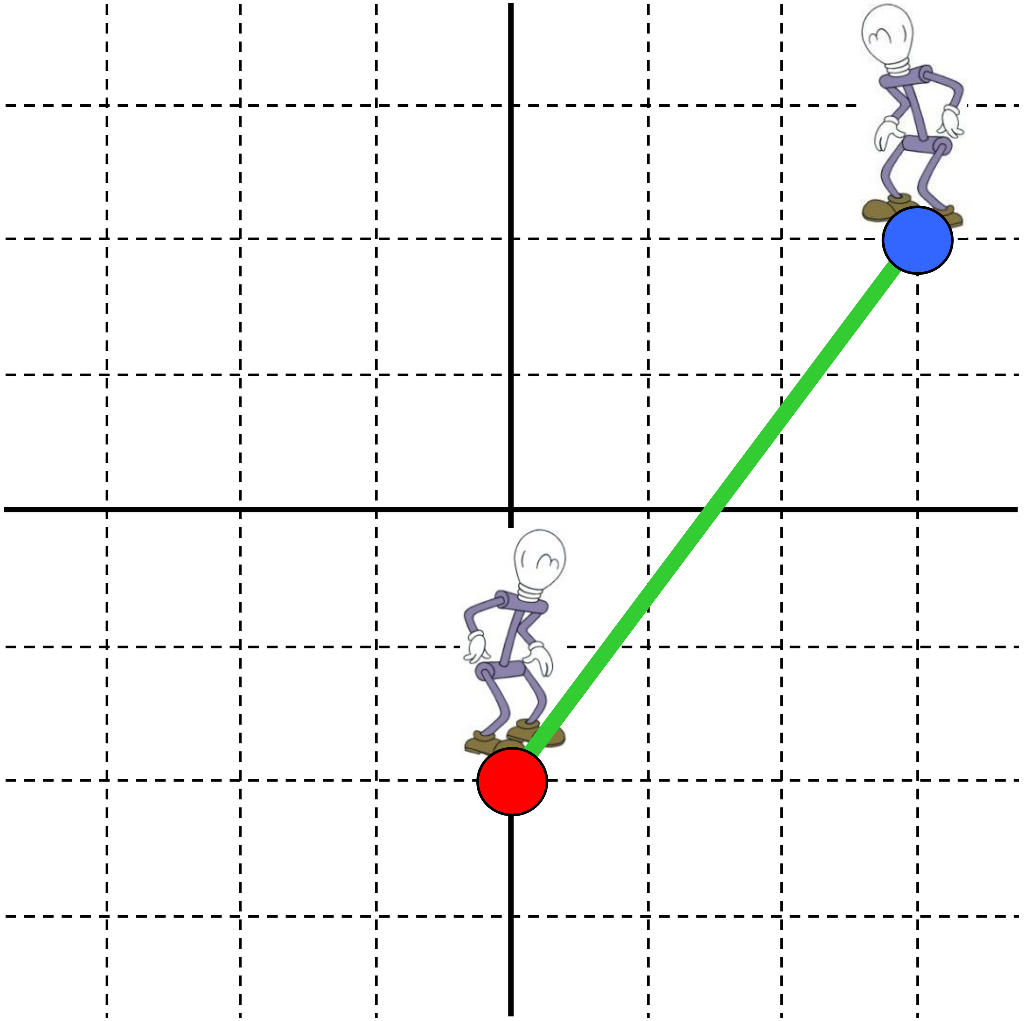
We found (in an earlier clicker question) that according to Lucy:

Ethel's clock on the right reads a later time than Ethel's clock on the left.

→ This problem can be solved with the Lorentz transformation (no need to draw trains every time…)

Spacetime interval

Remember this? (from 1st class)



The distance between the blue and the red ball is:

$$\sqrt{(3m)^2+(4m)^2}=\sqrt{25m}=5m$$

If the two balls are not moving relative to each other, we found that the distance between them was "**invariant**" under Galileo transformations...

…but not under Lorentz transformations! (Length contraction.) → need new definition for distance?

Spacetime interval

Say we have two events: (x_1,y_1,z_1,t_1) and (x_2,y_2,z_2,t_2).
Define the spacetime interval (sort of the "distance") **between two events** as:

$$(\Delta s)^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$$

Spacetime interval

With: $\Delta x = x_1 - x_2$

$$\Delta y = y_1 - y_2$$

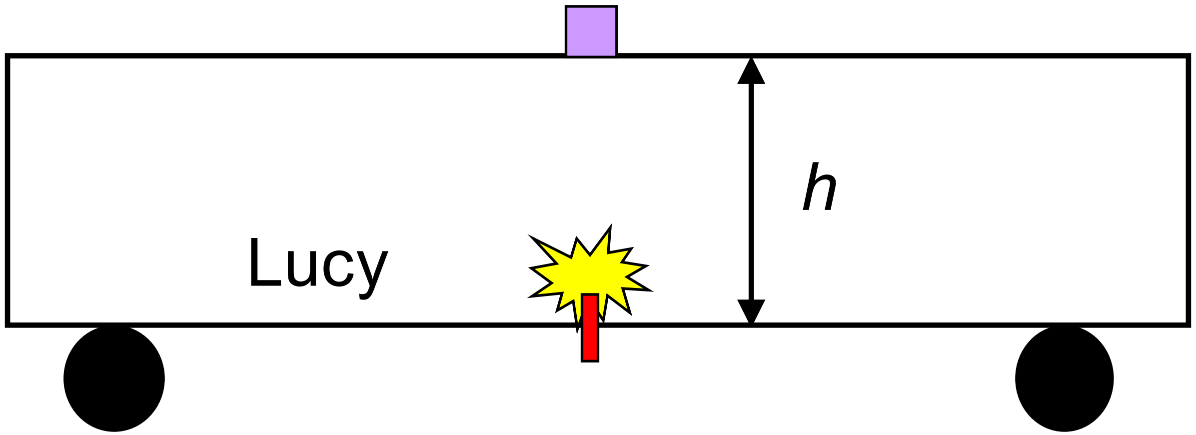
$$\Delta z = z_1 - z_2$$

$$\Delta t = t_1 - t_2$$

The spacetime interval has the same value in all inertial reference frames! I.e. Δs^2 is "invariant" under Lorentz transformations.

(Homework #3!)

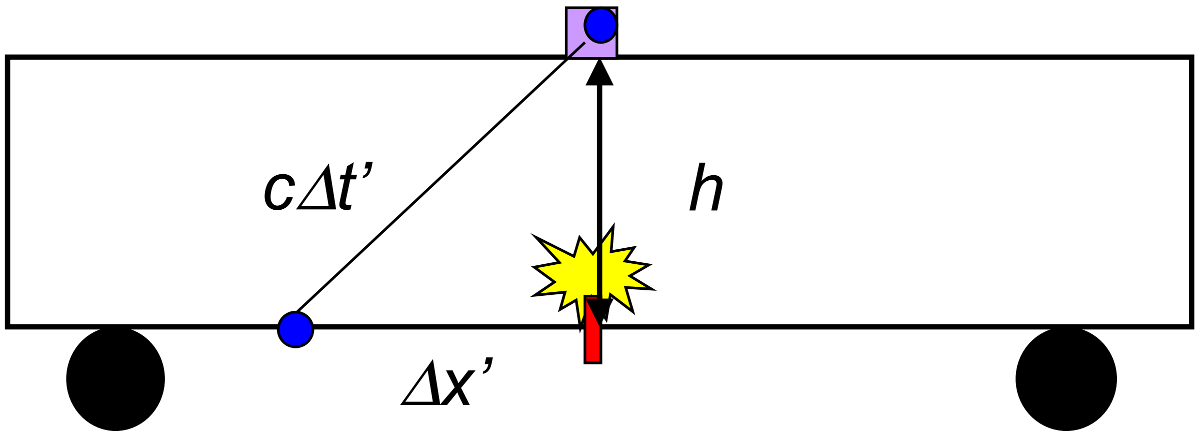
Remember Lucy?



Event 1 – firecracker explodes
Event 2 – light reaches detector
Geometrical distance between events is h.
Time between the two events is Δt.
And we know that:

$$h = c\Delta t \quad \text{or:} \quad 0 = (c\Delta t)^2 - h^2$$

Remember Ethel?



Event 1 – firecracker explodes
Event 2 – light reaches detector
Geometrical distance between events is $c\Delta t'$
Distance between x-coordinates is $\Delta x'$
and: $(c\Delta t')^2 = (\Delta x')^2 + h^2$

We can write:

$$0 = (c\Delta t')^2 - (\Delta x')^2 - (h)^2$$

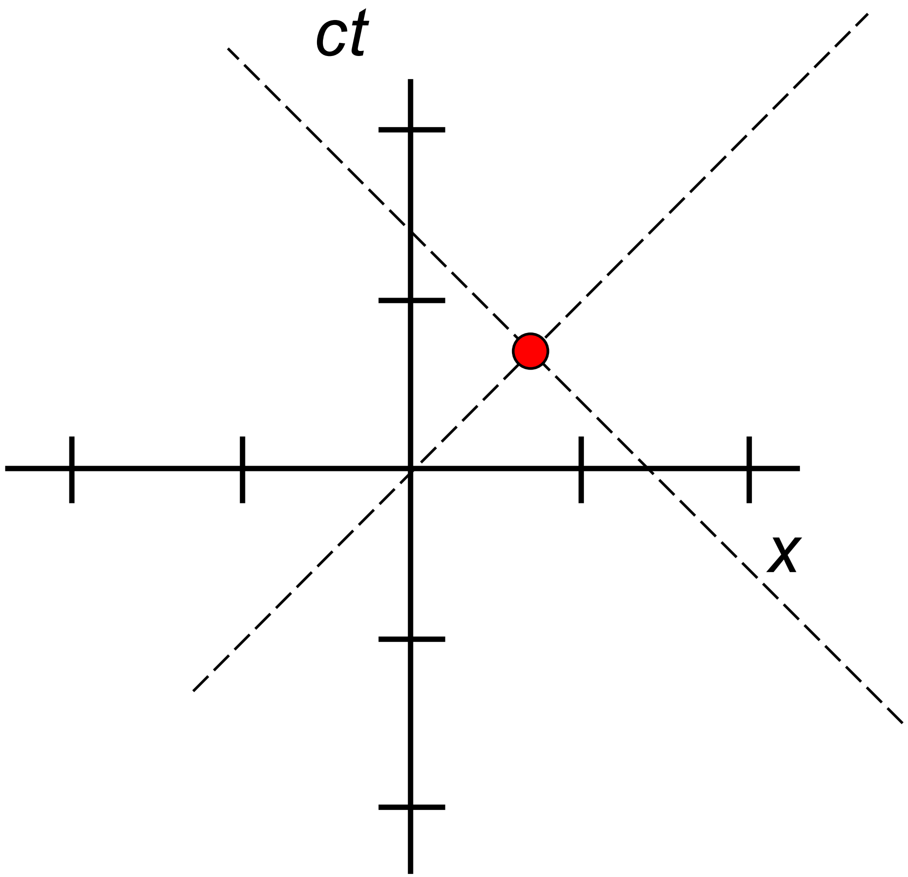
And Lucy got

$$0 = (c\Delta t)^2 - (\Delta x)^2 - (h)^2$$

since $\Delta x = 0$

or: $0 = (\Delta s)^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$ $, with: \Delta y = h, \Delta z = 0$

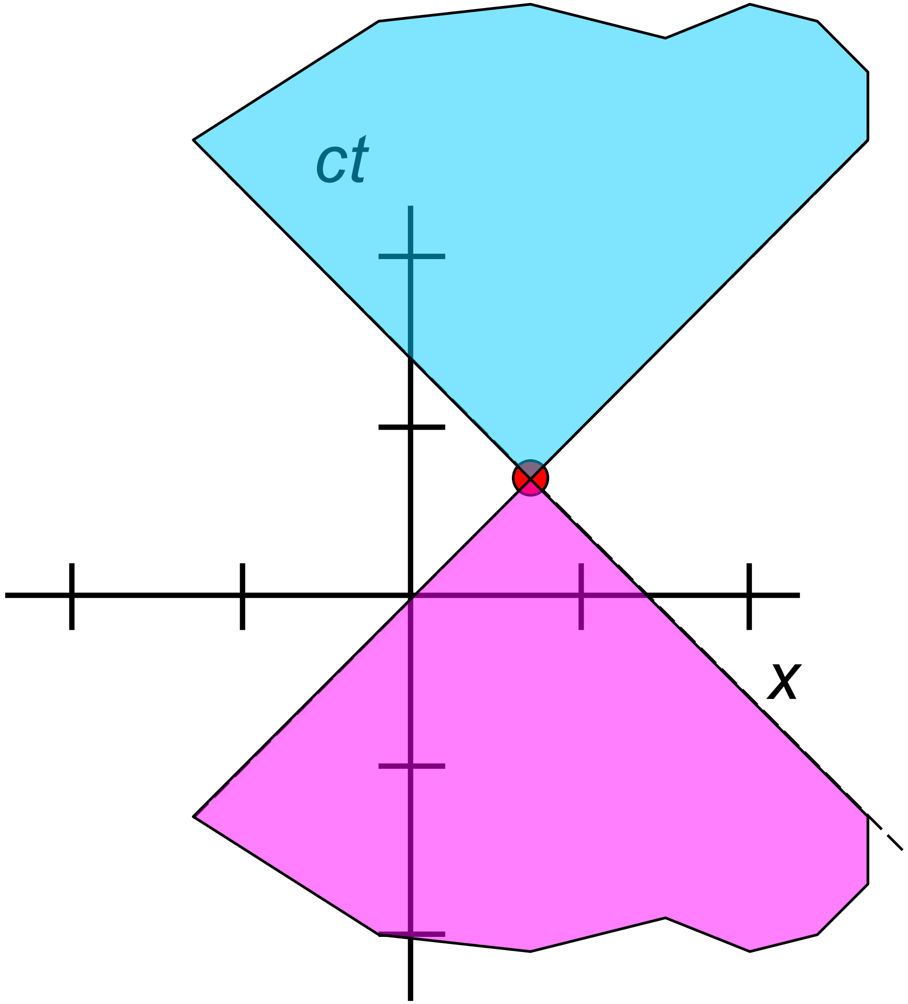
Spacetime



Here is an event in spacetime.

Any light signal that passes through this event has the dashed world lines. These identify the '*light cone*' of this event.

Spacetime

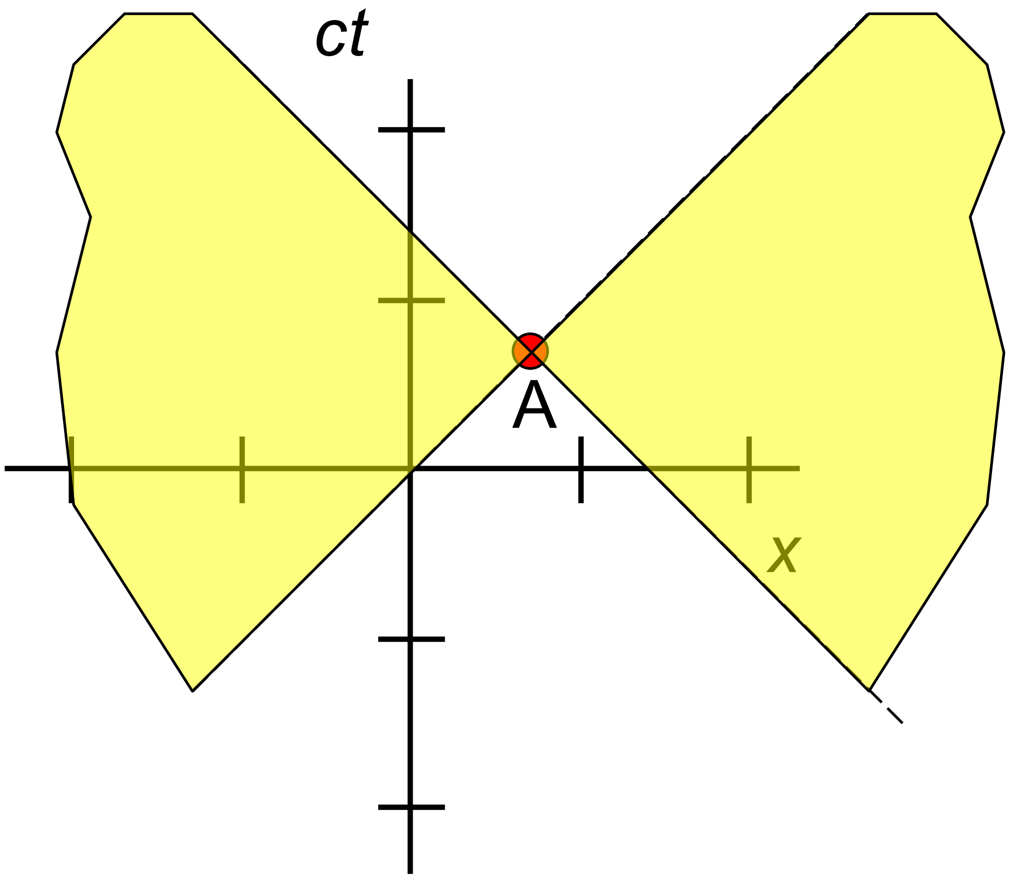


Here is an event in spacetime.

The blue area is the *future* on this event.

The pink is its *past*.

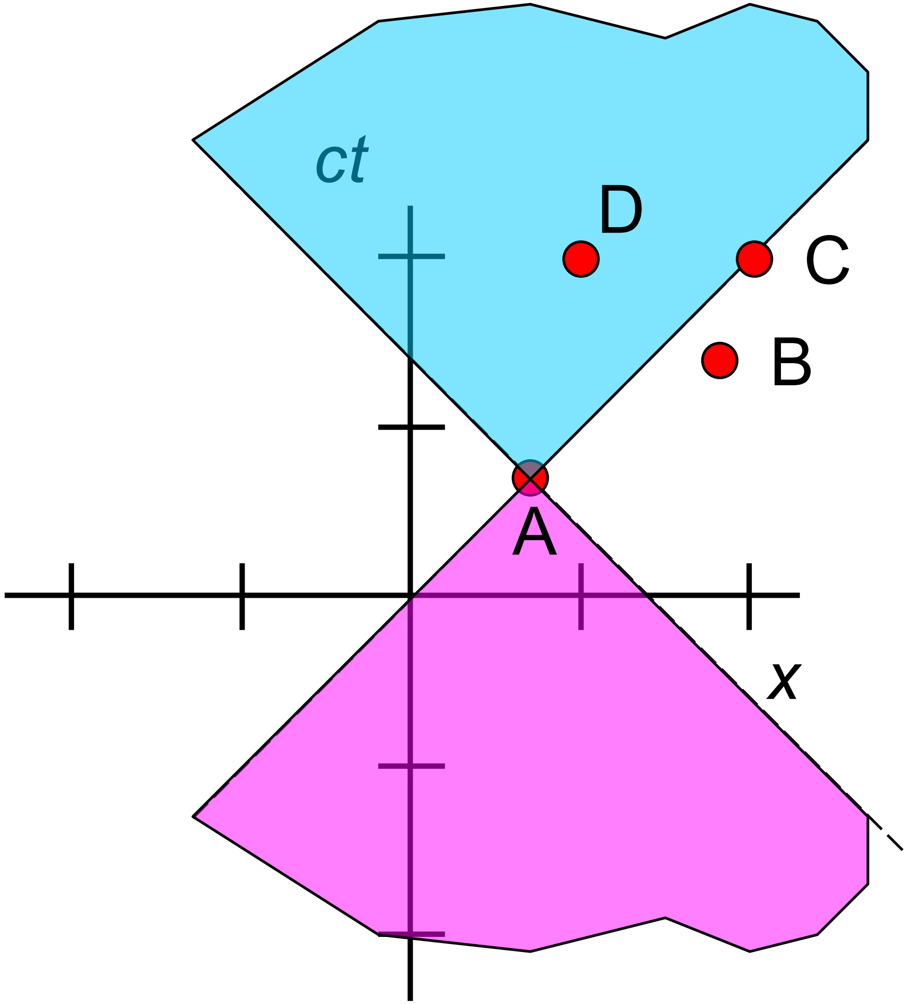
Spacetime



Here is an event in spacetime.

The yellow area is the "*elsewhere*" of the event. No physical signal can travel from the event to its elsewhere!

Spacetime



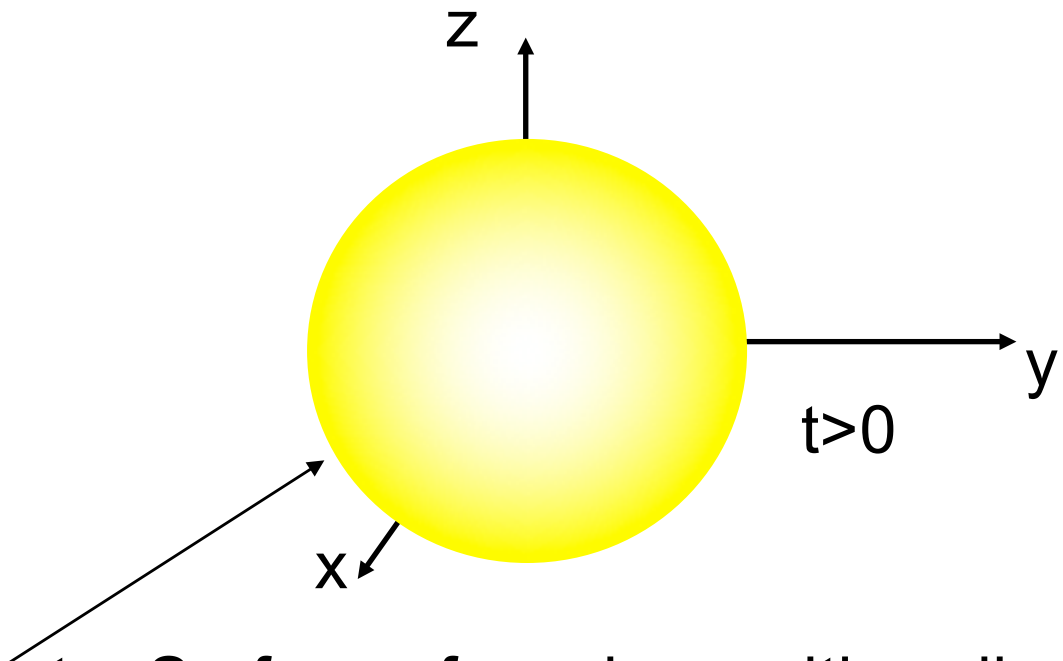
$(\Delta s)^2 > 0$: **Time-like events**
(A – D)

$(\Delta s)^2 < 0$: **Space-like events**
(A – B)

$(\Delta s)^2 = 0$: **Light-like events**
(A – C)

$(\Delta s)^2$ is invariant under Lorentz transformation.

Example: Wavefront of a flash



Wavefront = Surface of a sphere with radius ct:

$$(ct)^2 - x^2 - y^2 - z^2 = 0$$

Spacetime interval for light-like event: $(\Delta s)^2 = 0$

Einstein: 'c' is the same in all inertial systems.
Therefore: $(ct')^2 - x'^2 - y'^2 - z'^2 = 0$ in all inertial systems!
(Here we assumed that the origins of S and S' overlapped at t=0)

This is the end of our excursion to the relativistic spacetime.

Questions?