Lindblad superoperators from Wigner's phase space continuity equation

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ABSTRACT

For a simple quantum system weakly interacting with the environment, Wigner's 1932 formulation of quantum physics can be used to derive coupling to the environment using simple algebra. We show that the correct expressions, using coupling terms of "Lindblad form," are forced upon us. This is remarkable given that it took several decades before Lindblad's result was found in 1976.

I. INTRODUCTION

We prove that the time evolution of Wigner's quantum distribution W[1,2] obeys the continuity equation, $\frac{\partial}{\partial t} W = -\nabla \cdot \boldsymbol{J}$, with the Wigner current $\boldsymbol{J}$,[3,4] and use this to derive the Lindblad-form for a harmonic oscillator.

We approach coupling to the environment by naïvely using the bosonic ladder operators $\hat{a}^\dagger$ and $\hat{a}$ to describe quantum jumps between discrete levels and then combine the resulting expressions into the form of a current $\boldsymbol{J}$. The guidance provided by the continuity equation enforces the correct form of terms describing weak coupling to the environment: they are of Lindblad form.

In Sec. II, we review some history and further motivate this work. Then, in Sec. III, we remind the reader of Wigner's formulation of quantum theory and contrast it with the Schrödinger–von Neumann representation. For Hamiltonians that can be Taylor-expanded, Sec. IV proves that in the Wigner formulation a phase space current, $\boldsymbol{J}$, fulfilling a continuity equation, always exists. Using the jump operators $\hat{a}^\dagger$ and $\hat{a}$ naïvely, see Sec. V, leads to an instructive failure. In Sec. VI, we show how adhering to Wigner's continuity equation helps us to construct the correct Lindblad superoperator.

II. MOTIVATION

We learn in high school that natural phenomena are "quantum"; they are discrete. When an electron drops between discrete energy levels, it creates a photon, a discrete package of light. The electron has suffered a quantum jump, and so has the light field.

Such processes were studied in considerable detail starting in the 1930s.[5–10]

However, oftentimes we want to ignore the wealth of detail provided by such investigations and focus on fundamental (simple) structural aspects. That is the approach we take here.

A. From jumps to continuous evolution

With the inception of quantum mechanics, a statistical theory of quantum dynamics was given by Heisenberg and Schrödinger; in their representations, quantum jumps do not feature explicitly; the evolution of conservative systems is continuous.

However, a system coupled to the environment can be excited by it and can lose excitations to it. Here, we exclusively discuss the special case that such interactions are described, respectively, by the bosonic jump operators for system excitation, $\hat{a}^\dagger$, or de-excitation, $\hat{a}$. Where $\hat{a}^\dagger$ and $\hat{a}$ are the harmonic oscillator ladder operators obeying the bosonic commutation relation $[\hat{a}, \hat{a}^\dagger] = \hat{1}$. In quantum optics they are used to describe the addition, $\hat{a}^\dagger$, or removal, $\hat{a}$, of a photon from a light mode.[11]

The associated quantum jumps are observed when sufficiently discriminating measurements single out final states that have sufficiently small overlap with initial states of the system evolution.[12] In addition, this can be described using measurement theory.[13]

The measurement process, in particular, where to draw the boundary between system and measurement apparatus is, however, incompletely understood.

We, therefore, feel it is instructive to use the perspective of Wigner's formulation of quantum dynamics in order to review the connection between quantum jump operators, the coupling to the environment and the constraints imposed on such coupling descriptions.

Naïvely applying the operators $\hat{a}^\dagger$ and $\hat{a}$ does not work (see Sec. V). However, combining pieces from $\hat{a}^\dagger$ and $\hat{a}$ to construct a Wigner continuity equation guides us directly to the correct form, using Lindblad superoperators[14] (see Sec. VI).

We feel that this is an astonishingly simple and transparent "derivation" of such an important result that it is worth sharing. Lindblad derived it only in 1976,[15] and in the general case it is hardly straightforward, see Refs. 15 and 16. Yet, at least for the special case covered here,[14] Sec. VI shows the Lindblad form is "easy to guess" when we are guided by Wigner's continuity equation.

III. WIGNER DISTRIBUTION AND ITS CONTINUITY EQUATION

The conservative time-evolution of Wigner's quantum phase space distribution $W(x,p,t)$, for a one-dimensional continuous system, is governed by the phase space current, $\boldsymbol{J}$, and obeys a continuity equation[17]

$$\frac{\partial W(x,p,t)}{\partial t} + \nabla \cdot \boldsymbol{J}(x,p,t) = 0. \tag{1}$$

Here, $\nabla = (\partial/\partial_x, \partial/\partial_p)$ is the phase space divergence operator with respect to position x and momentum p, t is time, and $\boldsymbol{J} = (J_x, J_p)$ has two components and is a functional of W and the system Hamiltonian $H(x,p)$.

A. Why use the Wigner formulation?

The conventional approach of using Schrödinger's or von Neumann's equation forces us to investigate complex-valued wave functions to represent the state of the system; this already poses a stumbling block for the direct visualization of state changes.

Wigner's formulation of quantum theory, encapsulated by Eq. (1), has the great advantage of allowing us to visualize quantum dynamics in phase space more directly since W is real-valued.[18] In addition, Wigner's formulation permits the study of its real-valued phase space current, $\boldsymbol{J}$,[3] and formal line integrals along $\boldsymbol{J}$, which yield momentary snapshots of "field lines," reminiscent of classical phase portraits.[4]

Wigner's continuity Eq. (1) is the phase space equivalent of the conventional von Neumann's equation[17] [see Eq. (3) below].

This gives us a differential operator as a function of x, p, and t to describe infinitesimal changes in W in the explicit and highly constraining form $-\nabla \cdot \boldsymbol{J}$. It is this form that made us choose the Wigner distribution and its continuity equation as the starting point for our investigation.

IV. $J(x,p,t)$ FROM MOYAL'S BRACKET

We will now remind the reader of how Wigner's and Schrödinger's representations of quantum theory are connected mathematically.[19]

Consider a single-mode operator, $\hat{O}$, given in coordinate representation $\langle x-y|\hat{O}|x+y\rangle = O(x-y, x+y)$. To map to Wigner's phase space formulation, we employ the Wigner-transform, $\mathcal{W}[\hat{O}]$,[19–21]

$$\mathcal{W}[\hat{O}](x,p) = \int_{-\infty}^{\infty} dy\; O\left(x-\frac{y}{2}, x+\frac{y}{2}\right) e^{\frac{i}{\hbar}py}. \tag{2}$$

A settled terminology for these mappings does not exist in published literature.[2,19–24] Here, we use the convention that the Wigner transform $\mathcal{W}$ maps an operator $\hat{O}$ to its Wigner-symbol $\mathcal{W}[\hat{O}](x,p) = O(x,p)$,[19] which is a function on phase space. The inverse map, from a Wigner symbol to the associated normally ordered[20] operator $\hat{O}$, should be called its Weyl-symbol $\mathcal{W}^{-1}[O(x,p)] = \hat{O}$.[2]

If $\hat{O}$ is a normalized single-mode density matrix $\hat{\rho}$, then the associated *normalized* distribution in the Wigner representation is its Wigner symbol $W(x,p) \equiv \mathcal{W}[\hat{\rho}]/(2\pi\hbar)$.

Assuming that the Hamiltonian $\mathcal{W}[\hat{H}(\hat{x},\hat{p})] = H(x,p)$ is smooth enough, namely, has a global Taylor expansion, the Wigner transform of the von Neumann time evolution equation

$$\mathcal{W}\left[\frac{\partial\hat{\rho}}{\partial t} = \frac{1}{i\hbar}[\hat{H},\hat{\rho}]\right], \tag{3}$$

is the Wigner-symbol,

$$\frac{\partial W}{\partial t} = \{\!\{H, W\}\!\} = \frac{1}{i\hbar}(H \star W - W \star H), \tag{4}$$

where the Groenewold-$\star$-product[25] is the *Wigner symbol* of operator concatenation $\mathcal{W}[\hat{H}\circ\hat{\rho}]/(2\pi\hbar) = H \star W$. It is given by[19,25]

$$\star \equiv \exp\left[\frac{i\hbar}{2}\overleftrightarrow{\partial}\right] = \sum_{n=0}^{\infty}\frac{(i\hbar\overleftrightarrow{\partial})^n}{2^n n!}, \tag{5}$$

where we use the shorthand notations $\frac{\partial}{\partial_z} = \partial_z$ and $\overleftrightarrow{\partial} = \left(\overleftarrow{\partial_x}\overrightarrow{\partial_p} - \overleftarrow{\partial_p}\overrightarrow{\partial_x}\right)$ with arrows indicating the "directions" of differentiation: $f\overrightarrow{\frac{\partial}{\partial x}}g = g\overleftarrow{\frac{\partial}{\partial x}}f = f\frac{\partial}{\partial x}g$.

In other words, the non-commutative nature of the composition of Hilbert space operators manifests itself in the complicated and non-commutative structure of Groenewold's $\star$-product in the Wigner formulation.[19] The $\star$-product is the key-ingredient to map quantum behavior into phase space.[19,20]

Hence, "Moyal's bracket," $\{\!\{H, W\}\!\}$ of Eq. (4),[19,26] is of "Sine" form

$$\{\!\{f,g\}\!\} = \frac{2}{\hbar}f(x,p)\sin\left[\frac{\hbar}{2}\left(\frac{\overleftarrow{\partial}}{\partial x}\frac{\overrightarrow{\partial}}{\partial p} - \frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial x}\right)\right]g(x,p). \tag{6}$$

A series expansion of Eq. (4), using expression (6), reveals that to first order the Moyal bracket equals the Poisson bracket of classical mechanics,

$$\lim_{\hbar\downarrow 0}\{\!\{H, W\}\!\} = H\overleftrightarrow{\partial}W = \{H, W\}. \tag{7}$$

A. Proof that $\{\!\{H, W\}\!\} = -\nabla \cdot \boldsymbol{J}(x, p, t)$

If $H(x,p)$ has a global Taylor expansion, evolution Eq. (4) can be rewritten as the divergence of Wigner's phase space current, $\boldsymbol{J}$, of Eq. (1).[17,27]

Let us prove this statement by induction:

The lowest order (7) of Eq. (4) has the form $H\overleftrightarrow{\partial}W = (\partial_x H)(\partial_p W) - (\partial_p H)(\partial_x W) = \{H, W\} = \partial_p(W\partial_x H) - \partial_x(W\partial_p H) \equiv -\nabla\cdot J_1(x,p,t)$, yielding $\boldsymbol{J}_1 = (W\partial_p H, -W\partial_x H)^\top$.

Analogously, we can decompose terms of order $N+1$, in evolution Eq. (4), in terms of terms of order N: $H\overleftrightarrow{\partial}^{N+1}W = (\partial_x H)\overleftrightarrow{\partial}^{N}\partial_p W - (\partial_p H)\overleftrightarrow{\partial}^{N}\partial_x W = \partial_p((\partial_x H)\overleftrightarrow{\partial}^{N}W) - \partial_x((\partial_p H)\overleftrightarrow{\partial}^{N}W) \equiv -\nabla\cdot\boldsymbol{J}_{N+1}$, where $\boldsymbol{J}_{N+1} = ((\partial_p H)\overleftrightarrow{\partial}^{N}W, -(\partial_x H)\overleftrightarrow{\partial}^{N}W)^\top$.

This completes our proof.

Similarly, for multimode systems, tracing out degrees of freedom commutes with pulling out the gradient operator in front. For example, consider a two-mode system with modes a and b, and let us focus on mode a, tracing out b. This yields $\mathrm{Tr}_b\{H_{ab}\overleftrightarrow{\partial}_{ab}W_{ab}\} = \mathrm{Tr}_b\{-\nabla_{ab}\cdot \boldsymbol{j}_{ab}\} = -\nabla_a\cdot\boldsymbol{j}_a$, where $\boldsymbol{j}_a = \mathrm{Tr}_b\{\nabla_b \cdot \boldsymbol{j}_{ab}\}$ has two components in mode a, as desired. This result generalizes to terms of higher order in $\overleftrightarrow{\partial}_{ab}$ using the arguments presented in the previous paragraph (for an application, see Ref. 28).

What about non-conservative dynamics, as encountered in dissipative systems? The multimode considerations just given, tracing out (the bath) modes, cover this case. For a weakly coupled dissipative system, the Wigner current has been derived in analytical form[14] (and also been investigated experimentally[29]).

B. Remark on the "–" sign in the Moyal bracket $\{\!\{H, W\}\!\} = \frac{1}{i\hbar}(H \star W \text{ "–" } W \star H)$

It is the minus-sign in the Moyal bracket that is responsible for removing all terms from it that do not carry derivatives of W. This is, of course, inherited from the commutator in von Neumann's equation. However, it might appear to be less fundamental since it might be understood to "just arise" from the combination of phases of the unitary evolution operators for the bra- and ket-part that form the commutator.

Viewing it from the Wigner phase space perspective, knowing that $\partial_t W$ has to equal terms of the form $-\nabla\cdot\boldsymbol{J}$, reinforces that this is a fundamental feature.

V. PHOTON ADDITION: $\mathcal{W}[\hat{a}^\dagger\hat{\rho}\,\hat{a}]$ AND $J_{\hat{a}^\dagger\hat{\rho}\,\hat{a}}$

Let us now consider a bosonic single mode system, specifically a harmonic oscillator whose creation operator is $\hat{a}^\dagger(\hat{x},\hat{p}) = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} - \frac{i\hat{p}}{m\omega})$. This can describe an optical mode and its excitations.[11]

Setting $m = 1$, $\omega = 1$, and $\hbar = 1$, the corresponding Wigner symbols are $\mathcal{W}[\hat{a}^\dagger] = a^*(x,p) = (x - ip)/\sqrt{2}$ and $\mathcal{W}[\hat{a}] = a(x,p) = (x + ip)/\sqrt{2}$. The associated photon addition operator applied to a Wigner distribution (see Fig. 1), therefore, is

$$\frac{\mathcal{W}[\hat{a}^\dagger\hat{\rho}_0\,\hat{a}]}{2\pi\hbar} = a^* \star W_0 \star a = \frac{1}{2}\big(p^2 + x^2 + 1\big)W_0 +, \tag{8a}$$

$$+\left[-\frac{1}{2}\nabla\cdot\begin{pmatrix} xW_0 \\ pW_0\end{pmatrix} + \frac{\Delta W_0}{8}\right]. \tag{8b}$$

The differential operator in the square bracket (8b) obviously has the desired form of $\nabla\cdot\boldsymbol{J}$, where $\boldsymbol{J} = \frac{1}{8}([-4x + \partial_x]W_0, [-4p + \partial_p]W_0)^\top$ and $\Delta W_0 = \nabla\cdot(\nabla W_0)$ is the Laplacian in phase space.

We now seek a current $\boldsymbol{J}$ such that $\nabla\cdot\boldsymbol{J} \approx -\Delta\mathcal{W}[\hat{a}^\dagger\hat{\rho}_0\,\hat{a}]/\Delta t$ can be integrated over one unit of time, giving us expression (8): $W_0 - \int_0^1 dt\nabla\cdot\boldsymbol{J} = a^* \star W_0 \star a$.

We do not have in mind to use a trivial, unphysical interpolation such as $(1-t)W_0 + t\frac{\mathcal{W}[\hat{a}^\dagger\hat{\rho}\,\hat{a}]}{2\pi\hbar}$, with $t = 0,\ldots,1$.

The term (8a), $\frac{1}{2}(p^2 + x^2 + 1)W_0$, is troubling; it cannot be rendered in the form $\nabla\cdot\boldsymbol{J}(x,p)$.

If we attempt moving the polynomial terms of (8a) from the outside to the inside of a derivative operator, we end up increasing the order of the polynomials in x and p that are involved.

For instance, $p^2 W = \partial_x(p^2 x\, W) - p^2 x\partial_x W$. However, the second term shows an unwelcome increase to a higher order polynomial (here $p^2 x$). Its polynomial degree, if we similarly try to move it inside a derivative, will increase yet further, ad infinitum.

A. Compensating unphysical terms

Similarly to $a^* \star W \star a$, the phase space expression for photon removal from the light field W has the form

$$a \star W \star a^* = \frac{1}{2}\big(p^2 + x^2 - 1\big)W +, \tag{9a}$$

$$+\left[+\frac{1}{2}\nabla\cdot\begin{pmatrix} xW \\ pW\end{pmatrix} + \frac{\Delta W}{8}\right]. \tag{9b}$$

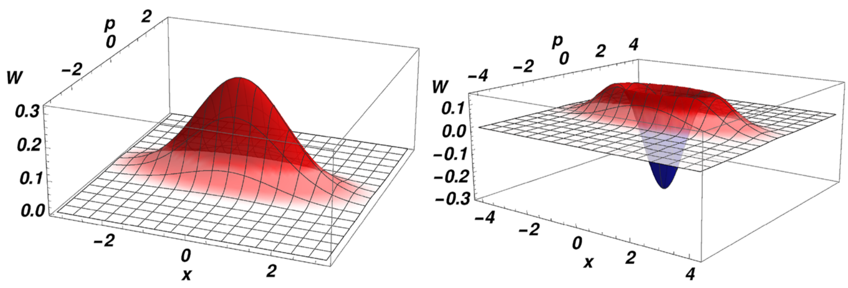

FIG. 1. Wigner distributions W_V for weakly squeezed pure states (left) and the same state after formal photon addition, $a^* \star W_V \star a$ (right), the value of the normalized x-variance is $V = 5/2$ (for further details, see the Appendix and Ref. 30).

Combining Eqs. (8a) and (9a), we find that neither $a^* \star W \star a + a \star W \star a^*$ nor $a^* \star W \star a - a \star W \star a^*$ can be given in the form of the divergence of a current.

Given that $a^*(x,p)$ and $a(x,p)$ are complex functions (representing the fact that $\hat{a}^\dagger$ and $\hat{a}$ are not Hermitian), we cannot use them by themselves; let us, therefore, consider the simplest extension possible for finding a current. We try quadratic combinations of a^* and a to compensate for the unphysical terms identified above.

We know that the phase space dynamics of quantum harmonic oscillators behaves classically,[17]

$$a^* \star a \star W - W \star a^* \star a = \nabla \cdot \begin{pmatrix} p\ W \\ -x\ W \end{pmatrix}. \tag{10}$$

Instead of subtracting such terms, adding them gives us

$$a^* \star a \star W + W \star a^* \star a =, \tag{11}$$

$$\left(p^2 + x^2 - 1\right) W - \frac{\Delta W}{4}, \tag{12}$$

and

$$a \star a^* \star W + W \star a \star a^* =, \tag{13}$$

$$\left(p^2 + x^2 + 1\right) W - \frac{\Delta W}{4}. \tag{14}$$

We see that expressions (12) and (14) can compensate the terms (8a) and (9a).

Namely, (only) the combinations

$$2a^* \star W \star a - (a \star a^* \star W + W \star a \star a^*) =, \tag{15}$$

$$\left[-\nabla \cdot \begin{pmatrix} xW \\ pW \end{pmatrix} + \frac{\Delta W}{2}\right], \tag{16}$$

and

$$2a \star W \star a^* - (a^* \star a \star W + W \star a^* \star a) =, \tag{17}$$

$$\left[\nabla \cdot \begin{pmatrix} xW \\ pW \end{pmatrix} + \frac{\Delta W}{2}\right], \tag{18}$$

can be rendered as $-\nabla \cdot \boldsymbol{J}_\mp$, respectively, where

$$\boldsymbol{J}_\mp = -\begin{pmatrix} \mp x + \frac{1}{2}\partial_x\ W \\ \mp p + \frac{1}{2}\partial_p\ W \end{pmatrix}. \tag{19}$$

Reinstating the variables m, ω, and $\hbar$, this becomes

$$2a^* \star W \star a - (a \star a^* \star W + W \star a \star a^*) =, \tag{20}$$

$$\left[-\nabla \cdot \begin{pmatrix} xW \\ pW \end{pmatrix} + \frac{\hbar}{2m\omega}\partial_x^2 W + \frac{\hbar m\omega}{2}\partial_p^2 W\right], \tag{21}$$

and

$$2a \star W \star a^* - (a^* \star a \star W + W \star a^* \star a) =, \tag{22}$$

$$\left[+\nabla \cdot \begin{pmatrix} xW \\ pW \end{pmatrix} + \frac{\hbar}{2m\omega}\partial_x^2 W + \frac{\hbar m\omega}{2}\partial_p^2 W\right]. \tag{23}$$

VI. LINDBLAD FORM

The standard Lindblad master equation terms

$$\begin{aligned}\frac{d\rho}{dt} = &-\frac{i}{\hbar}[H,\rho] + \frac{\gamma}{2}(\bar{n}+1)\left(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a\right) \\ &+ \frac{\gamma}{2}\bar{n}\left(2a^\dagger \rho a - aa^\dagger \rho - \rho a a^\dagger\right),\end{aligned} \tag{24}$$

are well known.[14]

Including natural units, the associated currents $\boldsymbol{J}_{\text{env}}$, associated with the γ-terms in (24) coupling to the environment, have the concise form[14]

$$\boldsymbol{J}_{\text{env}} = -\frac{\gamma}{2} W \begin{pmatrix} x \\ p \end{pmatrix} - \frac{\gamma}{2}\frac{\hbar}{\omega_0}\left(\overline{n} + \frac{1}{2}\right)\begin{pmatrix} \partial_x W \\ \partial_p W \end{pmatrix}, \tag{25}$$

$$\equiv \boldsymbol{J}_{\text{damp}} + \boldsymbol{J}_{\text{diff}}, \tag{26}$$

where $\boldsymbol{J}_{\text{damp}}$ is the classical damping current, whereas the diffusive current, $\boldsymbol{J}_{\text{diff}}(\overline{n} = 0)$, enforces Heisenberg's uncertainty principle.[29]

As such, our result for $\boldsymbol{J}_\mp$ of Eq. (19), or Eqs. (20)–(23), is not new; expressions of open systems dynamics in phase space have been derived by Wigner-transforming expressions such as (24).[14,31]

What is new, however, is that our derivation shows that the continuity equation of phase space quantum dynamics (1) enforces this form and that it does so in a transparent way.

A. Lindblad form put into context

For the specific coupling for photon creation (8) or annihilation (9), our analysis essentially gives a rederivation of the well-known Lindblad form (15) and (17) of a simple quantum system weakly coupled to a bath.

We think it is astonishing that the insistence to express dynamical terms as the negative divergence of Wigner's phase space current, $-\nabla \cdot \boldsymbol{J}$, should have such far-reaching and at the same time specific consequences.

This is an interesting observation since the standard consistency requirements imposed on the coupling terms,[15,16] when using the Schrödinger–von Neumann representation, do not bear resemblance to the requirement to be of the form of the divergence of a current.

Of course, our analysis was restricted to one case (photon addition to, or subtraction from, a light field) and, unsurprisingly, it is neither more general nor does it allow us to check on more subtle consistency requirements, such as those elucidated in Ref. 32. It also does not obviously enforce the fluctuation-dissipation theorem, namely, that all terms have to appear simultaneously, as in Eq. (25) (also see Ref. 29).

A good introductory text for the theory of quantum damping using terms of Lindblad-form is Ref. 33; a study giving visual insight is given in Ref. 34; also see the many references in Ref. 14. Recently, using the phase space approach on which this work is based, studies have been performed for various systems (see Refs. 14 and 31). Moreover, using phase space lends itself to studying dissipative dynamics with the help of the geometrical analysis provided by Wigner's phase space current, $\boldsymbol{J}$, see Refs. 14 and 28, including recent experimental work.[29]

VII. CONCLUSIONS AND OUTLOOK

We have found that the explicit requirement that quantum dynamics (in phase space) has to fulfill continuity Eq. (1) enforces that photon addition to, or subtraction from, a system has to obey a Lindblad form.[14,29] For this special case, the derivation consists of a few lines of transparent calculations of extreme simplicity.

Using the standard (von Neumann) approach yields more subtle implicit requirements, namely, trace-, hermiticity-, and complete positivity-conservation.[16]

When contrasting the two approaches, we wonder, in light of the simplicity of the phase space approach, whether the Lindblad form could have been discovered earlier if Wigner's phase space formulation would have been more widely adopted sooner.[19]

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Ole Steuernagel: Conceptualization (equal); Funding acquisition (equal); Writing – original draft (equal). **Ray-Kuang Lee**: Funding acquisition (equal); Project administration (equal); Writing – review & editing (equal).

DATA AVAILABILITY

Data underlying the results presented in this paper are not publicly available at this time but may be obtained from the authors upon reasonable request.

APPENDIX: WIGNER DISTRIBUTIONS DISPLAYED IN FIG. 1

In Fig. 1, we display the Wigner distribution W_V of a pure squeezed state in dimensionless units $m = 1$, $\omega = 1$, and $\hbar = 1$ in terms of variances V relative to the variance of the vacuum (harmonic oscillator ground) state. In other words, $V = 1$ is the vacuum state, whereas for values of $V > 1$, the state

$$W_V(x,p) = \frac{1}{\pi}\exp\left[-\frac{x^2}{V} - p^2 V\right], \tag{A1}$$

describes a pure squeezed state squeezed in the p-direction and anti-squeezed in the x-direction (compare Fig. 1, left panel). The associated form of the photon-added expression (8) for W_V (also see Ref. 30), displayed in the right panel of Fig. 1, is

$$a^* \star W_V(x,p) \star a = W_V(x,p)\left(2p^2 V - 1 + \frac{2x^2}{V}\right). \tag{A2}$$

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