EE 2015

(Partial) Differential Equations and Complex Variables

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Course Description:

★ **Time:** T5T6R5R6 (1:10PM-3:00PM, Tuesday and Thursday)

★ This course is one of "Engineering (Applied) Mathematics":

   ‣ Vector Calculus (required), [Textbook] PART B
   ‣ Linear Algebra (required), [Textbook] PART B
   ‣ Ordinary Differential Equations, ODEs, [Textbook] PART A
   ‣ Partial Differential Equations, PDEs, [Textbook] PART C
   ‣ Fourier Analysis (moved to "Signals and Systems," required)
   ‣ Complex Analysis, [Textbook] PART D
   ‣ Numeric Analysis, [Textbook] PART E
   ‣ Optimization and Graphs, [Textbook] PART F
   ‣ Probability and Statistics (required), [Textbook] PART G

★ **3 Credits, but 4 Hours**
★ No background, but "Calculus" is required.

★ Teaching Method: in-class lectures with examples.

★ Textbook:


★ Office hours: T78R78 at R523, EECS bldg.
Syllabus:

★ Course description and Introduction, 9/14

1. **Ordinary Differential Equations: 4 weeks**
   - First-order ODEs, Ch. 1: 9/16, 9/21
   - Second-order ODEs, Ch. 2: 9/23, 9/28, 9/30, 10/5, 10/7
   - Higher-order ODEs, Ch. 3: 10/12
   - Systems of ODES, Ch. 4: 10/14
   - 1st EXAM, 10/15 (Friday night)

2. **Transform Methods: 4 weeks**
   - Laplace Transforms, Ch. 6: 10/19 - 11/11
   - 2nd EXAM, 11/12 (Friday night)

3. **Series and Complex Variables: 9 weeks**
   - Power Series, Ch. 5: 11/16, 11/18
   - Fourier Series, Ch. 11: 11/23, 11/25, 11/30, 12/2
   - 3rd EXAM, 12/3 (Friday night)
   - PDE by Fourier Series, Ch. 12, 12/7 - 12/22
   - 4th EXAM, 12/23 (in Class)
   - Taylor and Laurent Series, Ch. 13-16: 12/28, 12/30
   - Complex and Residue Integrations, Ch. 16: 1/4 - 1/13
   - 5th EXAM, 1/14 (Friday night)
Evaluation:

1. **Homework**: 30%
2. **EXAMS**: 70%
   - 1st EXAM: 20%, **10/15** (Friday night)
     Ordinary Differential Equations, [Textbook] Ch.1 - Ch. 4
   - 2nd EXAM: 15%, **11/12** (Friday night)
     Laplace Transforms, [Textbook] Ch. 6
   - 3rd EXAM: 10%, **12/3** (Friday night)
     Power and Fourier Series, [Textbook] Ch. 5, Ch. 11
   - 4th EXAM: 10%, **12/23** (in class)
     Partial Differential Equations, [Textbook] Ch.12
   - 5th EXAM: 15%, **1/14** (Friday night)
     Complex Variables, [Textbook] Ch. 13 - Ch. 16
3. **Bonus**: 10%
   - Quiz and Questions in the classroom

3 Credits = 17 Weeks*4 Hours + Homework (>12) + 5 EXAMS + .......
Engineering (Applied) Mathematics

1. Modeling
2. Solving
3. Interpretation

- Analytical approach
- Numerical approach
Maxwell’s equations:

- Gauss’s law for the electric field:

\[ \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \iff \oint_S \mathbf{E} \cdot d\mathbf{A} = \frac{q}{\varepsilon_0}, \]

- Gauss’s law for magnetism:

\[ \nabla \cdot \mathbf{B} = 0 \iff \oint_S \mathbf{B} \cdot d\mathbf{A} = 0, \]

- Faraday’s law of induction:

\[ \nabla \times \mathbf{E} = -\frac{\partial}{\partial t} \mathbf{B} \iff \oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial}{\partial t} \Phi_B, \]

- Ampère’s circuital law:

\[ \nabla \times \mathbf{B} = \mu_0 (\mathbf{J} + \frac{\partial}{\partial t} \mathbf{D}) \iff \oint_C \mathbf{B} \cdot d\mathbf{l} = -\mu_0 (\mathbf{I} + \frac{\partial}{\partial t} \Phi_D) \]
QUIZ: Differential or Integral Equations?

<table>
<thead>
<tr>
<th>Differential</th>
<th>Integral</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{d}{dx} f(x) )</td>
<td>( \int f(x), dx )</td>
</tr>
<tr>
<td>Change Rate</td>
<td>Total Sum</td>
</tr>
<tr>
<td>Local Information</td>
<td>Global Information</td>
</tr>
<tr>
<td>Initial Value Problem</td>
<td>Boundary Condition</td>
</tr>
</tbody>
</table>
Newton’s mechanics: $m \ddot{a} = \vec{F}$, i.e.,

$$\frac{dv_y}{dt} = -g,$$

$$v_y(t) = v_0 - gt.$$
Solving: Projectile motion without Air Resistance

\[ \frac{dv_y}{dt} = -g \quad \Rightarrow \quad v_y(t) = v_0 \sin \theta - gt, \]
\[ \frac{dv_x}{dt} = 0 \quad \Rightarrow \quad v_x(t) = v_0 \cos \theta, \]

- **Analytically approach:**

\[ x(t) = \int_0^t v_x(t) \, dt = x(0) + v_0 \cos \theta \, t \]
\[ y(t) = \int_0^t v_y(t) \, dt = y(0) + v_0 \sin \theta \, t - \frac{1}{2} gt^2, \]

- **Numerical approach:**
  - Solve Differential Eq.: Finite-Difference, Finite-Element, ...
  - Solve Integral Eq.: Finite-Volume, Moment methods, ...
Assumption 1: that there exists an air drag force,

\[
\begin{align*}
    m \frac{dv_x}{dt} &= -|\vec{F}_x^D(v_x, v_y, t)|, \\
    m \frac{dv_y}{dt} &= -mg - |\vec{F}_y^D(v_x, v_y, t)|.
\end{align*}
\]
Modeling: Projectile motion with Air Resistance, cont.

Assumption 2:

- Assume that the magnitude of the air drag force $\vec{F}^D$ is approximately proportional to the square of the projectile’s speed relative to the air, i.e., $|\vec{F}^D| \approx v^2$, or

$$\vec{F}^D \equiv C |\vec{v}| \vec{v} = C \sqrt{v_x^2 + v_y^2} \vec{v},$$

where the constant $C$ depends on the density $\rho$ of air, the silhouette area $A$ of the body (its area as seen from the front), and a dimensionless constant $C_d$ called the drag coefficient that depends on the shape of the body, i.e., $C = C_d \rho A$.

\[
\frac{dv_x}{dt} = -\frac{C}{m} |\vec{v}| v_x = -\frac{C}{m} \sqrt{v_x^2 + v_y^2} v_x,
\]
\[
\frac{dv_y}{dt} = -g - \frac{C}{m} |\vec{v}| v_y = -\frac{C}{m} \sqrt{v_x^2 + v_y^2} v_y,
\]
**Modeling: Projectile motion with Air Resistance, cont.**

**Assumption 2:**

- **Model 1:** \( \frac{dv}{dt} = -\frac{C}{m}v^2, \)
- **Model 2:** \( \frac{dv}{dt} = -\frac{C}{m}v, \)
- **Model 3:** \( \frac{dv}{dt} = -\frac{C}{m} \sqrt{v}, \)
- **Model 4:** \( \frac{dv}{dt} = -\frac{C(t)}{m}f(v), \)

**QUIZ:** Which model supports the longest projectile motion distance?
Interpretation: Projectile motion with drag

QUIZ: The projectile angle to support a farthest projectile motion is the same as the case without a drag resistance?
Terminologies:

**ONE independent variable**

\[ y'(x) \equiv \frac{d}{dx} y(x) \]

**More than ONE independent variable**

\[ f_x \equiv \frac{\partial}{\partial x} f(x, y, \ldots) \]

**Ordinary Differential Equation, ODE**

Ch. 1 - 5

**Partial Differential Equation, PDE**

Ch. 12
First-order ODEs: Order

- If the \( n \)th derivative \( y^{(n)} = \frac{d^n y}{d x^n} \) of the unknown function \( y(x) \) is the highest occurring derivative, it is called an ODE of \( n \)th-order:

  \[
  F(x, y, y', \ldots, y^{(n)}) = 0, \quad \text{where} \quad y^{(n)} = \frac{d^n y}{d x^n},
  \]

- Linear \( n \)th-order ODE:

  \[
  y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = r(x),
  \]

- Explicit form:

  \[
  y' = f(x, y)
  \]

- Implicit form:

  \[
  F(x, y, y') = 0
  \]
Family of solutions:

\[ y' = \frac{dy}{dt} = \pm \gamma y, \]

- General solutions:
  \[ y(t) = ce^{\pm \gamma t}, \]
  \( \pm \gamma \) denotes the growth/decay rate.
  \( c \) is an arbitrary constant.
  \( y(t) = ce^{\pm \gamma t} \) is a family of solutions.

- Initial value problem, \( y(t = 0) \) is given.
- Boundary value problem, \( y(t_1) \) is given.
Family of solutions: Exponential Decay

\[ y' = -0.2y \]
First-order ODEs: Separable equations

\[ y' = f(x, y) = f_1(x) f_2(y), \quad \text{or equivalently} \]

\[ g(y) \, dy = f(x) \, dx, \quad \Rightarrow \quad \int_{y_0}^{y} g(y_1) \, dy_1 = \int_{x_0}^{x} f(x_1) \, dx_1. \]

Example:

\[ y' = 1 + y^2 \]

Hint:

Integrals:

\[ \int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \]

\[ \int \frac{1}{a^2 - x^2} \, dx = \frac{1}{a} \tanh^{-1} \frac{x}{a}, \]

Solution:

\[ y = \tan(x + c) \quad \text{or} \quad y = \tan x + c \]
First-order ODEs: Reducible to Separable Form

Example:

\[ 2xyy' = y^2 - x^2 \]

Hints:

1. Divide the given equation by \( 2xy \).
2. Define the new variable \( u \equiv \frac{y}{x} \), then reduce the Eq. into a separable form.

Solution:

\[ x^2 + y^2 = cx \]

Integrals:

\[
\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a},
\]

\[
\int \frac{x}{a^2 + x^2} \, dx = \frac{1}{2} \ln(a^2 + x^2),
\]
1. (25%) Solve

\[ yy' = (x - 1)e^{-y^2}, \quad y(0) = 1. \]  \hspace{1cm} (1)

2. (25%) Solve

\[ y' = \frac{2\sqrt{xy} - y}{x}, \quad \text{Hint: try } y = ux. \]  \hspace{1cm} (2)

3. (25%) Solve

\[ (\cos x \sin x - xy^2) \, dx + y(1 - x^2) \, dy = 0, \quad y(0) = 2. \]  \hspace{1cm} (3)

4. (25%) Show that any equation which is separable, that is, of the form:

\[ M(x) + N(y)y' = 0, \]

is also exact.
Homework #1:

1. Please do the homework Yourself!!

2. Homework is designed for your PRACTICE, Take It Easy ^.^

3. If you have any questions, please write an email to me or come to my office.

5. Please return the Homework by the Deadline:

Deadline Sep. 21 (next Tuesday), 1:00PM before the class!!
First-order ODEs: Exact Eq.

- Explicit form for a 1st-order ODE: \( y' = f(x, y) = -\frac{M(x, y)}{N(x, y)} \).
- Re-write 1st-order ODE:

\[
M(x, y) \, dx + N(x, y) \, dy = 0.
\]

The necessary and sufficient condition to have an exact differential equation is

\[
\frac{\partial^2 u(x, y)}{\partial x \partial y} = \frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x}.
\]

- If there is a function \( u(x, y) = c \), then the total differential of \( u(x, y) \) is

\[
du(x, y) = \frac{\partial u(x, y)}{\partial x} \, dx + \frac{\partial u(x, y)}{\partial y} \, dy = 0.
\]
Geometric meaning: Direction Field

\[ y' = xy \]
First-order ODEs: Exact Equation

- One can rewrite \( y' = f(x, y) \) as
  \[
  M(x, y) \, dx + N(x, y) \, dy = 0.
  \]

- If there is a function \( u(x, y) = c \), then the total differential of \( u(x, y) \) is
  \[
  du(x, y) = \frac{\partial u(x, y)}{\partial x} \, dx + \frac{\partial u(x, y)}{\partial y} \, dy = 0
  \]

- The necessary and sufficient condition to have an exact differential equation is
  \[
  \frac{\partial^2 u(x, y)}{\partial x \partial y} = \frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x}.
  \]

- Integrate \( M(x, y) \) or \( N(x, y) \) to get,
  \[
  u(x, y) = \int M(x, y) \, dx + k(y), \quad \text{or}\quad u(x, y) = \int N(x, y) \, dy + h(x).
  \]
First-order ODEs: Exact Equation, Example

Example:

\[ \cos(x + y) \, dx + \left[ 3y^2 + 2y + \cos(x + y) \right] \, dy = 0 \]

Hints:

1. Test for exactness.
2. Integrate \(dx\) then \(dy\), or Integrate \(dy\) then \(dx\).

Solution:

\[ u(x, y) = \sin(x + y) + y^3 + y^2 = c \]
First-order ODEs: Non-Exact Equations

- For the explicit form,

\[ M(x, y) \, dx + N(x, y) \, dy = 0, \]

- If the test for exactness fails, i.e.,

\[ \frac{\partial M(x, y)}{\partial y} \neq \frac{\partial N(x, y)}{\partial x}. \]
First-order ODEs: Integrating Factor

- Multiply a given non-exact equation by a function $F(x, y)$, an integrating factor,

$$F(x, y)M(x, y) \, dx + F(x, y)N(x, y) \, dy = 0,$$

- To result in an exact equation:

$$\frac{\partial}{\partial y} M(x, y) = \frac{\partial}{\partial x} N(x, y),$$

$$\frac{\partial}{\partial x} F(x, y)M(x, y)$$

**QUIZ:** Is there always an Integrating Factor to find?

- We can choose

$$F(x) = \exp \left[ \int \frac{1}{N} \left( \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \, dx \right]$$

or

$$F(y) = \exp \left[ \int \frac{1}{M} \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dy \right].$$
First-order ODEs: Integrating Factor

Example: 

\[(e^{x+y} + ye^y) \, dx + (xe^y - 1) \, dy = 0, \quad \text{with} \quad y(0) = -1\]

Hints:

- Find the Integrating factor,

\[F(x) = \exp \int \left[ \frac{1}{xey - 1} \left( e^{x+y} + e^y + ye^y - e^y \right) \right] \, dx, \quad \text{fails} \]

\[F(y) = \exp \int [-1] \, dy = e^{-y}. \]

Solution:

\[u(x, y) = e^x + xy + e^{-y} = 1 + e. \]
First-order ODEs: Linear ODEs

- Linear ODE:
  \[ y' + p(x)y = 0, \quad \text{homogeneous} \]
  \[ y' + p(x)y = r(x), \quad \text{non-homogeneous} \]

- \( y(x) = 0 \) is the \textit{trivial solution} for the homogeneous ODEs.

- Non-linear ODE:

**QUIZ:** Which one is a linear ODE?

- \( \Box \) \( y' + 3x^2y = 0, \)
- \( \Box \) \( y' + 3x^2y = 5 \cos(x^2), \)
- \( \Box \) \( y'^2 + 3x^2y = 5 \cos(x^2), \)
- \( \Box \) \( y' + 3x^2 \sin(y) = 5 \cos(x^2). \)
QUIZ: Why Linear Systems are so important?

1. Basis.
2. Vector space.

› Linear Algebra.
› Signals and Systems.
QUIZ: Does superposition principle apply to?

1. Homogeneous \textbf{Linear} ODEs?
2. Non-homogeneous \textbf{Linear} ODEs?
3. Non-linear ODEs?

$1 + 1 \neq 2$
First-order ODEs: Nonhomogeneous & Linear

- Non-homogeneous Linear ODE:
  \[ y' + p(x)y = r(x), \]

- The *general solution* of the homogeneous ODE is
  \[ y(x) = ce^{-\int p(x)\,dx} \equiv ce^{-h(x)}, \]

- The solution for the non-homogeneous ODE is
  \[
y(x) = e^{-h(x)} \int e^{h(x_1)} r(x_1) \, dx_1 + ce^{-h(x)},
  \]
  \[
  = \text{non-homogeneous solution} + \text{homogeneous solution}
  \]

Total Output = Response to the Input + Response to the Initial Data.
Let $y(t)$ be the hormone level at time $t$.

The removal rate is $Ky(t)$.

The input rate is $A + B \cos(2\pi t/24)$, where $A$ is the average input rate and $B$ is the amplitude of a sinusoidal input with a 24-hour period.

Modeling:

$$y' = -Ky + A + B \cos\left(\frac{1}{12}\pi t\right),$$

The initial condition for a particular solution is given by $y(t = 0) = y_0$. 

First-order ODEs: Hormone Level Problem
Modeling:

\[ y' = -Ky + A + B \cos\left(\frac{1}{12}\pi t\right), \]

Solving:

\[ y(t) = e^{-Kt} \int e^{Kt_1} [A + B \cos\left(\frac{\pi t_1}{12}\right)] \, dt_1 + ce^{-Kt}, \]

\[ = \frac{A}{K} + \frac{B}{144K^2 + \pi^2} \left[ 144K \cos\left(\frac{\pi t}{12}\right) + 12\pi \sin\left(\frac{\pi t}{12}\right) \right] + ce^{-Kt}. \]

Steady-State solution.

Entire solution is called Transient-State solution.
### Terminologies:

<table>
<thead>
<tr>
<th>ODE</th>
<th>PDE</th>
<th>number of independent variable</th>
</tr>
</thead>
<tbody>
<tr>
<td>Homogeneous</td>
<td>Non-homogeneous</td>
<td>RHS</td>
</tr>
<tr>
<td>General solution</td>
<td>Particular solution</td>
<td>Special solution</td>
</tr>
<tr>
<td>Linear Eq.</td>
<td>Nonlinear</td>
<td>Superposition</td>
</tr>
<tr>
<td>Exact Eq.</td>
<td>Non-exact Eq.</td>
<td>Exactness</td>
</tr>
<tr>
<td>Steady-state</td>
<td>Transient-state</td>
<td>$t \to \infty$</td>
</tr>
</tbody>
</table>
Homework #2:

1. (25%) Solve the non-exact Eq.:

\[(3xy + y^2) + (x^2 + xy)y' = 0. \tag{1}\]

2. (25%) Solve the non-homogeneous Eq.:

\[x^3y' + 3x^2y = 5 \sinh(10x), \tag{2}\]

Problem 17 in the [Textbook], at p.p. 32.

3. (25%) Solve the Bernoulli’s Eq.:

\[2yy' + y^2 \sin x = \sin x, \quad y(0) = \sqrt{2}, \tag{3}\]

Problem 23 in the [Textbook], at p.p. 33.

4. (25%) Solve a 1st-order ODE by using Richard’s method of iteration:

\[y' = y - 1, \quad y(0) = 2,\]
Homework #2: Richard’s method

1. To solve an initial-value problem:
\[ y' = f(x, y), \quad y(x_0) = y_0. \]

2. Integrate both sides with respect to \( x \) directly, with the initial value, i.e.
\[ y_1(x) = y_0 + \int_{x_0}^{x} f(x, y_0) \, dx, \]

3. Integrate both sides with respect to \( x \) directly again, but updating the value of \( y(x) \)
\[ y_2(x) = y_0 + \int_{x_0}^{x} f(x, y_1) \, dx, \]

4. Then you can find a sequence of functions:
\[ y_1(x), y_2(x), \ldots, y_n(x). \]

5. To the limit of \( y_n(x) \) as \( n \to \infty \), we have the exact solution for the given initial-value problem,
\[ y(x) = 1 + e^x. \]

This approach is called the Picard’s method of iteration.
Homework #2:

1. Please do the homework Yourself!!

2. Homework is designed for your PRACTICE, Take It Easy ^.^

3. If you have any questions, please write an email to me or come to my office.

5. Please return the Homework by the Deadline:

Deadline Sep. 23 (Thursday), 1:00PM before the class!!
Some nonlinear ODEs can be transformed to linear ODEs.

**Bernoulli equation:**

\[ y' + p(x)y = g(x)y^a, \quad a \text{ is a real number.} \]

If \( a = 0 \) or \( a = 1 \), it is linear; otherwise, it is nonlinear.

**Hint:**

\[ u(x) = [y(x)]^{1-a}, \]

The transformed ODE for \( u(x) \) is linear,

\[ u' + (1 - a)p(x)u(x) = (1 - a)g(x). \]
First-order ODEs: Bernoulli Equation, cont.

Example: Logistic Equation (Verhulst Equation)

\[ y' = Ay - By^2, \]

Hints:

Solution:

\[ y(t) = \frac{1}{ce^{-At} + \frac{B}{A}} \]
For *linear* 1st-order ODEs in the *initial value problem*,

\[ y' = f(x, y), \quad y(x_0) = y_0, \]

- \( f(x, y) \) is *continuous* and *bounded* at all points \((x, y)\) in some rectangle,

\[ R : |x - x_0| < a, \quad |y - y_0| < b \]

- there is a number \( K \) such that,

\[ |f(x, y)| \leq K, \quad \text{for all } (x, y) \text{ in } R \]

- Then the *initial value problem* has at least one solution \( y(x) \) in the sub-interval \( |x - x_0| < a(b/K) \).
For linear 1st-order ODEs in the initial value problem,

\[ y' = f(x, y), \quad y(x_0) = y_0, \]

Let \( f(x, y) \) and its partial derivative \( f_y = \partial f / \partial y \) is continuous and bounded at all points \((x, y)\) in some rectangle,

\[
|f(x, y)| \leq K, \\
|f_y(x, y)| \leq M, \quad \text{for all} \ (x, y) \ \text{in} \ R
\]

Then the initial value problem, IVP has at most one solution \( y(x) \).

Combine the Existence and Uniqueness theorems, the IVP has precisely one solution in the sub-interval \( |x - x_0| < \alpha \).
Existence and Uniqueness Theorem, Example 1

**Example:**

\[ y' = x - y + 1, \quad y(1) = 2 \]

**Hints:**

Both \( f(x, y) = x - y + 1 \) and \( f_y(x, y) = -1 \) are defined and continuous at all points \((x, y)\),

**Solution:**

The theorem guarantees a *unique* solution to the ODE exists in some open interval centered at 1.

\[ y(x) = x + ce^{-x}. \]
Existence and Uniqueness Theorem, Example 2

Example: \[ y' = 1 + y^2, \quad y(0) = 0 \]

Hints: Both \( f(x, y) = 1 + y^2 \) and \( f_y(x, y) = 2y \) are defined and continuous at all points \((x, y)\).

Solution: The theorem guarantees a unique solution to the ODE exists in some open interval centered at 0.

\[ y(x) = \tan(x + c), \]

which is defined for all \( x \neq (2n + 1)/\pi, \ n \) is an integer.
Example: \[ y' = \frac{2y}{x}, \quad y(x_0) = y_0 \]

Hints: Both \( f(x, y) = \frac{2y}{x} \) and \( f_y(x, y) = \frac{2}{x} \) are defined and continuous at all points \( x \neq 0 \).

Solution: The theorem guarantees a unique solution to the ODE exists in some open interval centered at \( x_0 \neq 0 \).

\[ y(x) = cx^2, \]

- No solution if \( x_0 = 0 \) and \( y_0 \neq 0 \);
- Infinitely many solutions if \( x_0 = 0 \) and \( y_0 = 0 \).
Homework #0: Projectile motion without Air Resistance

- Find the analytical solutions for the projectile motion,

\[
\begin{align*}
\frac{dv_y}{dt} &= -g, \\
\frac{dv_x}{dt} &= 0,
\end{align*}
\]

with the initial velocity \( v_x = v_0 \cos \theta \) and \( v_y = v_0 \sin \theta \).

- For a constant \( v_0 \), find the projectile angle \( \theta \) that gives the longest projectile distance.

- Based on Finite-Difference method, write a code to test your analytical results.
Finite Difference Approximation

\[
\frac{d}{dx}y(x)_{x=x_j} \approx \frac{y(x_j) - y(x_{j-1})}{x_j - x_{j-1}}
\]

• Taylor’s expansion:

\[
\begin{align*}
    u(x_{j+1}) &= u(x_j) + u'(x_j)\Delta x + \frac{u''(x_j)}{2}(\Delta x)^2 + \frac{u'''(x_j)}{3!}(\Delta x)^3 + \ldots, \\
    u(x_{j-1}) &= u(x_j) - u'(x_j)\Delta x + \frac{u''(x_j)}{2}(\Delta x)^2 - \frac{u'''(x_j)}{3!}(\Delta x)^3 + \ldots,
\end{align*}
\]

• Euler’s 2nd-order FD approximation:

\[
    u'(x_j) = \frac{u(x_{j+1}) - u(x_{j-1})}{2\Delta x} - \frac{u'''(x_j)}{2 \times 3!}(\Delta x)^2 + \ldots,
\]

\[
\approx \frac{u(x_{j+1}) - u(x_{j-1})}{2\Delta x} + O(\Delta x^2),
\]

• 4th-order FD method:
• Runge-Kutta method:
• Differential matrix:
First-order ODEs: Summary

- 1st-order
  - Modeling, Ch. 1.1
  - Direction Fields, Ch. 1.2
  - Separable Eq., Ch. 1.3
  - Exact Eq., Ch. 1.4
  - Integrating Factor, Ch. 1.4
  - Linear ODEs, Ch. 1.5
  - Non-homogeneous sol., Ch. 1.5
  - Bernoulli Eq., Ch. 1.5
  - Orthogonal Trajectory, Ch. 1.6
  - Existence and Uniqueness, Ch. 1.7
  - Numeric methods

- 2nd-order

- Higher-order

- Systems of ODEs