W-states from Bose-Einstein condensates in an optical cavity

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Abstract

We analytically investigate the ground state of a Bose-Einstein condensate of bosonic atoms with two hyperfine structures inside an optical cavity within the cavity quantum electrodynamics formalism. The system allows a series of quantum phase transitions. The critical coupling value at the first quantum phase transition of the system is calculated and the maximum shared bipartite entanglement of the condensate is studied numerically. We reveal the existence of a maximum entangled multipartite qubit state, a W-state, for certain parameters at the first non-vacuum phase region of the system due to finite size effects.

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Cavity quantum electrodynamics (cavity-QED), the study of atoms coherently interacting with quantized field cavity modes, has proved a versatile and controllable platform in quantum optics describing many interesting phenomena[1, 2]. The simplest light-atom system, *i.e.* the Jaynes-Cummings model, describes the coherent interaction of a quantized electromagnetic field with a single two-level atom [3]. As the number of two-level atoms increases, *i.e.* the Dicke or Tavis-Cummings model, collective effects lead to an intriguing many-body phenomena known as a superradiant phase [4, 5] where a state with photon and atomic excitations has a lower energy than the vacuum field and ensemble ground state at zero [6] and finite temperature [7, 8]. This quantum critical phenomenon has been associated with entanglement between the atomic ensemble and the field [9, 10] as well as bipartite entanglement among atomic components due to finite-size effects [11–13]. Entanglement is an essential resource for quantum information processing [14].

It is known that charge-only atomic ensembles in closed cavity-QED systems cannot yield a superradiant quantum phase transition [15–18]. Several implementation proposals allowing the Dicke superradiant phase transition have been presented, including open dynamical systems involving semiconductor quantum wells or quantum dots [19, 20], open dynamical cavity-QED systems with neutral atoms [21] and ions [22], and superconducting quantum devices [23, 24]. Arrays of coupled cavities have extended the application of Dicke model to the study of strongly interacting many-body systems in condensed matter physics [25–29].

Recently, substituting free atoms, weakly-interacting ultracold atoms in a Bose-Einstein condensate have been loaded into high-finesse optical cavities [30, 31]. It has been shown that, accounting for the center of motion of the atomic components, it is possible to realize the Dicke model in these BEC-cavity-QED systems where the superradiant phase corresponds to a periodical self-organized supersolid phase of the BEC [32, 33].

Motivated by the BEC-cavity-QED system, we study an extended Dicke Hamiltonian related to the coupling of two hyperfine levels of a single atomic species BEC to a quantized cavity field mode [30, 31]. Both the critical coupling and exact ground state are given in analytical closed-form for the two-parameter phase space defined by the field-ensemble coupling strength and the intra-atomic ensemble interaction parameter. For the sake of completeness, we also show the first critical coupling values including counter-rotating terms into the studied Hamiltonian and in the classical limit, *i.e.* infinitely large ensemble size. Besides the existences of intra-ensemble maximum shared bipartite concurrence and field-ensemble entropy of entanglement [13], a finite probability of finding the ground state in a W-state is revealed explicitly.

Since a multi-qubit W-state [34] maximizes pairwise entanglement of formation [35, 36] and is a robust source of entanglement [37, 38], which retains bipartite entanglement whenever any of the qubits is traced out, the results presented in this manuscript show the importance of BEC-cavity-QED systems for quantum information processing.

Within the two-mode approximation, we consider a single atomic species BEC with two hyperfine levels coupled to a quantized cavity field mode, which can be described by an extended Dicke Hamiltonian including intra-ensemble interactions [39], under the rotating wave approximation, in units of $\hbar$ from now on,

$$\hat{H} = \omega_f \hat{N} + \Delta \hat{J}_z + \kappa \hat{J}_z^2 + \frac{\lambda}{\sqrt{N_a}}(\hat{a}\hat{J}_+ + \hat{a}^\dagger \hat{J}_-), \tag{1}$$

where the total number of excitations operator is $\hat{N} = \hat{a}^\dagger\hat{a} + \hat{J}_z$. The quantized cavity field mode is described by the boson creation (annihilation) operators $\hat{a}$ ($\hat{a}^\dagger$) and the field frequency ω_f, while the atomic ensemble system is represented by the transition frequency ω_a and the orbital angular momentum operators $\hat{J}_i$, with $i = x, y, z$ and $\hat{J}_\pm = \hat{J}_x \pm \imath \hat{J}_y$. The transition and field frequencies define a detuning given by $\Delta = \omega_a - \omega_f$. The interaction parameter κ depends on the interaction between the two hyperfine-structure-defined modes and can be tuned by modifying the s-wave scattering lengths of aforementioned modes. The coupling strength λ is related to the coupling strength between one atom and the cavity field mode. Note that the number of atoms in the system for this physical realization is restricted by the two-mode approximation that sets the condition $N_a\rho \leq r_0$ [40], where the parameters ρ and r_0 are the typical scattering length of the atomic species and the radius of the trap, in that order.

Instead of solving the extended Dicke model in Eq.(1) in the large particle number regime [39], we partition the corresponding Hilbert space into subspaces with a mean total excitation number $\langle \hat{N} \rangle = n - N_a/2$ for $n = 0, 1, 2, \ldots$ due to conservation of the total number of excitations, $[\hat{H}, \hat{N}] = 0$. The ground state at the n-th subspace is given by $|\psi_g^{(n)}\rangle = \sum_{k=0}^{n} c_k^{(n)} |k\rangle_f |n - k - N_a/2\rangle$ such that $\hat{H}|\psi_g^{(n)}\rangle = E_g^{(n)}|\psi_g^{(n)}\rangle$ and $E_g^{(n)}$ is the lowest eigenvalue for the given subspace. The ket notation $|n\rangle_f$ refers to a Fock state of the field with n photons and the shorthand notation $|m\rangle \equiv |N_a/2, m\rangle$, $m = -N_a/2, 1 - N_a/2, \ldots, N_a/2 - 1, N_a/2$, is used for the Dicke state corresponding to the normalized superposition of all possible combinations of N_a atoms with $(N_a/2+m)$ of them in the excited state and the rest, $(N_a/2-m)$, in the ground state.

The ground state is found as the lowest eigenvalue for the set $\{H^{(n)} = H_O^{(n)} + H_I^{(n)}\}$ of square matrices, with rank equal or less than (N_a+1), defined by the auxiliary matrices,

$$H_O^{(n)} = \omega_f (n - N_a/2) \mathbb{I}_{\tilde{n}}, \tag{2}$$

$$\left(H_I^{(n)}\right)_{i,j} = \delta_{i,j} d_i + \frac{\lambda}{\sqrt{N_a}} \left(\delta_{i,j-1} o_j + \delta_{i,j+1} o_{j+1}\right), \tag{3}$$

where the identity matrix of rank d is given by $\mathbb{I}_d$, the row and column labels are in the range $i,j = \tilde{n}, \tilde{n}+1, \ldots, n$, where $\tilde{n} = \max(0, n - N_a)$, for the photon number $n = 0, 1, 2, \ldots$ The symbol $\delta_{i,j}$ stands for Kronecker delta and the diagonal and off-diagonal terms are defined as,

$$d_j = (n - j - N_a/2)\left[\Delta + \kappa(n - j - N_a/2)\right], \tag{4}$$

$$o_j = \left[j(N_a + j - n)(n - j + 1)\right]^{1/2}. \tag{5}$$

The auxiliary matrix $H_I^{(n)}$ is a tri-diagonal symmetric real matrix with positive off-diagonal terms, *i.e.* a Jacobi matrix, and its eigenvalues can be found analytically [41–44]. Furthermore, each and every phase transition can be located at the intersection of two ground state energies belonging to contiguous subspaces.

The first phase, which will be called vacuum phase from now on, corresponds to the vacuum field and the atomic ensemble ground state, $|\psi_g^{(0)}\rangle = |0\rangle| - N_a/2\rangle$.

The first phase transition is found at the critical coupling strength,

$$\lambda_{c1} = \left\{\left[\omega_a + (1 - N_a)\kappa\right]\omega_f\right\}^{\frac{1}{2}}, \tag{6}$$

with the corresponding first non-vacuum state

$$|\psi_g^{(1)}\rangle = c_0^{(1)}|0\rangle \left|1 - \frac{N_a}{2}\right\rangle + c_1^{(1)}|1\rangle \left|-\frac{N_a}{2}\right\rangle. \tag{7}$$

In our case, the amplitudes are given by the expressions $c_0^{(1)} = h/(h^2+1)^{1/2}$ and $c_1^{(1)} = 1/(h^2+1)^{1/2}$ related to the amplitude parameter,

$$h = \frac{(N_a - 1)\kappa + \Delta - \left\{4\lambda^2 + \left[(N_a - 1)\kappa - \Delta\right]^2\right\}^{\frac{1}{2}}}{2\lambda}. \tag{8}$$

Any extended Dicke model that conserves the total number of excitations has a first non-vacuum phase of the form given by Eq.(7), which includes the ground state of the atomic ensemble in the W-state $|1-N_a/2\rangle$. The relevant characteristic of the studied extended Dicke model is that as the amplitude parameter, Eq.(8), for a infinitely large interaction-coupling strength parameters ratio, $\kappa/\lambda_c \to \infty$, leads to the amplitude values $c_0^{(1)} \approx 1$ and $c_1^{(1)} \approx 1/|h| \approx 0$. Thus, there exists a finite probability of finding the ground state of the atomic ensemble in the W-state. Note that for a large number of atoms, $N_a \gg 1 \wedge N_a \gg \Delta/\lambda$, the amplitude parameter goes to zero, $\lim_{N_a \gg \Delta/\lambda} h \approx (N_a-1)\kappa - (N_a-1)\kappa = 0$, and the probability of finding the first non-vacuum ground state in a W-state is null.

The inclusion of counter-rotating terms yield

$$\hat{H}_{CR} = \omega_f \hat{a}\hat{a}^\dagger + \omega_a \hat{J}_z + \frac{\kappa}{N_a}\hat{J}_z^2 + \frac{2\lambda}{\sqrt{N_a}}(\hat{a}+\hat{a}^\dagger)\hat{J}_x. \tag{9}$$

By using the unitary transformation,

$$\hat{U} = e^{-\imath\xi(\hat{a}+\hat{a}^\dagger)\hat{J}_y}, \quad \xi = \frac{2\lambda}{\sqrt{N}(\omega_a+\omega_f)}, \tag{10}$$

in the weak coupling regime, $\lambda \ll \omega_a$ such that $\xi \ll 1$, it is possible to write the full Hamiltonian, Eq.(9) ,up to first order in ξ, as

$$\begin{aligned} \tilde{H}_{CR} &= \hat{U}^{-1}\hat{H}_{CR}\hat{U}, \\ &\approx \omega_f \hat{a}^\dagger\hat{a} + \left[\omega_a + \xi\lambda(\hat{a}+\hat{a}^\dagger)^2\right]\hat{J}_z + \kappa\hat{J}_z^2 - k\xi(\hat{a}+\hat{a}^\dagger)\left\{\hat{J}_z,\hat{J}_x\right\} + \omega_f\xi(\hat{a}\hat{J}_+ + \hat{a}^\dagger\hat{J}_-) \end{aligned} \tag{11}$$

Requiring weak intra-ensemble interaction, $\kappa \ll \omega_a$, it is possible to approximate

$$\tilde{H}_{CR} \approx \omega_f \hat{a}^\dagger\hat{a} + \omega_a\hat{J}_z + \kappa\hat{J}_z^2 + \frac{\tilde{\lambda}}{\sqrt{N}}(\hat{a}\hat{J}_+ + \hat{a}^\dagger\hat{J}_-), \tag{12}$$

where the auxiliary coupling strength is given by $\tilde{\lambda} = 2\omega_f\lambda/(\omega_a+\omega_f)$. This is the original Hamiltonian, *i.e.* the unitary transformation under weak coupling and intra-ensemble interaction acts as a rotating wave approximation, which has a first phase transition at the critical value on the weak coupling regime given by the expression,

$$\lambda_{WCR1} = \frac{(\omega_a+\omega_f)}{2}\left[\frac{\omega_a + (1-N)\kappa}{\omega_f}\right]^{\frac{1}{2}}. \tag{13}$$

Note that on-resonance the expression for the first critical coupling in the weak coupling regime, Eq.(13), is equal to the critical coupling in the rotating wave approximation, Eq.(6).

In the strong coupling regime, $\lambda \gg \omega_a$, it is possible to suppose the state of the system as the superposition of coherent states, $|\alpha\rangle_f |\theta,\phi\rangle_a$, with $|\theta,\phi\rangle_a = e^{-\imath\hat{J}_z\phi}e^{-\imath\hat{J}_y\theta}|N_a/2\rangle$. Under this representation it is possible to write [45],

$$\langle \hat{J}_z \rangle = \frac{\sqrt{N_a}}{2} \sin\theta, \tag{14}$$

$$\langle \hat{J}_\pm \rangle = \left[\frac{N_a}{2}\left(\frac{N_a}{2}+1\right) - \frac{N_a}{4}\sin^2\theta \pm \frac{\sqrt{N_a}}{2}\sin\theta \right]^{\frac{1}{2}} \left[B_{N_a/2}(\theta) e^{\pm\imath\phi} + \left(\frac{\sin\theta}{2}\right)^{N_a} \right], \tag{15}$$

with the coefficient

$$B_j(\beta) = \sum_m \begin{pmatrix} 2j \\ j+m \end{pmatrix}^{\frac{1}{2}} \begin{pmatrix} 2j \\ j+m+1 \end{pmatrix}^{\frac{1}{2}} \left(\cos\frac{\beta}{2}\right)^{2j+2m+1} \left(\sin\frac{\beta}{2}\right)^{2j-2m-1}. \tag{16}$$

Minimizing the mean value energy for the three coherent parameters, $\{\alpha, \theta, \phi\}$, to find the critical coupling in the strong limit is beyond the scope of this manuscript.

In the classical limit, *i.e.* an infinitely large atomic ensemble interacting with a coherent field, it has been shown that finite size and classical limit analysis for Hamiltonians related to Dicke model yield identical critical couplings [11–13]. By using the Holstein-Primakoff transformation [46], $\hat{J}_z = \hat{b}^\dagger\hat{b} - N_a/2$, $\hat{J}_+ = \sqrt{N_a}\hat{b}^\dagger\left(1 - \hat{b}^\dagger\hat{b}/N_a\right)^{1/2}$, $\hat{J}_+ = \hat{J}_-^\dagger$, and a Taylor expansion of the square root in terms of $(1/N_a)$, up to first order in $(1/N_a)$, $\left(1 - \hat{b}^\dagger\hat{b}/N_a\right)^{1/2} \approx 1 - \hat{b}^\dagger\hat{b}/2N_a$, up to a constant energy term of $N_a(\kappa N_a - 2\omega_a)/2$, the Hamiltonian in Eq.(1) transforms into the expression,

$$\hat{H} \approx \omega_f \hat{a}^\dagger\hat{a} + (\omega_a - \kappa N_a)\hat{b}^\dagger\hat{b} + \kappa(\hat{b}^\dagger\hat{b})^2 + \lambda\left[\hat{a}\hat{b}^\dagger\left(1 - \frac{\hat{b}^\dagger\hat{b}}{2N_a}\right) + \left(1 - \frac{\hat{b}^\dagger\hat{b}}{2N_a}\right)\hat{a}^\dagger\hat{b}\right]. \tag{17}$$

In a Fock basis, the problem transforms into diagonalizing the Hamiltonian in the one excitation subspace spanned by the states $\{|0,1\rangle, |1,0\rangle\}$. Thus, the ground state energy in the classical regime for the first non-vacuum phase is given by the expression,

$$E_{Cg}^{(1)} = \frac{1}{2}\left[\omega_a + \omega_f + (1-N_a)\kappa - \left\{4\lambda^2 + \left[\omega_a - \omega_f + (1-N_a)\kappa\right]^2\right\}^{\frac{1}{2}}\right]. \tag{18}$$

The first critical coupling can be found for $E_g^{(1)} = 0$ and yields a form identical to the exact first critical coupling strength found above in Eq.(6),

$$\lambda_{CC1} = \left\{\left[\omega_a + (1-N_a)\kappa\right]\omega_f\right\}^{\frac{1}{2}}. \tag{19}$$

For the Hamiltonian with counter-rotating terms, Eq.(9), by using the Holstein-Primakoff transformation and the expansion of the square root the Hamiltonian, as stated above, the following Hamiltonian is obtained,

$$\hat{H}_{CR} \approx \omega_f \hat{a}^\dagger \hat{a} + (\omega_a - \kappa N_a)\hat{b}^\dagger \hat{b} + \kappa(\hat{b}^\dagger \hat{b})^2 + \lambda \left(\hat{a} + \hat{a}^\dagger\right)\left(\hat{b} + \hat{b}^\dagger\right) - \frac{\lambda}{2N_a}\left(\hat{a} + \hat{a}^\dagger\right)\hat{b}^\dagger\left(\hat{b} + \hat{b}^\dagger\right)\hat{b}. \tag{20}$$

In order to find the phase transition, it is possible to use the coherent state basis $|\alpha\rangle_a |\beta\rangle_b$, such that the mean value for the energy is given by

$$\langle \hat{H}_{CR} \rangle \approx \omega_f |\alpha|^2 + (\omega_a - \kappa N_a)|\beta|^2 + \kappa |\beta|^4 + \left(\lambda - \frac{\lambda}{2N_a}|\beta|^2\right)(\alpha + \alpha^*)(\beta + \beta^*) \tag{21}$$

Optimizing with respect to the coherent parameters α and β, assuming them to be real numbers for the sake of simplicity, leads to a trivial solution and a non-trivial solution that requires $\lambda \geq [(\omega_a - N_a\kappa)\omega_f]^{1/2}/2$ for the coherent parameters to be real, thus a critical coupling strength for the Hamiltonian including counter rotating terms in the classical regime is found, as

$$\lambda_{CCRC1} = \frac{1}{2}\left[(\omega_a - N_a\kappa)\,\omega_f\right]^{\frac{1}{2}}. \tag{22}$$

This critical coupling, λ_{CCRC1}, is half of the critical strength found for the case without counter-rotating terms, λ_{CC1}, as $(1-N) \approx -N$ for $N \gg 2$. This is similar to what happens for the Dicke model in the classical limit result where accounting for counter-rotating terms halves the critical coupling found without the counter-rotating terms [47].

In order to verify our analytical results, in Fig. 1, we show a numerical phase diagram defined by the coupling strength, λ, and interaction parameter, κ, for a finite size atomic ensemble, $N_a = 5$. The maximum shared bipartite concurrence following the entangled web approach [35] and the field-ensemble entanglement probed through von Neumann entropy of the reduced two-level ensemble, also known as entropy of entanglement [48] are shown in Fig.(1) for the case within the rotating wave approximation and in Fig.(2) for the model including counter-rotating terms. The first and second critical couplings shown in dashed black lines in Fig.(1) are analytically exact obtained from the analysis presented above. Likewise, in Fig.(2) the first critical coupling including counter-rotating terms is shown. Analytical and numerical results are in agreement as shown in the figures.

A ground W-state is found for very small values of the coupling strength and near the second non-vacuum phase for ensembles consisting of a few atoms. Figures 1(a) and 1(b) show that it is possible to have a W-ground-state for a penta-partite ensemble. Even for five two-level systems, there exists a large parameter region at the first non-vacuum phase where maximal value for the maximum shared bipartite concurrence, $C_w = 2/5$, is reached. This value wanes subsequently as expected. The entropy of entanglement rises to a maximum after the first zone, a behavior shared by all studied cases, $N_a \in [2, 6]$, to decay for very large values of the coupling strength, $\lambda \gg \omega_a$.

In contrast to the Dicke model, where the first critical coupling is independent of the ensemble size, in this extended Dicke model the interaction-coupling strength parameters ratio at the first phase transition is found to be inversely proportional to the ensemble size, *e.g.* Eq.(6). In the Dicke model the vacuum phase remains fixed as the first critical coupling stays constant for an increasing number of atoms, while the phases beyond the vacuum phase reduce to allow just the vacuum and classical superradiant phases for an infinitely large atomic ensemble. In this BEC-cavity-QED model, the area of all the different phases reduces as the ensemble size increases.

In conclusion, we have shown a striking feature, a heralded maximal entangled atomic state, at the quantum phase transition of a experimentally feasible BEC-cavity-QED system that might pave the way for quantum information processing.

We have analytically derived the exact critical coupling value and ground states at the first quantum phase transition for a BEC-cavity-QED system in the rotating wave approximation, including counter-rotating terms, and in the classical limit. The most interesting feature of the finite-size ground states is the existence of a maximal entangled multi-qubit W-state, as a finite-size effect, in a certain phase space region within the first non-vacuum phase. Furthermore, the existence of such ground state is heralded by the presence of just the one photon in the cavity. The existence of this maximal entangled atomic ensemble state survives the addition of counter rotating terms. Numerical results for the maximum shared bipartite concurrence, following the entangled web approach, and the field-ensemble entanglement, probed through von Neuman entropy of the reduced two-level ensemble, were presented. Agreement between the exact analytical and the numerical results was obtained.

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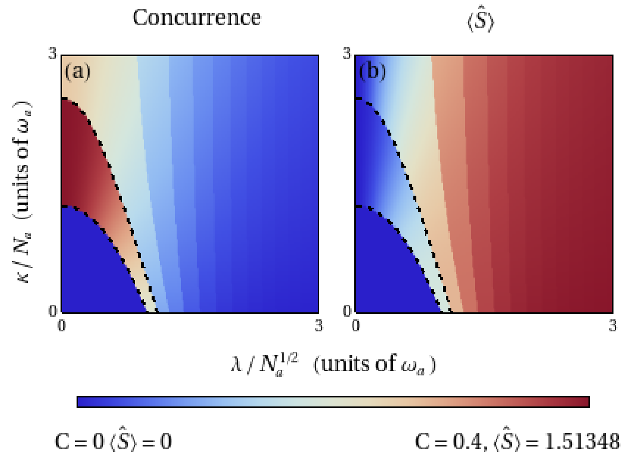

Figure 1: (Color online) The phase diagram of the finite size extended Dicke Hamiltonian for the case of a penta-partite two-level ensemble, $N_a = 5$, in the parameter space defined by the field-ensemble coupling strength, λ, and the intra-ensemble interaction strength, κ. (a) Maximum shared bipartite concurrence in the sense of entangled webs, C_w. (b) Entropy of entanglement, $\langle \hat{S} \rangle$, between the field and the atomic ensemble. The corresponding minima and maxima values for the color legend are shown below it. The dashed lines show the two first exact critical couplings.

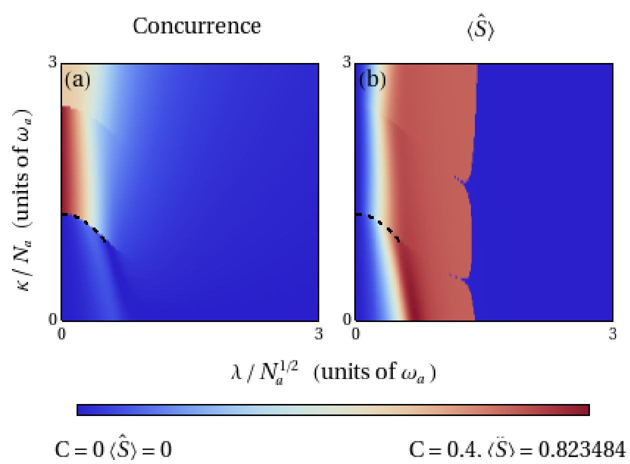

Figure 2: (Color online) Same as Fig.(1) but for the finite size extended Dicke Hamiltonian including counter rotating terms. The dashed line show the first exact critical coupling in the weak coupling regime.