Note for *Quantum Optics*: Input-Output formulation of optical cavity

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Reference:
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Ch. 13 in *"Elements of Quantum Optics,"* by P. Meystre and M. Sargent III.
Ch. 10 in *"Introductory Quantum Optics,"* by C. Gerry and P. Knight.
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"Theoretical Problems in Cavity Nonlinear Optics," by P. Mandel.

I. INPUT-OUTPUT FORMULATION OF OPTICAL CAVITY

In preceding chapters, we have used a *master* equation to calculate the photon statistics inside an optical cavity when the internal field is damped. In this approach, the field external to the cavity is treated as a *heat bath*, reservoir. The heat bath is simply a *passive* system with which the system gradually comes into equilibrium. Now we would explicitly treat the heat bath as the external field, and determine the effect of the intra-cavity dynamics on the quantum statistics of the output field.

Consider a single cavity mode interacting with an external field. The interaction Hamiltonian is

$$\hat{H}_I = i\hbar \int \mathrm{d}\omega g(\omega)[\hat{b}(\omega)\hat{a}^\dagger - \hat{a}\hat{b}^\dagger(\omega)], \tag{1}$$

where $\hat{a}$ is the annihilation operator for the intra-cavity field, with the commutation relations,

$$[\hat{a}, \hat{a}^\dagger] = 1, \tag{2}$$

where $\hat{b}(\omega)$ are the annihilation operators for the external field, with

$$[\hat{b}(\omega), \hat{b}^\dagger(\omega')] = \delta(\omega - \omega'). \tag{3}$$

In actual fact the physical frequency limits are $(0, \infty)$. However, for high frequency optical systems we may shift the integration to a frequency ω_0, the cavity resonance frequency. The integration limits are $(-\omega_0, \infty)$, as ω_0 is large, then we approximate $\int_{-\infty}^{\infty}$. The Heisenberg equation of motion for $\hat{b}(\omega)$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{b}(\omega) = -i\omega\hat{b}(\omega) + g(\omega)\hat{a}, \tag{4}$$

with the initial condition at time $t_0 < t$, the *input*,

$$\hat{b}(\omega) = e^{-i\omega(t-t_0)}\hat{b}_0(\omega) + g(\omega)\int_{t_0}^{t} \mathrm{d}t' e^{-i\omega(t-t')}\hat{a}(t'), \tag{5}$$

where $t_0 < t$ and $\hat{b}_0(\omega)$ is the value of $\hat{b}(\omega)$ at $t = t_0$, or with the final condition at time $t_1 > t$, the *output*,

$$\hat{b}(\omega) = e^{-i\omega(t-t_1)}\hat{b}_1(\omega) - g(\omega)\int_{t}^{t_1} \mathrm{d}t' e^{-i\omega(t-t')}\hat{a}(t'), \tag{6}$$

where $t < t_1$ and $\hat{b}_1(\omega)$ is the value of $\hat{b}(\omega)$ at $t = t_1$.

The system operator obeys the equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{a} = -\frac{i}{\hbar}[\hat{a}, \hat{H}_S] - \int_{-\infty}^{\infty} \mathrm{d}\omega g(\omega)\hat{b}(\omega). \tag{7}$$

In terms of the solutions with initial condition,

$$\hat{b}(\omega) = e^{-i\omega(t-t_0)}\hat{b}_0(\omega) + g(\omega)\int_{t_0}^{t} \mathrm{d}t' e^{-i\omega(t-t')}\hat{a}(t'), \tag{8}$$

then

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{a} = -\frac{i}{\hbar}[\hat{a}, \hat{H}_S] - \int_{-\infty}^{\infty} \mathrm{d}\omega g(\omega) e^{-i\omega(t-t_0)}\hat{b}_0(\omega) - \int_{-\infty}^{\infty} \mathrm{d}\omega g^2(\omega)\int_{t_0}^{t} \mathrm{d}t' e^{-i\omega(t-t')}\hat{a}(t'), \tag{9}$$

Here we define the *input field*,

$$\hat{a}_{IN}(t) = \frac{-1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}\omega e^{-i\omega(t-t_0)}\hat{b}_0(\omega), \tag{10}$$

which satisfy the commutation relation, $[\hat{a}_I(t), \hat{a}_I^{\dagger}(t')] = \delta(t-t')$. With the *Markovian* approximation:

$$\int_{-\infty}^{\infty} \mathrm{d}\omega g^2(\omega)\int_{t_0}^{t} \mathrm{d}t' e^{-i\omega(t-t')}\hat{a}(t') \;\approx\; g^2(\omega)\int_{-\infty}^{\infty} \mathrm{d}\omega \int_{t_0}^{t} \mathrm{d}t' e^{-i\omega(t-t')}\hat{a}(t'), \tag{11}$$

$$= \frac{\gamma}{2\pi}2\pi\int_{t_0}^{t} \mathrm{d}t'\delta(t-t')\hat{a}(t') = \frac{\gamma}{2}a(t), \tag{12}$$

we have

$$\int_{t_0}^{t} \mathrm{d}t'\delta(t-t')f(t') = \int_{t}^{t_1} \mathrm{d}t'\delta(t-t')f(t')\frac{1}{2}f(t), \quad (t_0 < t < t_1). \tag{13}$$

The the introduction of the *Input field*, the equation of motion for the system becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{a} \;=\; -\frac{i}{\hbar}[\hat{a}, \hat{H}_S] - \int_{-\infty}^{\infty} \mathrm{d}\omega g(\omega) e^{-i\omega(t-t_0)}\hat{b}_0(\omega) - \int_{-\infty}^{\infty} \mathrm{d}\omega g^2(\omega)\int_{t_0}^{t} \mathrm{d}t' e^{-i\omega(t-t')}\hat{a}(t'), \tag{14}$$

$$= -\frac{i}{\hbar}[\hat{a}, \hat{H}_S] - \frac{\gamma}{2}\hat{a}(t) + \sqrt{\gamma}\hat{a}_I(t). \tag{15}$$

This is a Langevin equation for the damped amplitude $\hat{a}(t)$, but with the noise term appears explicitly as the input field. The time reverse Langevin equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{a} = -\frac{i}{\hbar}[\hat{a}, \hat{H}_S] + \frac{\gamma}{2}\hat{a}(t) - \sqrt{\gamma}\hat{a}_O(t), \tag{16}$$

where

$$\hat{a}_O(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}\omega e^{-i\omega(t-t_1)}\hat{b}_1(\omega). \tag{17}$$

By the *input-output theorem*, the system operator obeys the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{a} \;=\; -\frac{i}{\hbar}[\hat{a}, \hat{H}_S] - \frac{\gamma}{2}\hat{a}(t) + \sqrt{\gamma}\hat{a}_I(t), \tag{18}$$

$$= -\frac{i}{\hbar}[\hat{a}, \hat{H}_S] + \frac{\gamma}{2}\hat{a}(t) - \sqrt{\gamma}\hat{a}_O(t). \tag{19}$$

The relation between the external field and the intra-cavity field may be obtained,

$$\hat{a}_O(t) + \hat{a}_I(t) = \sqrt{\gamma}\hat{a}(t), \tag{20}$$

which is a boundary condition relating each of the far-field amplitudes outside the cavity to the internal cavity field. It is easy to see that interference between the input and the cavity field may contribute to the observed output field,

II. LINEAR CAVITY

For a linear system, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{a}(t) = \mathbf{A}\mathbf{a} - \frac{\gamma}{2}\hat{a}(t) + \sqrt{\gamma}\mathbf{a}_I(t), \tag{21}$$

where

$$\mathbf{a}(t) = \begin{pmatrix} \hat{a}(t) \\ \hat{a}^\dagger(t) \end{pmatrix}, \quad \text{and} \quad \mathbf{a}_I(t) = \begin{pmatrix} \hat{a}_I(t) \\ \hat{a}_I^\dagger(t) \end{pmatrix}. \tag{22}$$

If we define the Fourier components of the intra-cavity field,

$$\mathbf{a}(\omega) = \begin{pmatrix} \hat{a}(\omega) \\ \hat{a}^\dagger(\omega) \end{pmatrix}, \quad \text{where} \quad \hat{a}(t) = \frac{1}{\sqrt{2\pi}}\int \mathrm{d}\omega e^{-i\omega(t-t_0)}\hat{a}(\omega), \tag{23}$$

then, the equation of motion in frequency domain becomes

$$[\mathbf{A} + (i\omega - \frac{\gamma}{2})\mathbf{1}]\mathbf{a}(\omega) = -\sqrt{\gamma}\mathbf{a}_I(\omega) \tag{24}$$

By the same way, one can find that

$$[\mathbf{A} + (i\omega + \frac{\gamma}{2})\mathbf{1}]\mathbf{a}(\omega) = +\sqrt{\gamma}\mathbf{a}_O(\omega), \tag{25}$$

or

$$\mathbf{a}_O(\omega) = -[\mathbf{A} + (i\omega + \frac{\gamma}{2})\mathbf{1}][\mathbf{A} + (i\omega - \frac{\gamma}{2})\mathbf{1}]^{-1}\mathbf{a}_I(\omega). \tag{26}$$

For example, consider an empty one-sided cavity. In this case, the only source of loss in the cavity is through the mirror which couples the input and output fields. The system Hamiltonian is $\hat{H}_S = \hbar\omega_0\hat{a}^\dagger\hat{a}$, and

$$\mathbf{A} = \begin{pmatrix} -i\omega_0 & 0 \\ 0 & i\omega_0 \end{pmatrix}, \tag{27}$$

then

$$\mathbf{a}_O(\omega) = \frac{\gamma/2 + i(\omega - \omega_0)}{\gamma/2 - i(\omega - \omega_0)}\mathbf{a}_I(\omega). \tag{28}$$

There is a frequency dependent phase shift between the output and input. The relationship between the input and the internal field is,

$$\mathbf{a}(\omega) = \frac{\sqrt{\gamma}}{\gamma/2 - i(\omega - \omega_0)}\mathbf{a}_I(\omega), \tag{29}$$

which leads to a *Lorentzian* of width $\gamma/2$ for the intensity transmission function.

III. TWO-SIDED CAVITY

A *two-sided* cavity has two partially transparent mirrors with associated loss coefficients γ_1 and γ_2. In this case there are two input ports and two output ports, and the equation of motion for the internal field is

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{a}(t) = -i\omega_0\hat{a}(t) - \frac{1}{2}(\gamma_1 + \gamma_2)\hat{a}(t) + \sqrt{\gamma_1}\hat{a}_I(t) + \sqrt{\gamma_2}\hat{b}_I(t). \tag{30}$$

The relationship between the internal and input field frequency components for an empty cavity is then

$$\mathbf{a}(\omega) = \frac{\sqrt{\gamma_1}\mathbf{a}_I(\omega) + \sqrt{\gamma_2}\mathbf{b}_I(\omega)}{\frac{\gamma_1+\gamma_2}{2} - i(\omega - \omega_0)}. \tag{31}$$

IV. TWO-TIME CORRELATION FUNCTION

There are two boundary conditions of the reservoir,

$$\hat{b}(\omega) = e^{-i\omega(t-t_0)}\hat{b}_0(\omega) + g(\omega)\int_{t_0}^{t} \mathrm{d}t' e^{-i\omega(t-t')}\hat{a}(t'), \quad \text{at time } t_0 < t, \text{ the } \textit{input}, \tag{32}$$

$$\hat{b}(\omega) = e^{-i\omega(t-t_1)}\hat{b}_1(\omega) - g(\omega)\int_{t}^{t_1} \mathrm{d}t' e^{-i\omega(t-t')}\hat{a}(t'), \quad \text{at time } t_1 > t, \text{ the } \textit{output}. \tag{33}$$

For The input and output fields, we have

$$\hat{a}_{IN}(t) = \frac{-1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}\omega e^{-i\omega(t-t_0)}\hat{b}_0(\omega), \tag{34}$$

$$\hat{a}_{OUT}(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}\omega e^{-i\omega(t-t_1)}\hat{b}_1(\omega), \tag{35}$$

or

$$\hat{a}_{IN}(t) = \frac{\sqrt{\gamma}}{2}\hat{a}(t) - \frac{1}{\sqrt{2\pi}}\int \mathrm{d}\omega\hat{b}(\omega, t), \tag{36}$$

$$\hat{a}_{OUT}(t) = \frac{\sqrt{\gamma}}{2}\hat{a}(t) + \frac{1}{\sqrt{2\pi}}\int \mathrm{d}\omega\hat{b}(\omega, t). \tag{37}$$

Let $\hat{c}(t)$ be any system operator, then

$$[\hat{c}(t), \sqrt{\gamma}\hat{a}_{IN}(t')] = \frac{\gamma}{2}[\hat{c}(t), \hat{a}(t')], \quad \text{for} \quad t = t', \tag{38}$$

$$[\hat{c}(t), \sqrt{\gamma}\hat{a}_{IN}(t')] = 0, \quad \text{for} \quad t' > t, \tag{39}$$

$$[\hat{c}(t), \sqrt{\gamma}\hat{a}_{OUT}(t')] = 0, \quad \text{for} \quad t' < t, \tag{40}$$

$$[\hat{c}(t), \sqrt{\gamma}\hat{a}_{IN}(t')] = \gamma[\hat{c}(t), \hat{a}(t')], \quad \text{for} \quad t' < t, \tag{41}$$

with

$$\hat{a}_O(t) + \hat{a}_I(t) = \sqrt{\gamma}\hat{a}(t). \tag{42}$$

The commutator for the output field is

$$[\hat{a}_O(t), \hat{a}_O^\dagger(t')] = [\hat{a}_I(t), \hat{a}_I^\dagger(t')]. \tag{43}$$

V. SPECTRUM OF SQUEEZING FOR THE PARAMETRIC OSCILLATOR

Below the threshold, the Hamiltonian for a parametric oscillator is

$$\hat{H}_S = \hbar\omega_0\hat{a}^\dagger\hat{a} + \frac{i\hbar}{2}(\epsilon\hat{a}^{\dagger 2} - \epsilon^*\hat{a}^2), \tag{44}$$

then

$$[\mathbf{A} + (i\omega - \frac{\gamma}{2})\mathbf{1}]\mathbf{a}(\omega) = -\sqrt{\gamma}\mathbf{a}_I(\omega), \tag{45}$$

$$[\mathbf{A} + (i\omega + \frac{\gamma}{2})\mathbf{1}]\mathbf{a}(\omega) = +\sqrt{\gamma}\mathbf{a}_O(\omega), \tag{46}$$

where

$$\mathbf{A} = \begin{pmatrix} -i\omega_0 & \epsilon \\ \epsilon^* & i\omega_0 \end{pmatrix}. \tag{47}$$

Take the Fourier components for the output field,

$$\hat{a}_O(\omega) = \frac{1}{(\frac{\gamma}{2} - i(\omega-\omega_0)^2 - |\epsilon|^2}\{[(\frac{\gamma}{2})^2 + (\omega-\omega_0)^2 + |\epsilon|^2]\hat{a}_I(\omega) + \epsilon\gamma\hat{a}_I^\dagger(-\omega)\}. \tag{48}$$