The internal nonlocality in $\mathcal{PT}$-symmetric systems

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In this paper, we investigate the properties of $\mathcal{PT}$-symmetric quantum systems, proposing a new way to illustrate its novelty and measure its departure from classical and Hermitian quantum systems. We argue that there exists some internal nonlocal structure when simulating $\mathcal{PT}$-symmetric quantum systems. This is done by unveiling its merit in a concrete scenario. In addition, by using the dilation method, it is clearly shown that there exists some bounds of the correlation expectations. Unusually, the bounds of classical and local Hermitian models coincide and is larger than the simulation model of $\mathcal{PT}$-symmetric systems. Moreover, it is shown that the exceptional point is some extreme point in this case. When the parameters are adjusted and the Hamiltonian tends to the exceptional point, it also tends to a largest departure from the usual Hermitian systems. This also provides us with a new perspective to discuss the properties of the exceptional point.

I. INTRODUCTION

In recent years, researchers have witnessed a growing interests in discussing non-Hermitian systems, especially in the field of dynamics and topology [1]. Lots of work have been done and many intriguing properties of non-Hermitian systems are revealed and discussed. The related topics, such as skin effect, attracts much increasing attentions [2–6].

As one of the most important classes of non-Hermitian systems, $\mathcal{PT}$-symmetric systems are of of interests both theoretically and experimentally. The systematic researches of such systems began in 1998, with Bender and his colleagues' discussion on the reality of the eigenvalues of a class of $\mathcal{PT}$-symmetric Hamiltonians [7]. Since then, lots of work have been done to investigate $\mathcal{PT}$-symmetric quantum systems. An important work is given by Mostafazadeh, which generalized $\mathcal{PT}$-symmetric theory to pseudo-hermitian theory [8–11]. Recently, there are also discussions on anti-$\mathcal{PT}$-symmetric systems.

Recently, in the field of quantum information, there are some new works discussing $\mathcal{PT}$-symmetric systems, but mostly are limited to some information quantities [12–14]. They can explain the properties of the systems and sometimes provides some physical intuitions, which might help in discussing some phenomena. However, most of these works does not deal with a physically realizable scenario and thus the discussions are made in an indirect way. Thus direct ways, especially discussions which can connect $\mathcal{PT}$-symmetric systems to concrete quantum information scenarios are still of interests and needed.

Despite the initial motivation to establish a new and more physical framework of quantum theory, researchers also view $\mathcal{PT}$-symmetric systems as effective descriptions of large Hermitian systems in some subspaces. Such a viewpoint is also natural and provides us with some useful tools to discuss

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Hermitian quantum systems, which also has deep connections with topics such as Feshbach formalism in thought and theoretical background. The first attempt in this approach is Günther and Samsonov's work, in which a class of two dimensional unbroken $\mathcal{PT}$-symmetric systems are embedded in four dimensional spaces, by using the Naimark dilation method [15]. Actually, by improving the techniques and further scrutinizing the mathematical essence, later researches showed that any finite dimensional unbroken $\mathcal{PT}$-symmetric systems can be dilated in this sense [16, 17]. By dilating the system to a large Hermitian one and projecting out the ancillary system, this paradigm successfully simulates the evolution of unbroken $\mathcal{PT}$-symmetric Hamiltonians. Such a way, inspired by Naimark dilation and typical ideas in quantum simulation, endows direct physical meaning of $\mathcal{PT}$-symmetric quantum systems in the sense of open systems. As for the broken $\mathcal{PT}$-symmetric case, there are also different approaches. One way is utilizing weak measurement, which can be viewed as an approximation paradigm of broken $\mathcal{PT}$-symmetric systems [18]. Another impressive work is using time dependent Hamiltonians, which can simulates time dependent evolution of broken $\mathcal{PT}$-symmetric systems, which can also help in discussing the topology and dynamics [19].

In this paper, we propose a new way to discuss $\mathcal{PT}$-symmetric systems, as well as to measure its departure from classical and Hermitian quantum systems. We argue that there exists some internal nonlocal structure when simulating $\mathcal{PT}$-symmetric quantum systems. A concrete scenario and three models for discussing such correlations is given. In addition, by using the dilation method, it is clearly shown that in this scenario, there exist some bounds of the correlation expectations. Interestingly, the bounds of classical and local Hermitian models coincide and are larger than the simulation model. Moreover, it is shown that the exceptional point is some extreme point in this case, which means a largest departure from the usual Hermitian systems.

II. PRELIMINARIES

A. Basic notions of $\mathcal{PT}$-symmetric systems

As our discussions are limited to finite dimensional spaces, some basic notions are briefly introduced as follows.

A parity operator $\mathcal{P}$ is a linear operator such that $\mathcal{P}^2 = \mathcal{I}_d$, where $\mathcal{I}_d$ is the identity operator on $\mathbb{C}^d$.

A time reversal operator $\mathcal{T}$ is an anti-linear operator such that $\mathcal{T}^2 = \mathcal{I}_d$. Moreover, it is demanded that $\mathcal{PT} = \mathcal{TP}$.

A linear operator $\mathcal{H}$ on $\mathbb{C}^d$ is said to be $\mathcal{PT}$-symmetric if $\mathcal{HPT} = \mathcal{PTH}$.

In finite dimensional case, a linear operator corresponds uniquely to a matrix and an anti-linear operator corresponds to the composition of a matrix and a complex conjugation [20]. Let A be a matrix, with $\overline{A}$ the complex conjugation of A and $A^\dagger$ the transpose of $\overline{A}$. Let P, T and H be the matrices of $\mathcal{P}$, $\mathcal{T}$ and $\mathcal{H}$, respectively. Then the definition conditions of $\mathcal{P}$, $\mathcal{T}$, $\mathcal{H}$ are $P^2 = T\overline{T} = I$, $PT = T\overline{P}$ and $HPT = PT\overline{H}$.

By considering the spectral property of H, $\mathcal{PT}$-symmetric systems can be classified into two classes:

A $\mathcal{PT}$-symmetric operator $\mathcal{H}$ is said to be unbroken if H is similar to a real diagonal matrix;

A $\mathcal{PT}$-symmetric operator $\mathcal{H}$ is said to be broken if H cannot be diagonalised or has complex eigenvalues.

B. Dilation method

We first recall the definition of dilation [16]. Let $\mathcal{H}$ be a $\mathcal{PT}$-symmetric operator on $\mathbb{C}^n$ and $\hat{\mathcal{H}}$ be a

Hermitian operator on $\mathbb{C}^m$, where $m > n$. $\mathcal{P}_1$ is an operator defined by $\mathcal{P}_1 : \mathbb{C}^m \to \mathbb{C}^n$, $\mathcal{P}_1 \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \phi_1$, where $\phi_1 \in \mathbb{C}^n$ and $\phi_2 \in \mathbb{C}^{m-n}$.

Let $X_{\hat{\mathcal{H}}} = \{x : x \in \mathbb{C}^m, \mathcal{P}_1 \hat{\mathcal{H}} x = \mathcal{H}\mathcal{P}_1 x, \mathcal{P}_1 e^{-it\hat{H}} x = e^{-itH}\mathcal{P}_1 x\}$.

If $\mathcal{P}_1 X_{\hat{\mathcal{H}}} = \mathbb{C}^n$, then we say that $\mathcal{H}$ can be dilated to $\hat{\mathcal{H}}$ and $\hat{\mathcal{H}}$ is a Hermitian dilation of $\mathcal{H}$.

The meaning of this definition is that by evolving a Hermitian Hamiltonian on a large space, the $\mathcal{PT}$-symmetric evolution can be realized in the subspace. It can be proved that only unbroken $\mathcal{PT}$-symmetric operators can be dilated in this way. Exactly, this definition actually gives the following equations (unnormalised for convenience),

$$\begin{aligned} &\tilde{H}(|0\rangle|\psi\rangle + |1\rangle|\tau\psi\rangle) \\ &= |0\rangle|H\psi\rangle + |1\rangle|\tau H\psi\rangle, \qquad (1) \\ &\tilde{U}(t)(|0\rangle|\psi\rangle + |1\rangle|\tau\psi\rangle) \\ &= |0\rangle|U(t)\psi\rangle + |1\rangle|\tau U(t)\psi\rangle. \qquad (2) \end{aligned}$$

The above equations do not determine $\tilde{H}$ uniquely. To determine $\tilde{H}$, one can further add the following conditions,

$$\begin{aligned} &\tilde{H}(|0\rangle| - \tau\psi\rangle + |1\rangle|\psi\rangle) \\ &= |0\rangle| - \tau H\psi\rangle + |1\rangle|H\psi\rangle, \qquad (3) \\ &\tilde{U}(t)(|0\rangle| - \tau\psi\rangle + |1\rangle|\psi\rangle) \\ &= |0\rangle| - U(t)\tau\psi\rangle + |1\rangle|U(t)\psi\rangle. \qquad (4) \end{aligned}$$

It can be shown that there exists $\tilde{H}$ satisfy all the above four equations. Apparently, $\tilde{H}$ has a two folded structure, that is, it has the same eigenvalues as H, with multiplicity of two.

It should be noted that the constructions of $\tilde{H}$ actually allows us to use measurements on the large systems to simulate measurements on the $\mathcal{PT}$-symmetric system in the subspace. For more details, see [16, 17].

C. Two dimensional model

A typical two dimensional $\mathcal{PT}$-symmetric Hamiltonian was first given by [21]. In [15], the authors considered a special case of that in [21], which is

$$H = E_0 I_2 + s \begin{bmatrix} i\sin\alpha & 1 \\ 1 & -i\sin\alpha \end{bmatrix}. \qquad (5)$$

The eigenvalues are $\lambda_\pm = E_0 \pm s\cos\alpha$. In this case, $\alpha = \frac{\pi}{2}$ is the exceptional point. When α takes other values, $\mathcal{PT}$-symmetry is unbroken.

The discussions of dilation method also contains this Hamiltonian as a special case. The concrete form of the four dimensional Hermitian dilation Hamiltonian $\tilde{H}$ of the $\mathcal{PT}$-symmetric Hamiltonian H in Eq. (5) is as follows [15],

$$\begin{aligned} \tilde{H} &= f^2[I_2 \otimes (H\eta^{-1} + \eta H) + i\sigma_y \otimes (H - H^\dagger)] \\ &= I_2 \otimes \Lambda + i\sigma_y \otimes \Omega, \qquad (6) \\ \Lambda &= E_0 I_2 + \frac{\omega_0}{2}\cos\alpha\sigma_x, \qquad (7) \\ \Omega &= i\frac{\omega_0}{2}\sin\alpha\sigma_z, \qquad (8) \end{aligned}$$

where $\omega_0 = 2s\cos\alpha$.

III. THE INTERNAL NON-LOCALITY OF PT-SYMMETRIC HAMILTONIANS

To experimentally simulate a $\mathcal{PT}$-symmetric quantum system, a large system is needed, with a non-separable Hermitian dilation Hamiltonian $\tilde{H}$ on it. In addition, note that $\tilde{H}$ is an observable and measuring it will give rise to some randomness. Hence $\tilde{H}$ can be viewed as some resource in simulating a $\mathcal{PT}$-symmetric system. Moreover, this naturally implies the possibility of investigating $\mathcal{PT}$-symmetry from the perspective of correlations or non-locality. To be exact, what we want to see is the internal correlation of the large Hermitian Hamiltonian and its effect on the $\mathcal{PT}$-symmetric system and the exceptional point.

To this end, let Alice and Bob be two observers and share some randomness given by the Hermitian dilation Hamiltonian $\tilde{H}$. Now, to see the internal correlations, Alice and Bob only need to "make local measurements". Then by investigating the products of Alice's and Bob's results, one can discuss the internal correlations of $\tilde{H}$ and the simulation procedures. The following rule is key to the construction of a concrete scenario and its classical and quantum models:

The evolution of the Hermitian dilation Hamiltonian $\tilde{H}$ is used to simulate the evolution of $\mathcal{PT}$-symmetric Hamiltonian $\tilde{H}$, the measurements on the large space are used to simulate the measurements of the $\mathcal{PT}$-symmetric system in the subspace.

The above rule means that our scenario should be consistent with the simulation of $\mathcal{PT}$-symmetric systems.

For convenience, take Eq.(5) as an example. The concrete scenario and its classical model is as follows. Let Alice and Bob share some randomness, which is given by the "resource for simulation". Note that here we do not state the resource is $\tilde{H}$ since we are describing a generic scenario. Now suppose Alice can choose to make two measurements A_0 and A_1 and both of A_i may have several random results. Similarly, Bob make two measurements B_0 and B_1. Then consider the following expectation of correlations,

$$\sum(-1)^{i+j+ij}\langle AB\rangle_{ij}. \tag{9}$$

The readouts of A_i and B_j are still untouched. Note that our scenario should be consistent with the simulation, hence it is natural to assume that either Alice or Bob has a "$\mathcal{PT}$-symmetric like" system and either A_i or B_i are measuring the energy of the local systems. That is, one of Alice or Bob is making measurements of some Hamiltonian whose two eigenvalues are just $\lambda_\pm$, the same as the $\mathcal{PT}$-symmetric Hamiltonian H. Still according to the rule above, we are using measurements of $\tilde{H}$ to simulate the measurement of H. Hence the results of the measurements on the other side should be 1, such that the product of Alice's and Bob's result trivially gives out the eigenvalues of $\tilde{H}$. Now that the A_i and B_i are determined, the scenario is completed. Moreover, since one of Alice's or Bob's results is always 1, it is apparently the two observers' results and probability distributions are independent. Thus the above model is a classical local one.

Now consider the value of Eq. (9). Without loss of generality, one can assume that A_i take value of 1 and B_i are measurements of the eigenvalues,

$$\begin{aligned}
&\sum(-1)^{i+j+ij}\langle AB\rangle_{ij}\\
&=\int[A_0(\nu)(B_0+B_1)(\nu)+A_1(\nu)(B_0-B_1)(\nu)]d\nu\\
&=\int[(B_0+B_1)(\nu)+(B_0-B_1)(\nu)]d\nu\\
&=2E_0+\omega_0(p_+-p_-).
\end{aligned} \tag{10}$$

where $p_\pm$ are the probabilities that the results of B_0 are $\lambda_\pm$.

Now we consider the quantum values of Eq (9). First, it should be noted that there are two such quantum values since there are two different physical mechanism, one is the simulation scenario while the other is local Hamiltonians.

First consider the simulation scenario. In this case, the randomness comes from the measurements of the Hermitian dilation Hamiltonian $\tilde{H}$, shared by Alice and Bob. Now what to do is just replace measurements A_i and B_i with some states. Suppose Alice uses two local states $\{|0\rangle,|1\rangle\}$ instead of A_0, A_1 and Bob can use local states $|u_+\rangle = u|0\rangle + v|1\rangle, |u_-\rangle = v|0\rangle - u|1\rangle$ instead of B_0, B_1 for measurements.

The four expectations are

$$\langle A_0B_0\rangle = Tr|0\rangle\langle 0|\otimes|u_+\rangle\langle u_+|\tilde{H}, \tag{11}$$
$$\langle A_0B_1\rangle = Tr|0\rangle\langle 0|\otimes|u_-\rangle\langle u_-|\tilde{H}, \tag{12}$$
$$\langle A_1B_0\rangle = Tr|1\rangle\langle 1|\otimes|u_+\rangle\langle u_+|\tilde{H}, \tag{13}$$
$$\langle A_1B_1\rangle = Tr|1\rangle\langle 1|\otimes|u_-\rangle\langle u_-|\tilde{H}. \tag{14}$$

The concrete expression of $\sum(-1)^{i+j+ij}\langle AB\rangle_{ij}$ is

$$\begin{aligned}&\sum(-1)^{i+j+ij}\langle AB\rangle_{ij}\\ =\ & Tr[|0\rangle\langle 0|\otimes I_2+|1\rangle\langle 1|\otimes 2uv(|0\rangle\langle 1|+|1\rangle\langle 0|)]\tilde{H}\end{aligned} \tag{15}$$

Direct calculations show that the absolute value is

$$\begin{aligned}&= Tr|0\rangle\langle 0|\otimes\Lambda+Tr|1\rangle\langle 1|\otimes 2uv(|0\rangle\langle 1|+|1\rangle\langle 0|)\Lambda\\ &= 2E_0+2uv\omega_0\cos\alpha\end{aligned} \tag{16}$$

Since $\omega_0=2s\cos\alpha$, we have

$$|2uv\omega_0\cos\alpha|\leq|2s\cos^2\alpha| \tag{17}$$

The identity holds if and only if $u=v=\pm\frac{1}{\sqrt{2}}$. That is, the boundary can be reached for a maximally entangled state.

As $\alpha\to\frac{\pi}{2}$, Eq. (16) tends to $2E_0$. As $\alpha\to 0$, Eq. (16) tends to $2E_0\pm 2s$. It means that the unbroken $\mathcal{PT}$-symmetric Hamiltonians can be viewed as a intermediate case between the Hermitian and broken $\mathcal{PT}$- symmetric case.

Now we consider how about the boundary of local Hermitian Hamiltonians. Note that $\tilde{H}$ has a two folded structure. Hence it is reasonable to take the global Hamiltonian $\tilde{H}'$ as $I\otimes H_h$, a tensor product of two local Hamiltonians, where $H_h=\lambda_+|s_+\rangle\langle s_+|+\lambda_-|s_-\rangle\langle s_-|$ and $|s_\pm\rangle$ are two orthogonal states.

Similar to the above, replace $\tilde{H}$ with $\tilde{H}'$ in Eqs. (11-14), we have the expectation as follows

$$\begin{aligned}&\sum(-1)^{i+j+ij}\langle AB\rangle_{ij}\\ =\ & Tr(I\otimes|u_+\rangle\langle u_+|)(I\otimes H_h)\\ +\ & Tr[(|0\rangle\langle 0|-|1\rangle\langle 1|)\otimes|u_-\rangle\langle u_-|](I\otimes H_h)\end{aligned}$$

which will reduce to

$$2\langle u_+|H_h|u_+\rangle=2\lambda_+|\langle u_+|s_+\rangle|^2+2\lambda_-|\langle u_+|s_-\rangle|^2. \tag{18}$$

Note that $\lambda_\pm=E_0\pm\frac{\omega_0}{2}$ and denote $p_\pm=|\langle u_+|s_\pm\rangle|^2$. We have

$$2E_0+\omega_0(p_+-p_-), \tag{19}$$

which shows that the classical and local quantum boundaries coincide. Compare Eq.(19) with Eq.(16) and Eq.(10). All the three values contain two terms, one is $2E_0$, which is the sum of the two operators λ_+ and λ_-, and the other is a perturbation term. The perturbation terms of classical and local Hamiltonian models are the same and we have,

$$|\omega_0(p_+-p_-)|=|2s(p_+-p_-)\cos\alpha|\leq|2s\cos\alpha|, \tag{20}$$

which means that they have larger bound than the simulation case of Eq. (17).

IV. DISCUSSIONS

Now we further explain the physical implications behind our results.

It should be first noted that our scenario is essentially different from other nonlocality discussions like CHSH scenario. In CHSH scenario, the two observers share some entangled state and make local measurements to explore their correlations. In our discussions, the resource of correlations is the Hermitian dilation Hamiltonian rather than states. This also leads to other subtle differences. In CHSH scenario, the observers do make several local measurements, for example, Alice can measure the spin in X or Z directions. However, in our case, Alice make two "local measurements" with two orthogonal local states $|0\rangle$ and $|1\rangle$. In the usual sense, two states can only represent one measurement rather than two. However, our randomness and correlations come from the global Hamiltonian. Hence Alice and Bob can obtain "measurement results" simply by inputting different states, reaching similar effect of measurements in CHSH scenario. The most significant distinction between our discussions and CHSH's is that our scenario is concretely constructed and logically derived by a priori rule, which reflects the natural ideas and requirements in simulations of $\mathcal{PT}$-symmetric systems. This also explains why the measurement results are a posteriori determined by the rule even for the classical case, while in CHSH case they are a priori known. And the reason for taking such a way is that what we want to discuss is the internal correlations of the Hamiltonian.

This also explains why the classical and local Hamiltonian models have the same boundaries. Our scenario is constructed according to the rule that "the measurements on the large space are used to simulate the measurements on the subspace", which can actually be viewed as a common property of the local Hamiltonian and Hermitian dilation model. Hence the classical model reflects some merits of the other two models. In addition, its constructions implies some correlation that when considering Alice and Bob as an entity, the result of the measurement only depends on one side. This is a property of measurements of the global system, making it suitable for discussing the other two models. Obtaining the same results means that the boundaries of the local Hamiltonian model can be determined by using the correlation behaviour in the classical model. In this sense, the classical model gives some concrete interpretation to the local Hamiltonian.

Another problem is that the classical and local Hamiltonian model has a larger range than the simulation case. At first sight, it is unreasonable as the latter has some correlations. For example, in the discussions like CHSH or network nonlocality, the nonlocal correlations usually renders a larger range, while the correlations renders a smaller range in our discussion. But in fact, this might be a natural result. Note that our scenario is based on the rule that utilizing $\tilde{H}$ to simulate H, and the measurements of $\tilde{H}$ to simulate measurements of H. Hence the expectation in Eqs. (10), (16) and (19) are all essentially characterizing the average departure form the mean value $2E_0$ in the measuring process. The internal correlations in the Hermitian dilation Hamiltonian impose some constraints on the system and thus the departure is limited as opposed to the local model. At the exceptional point, the two eigenvalues and eigenvectors coalescence, thus the it is reasonable that the average departure is limited to the lowest extent.

From the perspective of simulation, Eq. (16) gives some description of the range in which the $\mathcal{PT}$-symmetric systems can be simulated and the range in which the local Hamiltonians that can be used for simulation. Note that the two folded structure of $\tilde{H}$ implies that the simulation of $\mathcal{PT}$-symmetric systems can be viewed as the effect of a local Hamiltonian and some internal correlations between different spaces. It implies that $\mathcal{PT}$-symmetric systems can be simulated by adding resource to the local Hamiltonians. Hence it might be also reasonable that the range of simulation model is smaller than that of local Hamiltonians on average.

The Eqs.(1)-(4) can hold in generic finite dimensional spaces. Hence a two folded structure can also exist in the simulation, making it possible to generalize the discussions in this paper to any finite dimensional case. In fact, Hamiltonians with such a structure is also the most typical one for simulation, which is easier to prepare and has clear physical meaning. That is why such Hermitian dilation Hamiltonians are of specific interests and taken as examples for illustration in this paper.

There are some works regarding CHSH inequality of $\mathcal{PT}$-symmetric quantum systems, considering a $\mathcal{PT}$-symmetric system interacting with other systems and a transferring to the η or $\mathcal{CPT}$ inner product [22]. Such works differ from motivations to the concrete scenario and discussions. Hence that is a completely different problem in essence.

In summary, we propose a new way to explore the properties of $\mathcal{PT}$-symmetric systems, by constructing a concrete scenario. Such a scenario is consistent with the simulation process and give some method to show the internal correlations of the Hermitian dilation Hamiltonians and the simulation. The range of different models clearly shows the departure of $\mathcal{PT}$-symmetric systems from classical and Hermitian quantum systems. Interestingly, the classical model also gives interpretations to local Hamiltonians. The extremal property of the exceptional point is obtained in the simulation model, which might help in other problems. $\mathcal{PT}$-symmetric systems are governed by their Hamiltonians, such a new way to discuss $\mathcal{PT}$ and their Hermitian dilation Hamiltonians is natural and believed to be instructive for future research.

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