Simulating broken $\mathcal{PT}$-symmetric Hamiltonian systems by weak measurement

Minyi Huang,[1,*] Ray-Kuang Lee,[2,†] Lijian Zhang,[3,‡] Shao-Ming Fei,[4,5,§] and Junde Wu[6,¶]

[1] School of Mathematical Sciences, Zhejiang University, Hangzhou 310027, China
[2] Department of Physics, National Tsing Hua University, Hsinchu 300, Taiwan
[3] College of Engineering and Applied Sciences, Nanjing University, Nanjing 210093, China
[4] School of Mathematical Sciences, Capital Normal University, Beijing 100048, China
[5] Max-Planck-Institute for Mathematics in the Sciences, 04103 Leipzig, Germany
[6] School of Mathematical Sciences, Zhejiang University, Hangzhou 310027, PR China

By embedding a $\mathcal{PT}$-symmetric (pseudo-Hermitian) system into a large Hermitian one, we disclose the relations between $\mathcal{PT}$-symmetric Hamiltonians and weak measurement theory. We show that the amplification effect in weak measurement on a conventional quantum system can be used to effectively simulate a local broken $\mathcal{PT}$-symmetric Hamiltonian system, with the pre-selected state in the $\mathcal{PT}$-symmetric Hamiltonian system and its post-selected state resident in the dilated Hamiltonian system.

arXiv:1809.09918v1 [quant-ph] 26 Sep 2018

Introduction Generalizing the conventional Hermitian quantum mechanics, Bender and his colleagues established the Parity ($\mathcal{P}$)-time ($\mathcal{T}$)-symmetric quantum mechanics in 1998 [1]. With the additional degree of freedom from a non-conservative Hamiltonian, as well as the existence of exceptional points between unbroken and broken $\mathcal{PT}$-symmetries, optical $\mathcal{PT}$-symmetric devices have been demonstrated with many useful applications [2–7]. Although calling for more caution on physical interpretations, especially on the consistency problem of local $\mathcal{PT}$-symmetric operation and the no-signaling principle [8], $\mathcal{PT}$-symmetric quantum mechanics has been stimulating our understanding on many interesting problems such as spectral equivalence [9], quantum brachistochrone [10] and Riemann hypothesis [11].

Compared with the Dirac inner product in conventional quantum mechanics, $\mathcal{PT}$-symmetric quantum theory can be well manifested by the η-inner product [12, 13]. In the broken $\mathcal{PT}$-symmetry case, the η-inner product of a state with itself can be negative, which makes the broken $\mathcal{PT}$-symmetric quantum systems a complete departure from conventional quantum mechanics. While in the unbroken $\mathcal{PT}$-symmetry case, the η-inner product presents a completely analogous physical interpretation to the Dirac inner product, giving rise to many similar properties between $\mathcal{PT}$-symmetric and conventional quantum mechanics. Recent works also show that the η-inner product is tightly related to the properties of superposition and coherence in conventional quantum mechanics [14].

Concerning the relations between the $\mathcal{PT}$-symmetric and conventional quantum mechanics, a natural question is to simulate $\mathcal{PT}$-symmetric quantum systems with conventional quantum theory. On this issue, one should answer the question in what sense a quantum system can be viewed as $\mathcal{PT}$-symmetric. It was showed by Günther and Samsonov that an unbroken $\mathcal{PT}$-symmetric Hamiltonian H on $\mathbb{C}^2$ can be embedded into a Hermitian Hamiltonian $\tilde{H}$ on $\mathbb{C}^4$ in terms of the Naimark dilation [15]. Their paradigm ensures that the evolution $U(t) = e^{-itH}$ is visualized in a subspace. Indeed, such an approach is always possible for any unbroken $\mathcal{PT}$-symmetric systems in finite dimensional spaces [16, 17]. Hence the simulation problem has a satisfactory answer for unbroken $\mathcal{PT}$-symmetry. However, this paradigm fails for the broken $\mathcal{PT}$-symmetric systems, due to the essential distinctions between broken $\mathcal{PT}$-symmetric and conventional systems.

In this Letter, we illustrate the simulation for broken $\mathcal{PT}$-symmetric systems based on weak measurement [18]. For a system weakly coupled to the apparatus, the pointer state will be shifted by the weak value when a weak measurement is performed. The weak value, tightly related to the non-classical features of quantum mechanics, such as the Hardy's paradox [19], three box paradox [20] and Leggett-Grag inequalities [21], can take values beyond the expected values of an observable, and even be a complex number. The weak measurement theory has provided new ways to measure geometric phases [22–25] and non-Hermitian operators [26], as well as to amplify signals as a sensitive estimation of small evolution parameters [27–29]. Our aim is to propose a concrete scenario in which the quantum system can be viewed as $\mathcal{PT}$-symmetric by utilizing the weak measurement. Our result reveals the connections between $\mathcal{PT}$-symmetry and the weak measurement theory, providing the important missing point for the simulation problem of broken $\mathcal{PT}$-symmetric quantum systems.

Generalized embedding of $\mathcal{PT}$-symmetric systems Consider n-dimensional discrete quantum systems. A linear operator P is said to be a parity operator if $P^2 = I$, where I denotes the $n \times n$ identity matrix. An antilinear operator T is said to be a time reversal operator if $T\overline{T} = I$ and $PT = T\overline{P}$, where $\overline{T}$ ($\overline{P}$) stands for the complex conjugation of T (P). A Hamiltonian H is said to be PT-symmetric if $HPT = PT\overline{H}$ [30]. H is called unbroken $\mathcal{PT}$-symmetric if it is diagonalizable and all of its eigenvalues are real. Otherwise, H is called broken $\mathcal{PT}$-symmetric.

In quantum mechanics, a Hamiltonian H gives rise to a unitary evolution of the system. Let ϕ_1 and ϕ_2 be two states. On can introduce a Hermitian operator η to define the η-inner product by $\langle\phi_1|\phi_2\rangle_\eta = \langle\phi_1|\eta|\phi_2\rangle$. With respect to the η-inner product, H presents a unitary evolution if and only if $H^\dagger\eta = \eta H$ [12, 13, 31–33], where $H^\dagger$ denotes the conjugation and transpose of H. Here, η is said to be the metric operator of H. The following theorem gives a useful property of $\mathcal{PT}$-symmetric Hamiltonians.

Theorem 1. *Let H be an $n\times n$ $\mathcal{PT}$-symmetric matrix and η be the metric matrix of H. Let J and S be the canonical forms of H and η, respectively. Then, there exist $n\times n$ invertible matrices Ψ, Ξ, Σ and a $2n\times 2n$ Hermitian matrix $\tilde{H}$ such that for $\tilde{\Psi} = \begin{pmatrix}\Psi\\ \Xi\end{pmatrix}$ and $\tilde{\Phi} = \begin{pmatrix}\Psi\\ \Sigma\end{pmatrix}$, the following equations hold,*

$$\tilde{\Phi}^\dagger\tilde{\Psi} = S, \quad \tilde{\Phi}^\dagger\tilde{H}\tilde{\Psi} = SJ. \tag{1}$$

Proof. A $\mathcal{PT}$-symmetric operator H is similar to its canonical form [16],

$$J = diag(J_{n_1}(\lambda_{n_1},\overline{\lambda}_{n_1}),...,J_{n_p}(\lambda_{n_p},\overline{\lambda}_{n_p}),J_{n_q}(\lambda_{n_q}),...,J_{n_r}(\lambda_{n_r}), \tag{2}$$

where $J_{n_k}(\lambda_{n_k},\overline{\lambda}_{n_k}) = J_{n_k}(\lambda_{n_k}) \oplus J_{n_k}(\overline{\lambda_{n_k}}) = \begin{pmatrix} J_{n_k}(\lambda_{n_k}) & 0\\ 0 & J_{n_k}(\overline{\lambda_{n_k}})\end{pmatrix}$, $J_{n_j}(\lambda_{n_j})$ is the Jordan block, $\lambda_{n_q},\cdots,\lambda_{n_r}$ are real numbers.

For each $\mathcal{PT}$-symmetric operator H, there exists a matrix Ψ' such that $\Psi'^{-1}H\Psi' = J$ and

$$\Psi'^\dagger\eta\Psi' = S = diag(S_{2n_1},...,S_{2n_p},\epsilon_{n_q}S_{n_q},...,\epsilon_{n_r}S_{n_r}), \tag{3}$$

where n_i denotes the order of Jordan blocks in Eq. (2), i.e., $S_k = \begin{pmatrix} & & 1\\ & \iddots & \\ 1 & & \end{pmatrix}_{k\times k}$ and $\epsilon_i = \pm1$ is uniquely determined by η [34]. For convenience, we only consider the situations in which $\epsilon_i = 1$. In this case, S is a permutation matrix and $S^2 = I$. Note that S can be equal to I if and only if H is unbroken $\mathcal{PT}$-symmetric [16]. Henceforth we always assume $S = I$ in the unbroken case for convenience.

Since $\Psi'^\dagger\Psi' > 0$, there always exits a positive number c such that $c^2\Psi'^\dagger\Psi' > 1$. Set $\Psi = c\Psi'$. Let λ_{max} and λ_{min} denote the maximal and minimal eigenvalues of a Hermitian matrix. Then $\lambda_{min}(\Psi^\dagger\Psi - S) \geqslant \lambda_{min}(\Psi^\dagger\Psi) - \lambda_{max}(S) > 0$. Hence $\Psi^\dagger\Psi - S$ is invertible.

Let Ξ be an $n\times n$ invertible matrix. Taking $\Sigma = (\Xi^{-1})^\dagger(S - \Psi^\dagger\Psi)$, $\eta = (\Psi^{-1})^\dagger S\Psi^{-1}$, $H_1 = \eta H$, $H_2 = (\Psi^\dagger)^{-1}(\Xi)^\dagger$ and $H_4 = -H_2^\dagger\Psi\Xi^{-1} - (\Sigma^\dagger)^{-1}\Psi^\dagger H_2$, one can directly verify that $\tilde{H} = \begin{pmatrix} H_1 & H_2\\ H_2^\dagger & H_4\end{pmatrix}$ is Hermitian and Eq. (1) holds. □

Theorem 1 actually gives out the inner product structure of H in a subspace. Note that the matrix Ψ in Theorem 1 can be written as $\Psi = (|\psi_1\rangle,\cdots,|\psi_n\rangle)$, where the column vectors $\{|\psi_i\rangle\}$ form a linear basis of $\mathbb{C}^n$. Similarly, $\Xi = (|\xi_1\rangle,\cdots,|\xi_n\rangle)$ and $\Sigma = (|\sigma_1\rangle,\cdots,|\sigma_n\rangle)$. Correspondingly we have $\tilde{\Psi} = (|\tilde{\psi}_1\rangle,\cdots,|\tilde{\psi}_n\rangle)$ and $\tilde{\Phi} = (|\tilde{\phi}_1\rangle,\cdots,|\tilde{\phi}_n\rangle)$, where $\tilde{\psi}_i = \begin{pmatrix}|\psi_i\rangle\\ |\xi_i\rangle\end{pmatrix}$ and $\tilde{\phi}_i = \begin{pmatrix}|\psi_i\rangle\\ |\sigma_i\rangle\end{pmatrix}$. Moreover, $\tilde{\Phi}S = (|\tilde{\mu}_1\rangle,\cdots,|\tilde{\mu}_n\rangle) = (|\tilde{\phi}_{s(1)}\rangle,\cdots,|\tilde{\phi}_{s(n)}\rangle)$, where S is the permutation matrix in Theorem 1, and s is the permutation induced by S. Similarly, we can write $\Psi S = (|\mu_1\rangle,\cdots,|\mu_n\rangle)$, where $|\mu_i\rangle = |\psi_{s(i)}\rangle$. From the definition of $|\tilde{\mu}_i\rangle$, we have $\langle\tilde{\mu}_i|\tilde{\psi}_j\rangle = (S\tilde{\Phi}^\dagger\tilde{\Psi})_{ij}$ and $\langle\tilde{\mu}_i|\tilde{H}|\tilde{\psi}_j\rangle = (S\tilde{\Phi}^\dagger\tilde{H}\tilde{\Psi})_{ij}$. According to Eq. (1), we have

$$\langle\tilde{\mu}_i|\tilde{\psi}_j\rangle = \delta_{i,j}, \quad \langle\tilde{\mu}_i|\tilde{H}|\tilde{\psi}_j\rangle = J_{i,j}, \tag{4}$$

where $J_{i,j}$ is the (i,j)-th entry of J.

On the other hand, note that the metric matrix η of H is $(\Psi^\dagger)^{-1}S\Psi^{-1}$. Thus we have the following relations between the Dirac and η-inner products

$$\langle\tilde{\mu}_i|\tilde{\psi}_j\rangle = \langle\mu_i|\psi_j\rangle_\eta, \tag{5}$$

$$\langle\tilde{\mu}_i|\tilde{H}|\tilde{\psi}_j\rangle = \langle\mu_i|H|\psi_j\rangle_\eta, \tag{6}$$

where $\langle\mu_i|H|\psi_j\rangle_\eta = \langle\mu_i|\eta H|\psi_j\rangle$. The results show that there exist two different sets of basis vectors with the same projections onto the subspace of the $\mathcal{PT}$-symmetric system, with respect to the η-inner product. When confined to the subspace, the Hermitian Hamiltonian $\tilde{H}$ in large space has the same effect as a $\mathcal{PT}$-symmetric Hamiltonian H, in the sense of this η-inner product.

Simulation of $\mathcal{PT}$-symmetric Hamiltonian systems To infer a quantum system is $\mathcal{PT}$-symmetric, it is sufficient to identify the Hamiltonian and its inner product structure. In the weak measurement formalism, one starts by pre-selecting an initial state $|\varphi_i\rangle$. The target system is coupled to the measurement apparatus, which is in a pointer state $|P\rangle$. Let A be an observable of the system and M be that of the apparatus. The interaction Hamiltonian between the system and apparatus is $H_{int} = f(t)A\otimes M$, with sufficiently small interaction strength $g = \int f(t)dt$. The state evolves as $|\varphi_i\rangle\otimes|P\rangle \to e^{-igA\otimes M}|\varphi_i\rangle\otimes|P\rangle \approx (I - igA\otimes M)|\varphi_i\rangle\otimes|P\rangle$. For a post-selected state $|\varphi_f\rangle$ that $\langle\varphi_f|\varphi_i\rangle \neq 0$, one has $\langle\varphi_f|e^{-igA\otimes M}|\varphi_i\rangle|P\rangle \approx \langle\varphi_f|\varphi_i\rangle e^{-ig\langle A\rangle_w M}|P\rangle$, where $\langle A\rangle_w = \frac{\langle\varphi_f|A|\varphi_i\rangle}{\langle\varphi_f|\varphi_i\rangle}$ is the so-called weak value. Meanwhile, the pointer state $|P\rangle$ changes to $e^{-ig\langle A\rangle_w M}|P\rangle$, with the post-selection probability $p = |\langle\varphi_f|\varphi_i\rangle|^2(1 + 2gIm\langle A\rangle_w\langle P|M|P\rangle)$. The amplification effect of weak measurement ensures that the state is shifted by the weak value which can be read out experimentally [35].

From Eq. (4), we have $\lambda_i = J_{i,i} = \langle\tilde{\mu}_i|\tilde{H}|\tilde{\psi}_i\rangle = \frac{\langle\tilde{\mu}_i|\tilde{H}|\tilde{\psi}_i\rangle}{\langle\tilde{\mu}_i|\tilde{\psi}_i\rangle}$. Therefore, the eigenvalues of H can be obtained via a weak measurement, by pre-selecting the vector $|\tilde{\psi}_i\rangle$ and post-selecting the vector $|\tilde{\mu}_i\rangle$. This observation implies that one can use weak measurement to simulate the measurements on a $\mathcal{PT}$-symmetric system.

In conventional quantum mechanics, the expectation value of an Hermitian Hamiltonian $H_0 = \sum_i \lambda_i|u_i\rangle\langle u_i|$ with respect to a sate $|\psi_0\rangle = \sum_i d_i|u_i\rangle$ is given by the inner product $\langle\psi_0|H_0|\psi_0\rangle$. For $\mathcal{PT}$-symmetric Hamiltonian system with the metric matrix η, the expectation value of a Hamiltonian H with respect to a state $|u\rangle = \sum_i a_i|\psi_i\rangle$ of the $\mathcal{PT}$-symmetric system is instead given by $\langle u|H|u\rangle_\eta$. Given two vectors $|v\rangle = \sum_i b_i|\mu_i\rangle$ and $|w\rangle = \sum_i c_i|\psi_i\rangle$ of the $\mathcal{PT}$-symmetric system. Let $|\tilde{v}\rangle = \sum_i b_i|\tilde{\mu}_i\rangle$ (unnormalized for convenience) and $|\tilde{w}\rangle = \sum_i c_i|\tilde{\psi}_i\rangle$ be two vectors in the extended system. It follows from Eq. (6) that $\langle v|H|w\rangle_\eta = \langle\tilde{v}|\tilde{H}|\tilde{w}\rangle$. Assume that $|u\rangle$ satisfies the condition $\langle u|u\rangle_\eta = \pm 1$. Now take two states $|\tilde{u}_1\rangle = \sum_i a_{s(i)}|\tilde{\mu}_i\rangle$ and $|\tilde{u}_2\rangle = \sum_i a_i|\tilde{\psi}_i\rangle$, whose projections to the $\mathcal{PT}$-symmetric subspace are both $|u\rangle$. Then we have

$$\frac{\langle u|H|u\rangle_\eta}{\langle u|u\rangle_\eta} = \frac{\langle\tilde{u}_1|\tilde{H}|\tilde{u}_2\rangle}{\langle\tilde{u}_1|\tilde{u}_2\rangle}. \tag{7}$$

Therefore, confined to the $\mathcal{PT}$-symmetric subspace, a weak measurement can completely describe the expectations of H.

In conventional quantum mechanics, when an eigenvalue is detected, the measured state collapses to the corresponding eigenstate. However, the problem in $\mathcal{PT}$-symmetric system is subtle. According to Eq. (5), $\langle\psi_i|\psi_i\rangle_\eta \neq 0$ only if $i = s(i)$. This observation makes it reasonable to assume that for any vector $|u\rangle = \sum_i a_i|\psi_i\rangle$ satisfying $\langle u|u\rangle_\eta \neq 0$, if $a_i \neq 0$, then $a_{s(i)} \neq 0$. That is, if $\langle u|u\rangle_\eta \neq 0$, its vector components of $|\psi_i\rangle$ and $|\psi_{s(i)}\rangle$ take zero or nonzero values simultaneously, while the eigenvalues associated with ψ_i and $\psi_{s(i)}$ are either equal or complex conjugations. In this case, one can generalize the detection of an eigenvalue of λ_i in conventional quantum mechanics to the following. For $|u\rangle = \sum_i a_i|\psi_i\rangle$, if the value of

$$\frac{a_i\overline{a_{s(i)}}\lambda_i + \overline{a}_i a_{s(i)}\overline{\lambda}_i}{a_i\overline{a_{s(i)}} + \overline{a}_i a_{s(i)}}$$

is detected [36], the state $|u\rangle$ will collapse to

$$\frac{a_i|\psi_i\rangle + a_{s(i)}|\psi_{s(i)}\rangle}{|a_i\overline{a_{s(i)}} + a_{s(i)}\overline{a_i}|^{\frac{1}{2}}}.$$

Apparently, when $i = s(i)$, the state $|u\rangle$ will collapse to $|\psi_i\rangle$, similar to the case of conventional quantum mechanics. Note that $i = s(i)$ only if the system is unbroken $\mathcal{PT}$-symmetric, for which it is analogous to conventional quantum mechanics and such an analogy in state collapse is not unexpected.

By pre- and post-selecting the states, we see that the weak measurements can successfully simulate an arbitrary η-inner product. Furthermore, when confined to the subspace, the measurement results actually abstract the same information as a $\mathcal{PT}$-symmetric Hamiltonian system. Such information help us eventually infer that the subsystem is $\mathcal{PT}$-symmetric.

Discussions and conclusion We further discuss the mechanism and physical implications related to the weak measurement paradigm, by comparing it with the embedding paradigm [15, 16]. The essence of the embedding paradigm is to realize the evolution of a $\mathcal{PT}$-symmetric Hamiltonian in the subspace, by evolving the state under the Hermitian Hamiltonian in the large space. The problem can be mathematically described as follows [16]: For a given $n \times n$ unbroken $\mathcal{PT}$-symmetric Hamiltonian H, find a $2n \times 2n$ Hermitian matrix $\tilde{H}$, $n \times n$ invertible matrices Ψ, Ξ so that $\tilde{\Psi}^\dagger\tilde{\Psi} = I$ and the following equations

$$e^{-it\tilde{H}}\tilde{\Psi} = \tilde{\Psi}e^{-itJ}, \quad e^{-itH}\Psi = \Psi e^{-itJ} \tag{8}$$

hold, where $\tilde{\Psi} = \begin{pmatrix}\Psi \\ \Xi\end{pmatrix}$. The equations are actually equivalent to the following conditions [37]:

$$\tilde{\Psi}^\dagger\tilde{\Psi} = I, \;\; \tilde{H}\tilde{\Psi} = \tilde{\Psi}J, \;\; H\Psi = \Psi J. \tag{9}$$

Equation (8) ensures that the unitary evolution $\tilde{U}(t)$ gives the evolution $U(t)$ of a $\mathcal{PT}$-symmetric Hamiltonian H in a subspace. In this sense, the embedding paradigm gives a natural way of simulation. Nevertheless, in the broken $\mathcal{PT}$-symmetric case, the solutions do not exist [16]. In fact, Eq. (1) is mathematically a generalization of Eq. (9) [38]. Like the case of the embedding paradigm, it is natural to further require that $\tilde{\Phi}^\dagger e^{-it\tilde{H}}\tilde{\Psi} = Se^{-itJ}$, so that $e^{-it\tilde{H}}$ gives the same effect as e^{-itH} in the subspace. However, such a requirement cannot be satisfied for broken $\mathcal{PT}$-symmetry, which is obvious by the unboundedness of Se^{-itJ}.

However, consider sufficiently small time $t \in [0, \epsilon]$. We have $|\tilde{u}(t)\rangle\rangle = e^{-it\tilde{H}}|\tilde{u}\rangle = (I - it\tilde{H})|\tilde{u}\rangle$. On the other hand, $|u(t)\rangle\rangle = e^{-itH}|u\rangle = (I - itH)|u\rangle$. Now equations Eqs. (5) and (6) insure that when confined to the subspace, $|\tilde{u}(t)\rangle$ is equivalent to $|u(t)\rangle$ in the sense of η-inner product. This observation implies that $\mathcal{PT}$-symmetric quantum systems can be well approximated in a sufficiently small time evolution, by choosing two different sets of basis $\{|\tilde{\phi}\rangle\}$ and $\{|\tilde{\psi}\rangle\}$ with the same components in the subspace, which can be realized by weak measurement. Here instead of the time interval, the sufficiently small interaction strength g in weak measurement ensures the first order approximation. And the choice of the two basis is realized by the pre- and post-selection processes. The weak measurement paradigm can be viewed as a generalization of the embedding paradigm, due to the fact that Eq. (9) is a special case of Eq. (1) in the $\mathcal{PT}$-symmetric unbroken case. Hence, the Hamiltonian $\tilde{H}$ in the embedding paradigm can also be utilized in the weak measurement approach, although the embedding paradigm itself does not work. Comparing our approach with that in [26], where one obtains the expected value of a Hamiltonian in the Dirac inner product by using the polar decomposition, our method lays emphasis on the properties of a $\mathcal{PT}$-symmetric Hamiltonian with respect to the η-inner product.

In summary, we have proposed a weak measurement paradigm to investigate the behaviors of broken $\mathcal{PT}$-symmetric Hamiltonian systems. By embedding the $\mathcal{PT}$-symmetric system into a large Hermitian system and utilizing weak measurements, we have shown how a $\mathcal{PT}$-symmetric Hamiltonian can be simulated. Our paradigm may shine new light on the study of $\mathcal{PT}$-symmetric quantum mechanics and its physical implications and applications.

This work is supported by National Natural Science Foundation of China (11171301, 11571307, 11690032, 61490711 and 11675113) and the NSF of Beijing under Grant No. KZ201810028042.

* 11335001@zju.edu.cn
† rklee@mx.nthu.edu.tw
‡ lijian.zhang@nju.edu.cn
§ feishm@cnu.edu.cn
¶ wjd@zju.edu.cn

[1] C. M. Bender and S. Boettcher, Phys. Rev. Lett. 80, 5243 (1998).
[2] R. El-Ganainy, K. Makris, D. Christodoulides, and Z. H. Musslimani, Opt. Lett. 32, 2632 (2007).
[3] K. G. Makris, R. El-Ganainy, D. Christodoulides, and Z. H. Musslimani, Phys. Rev. Lett. 100, 103904 (2008).
[4] A. Guo, G. Salamo, D. Duchesne, R. Morandotti, M. Volatier-Ravat, V. Aimez, G. Siviloglou, and D. Christodoulides, Phys. Rev. Lett. 103, 093902 (2009).
[5] C. E. Ruter, K. G. Makris, R. El-Ganainy, D. N. Christodoulides, M. Segev, and D. Kip, Nat. Phys. 6, 192 (2010).
[6] L. Chang, X. Jiang, S. Hua, C. Yang, J. Wen, L. Jiang, G. Li, G. Wang, and M. Xiao, Nat. Photon. 8, 524 (2014).
[7] J.-S. Tang, Y.-T. Wang, S. Yu, D.-Y. He, J.-S. Xu, B.-H. Liu, G. Chen, Y.-N. Sun, K. Sun, Y.-J. Han, C.-F. Li, and G.-C. Guo, Nat. Photon. 10, 642 (2016).
[8] Y.-C. Lee, M.-H. Hsieh, S. T. Flammia, and R.-K. Lee, Phys. Rev. Lett. 112, 130404 (2014).
[9] P. Dorey, C. Dunning, and R. Tateo, J. Phys. A: Math. Theor. 34, 5679 (2001); *ibid* 40, R205 (2007).
[10] U. Gunther and B. F. Samsonov, Phys. Rev. A 78, 042115 (2008).
[11] C. M. Bender, D. C. Brody, and M. P. Muller, Phys. Rev. Lett. 118, 130201 (2017).
[12] A. Mostafazadeh, J. Math. Phys. 43, 205, 2814, and 3944 (2002).
[13] A. Mostafazadeh, Int. J. Geom. Meth. Mod. Phys. 7, 1191 (2010).
[14] M. Huang, R.-K. Lee and J. Wu, J. Phys. A: Math. Theor. 51, 414004 (2018).
[15] U. Gunther and B. F. Samsonov, Phys. Rev. Lett. 101, 230404 (2008).
[16] M. Huang, A. Kumar and J. Wu, Phys. Lett. A 382, 2578 (2018).
[17] K. Kawabata, Y. Ashida, and M. Ueda, Phys. Rev. Lett. 119, 190401 (2017).
[18] Y. Aharonov, D. Z. Albert, and L. Vaidman, Phys. Rev. Lett. 60, 1351 (1988).
[19] Y. Aharonov, A. Botero, S. Popescu, B. Reznik, and J. Tollaksen, Phys. Lett. A 301, 130 (2002).
[20] K. J. Resch, J. S. Lundeen, and A. M. Steinberg, Phys. Lett. A 324, 125 (2004).
[21] A. Palacios-Laloy, F. Mallet, F. Nguyen, P. Bertet, D. Vion, D. Esteve, and A. N. Korotkov, Nat. Phys. 6, 442 (2010).
[22] O. Hosten and P. Kwiat, Science 319, 787 (2008).
[23] E. Sjoqvist, Phys. Lett. A 359, 187 (2006).
[24] H. Kobayashi, S. Tamate, T. Nakanishi, K. Sugiyama, and M. Kitano, Phys. Rev. A 81, 012104 (2010).
[25] L. Zhang, A. Datta, and I. A. Walmsley, Phys. Rev. Lett. 114, 210801 (2015)
[26] A. K. Pati, U. Singh, and U. Sinha, Phys. Rev. A 92, 052120 (2015).
[27] J. S. Lundeen, B. Sutherland, A. Patel, C. Stewart, and C. Bamber, Nature 474, 188 (2011).
[28] D. J. Starling, P. B. Dixon, A. N. Jordan, and J. C. Howell, Phys. Rev. A 82, 063822 (2010).
[29] N. Brunner and C. Simon, Phys. Rev. Lett. 105, 010405 (2010).
[30] The definitions of $\mathcal{PT}$-symmetry in this paper is actually equivalent to the pseudo-Hermitian. Hence we will not distinguish between the two concepts.
[31] J.-w. Deng, U. Guenther, and Q.-h. Wang, arXiv:1212.1861 (2012).
[32] P. D. Mannheim, Phil. Trans. Royal Soc. London A: Math. Phys. Eng. Sci. 371, 20120060 (2013).
[33] R. A. Horn and C. R. Johnson, *Matrix analysis*, (Cambridge University, 2012).
[34] I. Gohberg, P. Lancaster, and L. Rodman, *Matrices and indefinite scalar products*, vol. 8 (1983).
[35] J. Dressel, M. Malik, F. M. Miatto, A. N. Jordan, and R. W. Boyd, Rev. Mod. Phys. 86, 307 (2014).
[36] Actually, this value is $\frac{\langle a_i\psi_i+a_{s(i)}\psi_{s(i)}|H|a_i\psi_i+a_{s(i)}\psi_{s(i)}\rangle_\eta}{a_i\overline{a_{s(i)}}+\overline{a_i}a_{s(i)}}$, which reduces to $\langle\psi_i|H|\psi_i\rangle_\eta = \lambda_i$ if $i = s(i)$ (only one vector ψ_i considered).
[37] Equation (8) is actually the matrix version of the embedding in [16]. Denote $\Xi = \tau\Psi$. Then (9) reduces to $H_1 + H_2\tau = H$ and $H_2^\dagger + H_4\tau = \tau H$, which gives the equivalent description of the embedding property. A concrete solution to (8) can also be found in [17].
[38] Apparently, if we take $\Xi = \Sigma$ and $S = I$, which is always possible in the unbroken $\mathcal{PT}$ symmetric case, Eq. (9) is a special case of Eq. (1).